MATHEMATICS

REMARKS ON SOME PROBLEMS IN LINEAR TOPOLOGICAL SPACES OVER FIELDS WITH NON-ARCHIMEDEAN VALUATION

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In these notes I will make some remarks on problems in linear topological spaces over fields with non-archimedean valuation, especially in n.a. normed spaces. They concern mainly metrical problems which are well known in spaces over the real field R and about which there are many results in the literature. They were not considered in the non-archimedean case; a comparative study of the two cases may be interesting. The problems concern (i) questions of approximation (ii) definitions of convexity (iii) spherical completeness.

ξ 1. *Problems of approximation.*

By K I design a field with a non-trivial non-archimedean (n.a.) valuation which is supposed to be complete under the topology derived from the valuation.

1.1. *Best approximation.*

Let E be a n.a. normed space over K. Let $V \subset E$ be a closed linear subspace. For a given $x \in E$, a best approximation of x in V is defined to be an element $\xi \in V$ such that

$$\|x-\xi\| = \inf_{g \in V} \|x-g\|.$$

This is the same definition as in space over R. There are the following problems.

(i) The existence of a best approximation for an arbitrary $x \in E$ for all $V \subset E$ or for a given $V \subset E$.

(ii) The problem of the uniqueness of best approximations.

The solution of the second problem is simple.

Theorem 1. *A best approximation of $x \in E$, $x \notin V$, in V when it exists is never uniquely determined unless $V = \{0\}$, that is to say: V is not a Chebyshev set unless $V = \{0\}$.*

Proof. Let

$$\|x-\xi\| = \inf_{g \in V} \|x-g\| > 0.$$

Then

$$\|x-\eta\|=\|x-\xi\|$$

for all η such that $\|\eta-\xi\|<\|x-\xi\|$. So every $\eta \in V$ satisfying this inequality is also a best approximation.

It is a consequence of this theorem that the problem of best approximation leads essentially to the problem of the existence and then to a study of the structure of the set of best approximations.

In spaces over R, a necessary and sufficient condition for uniqueness of the best approximation is that the norm be strict, that is if $\|x+y\|=\|x\|+\|y\|$, $x\neq 0$ imply $y=tx$ for some $t\geqslant 0$. An equivalent condition is that E be strictly convex, this means that

$$\|x\|=\|y\|=1,\ x\neq y,\ x, y\in E$$

imply

$$\|\lambda x+(1-\lambda)y\|<1 \quad (0<\lambda<1).$$

A similar definition of a strict norm in n.a. normed spaces has evidently no sense.

If strict convexity of a space E over K is defined in an analogous way, replacing the condition $0<\lambda<1$ by $0<|\lambda|<1$, as is reasonable by the definition of convexity in such spaces, it must be remarked that

$$\|\lambda x+(1-\lambda)y\|=1$$

for all λ such that $0<|\lambda|<1$.

In this way no space E should be strictly convex; one could say that this is in agreement with theorem 1. The difficulties of these definitions are related to the problems concerning the boundary of convex sets in n.a. spaces.

1.2. With regard to the existence of best approximation there is the following theorem (for a proof see [9]):

Theorem 2. *If V is spherically complete every $x\in E$ has a best approximation in V.*

Corollary. *Suppose K is spherically complete. Then there is a best approximation in every finite dimensional subspace.*

As is well known, reflexivity of the space is a sufficient condition for the existence of best approximations on every closed subspace for the class of normed spaces over R. N.a. normed spaces over a spherically complete field K are never reflexive unless they are finite dimensional. Thus, for the existence problem, considered for the general case, reflexivity is not a necessary condition.

Spherical completeness of the field K is not necessary. A counterexample can be found in a paper of GERRITZEN and GÜNTZER [3]. This is remarkable since, in the problem of existence of best approximation operators, the theorem of Hahn-Banach for operators plays a role which is valid only under conditions on spherical completeness.

As in the real case, there is a relation between the problem of existence of orthogonal projections on V and the existence of best approximations. For orthogonality in n.a. spaces see [9]. An orthogonal projection P in a space E over K is defined to be a linear continuous operator P such that Px is, for all $x \in E$, a uniquely determined best approximation of x in a given closed linear subspace V. Earlier I proved

(i) Let T be an idempotent linear continuous operator in E with norm 1; then T is an orthogonal projection;

(ii) Let V be a closed spherically complete linear subspace. Then there exist orthogonal projections on V.

Compare the work of HIRSCHFELD [6] concerning approximation operators in the real case (characterizations of the Hilbert space).

Concerning the existence of best approximations, one can put the following problem.

Let V be a closed subspace of E. Let $x_0 \in E$, $x_0 \notin V$. Suppose best approximations of x_0 in V exist.

It is easy to prove that then every element of the linear space, generated by V and x_0 has a best approximation in V.

Problem 1. *Under these conditions, does every $x \in E$ have a best approximation in V?*

1.3. We give some theorems on the structure of the set of best approximations.

Theorem 3. *Let E be a n.a. normed space over K. Let $V \subset E$ be a closed subspace. Suppose the set G_x of best approximations of x is not empty. Then G_x is K-convex.*

Proof. We can suppose that $0 \in G_x$ (apply a translation). We can take evidently $x \notin V$. Let $\xi, \eta \in G_x$; this means

$$\|x-\xi\| = \|x-\eta\| = \inf_{g \in V} \|x-g\| = d > 0.$$

Then

$$\|x-\lambda\xi-(1-\lambda)\eta\| \leqq \max(\|x-\eta\|, |\lambda| \cdot \|\xi-\eta\|) \leqq d$$

for all $|\lambda| \leqq 1$ because

$$\|\xi-\eta\| \leqq \max(\|\xi-x\|, \|\eta-x\|) \leqq d.$$

Now, it follows from the definition of d that it is not possible that $\|x-\lambda\xi-(1-\lambda)\eta\| < d$. This means that $\lambda\xi+(1-\lambda)\eta \in G_x$ and G_x is K-convex.

Theorem 4. *Let G_x be the set of theorem* 3. *Let $\xi \in G_x$. Then*

(i) *$G_x - \xi$ is absorbing in V.*
(ii) *$G_x - \xi$ is bounded,*
(iii) *$G_x - \xi$ is circled.*
(iv) *$G_{ax} = aG_x$ for all $a \in K$.*

Proof. The proof of (i), (ii) and (iv) is rather trivial (compare the proof of theorem 1).

(iii) Suppose again that $0 \in G_x$. Let $\xi \in G_x$, this means $\|x-\xi\| = \inf \|x-g\|$. Because $0 \in G_x$, one has $\|x-\xi\| = \|x\|$ and $\|\xi\| \leqq \|x-\xi\|$. Then

$$\|x-\lambda\xi\| = \|x-\xi+(1-\lambda)\xi\| \leqq \max(\|x-\xi\|, |1-\lambda|\cdot\|\xi\|) \leqq \|x-\xi\|$$

for all $|\lambda| \leqq 1$. Because $\|x-\xi\|$ is the shortest distance to V it follows $\|x-\lambda\xi\| = \|x-\xi\|$, this means $\lambda\xi \in G_x$.

Now we have the following problem.

Problem 2. *Characterize those convex sets in a closed linear subspace $V \subset E$ which are the set of best approximations of an element $x \in E$.*

I mention for instance: can G_{x+y} be expressed in terms of G_x and G_y?

It seems that this problem is related mainly to those properties of G_x which are invariant under norm-isomorphisms.

1.4. The notion of best approximation can be defined in the same way with regard to any non-empty closed subset $A \subset E$. It is evident how the notion of spherical completeness has to be defined in this case.

Definition. *The closed set A is called spherically complete when the following condition is satisfied: let $B_1 \supset B_2 \supset \ldots$ be a sequence of spheres in E; then $\cap B_i \cap A \neq \emptyset$.*

Theorem 2 remains true in this general case.

1.5. The fact that n.a. normed spaces are not reflexive and that, however, reflexive locally K-convex spaces over K exist, leads to the question whether it is possible to define the notion of best approximation in locally K-convex spaces. One could think of a definition with the aid of the n.a. semi-norms which define the topology.

1.6. Grothendieck [5] introduced in real Banach spaces a notion which he calls the *metrical approximation property* (M.A.P.). The definition remains the same in n.a. normed spaces over a field K:

Definition. *Let E be a n.a. normed space over K. Then E has* M.A.P. *if for every finite set $\{x_i\}$ $1 \leqq i \leqq n$, $x_i \in E$, and every $\varepsilon > 0$ there is a linear operator T with norm* 1 *from E into a finite dimensional linear subspace of E such that*

$$\|Tx_i - x_i\| \leqq \varepsilon \text{ for } 1 \leqq i \leqq n.$$

This property becomes now rather trivial in our case.

Theorem 5. *Suppose K is spherically complete. Then every space E has the* M.A.P.

Proof. Let V be the closed subspace, generated by $\{x_i\}$. Then V is spherically complete. So there is a projection T with norm 1 on V. This projection satisfies the condition. Even, $Tx_i = x_i$, $1 \leqq i \leqq n$.

Thus, one has

Corollary. *There exist non-reflexive normed spaces which have* M.A.P.

In the literature about the M.A.P. reflexivity plays a role (see for instance Lindenstrauss [8[). So there is reason to study a similar notion in locally K-convex spaces, among which there are spaces which are reflexive. Compare a topological approximation property, introduced by Grothendieck, where the metrical inequalities are replaced by relations of the form $Tx - x \in U$, where U are neighborhoods of 0.

1.7. In discussing approximation problems in spaces over a n.a. valued field, the literature about the general non-metric approximations in topological spaces over R, leading to theorems on the existence of a basis, must be mentioned. In n.a. normed spaces over a field K, the problem of basis has found a solution in the theory of orthogonal families of elements in such a space.

It seems worthwhile to study this problem in topological or locally K-convex spaces over a field K. Is it possible to generalize orthogonal families to these spaces? See for instance [14] and [15].

§ 2. *Definitions of convexity.*

Let again K be a field with a non-trivial n.a. valuation; K is complete. I make some remarks about definitions of convexity in linear topological spaces over K.

There is a definition of so called K-convex sets in linear spaces over K [10]. This definition is valid in spaces without any topology; only the valuation of K is used. An ordering of the field is not used.

The next step is to bring more structure in the space by means of defining a topology. In particular one considers locally K-convex spaces. This gives then properties of K-convex sets which are invariant under topological isomorphisms; one gets for instance open or closed K-convex sets. Then there are already difficulties, because in many cases K-convex sets are open as well as closed; so there are difficulties in defining notions where – in analogy to the case of spaces over R – the notion of boundary seems to play an essential role.

A further step is to bring still more structure in the space, namely by considering K-convex sets in n.a. normed spaces. One should expect that in these spaces – which have a richer structure – K-convex sets will have properties which are invariant under norm-isomorphisms. These will be essentially metrical properties.

As is well known this leads in spaces over R to refinements of the notion of convexity, such as strict convexity, uniform convexity and so on. In these notions the boundary is important, for instance the boundary $\{x| \, \|x\|=1\}$ of the unit sphere $\{x| \, \|x\|\leqslant 1\}$.

Unfortunately in spaces over K, the boundary will in general be empty. Because of this there is only very little progress in the study of metrical properties of K-convex sets in these spaces; in § 1 I mentioned already these difficulties. So there is the following problem.

Problem 3. *Is it possible to give properties of K-convex sets in n.a. normed spaces which are analogous to the well known refinements in real spaces?*

I can only make some suggestions about this problem.

2.1. In a n.a. normed space every point of a sphere can be taken as centre of the sphere. This led me to define in a previous paper [11] the notion of centered K-convex set as a pair (S, x) of a K-convex set S and a fixed point $x \in S$, to be considered as centre. By means of this point the K-convex set obtains more structure. It is then possible to define a notion of "boundary" of such a pair (S, x) with metrical methods.

The question can be put whether it is possible to define extremal points, by means of a good choice of the centre?

2.2. The question naturally rises whether the definition of a K-convex set, such as was given hitherto [10], is a good one for the questions I mentioned or whether other definitions should be tried. It is naturally therefore to compare the situation with the various definitions for convexity in spaces over R.

2.2.1. *Quasi-convexity.*

Generalizing a well known definition in spaces over R, one might try the following definition.

Let J be a subset of the ring of integers I of K; E is a vector space over K.

Definition. *A set $S \subset E$ is called quasi-convex with respect to J if*

$$x, y \in S \Rightarrow \lambda x + (1-\lambda)\, y \in S \text{ for all } \lambda \in J.$$

When $J=I$ and the residue field $k \neq F_2$ (the field with 2 elements), this definition agrees with the previous definition; if $J=I$ and $k=F_2$ the definitions are not equivalent unless dim $E=1$.

In *real* vector spaces, *midpoint convexity* is a special case of the analogous definition; this means that S is called convex if $x, y \in S \Rightarrow \frac{1}{2}\,(x+y) \in S$.

Now in spaces over K, one can take for instance for J a set, consisting of one fixed element λ_0 with $|\lambda_0| \leqq 1$. It seems that $|\lambda_0| < 1$ will give the same difficulties concerning the boundary; so perhaps one should take $|\lambda_0|=1$.

For quasi-convexity see [4], treating spaces over R, where also relations of midpoint convexity and solutions of the functional equation $\phi: E \to E$, $\phi(x+y)=\phi(x)+\phi(y)$, are studied.

2.2.2. A generalization of midpoint convexity might perhaps be obtaîned in the following way.

Let T be a mapping from $E \times E$ into E. This mapping shall have to satisfy suitable chosen axioms.

A set $S \subset E$ is then called T-convex if $x, y \in S$ imply $T(x, y) \in S$.

The following axioms seem reasonable:

1. $T(x, x)=x$
2. $T(x, y) \neq T(x, z)$ for all $x \in E$ if $y \neq z$
   $T(x, y) \neq T(z, y)$ for all $y \in E$ if $x \neq z$
3. $T(x+z, y+z)=z+T(x, y)$ for all $z \in E$
4. $T(ax, ay)=aT(x, y)$ for all $a \in K$.

Perhaps it is useful to add an axiom of symmetry:

$$T(x, y)=T(y, x).$$

Besides of these general axioms it will be necessary to put some axioms concerning continuity or boundedness on T. I suggest an axiom which can be formulated in n.a. normed spaces over K, namely

5. $\|T(x, y)\| \leqslant \varrho \max (\|x\|, \|y\|)$.

Put

$$\|T\|= \inf \varrho.$$

Axiom 1 implies then $\|T\| \geqq 1$.

It seems that mappings such that $\|T\|>1$ will not give a good definition because in this case there seem to be too many unbounded T-convex sets; the iteration process may be divergent in this case (perhaps all T-convex sets are unbounded when $\|T\|>1$).

When convexity is defined in this way, simple properties which hold by the usual definition remain true. For instance:

a. The intersection of T-convex sets is T-convex.
b. Invariance of T-convexity by translations and homothetic transformation.
c. Existence of the T-convex hull CA of a given set A.

I give some examples.

(i) The sphere $\{x \mid \|x\| \leqq 1\}$ is T-convex for all T with $\|T\|=1$.

(ii) Take for K a p-adic field with $p \neq 2$. Then the mapping

$$(x, y) \xrightarrow{T} \frac{x+y}{2}$$

satisfies the axioms and $\|T\|=1$.

(iii) Let $\|T\|=1$. Take $x_0 \in E$, $x_0 \neq 0$. Let U be the T-convex hull of the set $\{0, x_0\}$. Put

$$B=\{x \mid \|x\| \leqq \|x_0\|\}.$$

One sees that $U \subset B$.

Can it be proved that $U=B$?

(iv) Let A be a circled set. One proves easily that the T-convex hull of A is also circled.

Problem 4. *Study the properties of such a general definition of convexity in spaces over a field K, in particular in n.a. normed spaces.*

2.2.3. In none of the definitions or suggestions of definitions of convexity in linear topological spaces over a field K the notion of order is used.

One can also consider spaces over a field K on which is defined an order as well as a valuation. If we want to exclude spaces over R, the order must then be non-archimedean. One can then define convexity on the one hand by means of the order of K and on the other hand by means of the valuation. Then there will be the problem of equivalence of both definitions [1]).

For literature on definitions of convexity see the Proceedings of a conference on convexity (1963), especially the articles by DANZER, GRÜNBAUM and KLEE [2] and MOTZKIN [13].

§ 3. *Spherical completeness of a space.*

3.1. A n.a. normed space E over K is called spherically complete if every decreasing sequence of spheres has a non-empty intersection. This notion is important in n.a. analysis, that is analysis in n.a. valued fields. Spherically complete fields seem there to be essential. This notion can be discussed from two points of view.

A. It is reasonable to compare spherical completeness with the following classical theorem of Cantor: among the metric spaces the complete spaces are characterized by the following property: every sequence of closed sets $A_1 \supset A_2 \supset \ldots$, of which the diameters tend to 0, has a non-empty intersection.

Now the following problems rise when we omit the condition that the diameters tend to 0. Let E be a *metric space.*

Problem 5.

(i) *Characterize those metric spaces on which every decreasing sequence of closed spheres has a non-empty intersection.*

(ii) *The same problem but changing spheres into closed sets.*

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[1]) In [2], p. 156, a study is mentioned of convexity in spaces over non-archimedean ordered fields. In that study, however, a theory is given on convexity in fields with a n.a. valuation; ordering is nowhere used.

(iii) *Let E be a normed linear space (over R or over K). Characterize those spaces in which every decreasing sequence of (closed) convex sets has a non-empty intersection.*

SIERPINSKY [16] gives the following condition: E is spherically complete if for all $x, y \in E$ and $r > 0$ there is $z \in E$ such that $d(x, y) + d(y, z) = d(x, z)$ and $d(y, z) = r$. Clearly this condition loses its sense if the metric is n.a. Compare the definition of convexity in distance geometry (BLUMENTHAL [1]).

B. In a n.a. normed space it is known that for any couple of spheres with non-empty intersection, one of the spheres is contained in the other.

Taking this into account, the property of spherical completeness belongs to the field of so called intersection properties of sets in linear topological spaces; one can consider closed sets, convex sets, spheres and so on. In finite dimensional spaces we have the classical theorem of Helly, but there are infinite dimensional generalizations; see KLEE [7].

I mention the so called *finite intersection property*.

It would be worth while to study spherical completeness in n.a. normed spaces from this point of view.

3.2. There are generalizations of spherical completeness to locally K-convex spaces over a field K. The definition makes use of the family Γ of n.a. semi-norms which defines the topology as follows. Let $p \in \Gamma$. The set $\{x \in E | p(x - x_0) \leqq r\}$ is called a p-sphere. The space E is called p-spherically complete if every decreasing sequence of p-spheres has a non-empty intersection. This notion is useful in n.a. analysis (for instance for the extension theorem for mappings in locally K-convex spaces). This definition can be generalized in the following way. Suppose that the topology of the locally K-convex space over K is defined by the family Γ of semi-norms p_i where i is in an index set I. Let $\Delta \subset I$. A Δ-sphere is defined by

$$\{x \in E | p_i(x - x_0) \leqq r,\ x_0 \in E,\ i \in \Delta\}.$$

Definition. *The space E is called Δ-spherically complete if every decreasing sequence of Δ-spheres has a non-empty intersection.*

If Δ consists of a single element, one gets p-spherical completeness.

With regard to the set of problems under A remark that this definition can be given for spaces over K, as well as for spaces over R. Note that in the first case the centre x_0 may be replaced by any y_0 such that $p_i(y_0 - x_0) \leqq r$ for any $i \in \Delta$. With this definition there are the following problems.

Problem 6.

(i) *If the space E is p-spherically complete for every $p_i \in \Gamma$, $i \in \Delta$, is it then Δ-spherically complete?*

(ii) *Characterize the spaces which are I-spherically complete.*
(iii) *Characterize the spaces which are Δ-spherically complete for all $\Delta \subset I$.*

If a space satisfies (iii), it satisfies (ii) but the converse is not at all certain.

3.3. For locally convex spaces (over R or over K) the classical theorem of Cantor can be generalized as follows.

Let E be a locally convex complete Hausdorff space; the topology is determined by the set Γ of semi-norms. For a set $A \subset E$ define the diameter by

$$d(A) = \sup_{\substack{p \in \Gamma \\ x,\, y \in A}} p(x-y).$$

The following properties are evident.

1. $d(A) > 0$ if A contains more than 1 point.
2. $A_1 \supset A_2 \Rightarrow d(A_1) \geqq d(A_2)$.

Theorem. *Let A_i be closed; $A_1 \supset A_2 \supset \dots$ Suppose*

$$\lim_{n \to \infty} d(A_n) = 0.$$

Then $\cap A_i \neq \emptyset$ and the intersection consists of just one point.

The proof is of course simple. Take a sequence $\{x_i\}$, $x_i \in A_i$. Then $\{x_i\}$ is a Cauchy sequence. Indeed, for $\varepsilon > 0$, there is $N(\varepsilon)$ such that $d(A_n) \leqslant \varepsilon$, $n \geqslant N(\varepsilon)$.

This means

$$p(x_n - x_m) \leqq d(A_n) \leqslant \varepsilon$$

for all $p \in \Gamma$, n, $m > N$.

Taking into account the definition of the uniform structure on E by means of the semi-norms, this implies that $\{x_i\}$ is a Cauchy sequence and E being complete there is a limit x; $x \in A_i$ for all i.

3.4. I consider now n.a. Banach spaces E over K. There is a possibility that there is a relation between spherical completeness and fixed-point properties of contractions.

Consider the spheres:

$$B_1 = \{x \mid \|x\| \leqq r_1\}$$
$$B_i = \{x \mid \|x - \xi_i\| \leqq r_i\}_{i=2,3,\dots}$$

where ξ_i are fixed points, $\xi_1 = 0$, $\xi_i \in B_{i-1}$, $r_i < r_{i-1}$ [1]). We have then

$$B_1 \supset B_2 \supset \dots$$

[1]) Remark that $r_i = r_{i-1}$ implies $B_i = B_{i-1}$; we exclude this trivial case.

Conversely, every decreasing sequence of spheres can be got in this way (up to a translation).

Suppose that for all i there is $\lambda_i \in K$ such that $|\lambda_i| = r_i r_{i+1}^{-1}$. This imposes a condition on the valuation of K or on the r_i. Remark that we can suppose that the norm $\|\cdot\|$ is not discrete because spaces with descrete norm are spherically complete and there is no problem at all.

Define a sequence of affine mappings $E \to E$ in the following way.

$$
\begin{aligned}
&y = T_1\, x:\, x = \lambda_1(y - \xi_2)\\
&y = T_2\, x:\, x - \xi_2 = \lambda_2(y - \xi_3)\\
&y = T_i\, x:\, x - \xi_i = \lambda_i(y - \xi_{i+1}).
\end{aligned}
$$

It follows from the definition of λ_i that $B_i \xrightarrow{T_i} B_{i+1}$ for all i.

We have the following properties.

(i) Because every ξ_i can be taken as centre of B_1 one sees easily (using the strong triangle inequality) that

$$x \in B_1 \Rightarrow T_i\, x \in B_1 \text{ for all } i.$$

Sharper, we have even for $x \in B_1$

$$T_i\, x \in \{x|\ \|x\| < r_1\},$$

because $|\lambda_i| < 1$.

Note that $\{x|\ \|x\| < r_1\}$ is just the interior of the sphere B_1 with respect to the centre 0, in the way as I introduced earlier (see 2.1.).

Thus $\{T_i\}$ is a sequence of affine mappings of B_1 into itself.

(ii) *Every T_i is a contraction.*

$$\|T_i\, x_1 - T_i\, x_2\| = |\lambda_i|^{-1}\|x_1 - x_2\| < \|x_1 - x_2\|.$$

If there is $0 < \mu < 1$ such that

$$|\lambda_i|^{-1} < \mu < 1 \text{ for all } i$$

then we have even

$$\|T_i\, x_1 - T_i\, x_2\| < \mu\|x_1 - x_2\|.$$

There is an extensive literature about these mappings (but not for n.a. normed spaces). However, this case is not interesting for us, because then $r \Rightarrow 0$ and so we can apply the classical theorem of Cantor.

(iii) If the restrictions of T_i to B_i have a common fixed point, that is to say if there is $x \in B_1$ such that

$$T_i\, x = x \text{ for all } i.$$

then $x \in \cap B_i$. *So E is spherically complete under this assumption.*

Conversely, if $\cap B_i \neq \emptyset$ and $x_0 \in \cap B_i$, one can ask whether there is a common fixed point for every T_i.

It is trivial that in this case one can construct a sequence of affine mappings with a common fixed point as above. Indeed, one may suppose $x_0 = 0$. Without changing the spheres B_i one can take $\xi_i = 0$ because the norm is n.a. Then 0 is a common fixed point; the T_i are homogenous.

Usually the fixed points theorems concern compact sets. It is not likely that in such a case fixed points theorems will be of use for studying spherical completeness, because for compact spaces spherical completeness is trivial.

On the other hand spherical completeness is a notion which belongs to the domain of compactness properties. So there is a possibility that spherical completeness may be useful for the study of fixed point theorems in n.a. normed spaces. There are only very few results on such theorems in n.a. spaces (see [12]).

The sequence of mappings we used above must be compared with the theorem of Markoff-Kakutani. Note that in that theorem the affine mappings are commuting. It is not to be expected that fixed point theorems for real normed spaces will have an exact analogue for spaces over a field K, because in such spaces "there are too many continuous functions". In many cases the proofs are based on the classical theorem of Brouwer. Such a method cannot be applied in n.a. spaces. New ways will be necessary.

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