ultrapowers $\mathcal{F}/\mathcal{U}$ for which $\mathcal{F}$ is the class of unary recursive functions and $\mathcal{U}$ is a free ultrafilter in the Boolean algebra of recursive sets). The special subsemirings of $\Lambda$ that are referred to here are those of the form $\mathbb{N}(A) = \langle (f_A(A) \mid f \in \mathcal{F}), +, \cdot \rangle$, where $f_A$ is the Nerode extension of $f$ to $A$ and $A$ is an infinite isol such that $f_A(A)$ is defined for all $f \in \mathcal{F}$. (For the existence of such $A$, see [1] or [18].) These structures provide a kind of ‘simple-structure framework’ for the study of what is called $\Pi^0_2$ Arithmetic: recursive functions on $ω$ lift naturally to recursive (i.e., $\Sigma^0_1$-definable) functions on $N(A)$; $\mathbb{N}(A)$ is generated by $A$, under the action of the recursive functions on $N(A)$; $\mathbb{N}(A) \models \Pi^0_2$ Arithmetic; each countable model of $\Pi^0_2$ Arithmetic is the union of a monotone chain $\langle N_i \mid i \in ω \rangle$ of isomorphs of Nerode semirings; and each countable model of $\Pi^0_2$ Arithmetic is embeddable, in an existentially closed fashion, in a Nerode semiring. (See [7, 10, 11] for details.)

Seven of the eight problems that I shall offer here are directly concerned with Nerode semirings, while a tempting but futile approach to the eighth one (Problem 2) leads, as will be seen, to the inclusion of Problem 3. Thus, these problems are problems in and around the topic of recursive ultrapowers: it is hoped that this will provoke some interest on the part of nonisoltheorists.

1. A problem about the elementarity level of an embedding

In [13, §2] it was shown that each of the following is possible, for a proper embedding

$$\omega \models \mathbb{N}(A) \preceq^\psi \mathbb{N}(B)$$

of one Nerode semiring in another:

$$\psi(\mathbb{N}(A)) \not\models \mathbb{N}(B),$$
$$\psi(\mathbb{N}(A)) \not\models \mathbb{N}(B) \land \psi(\mathbb{N}(A)) \not\models \mathbb{N}(B),$$
$$\psi(\mathbb{N}(A)) \not\models \mathbb{N}(B) \land \psi(\mathbb{N}(A)) \not\models \mathbb{N}(B).$$

(Here, of course, “$X <_n Y$” means that $X$ is an $n$-elementary substructure of $Y$.)

It was left unanswered in [13] whether

$$\omega \models \mathbb{N}(A) \preceq^\psi \mathbb{N}(B) \land \psi(\mathbb{N}(A)) \not\models \mathbb{N}(B)$$

is possible. Using the fact (see [7]) that there is a fixed $\Pi^0_2$ predicate $ϕ(x)$ such that $ϕ(x)$ defines $ω$ in each recursive ultrapower, and hence in each Nerode semiring, we can easily show that

$$\omega \models \mathbb{N}(A) \preceq^\psi \mathbb{N}(B) \land \psi(\mathbb{N}(A)) \not\models \mathbb{N}(B)$$
Eight problems about Nerode semirings (recursive ultrapowers)

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Abstract


Several problems that pertain to certain arithmetically well-behaved countable subsemirings of \( \mathbb{A} \), the semiring of isols, are discussed. This is relevant to the present volume memorializing the late John Myhill, in that Myhill was an early co-developer of the theory of \( \mathbb{A} \).

0. Introduction

One of the many areas of ‘Foundations’ in which John Myhill made contributions, an area that he in fact helped to establish, is isol theory: he and Dekker (who first put forward the notion of an isol) labored together to produce the first systematic discussion of the algebra of isols [3], later augmented and uniformized by Nerode’s classic papers [16, 17].

Isol theory per se has not proved to be an especially popular area of research. At its zenith, in the mid sixties, the subject might have commanded the use of an ordinary-sized meeting room for a convention of its active enthusiasts; today, the proverbial phone booth would almost suffice. But as Nerode once remarked in defense of isols, it is often not so much the specific content of a mathematical theory that matters, as it is the body of techniques characteristically employed in pursuing its study. Since isol-theoretic techniques center around the extension of relations from smaller to larger domains, they should remain of interest even if the isols themselves have come to be viewed as something of a curiosity.

When one turns from the entire semiring \( \mathbb{A} \) of isols to certain countable subsemirings, which I have attempted to persuade people to call Nerode semirings, the structure of the systems under examination is seen to be of a kind that should be at least moderately pleasing to logicians at large: these systems are, to within isomorphism, precisely the recursive ultrapowers (that is, those...
Now, either the correct answer to this question is ‘yes’, or else there are countable $\Pi^0_2$-correct subsemirings $\mathcal{F}$ of $\Lambda$ that cannot be extended to larger $\Pi^0_2$-correct subsemirings. For, by Zorn’s Lemma and the inductivity [7] of $\Pi^0_2$ Arithmetic, any $\Pi^0_2$-correct subsemiring of $\Lambda$ can be extended to a maximal one. There is, however, a small amount of information, and an additional problem, to be gained by pretending that we do not know this. The perverse yet modestly fruitful idea for getting a negative solution to Problem 2 goes like this: start with a (supposed) $\mathcal{K}_1$-like, $\Pi^0_2$-correct semiring $\Gamma \subseteq \Lambda$, and ‘rebuild it from the inside’, so as to obtain a contradiction to the fact [14, Theorem 4.12] that if a Nerode semiring $\mathbb{N}(B)$ is a noncofinal extension of another such semiring $\mathbb{N}(A)$, then $\mathbb{N}(B)$ must be a dense extension of $\mathbb{N}(A)$ in the sense that $\omega = \text{the common initial segment of } \mathbb{N}(A) \text{ and } \mathbb{N}(B)$. (The definition of ‘density’ is somewhat differently formulated in [14]; but the formulation of [14] is easily seen to be equivalent to $\text{Init}(\mathbb{N}(A), \mathbb{N}(B)) = \omega$.) One picks an infinite element $A_1$ of $\Gamma$ and with it generates a Nerode subsemiring $\mathbb{N}(A_1)$ of $\Gamma$. Then one chooses $A_2 \in \Gamma - \mathbb{N}(A_1)$ and generates $\mathbb{N}(2^{A_1 A_2})$, etc. Having achieved a monotone sequence $\langle S_i \mid i \in \omega \rangle$ of Nerode subsemirings of $\Gamma$ in this way, one takes their union, $S$, and observes that since $S$ is a countable model of $\Pi^0_2$ Arithmetic (the latter theory being inductive), $S$ is isomorphic to a proper substructure of a Nerode semiring. One then makes a great (and ill-founded) leap of hope and says: “let us suppose that $S$ is, in fact, properly contained in a Nerode semiring $\mathbb{N}(A)$, and even properly contained in one that is itself contained in $\Gamma$”. Now the construction can continue, and one ends up in the $\mathcal{K}_1$-run with a contradiction to the covering constraint [14, Theorem 4.12] on pairs of Nerode semirings.

The “let us suppose . . .” hypothesis in the foregoing fantasy is of course wrong. The interesting thing is that it can be shown wrong by the very same line of argument that was just indicated for taking advantage of it, were it correct! This last remark will be clarified in the next section, by the proof given for a proposition about nonextendability in $\Lambda$, and from that proposition and its proof a further problem will be extracted.

3. Are there any maximal Nerode semirings?

**Proposition.** Let $S$ be any countable semiring of isols, with $\omega \subseteq S$, such that $S \models \Pi^0_2$ Arithmetic. Then there is a countable, $\Pi^0_2$-correct subsemiring $\Gamma$ of $\Lambda$ such that (i) $S \subseteq \Gamma$, and (ii) $\Gamma$ cannot be properly extended to a Nerode semiring.

**Proof.** Suppose the contrary. We shall construct a monotone increasing sequence $\langle S_\alpha \mid \alpha < \mathcal{K}_1 \rangle$ whose union is an $\mathcal{K}_1$-like, $\Pi^0_2$-correct subsemiring of $\Lambda$, but whose structure contradicts [14, Theorem 4.12]. At Step 0, set $S_0 = S$. At a successor ordinal $\xi < \mathcal{K}_1$, let $S_\xi$ be a Nerode semiring properly containing $S_{\xi-1}$. At a limit ordinal $\lambda < \mathcal{K}_1$, let $S_\lambda = \bigcup \{ S_\beta \mid \beta < \lambda \}$; since $\Pi^0_2$ Arithmetic is an inductive
Problems about Nerode semirings

is not possible: letting $V(y, x, z)$ be (in $\omega$) a universal partial recursive function (i.e., $\forall e, x, w \ (\phi_e(x) = w \iff V(e, x, w))$, we have that

$$\forall z \exists e [\phi(e) \land \forall x \exists w V(e, x, w) \land V(e, \psi(A), z)]$$

is a $\Pi^0_2$ sentence $\tau$ with parameter $\psi(A) \in \psi(\mathbb{N}(A))$ such that $\psi(\mathbb{N}(A)) \models \tau$, while (since the embedding of $\mathbb{N}(A)$ in $\mathbb{N}(B)$ via $\psi$ is proper) $\mathbb{N}(B) \not\models \tau$. (The following facts are being applied here: $\psi(A)$ is a generator for $\psi(\mathbb{N}(A))$ under the functions that are total recursive in the sense of $\psi(\mathbb{N}(A))$, and the latter functions are both (a) the restrictions to $\psi(\mathbb{N}(A))$ of the functions total recursive in the sense of $\mathbb{N}(B)$, and (b) the extensions to $\psi(\mathbb{N}(A))$ of the total functions $\phi_e(x)$ on $\omega$.)

So, we pose the first problem.

**Problem 1.** Is

$$\mathbb{N}(A) \not\leq^\psi \mathbb{N}(B) \land \psi(\mathbb{N}(A)) \prec \mathbb{N}(B), \quad \psi \text{ an isomorphism},$$

possible? If so, what about

$$\mathbb{N}(A) \not\leq^\psi \mathbb{N}(B) \land \psi(\mathbb{N}(A)) \prec \mathbb{N}(B)?$$

(It might appear that the $\prec$ question should be easy to answer in the negative, in the same way as just done for $\leq$, since there is a 'fixed' $\Sigma^2_1$ predicate $\xi(x)$, albeit a parametric one, that defines $\omega$ in each $\mathbb{N}(X)$. The parameter in $\xi(x)$, however, is subject to change as we pass from the smaller semiring $\psi(\mathbb{N}(A))$ to the larger one $\mathbb{N}(B)$. Indeed, the 'natural' change here is from $\psi(A)$ to $B$.)

2. How big can a $\Pi^0_2$-correct subsemiring of $\Lambda$ be?

All Nerode semirings are models of $\Pi^0_2$ Arithmetic; i.e., they are all $\Pi^0_2$-correct. They are also all countable. As noted in Section 0, any countable model of $\Pi^0_2$ Arithmetic can be embedded in a Nerode semiring; so, at the countable level, there are $\Pi^0_2$-correct subsemirings of $\Lambda$ of all possible kinds.

On the other hand, as noted in [11], the possibilities for an uncountable $\Pi^0_2$-correct subsemiring $\Gamma$ of $\Lambda$ are severely limited: $\Gamma$ would have to be '$\aleph_1$-like' in terms of its order structure. (This is because no element of $\Lambda$ has more than $\aleph_0$ isolic predecessors. It should be noted that this limitation on order type does not in and of itself preclude the existence of an uncountable $\Gamma$; it is a theorem of Ellentuck that any $\aleph_1$-like linear order is embeddable, simply as an ordered set, in $\Lambda$.) Thus, we can state the next problem.

**Problem 2.** Are there any $\aleph_1$-like, $\Pi^0_2$-correct subsemirings of $\Lambda$?
result for Nerode semirings. (Not all Nerode semirings are existentially complete, nor is every singly-generated, existentially complete model of $\Pi^0_2$ Arithmetic isomorphic to a Nerode semiring.) Call a structure totally rigid if it is rigid and cannot be mapped isomorphically onto a proper subset of itself. As noted in [14], a Nerode semiring cannot be isomorphic to a proper initial segment of itself (whereas, by [4], any countable nonstandard model of $\Pi^0_n$ Arithmetic, $n \geq 3$, is isomorphic to one of its own proper cuts). To the best of my knowledge, however, the question of total rigidity remains open for Nerode semirings, just as it was pronounced open in [8] (and seems still to be so) for the related case of one-generator, existentially complete models of $\Pi^0_2$ Arithmetic. Thus we have the next problem.

**Problem 4.** Are all Nerode semirings totally rigid? (Some of them, of course, are; e.g., those that are minimal as described in the next section.)

5. Nontrivial minimal subsemirings

A Nerode semiring is termed minimal if it has no proper substructure other than $\omega$ that is a model of $\Pi^0_2$ Arithmetic. For $n \geq 3$, it is a consequence of [4] that minimal models of $\Pi^0_n$ Arithmetic, in the corresponding sense, do not exist; however, it was indicated by Hirschfeld [7] how to construct a multitude of recursive ultrapowers (and hence a corresponding multitude of Nerode semirings) having no proper, $\Pi^0_2$-correct submodels other than $\omega$. (For details, see [14, Section 3].) On account of the Davis-Putnam-Robinson-Matijasevic theorem on diophantine representation, an equivalent definition of minimal Nerode semiring is this: $\mathbb{N}(A)$ is minimal if and only if it is generated, under the action of the $\mathbb{N}(A)$-recursive functions, by each of its infinite elements.

As noted in [12], the intersection of any two $\Pi^0_2$-correct subsemirings of $\Lambda$ is again a $\Pi^0_2$-correct semiring. Consequently, if $\Gamma$ is a $\Pi^0_2$-correct subsemiring of $\Lambda$ and $\mathbb{N}(A)$ is a minimal Nerode semiring, then either $\mathbb{N}(A) \subseteq \Gamma$ or $\Gamma \cap \mathbb{N}(A) = \omega$.

**Problem 5.** Does every Nerode semiring have a minimal Nerode sub-semiring? To put it the other way around, is there a Nerode semiring whose intersection with every minimal Nerode semiring is $\omega$?

6. ‘Tame models’

Among the minimal Nerode semirings there are certain (presumably) special ones that are particularly well-behaved in terms of the way in which they are generated; these, the so-called tame models, are the Nerode semirings of the form $(\{g_{\Lambda}(A) \mid g \text{ is a nondecreasing recursive function}\}, +, \cdot)$. The observation that all
Problems about Nerode semirings

Problems about Nerode semirings

theory, $S_{\alpha}$ is a countable, $\Pi^0_2$-correct semiring. Our assumption that every countable, $\Pi^0_2$-correct subsemiring of $\Lambda$ is properly contained in a Nerode semiring pushes this construction all the way through $\aleph_1$; and if we set $I_0 = \bigcup \{ S_\alpha \mid \alpha < \aleph_1 \}$, then $I_0$ is a $\Pi^0_2$-correct, $\aleph_1$-like subsemiring of $\Lambda$. ($\aleph_1$-likeness follows from the fact that no isol has more than $\aleph_0$ predecessors.) But now there is trouble. Since $I_0$ is $\aleph_1$-like, there must be a step $\beta$ of the construction such that, subsequent to step $\beta$, no new element enters $I_0$ that is less than an element of $S_\alpha = S$. Yet at any successor step $\gamma + 1 > \beta$, the portion $S_{\gamma + 1}$ of $I_0$ defined at that step is a Nerode semiring extending $S$. At some still later successor step, say step $\tau + 1$, we obtain yet another Nerode semiring $S_{\tau + 1}$ such that $S_{\gamma + 1}$ is not cofinal in $S_{\tau + 1}$ (on the grounds that the construction must by step $\tau$ have stopped putting isolic predecessors of elements of $S_{\gamma + 1}$ into $I_0$). But $S_{\tau + 1}$ also extends $S_{\gamma + 1}$ nondensely, since $S_{\tau + 1}$ can contribute nothing new below elements of $S_0 = S$. The pair $S_{\gamma + 1}, S_{\tau + 1}$ therefore presents a contradiction to [14, Theorem 4.12]. So in fact there must be a least ordinal $\alpha < \aleph_1$ for which the construction ‘stalls’; i.e., $S_\alpha$ admits no proper extension to a Nerode semiring. Setting $\Gamma = S_\alpha$, we have the proposition. 

The foregoing proof is utterly nonconstructive; in fact, it is about as uninformative as possible, granted that it does yield the proposition. For instance, it provides no clue as to whether the ‘stalling point’ $\alpha$ can be taken to be a successor ordinal (in which case maximal Nerode semirings, with respect to inclusion, exist) or must, on the contrary, be a limit: this proof simply gives no information as to ‘what $S_\alpha$ looks like’. Even if $\alpha$ is necessarily a limit ordinal because no maximal Nerode semirings exist, the maximality of $S_\alpha$, as a countable $\Pi^0_2$-correct subsemiring of $\Lambda$, has not been shown. We are therefore led to pose the following problem.

**Problem 3. Is there a countable, maximal, $\Pi^0_2$-correct subsemiring $S$ of $\Lambda$? If so, can such an $S$ be a Nerode semiring?**

As was noted in the previous section, either Problem 2 or the first part of Problem 3 (and possibly each) admits the affirmative solution.

4. Are Nerode semirings totally rigid?

The first place I saw the statement of a ‘rigidity’ result, for structures of the general sort being discussed in this paper, was in [8, Chapter 9]; there Hirschfeld indicated how to show that any singly-generated, existentially complete model of $\Pi^0_2$ Arithmetic is rigid, i.e., admits only the identity automorphism. Following Hirschfeld’s outline, I dutifully wrote down in [13] a proof of the corresponding
The arithmetical properties of $D(H)$ depend greatly on how $H$ is chosen: if $H$ is a 'universal' isol (see [5] or [9, Chapter 19]), then the universal theory $T_U^{D(h)}$ of $D(H)$, in the 'Nerode language' $L_N$, for isol theory, is the same as that of $A$ itself; and that is a very weak theory. If, on the other hand, $H$ is hereditarily odd-even (i.e., if each predecessor of $H$ is either even or odd), then $T_U^{D(h)} = \text{the universal theory of } \omega$ (see [9, Chapter 20]). Nerode semirings, by contrast and as already noted, all satisfy $IT^0_2$ Arithmetic; one can show this to be true also for the extended language $L_N$. On the other hand, as shown in [1], Nerode semirings need not be closed under predecessor; whereas, by their very definition, all Dekker semirings are so closed. One is thus naturally led to the following question (first raised by Ellentuck in [6]).

**Problem 7.** Are there any countable semirings $S \subseteq A$ that are both Nerode and Dekker?

If the answer to this question is, as one would hope, ‘yes’, so that $\mathbb{N}(A) = D(B)$ for suitably chosen infinite regressive isols $A$ and $B$, then, by [9, Theorem 20.14], [10, Proposition 3.5] and [11, Theorems 1 and 2], it is necessarily the case that $B$ is both hereditarily odd-even and in the domain of $f_a$ for all unary recursive $f$. Thus a minimum requirement for an affirmative solution to Problem 7 is the existence of such isols $B$. A claim occurring at the end of [10] notwithstanding, I do not at present have a proof of even this much. It would appear (as it has already, several times, ‘appeared’ to the present writer) that another turn or two of the combinatorial crank will yield a somewhat long and messy proof of the existence of such $B$'s, via the suitable brute-force amalgamation of Nerode’s ‘compactness’ argument [18, Theorem 3.1] with the basic tree-squeezing construction of a hereditarily odd-even infinite regressive isol [9, Chapter 20]. In this connection, let it be noted that Myhill could be quite restrained in his enthusiasm for long, messy proofs (though of course there is sometimes no apparent alternative). One of his more memorable remarks, concerning [15, Theorem 9], goes as follows: “I will not even attempt to outline the proof of this theorem; it is a quite nasty mixture of algebraic computations and category arguments. I will not publish it until I have it in a form that yields more insight” [15, p. 103]. I will be even more reticent than Myhill, and say: I think I am near to having an ugly proof; but even if I were certain I had it, it would be too cumbersome for inclusion here.

**8. Nerode semirings and extensions of $\Sigma^0_1$ relations**

Nerode proved in [18] that every countable model of true arithmetic has an isomorphic copy $R \leq A$ such that if $\phi(x_1, \ldots, x_n)$ is any $\Sigma^0_1$ predicate in the
tame models are minimal is due to the late Erik Ellemtuck (private communication); for details see [14, Section 3]. The original definition of ‘tame model’, in [1], looks somewhat different from the above, involving as it does the assumption that $A$ is a regressive isol along with the stipulation of a couple of special properties concerning predecessors and no mention of Nerode semiringhood. It follows from results and observations in [1, 10, 18], however, that the definition I have given is equivalent (to within isomorphisms) to Barback’s formulation; indeed, it is equivalent (modulo isomorphism) to the following formulation in terms of ultrapowers (see [14, Section 3]).

**Definition.** Let a free ultrafilter $U$ in the Boolean algebra $B$ of recursive sets be called **tame** if for every recursive function $f : \omega \rightarrow \omega$, there exists $U \in \mathcal{U}$ such that $f | U$ is nondecreasing. Letting $\mathcal{F}$ denote the class of all unary recursive functions on $\omega$, we say that $\mathcal{F}/U$ is a **tame ultrapower** if and only if $U$ is a tame ultrafilter in $B$.

There is a 1–1 correspondence, via isomorphisms, between tame Nerode semirings $\mathbb{N}(A)$, as defined at the beginning of this section, and corresponding tame ultrapowers $\mathcal{F}/\mathcal{U}_{\mathbb{N}(A)}$; see [14, Theorem 3.10] (where tame recursive ultrapowers are called ‘tame 1-models’). The advantage of viewing tame models as tame ultrapowers is that it makes perfectly clear where the issue lies, as regards the following problem.

**Problem 6.** Is every minimal Nerode semiring a tame model?

One is certainly tempted to conjecture in the negative. While it is rather easy (see [14, Lemma 3.4]) to ‘turn nondecreasing functions into strictly increasing or constant ones’, within the confines of a given tame ultrafilter, it is another matter entirely to transform one-to-one functions into nondecreasing ones modulo a given free ultrafilter $U \subseteq B$ for which $\mathcal{F}/U$ is minimal; and it is just this latter feat that one needs to accomplish in order to prove that all minimal Nerode semirings are tame.

7. Nerode semirings versus Dekker semirings

Nerode semirings were not the first special subsemirings of $\Lambda$ to receive attention in the literature; preceding them were certain naturally-defined semirings introduced by Dekker [2].

**Definition.** Let $H$ be an infinite regressive isol. Then by the **Dekker semiring** $\mathbb{D}(H)$ is meant the structure $\langle \{Y \in \Lambda \mid \exists f : \omega \rightarrow \omega \text{ (f is recursive combinatorial and } Y \leq f, \lambda f(H))\}, +, \cdot \rangle$. (Here ‘$\leq$’ denotes the predecessor relation among isols. It is verified in [2] that $\mathbb{D}(H)$ is a semiring.)
(Here $T_A$ is as defined in [18].) This entails that $X_1 \in M_A$. On the other hand, there can be no $\Delta_1^0$ subset $S$ of $M$ such that $X_1 \in S_A$, lest $S \in \mathcal{F}$. (It seems likely that this example is well known, or at any rate known; but I do not know how to attribute it.)

I could of course be wrong, but it seems to me that the issue of whether $\mathcal{N}(X) \vdash \forall \bar{x} (R_A(\bar{x}) \rightarrow \phi(\bar{x}))$ is one of some delicacy.

9. Postscript

If it is the case, as many people believe, that the spirit survives the body, then I hope that John Myhill is *not* 'resting in peace': I hope he is doing mathematics.

Note added (22 July 1991)

Problem 8 has now been solved in the negative. The solution, which has to do with existential incompleteness, will appear in a paper by the author forthcoming in the Proceedings of the American Mathematical Society.

References

language of arithmetic, then, for all $X_1, \ldots, X_n \in \mathcal{R}$,

$$[\mathcal{R} \models \phi(X_1, \ldots, X_n)] \iff \langle X_1, \ldots, X_n \rangle \in R_\lambda,$$

where $R \subseteq \omega^n$ is the relation defined by $\phi$ in $\omega$ [18, Theorem 5.3]. Careful inspection of Nerode's proof reveals that it applies to all countable models of mere $\Pi^0_2$ Arithmetic, from which it follows readily that any Nerode semiring $\mathcal{N}(\Lambda)$ has an isomorphic copy $\mathcal{N} \subseteq \Lambda$ that behaves as does $\mathcal{R}$ above, with respect to arbitrary $\Sigma^0_1$ predicates. But there is no reason to suppose that the isomorphism from $\mathcal{N}(\Lambda)$ to $\mathcal{N}$ can be taken to be the identity, and so the following problem, previously posed in [12], suggests itself.

**Problem 8.** Let $\mathcal{N}(\Lambda)$ be an arbitrary Nerode semiring. Is it the case that if $\phi(x_1, \ldots, x_n)$ is a $\Sigma^0_1$ predicate in the language of arithmetic and $X_1, \ldots, X_n \in \mathcal{N}(\Lambda)$, then

$$[\mathcal{N}(\Lambda) \models \phi(X_1, \ldots, X_n)] \iff \langle X_1, \ldots, X_n \rangle \in R_\lambda?$$

As was noted in [12], one of the two implications involved here, namely

$$[\mathcal{N}(\Lambda) \models \phi(X_1, \ldots, X_n)] \Rightarrow \langle X_1, \ldots, X_n \rangle \in R_\lambda,$$

is fairly easy to verify. To see this, suppose $\phi(\bar{x})$ is $\Sigma^0_1$; by Davis–Putnam–Robinson–Matijasevic, it costs us no generality to suppose that $\phi(\bar{x})$ is of the form $\exists \bar{z}(P(\bar{x}, \bar{z}) = Q(\bar{x}, \bar{z}))$, $P$ and $Q$ being polynomials with coefficients in $\omega$. Then if $\phi(\bar{x})$ defines $R(\bar{x})$ in $\omega$, we have

$$\mathcal{N}(\Lambda) \models \forall \bar{x} \left( \phi(\bar{x}) \rightarrow R_\lambda(\bar{x}) \right)$$

by the general form of the Basic Nerode Metatheorem discussed in [9, Chapter 18, pp. 294, 295]. The other direction is a good deal less clear, since $\forall \bar{x} \left( R_\lambda(\bar{x}) \rightarrow \phi(\bar{x}) \right)$ is not a universal Horn sentence and I know of no proof of [11, Theorem 2] that does not depend upon all relation symbols denoting $\Delta^0_1$ relations. This last would not be an obstacle if we had a general theorem asserting that, for a $\Sigma^0_1$ relation $R(\bar{x})$, $R_\lambda(\bar{x})$ true implies $S_\lambda(\bar{x})$ true for some $\Delta^0_1$ subrelation $S$ of $R$. Unfortunately (at least for present purposes), such is not the case. For (as an example to the contrary) let $M$ be a maximal $\Sigma^0_1$ subset of $\omega$ (in the usual recursion-theoretic sense), let $\mathcal{M}$ be the collection of all those $\Sigma^0_1$ subsets $\Lambda$ of $\omega$ such that $\omega - M$ is almost contained in $\Lambda$, and let $\mathcal{F}$ be an extension of $\mathcal{M} \cup \{M\}$ to a maximal collection of infinite $\Sigma^0_1$ subsets of $\omega$ closed under finite intersections. Then no infinite $\Delta^0_1$ subset of $M$ can be in $\mathcal{F}$. Let $\mathcal{F}^*$ be the collection of all 'finitary recursively enumerable' relations $T^*$ (in the sense of [18]) such that (a) $1$ is a support for $T$, and (b) $T \upharpoonright 1 \in \mathcal{F}$; thus, $\mathcal{F}^* \upharpoonright 1 = \mathcal{F}$. It is easily seen that $\mathcal{F}^*$ is a realizability filter in the sense of [18]; hence, by [18, Theorem 3.1], there is a sequence $\langle X_1, X_2, \ldots \rangle$ of infinite isols such that for every finitary recursively enumerable relation $T$, $\langle X_1, X_2, \ldots \rangle \in T_\lambda \iff T \in \mathcal{F}^*$.\]