Discrete Mathematics 311 (2011) 362–386

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Duality theory and propagation rules for higher order nets

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ARTICLE INFO

Article history:
Received 27 January 2010
Received in revised form 6 October 2010
Accepted 3 November 2010
Available online 26 November 2010

Keywords:
(t, m, s)-net
Propagation rule
Duality theory
Quality parameter
Quasi-Monte Carlo

ABSTRACT

Higher order nets and sequences are used in quasi-Monte Carlo rules for the approximation of high dimensional integrals over the unit cube. Hence one wants to have higher order nets and sequences of high quality.

In this paper we introduce a duality theory for higher order nets whose construction is not necessarily based on linear algebra over finite fields. We use this duality theory to prove propagation rules for such nets. This way we can obtain new higher order nets (sometimes with improved quality) from existing ones. We also extend our approach to the construction of higher order sequences.

1. Introduction

The concept of (t, m, s)-nets and (t, s)-sequences in base b was introduced by Niederreiter [10] as a general framework for constructing point sets and sequences which can be used as quadrature points for quasi-Monte Carlo (qMC) rules. Such nets (and sequences) are point sets (and sequences of points) in the unit cube $[0, 1)^s$. Throughout the paper a point set is always understood as a multiset, i.e., points may occur repeatedly.

In general, qMC rules are of the form $\frac{1}{N}\sum_{n=0}^{N-1} f(\boldsymbol{x}_n)$, where $\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}$ are the quadrature points, which can be used to approximate integrals of the form $\int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$. Typically one obtains a convergence of the integration error of $\mathcal{O}(N^{-1}(\log N)^s)$ for such methods [8,10].

The notion of $(t, \alpha, \beta, n, m, s)$-nets and $(t, \alpha, \beta, \sigma, s)$-sequences in base b on the other hand was introduced in [6]. These so-called higher order nets and sequences are used for accurately evaluating high dimensional integrals of smooth functions. The main objective of this paper is twofold. One is to develop a duality theory which also applies to nonlinear constructions of nets and sequences. The second one is to present rules which show how to obtain new higher order nets and sequences from existing ones and how the parameters of these point sets propagate under these rules.

We give the definitions and some properties of $(t, \alpha, \beta, n, m, s)$-nets and $(t, \alpha, \beta, \sigma, s)$-sequences in base b. To this end some notation has to be fixed which is used throughout the paper.

Let $n, s \geq 1, b \geq 2$ be integers. For $\boldsymbol{\nu} = (\nu_1, \dots, \nu_s) \in \{0, \dots, n\}^s$ let $|\boldsymbol{\nu}|_1 = \sum_{j=1}^{s} \nu_j$ and define $\boldsymbol{i}_{\boldsymbol{\nu}} = (i_{1,1}, \dots, i_{1,\nu_1}, \dots, i_{s,1}, \dots, i_{s,\nu_s})$ with integers $1 \leq i_{j,\nu_j} < \cdots < i_{j,1} \leq n$ in case $\nu_j > 0$ and $\{i_{j,1}, \dots, i_{j,\nu_j}\} = \emptyset$ in case $\nu_j = 0$, for $j = 1, \dots, s$. For given $\boldsymbol{\nu}$ and $\boldsymbol{i}_{\boldsymbol{\nu}}$ let $\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0, \dots, b-1\}^{|\boldsymbol{\nu}|_1}$, which we write as $\boldsymbol{a}_{\boldsymbol{\nu}} = (a_{1,i_{1,1}}, \dots, a_{1,i_{1,\nu_1}}, \dots, a_{s,i_{s,1}}, \dots, a_{s,i_{s,\nu_s}})$.

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0012-365X/$ – see front matter
doi:10.1016/j.disc.2010.11.002

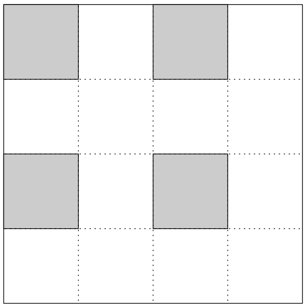

Fig. 1. Unit interval $[0, 1]^2$ and shaded the generalized elementary interval $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ for $b = 2$, $\nu_1 = \nu_2 = 1$, $i_{1,1} = i_{2,1} = 2$, and $a_{i_{1,1}} = 0$ and $a_{i_{2,1}} = 1$.

A *generalised elementary interval in base b* is a subset of $[0, 1)^s$ of the form

$$J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}) = \prod_{j=1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l\in\{1,\dots,n\}\setminus\{i_{j,1},\dots,i_{j,\nu_j}\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n} + \frac{1}{b^n}\right),$$

where $\{i_{j,1}, \dots, i_{j,\nu_j}\} = \emptyset$ in case $\nu_j = 0$ for $1 \le j \le s$. Note that a generalised elementary interval is actually not an interval anymore, but a union of intervals. An example of a generalised elementary interval in base 2 is shown in Fig. 1. An intuitive motivation for the definition of generalised elementary intervals is given in [5].

From [6, Lemmas 1 and 2] it is known that for $\boldsymbol{\nu} \in \{0, \dots, n\}^s$ and $\boldsymbol{i}_{\boldsymbol{\nu}}$ defined as above and fixed, the generalised elementary intervals $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ where $\boldsymbol{a}_{\boldsymbol{\nu}}$ ranges over all elements from the set $\{0, \dots, b-1\}^{|\boldsymbol{\nu}|_1}$ form a partition of $[0, 1)^s$ and the volume of $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ is $b^{-|\boldsymbol{\nu}|_1}$.

We can now give the definition of $(t, \alpha, \beta, n, m, s)$-nets based on [6, Definition 4].

Definition 1.1. Let $n, m, s, \alpha \ge 1$ be natural numbers, let $0 < \beta \le \min(1, \alpha m/n)$ be a real number, and let $0 \le t \le \beta n$ be an integer. Let $b \ge 2$ be an integer and $\mathcal{P} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{b^m-1}\}$ be a multiset in $[0, 1)^s$. We say that $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-*net in base b*, if for all integers $1 \le i_{j,\nu_j} < \cdots < i_{j,1}$, where $0 \le \nu_j \le n$, with

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \le \beta n - t,$$

where for $\nu_j = 0$ we set the empty sum $\sum_{l=1}^{0} i_{j,l} = 0$, the generalised elementary interval $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains exactly $b^{m-|\boldsymbol{\nu}|_1}$ points of $\mathcal{P}$ for each $\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0, \dots, b-1\}^{|\boldsymbol{\nu}|_1}$.

A $(t, \alpha, \beta, n, m, s)$-net in base b is called a *strict* $(t, \alpha, \beta, n, m, s)$-*net in base b*, if it is not a $(u, \alpha, \beta, n, m, s)$-net in base b with $u < t$.

Informally we refer to $(t, \alpha, \beta, n, m, s)$-nets as higher order nets.

Note that in the definition above the specific order of elements of a multiset is not important. The parameter t is often referred to as the *quality parameter* of the net. By the *strength* of the net one means the quantity $\beta n - t$. A geometric interpretation and an intuitive explanation of the definition of higher order nets is given in [3].

Remark 1.1. We obtain the definition of a classical (t, m, s)-net in base b due to Niederreiter [10, Definition 4.1] from Definition 1.1 by setting $\alpha = \beta = 1$, $n = m$, and considering all $\nu_1, \dots, \nu_s \ge 0$ so that $\sum_{j=1}^{s} \nu_j \le m - t$, where we set $i_{j,k} = \nu_j - k + 1$ for $k = 1, \dots, \nu_j$. In this case the definition can be simplified to the following. A multiset $\mathcal{P} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{b^m-1}\}$ whose elements belong to $[0, 1)^s$ is a (t, m, s)-net in base b if for all integers $d_1, \dots, d_s \ge 0$ with $d_1 + \cdots + d_s = m - t$ each elementary interval $J = \prod_{j=1}^{s} [\frac{a_j}{b^{d_j}}, \frac{a_j+1}{b^{d_j}})$ with integers $0 \le a_j < b^{d_j}$ for $1 \le j \le s$ and of volume b^{t-m} contains exactly b^t elements of $\mathcal{P}$. Hence a $(t, 1, 1, m, m, s)$-net is a (t, m, s)-net.

Remark 1.2. Let $n, m, s, \alpha \ge 1$ be natural numbers and let $0 < \beta \le 1$ be a real number. It follows from Definition 1.1 that any multiset consisting of b^m points in $[0, 1)^s$ is a $(\lfloor \beta n \rfloor, \alpha, \beta, n, m, s)$-net in base b.

Remark 1.3. Note that $b^{m-|\boldsymbol{\nu}|_1} = b^m \mathrm{Vol}(J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}))$. Hence Definition 1.1 states that the proportion of points of $\mathcal{P}$ in $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ equals the volume of $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$, i.e.,

$$\frac{|\{0 \le h < b^m : \boldsymbol{x}_h \in J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})\}|}{b^m} = \mathrm{Vol}(J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})).$$

We also give the definition of $(t, \alpha, \beta, \sigma, s)$-sequences from [6].

Definition 1.2. Let $\sigma, s, \alpha \geq 1$ be natural numbers, let $0 < \beta \leq 1$ be a real number, and let $t \geq 0$ be an integer. Let $b \geq 2$ be an integer and $\omega = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ be an infinite sequence in $[0, 1)^s$. We say that ω is a $(t, \alpha, \beta, \sigma, s)$-*sequence in base b*, if for all integers $k \geq 0$ and $m > t/(\beta\sigma)$ we have that the finite subsequence $\{\boldsymbol{x}_{kb^m}, \boldsymbol{x}_{kb^m+1}, \ldots, \boldsymbol{x}_{(k+1)b^m-1}\}$ is a $(t, \alpha, \beta, \sigma m, m, s)$-net in base b.

A $(t, \alpha, \beta, \sigma, s)$-sequence in base b is called a *strict* $(t, \alpha, \beta, \sigma, s)$-*sequence in base b* if it is not a $(u, \alpha, \beta, \sigma, s)$-sequence in base b with $u < t$.

Informally we refer to $(t, \alpha, \beta, \sigma, s)$-sequences as higher order sequences.

Note that in the definition above the specific order of elements of an infinite sequence is of importance.

A geometric interpretation and an intuitive explanation of the definition of higher order sequences is given in [6].

Of particular importance are $(t, \alpha, \beta, \sigma, s)$-sequences for which $\alpha = \beta\sigma$ since these are in some sense optimal, see [6]. For any $1 \leq \delta < \infty$ constructions of higher order sequences for which $\alpha = \beta\sigma$ for all $1 \leq \alpha \leq \delta$ are given in [4] (note that β and σ generally depend on α).

Remark 1.4. We obtain the definition of a classical (t, s)-sequence in base b due to Niederreiter [10, Definition 4.2] from Definition 1.2 and Remark 1.1 by setting $\alpha = \beta = \sigma = 1$. Hence a $(t, 1, 1, 1, s)$-sequence in base b is a (t, s)-sequence in base b.

Explicit constructions of $(t, \alpha, \beta, n, m, s)$-nets, respectively $(t, \alpha, \beta, \sigma, s)$-sequences, in prime power bases b are known using the digital construction scheme. Nets (and sequences) constructed in this manner are referred to as digital $(t, \alpha, \beta, n \times m, s)$-nets (and digital $(t, \alpha, \beta, \sigma, s)$-sequences) over a finite field $\mathbb{F}_b$. For more information we refer to [4, Section 4.4] and [7]. The proof that digital $(t, \alpha, \beta, n \times m, s)$-nets, respectively digital $(t, \alpha, \beta, \sigma, s)$-sequences, over $\mathbb{F}_b$ are in fact special cases of $(t, \alpha, \beta, n, m, s)$-nets, respectively $(t, \alpha, \beta, \sigma, s)$-sequences, in base b can be found in [6, Theorem 3.5].

The advantage of the more general concept due to [6] (in comparison to classical (t, m, s)-nets) is that $(t, \alpha, \beta, n, m, s)$-nets and $(t, \alpha, \beta, \sigma, s)$-sequences in base b can exploit the smoothness α of a function f (which is not the case for the classical concepts of (t, m, s)-nets and (t, s)-sequences). More precisely, we have the following theorem from [1].

Theorem 1.1. *Let* $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ *be a* $(t, \alpha, \beta, n, m, s)$*-net in base b. Let* $f : [0, 1]^s \to \mathbb{R}$ *have mixed partial derivatives up to order* $\alpha \geq 2$ *in each variable which are square integrable. Then*

$$\left| \int_{[0,1]^s} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} - \frac{1}{b^m} \sum_{h=0}^{b^m-1} f(\boldsymbol{x}_h) \right| = \mathcal{O}\left(b^{-(1-1/\alpha)(\beta n - t)} (\beta n - t)^{\alpha s} \right).$$

Additionally, the following results are known. If $\alpha = \beta = 1$ and $n = m$, then the integration error is of $\mathcal{O}(b^{-m+t} m^s)$, see [10]. If $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ is a digital $(t, \alpha, \beta, n \times m, s)$-net, then one obtains an integration error of $\mathcal{O}(b^{-(\beta n - t)} (\beta n - t)^{\alpha s})$, see [3,4].

Hence it is important to have explicit constructions of higher order nets with a large value of $\beta n - t$.

Special constructions of such point sets are based on the digital construction scheme introduced by Niederreiter [10] and generalised in [3,4]; the resulting point sets are referred to as digital nets and sequences. Nowadays many propagation rules for classical nets and sequences, digital or not, and also for *digital* higher order nets (see [7]) are known. Roughly speaking, propagation rules are methods by which one can construct new nets and sequences from existing ones (sometimes the net or sequence does not change, only the parameters change and the net or sequence with such parameters might not have been known before).

1.1. Six elementary propagation rules

Some simple propagation rules for $(t, \alpha, \beta, n, m, s)$-nets, respectively $(t, \alpha, \beta, \sigma, s)$-sequences, in base b were already listed in [6]. For completeness, we repeat them here. We also add some further trivial propagation rules in the following list:

Theorem 1.2 (*Propagation Rules 1–6*). *Let* $\mathcal{P}$ *be a* $(t, \alpha, \beta, n, m, s)$*-net in base b and let* ω *be a* $(t, \alpha, \beta, \sigma, s)$*-sequence in base b. Then we have the following:*

(1) *Weakening of strength:* $\mathcal{P}$ *is a* $(t', \alpha, \beta', n, m, s)$*-net in base b for all* $0 < \beta' \leq \beta$ *and all* $t \leq t' \leq \beta' n$*, and* ω *is a* $(t', \alpha, \beta', \sigma, s)$*-sequence in base b for all* $0 < \beta' \leq \beta$ *and all* $t \leq t'$.
(2) *Change of* α*:* $\mathcal{P}$ *is a* $(t', \alpha', \beta', n, m, s)$*-net in base b for all* $\alpha' \geq 1$ *where* $\beta' = \beta \min(\alpha, \alpha')/\alpha$ *and* $t' = \lceil t \min(\alpha, \alpha')/\alpha \rceil$*, and* ω *is a* $(t', \alpha', \beta', \sigma, s)$*-sequence in base b for all* $\alpha' \geq 1$ *where* $\beta' = \beta \min(\alpha, \alpha')/\alpha$ *and where* $t' = \lceil t \min(\alpha, \alpha')/\alpha \rceil$.
(3) *Reduction of resolution n: Consider the point set* $\mathcal{P}'$ *obtained by truncating the base b representation of each coordinate of each element of* $\mathcal{P}$ *after* n' *digits,* $1 \leq n' \leq n$*. The resulting point set is a* $(t', \alpha, \beta, n', m, s)$*-net in base b, where* $t' = \max(t - \beta(n - n'), 0)$.
(4) *Increasing of resolution n: Consider the point set* $\mathcal{P}'$ *obtained by truncating the base b representation of each coordinate of each element of* $\mathcal{P}$ *after n digits and adding* $n' - n$ *extra digits to every element, all of which are zero,* $n' \geq n$*. The resulting point set is a* $(t, \alpha, \beta', n', m, s)$*-net, where* $\beta' = \beta n/n'$.

(5) *Lower dimensional projection: The point set obtained by projecting $\mathcal{P}$ onto the coordinates in* $\mathfrak{u}$, *where* $\mathfrak{u} \subseteq \{1, \ldots, s\}$, *is a* $(t_{\mathfrak{u}}, \alpha, \beta, n, m, |\mathfrak{u}|)$*-net in base b, where* $t_{\mathfrak{u}} \le t$.

(6) *Multiset union of nets: Let* $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{b^r}$ *be* $(t, \alpha, \beta, n, m, s)$*-nets in base b. Then the multiset obtained from the union of the elements of* $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{b^r}$ *is a* $(t, \alpha, \beta, n, m + r, s)$*-net in base b.*

We remark that these propagation rules are analogous to Propagation Rules I–VI in [7] for digital higher order nets.

In this paper we introduce the new concept of a duality theory for *not* necessarily digital nets (in the digital case, duality theory is already well known to be an important tool for the analysis and construction of digital nets). This duality theory is in some cases an important tool to generalise certain propagation rules. Generalising propagation rules is then the second objective of the paper. In particular, we generalise the following propagation rules, which appeared in [7] for the special case of digital higher order nets: The direct product of two digital higher order nets, the $(u, u+v)$-construction, the matrix product construction, the double m-construction, base change propagation rules and the higher order to higher order construction. The paper is organised as follows. The new concept of duality theory for not necessarily digital nets is presented in Section 2 and propagation rules for $(t, \alpha, \beta, n, m, s)$-nets and $(t, \alpha, \beta, \sigma, s)$-sequences are presented in Sections 3 and 4 respectively.

Throughout the paper $\mathbb{N}_0$ is used to denote nonnegative integers and $\mathbb{N}$ is used to denote positive integers.

2. Duality theory

Duality theory, as introduced by Niederreiter and Pirsic [12] (see also [7]), is a helpful tool in the analysis and construction of digital nets. In [7] it was extended to digital higher order nets. Here we introduce a duality theory for higher order nets which also applies to point sets not obtained by the digital construction scheme. The theory developed below might also be adapted and of interest in the context of orthogonal arrays and error correcting codes. The basic tools are Walsh functions in integer base $b \ge 2$ whose definition and basic properties are recalled in the following.

Definition 2.1. Let $b \ge 2$ be an integer and represent $k \in \mathbb{N}_0$ in base b, $k = \kappa_{a-1} b^{a-1} + \cdots + \kappa_0$, with $\kappa_i \in \{0, \ldots, b-1\}$. Further let $\omega_b = \mathrm{e}^{2\pi \mathrm{i}/b}$ be the bth root of unity. Then the kth *b-adic Walsh function* ${}_b\mathrm{wal}_k(x) : [0, 1) \to \{1, \omega_b, \ldots, \omega_b^{b-1}\}$ is given by

$$ {}_b\mathrm{wal}_k(x) = \omega_b^{\xi_1 \kappa_0 + \cdots + \xi_a \kappa_{a-1}}, $$

for $x \in [0, 1)$ with base b representation $x = \xi_1 b^{-1} + \xi_2 b^{-2} + \cdots$ (unique in the sense that infinitely many of the ξ_i are different from $b-1$).

For dimension $s \ge 2$, $\boldsymbol{x} = (x_1, \ldots, x_s) \in [0, 1)^s$, and $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$, we define ${}_b\mathrm{wal}_{\boldsymbol{k}} : [0, 1)^s \to \{1, \omega_b, \ldots, \omega_b^{b-1}\}$ by

$$ {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^{s} {}_b\mathrm{wal}_{k_j}(x_j). $$

The following notation will be used throughout the paper: By $\oplus$ we denote the digitwise addition modulo b, i.e., for $x, y \in [0, 1)$ with base b expansions $x = \sum_{l=1}^{\infty} \xi_l b^{-l}$ and $y = \sum_{l=1}^{\infty} \eta_l b^{-l}$, we define

$$ x \oplus y = \sum_{l=1}^{\infty} \zeta_l b^{-l}, $$

where $\zeta_l \in \{0, \ldots, b-1\}$ is given by $\zeta_l \equiv \xi_l + \eta_l (\mathrm{mod}\ b)$. Let $\ominus$ denote the digitwise subtraction modulo b (for short we use $\ominus x := 0 \ominus x$). In the same fashion we also define the digitwise addition and digitwise subtraction for nonnegative integers based on the b-adic expansion. For vectors in $[0, 1)^s$ or $\mathbb{N}_0^s$, the operations $\oplus$ and $\ominus$ are carried out componentwise. Throughout the paper, we always use the same base b for the operations $\oplus$ and $\ominus$ as is used for the Walsh functions. Further, we call $x \in [0, 1)$ a b-adic rational if it can be written in a finite base b expansion. The following simple properties of Walsh functions will be used several times; see [8, Appendix A].

For all $k, l \in \mathbb{N}_0$ and all $x, y \in [0, 1)$, with the restriction that if x, y are not b-adic rationals, then $x \oplus y$ is not allowed to be a b-adic rational, we have ${}_b\mathrm{wal}_k(x) \cdot {}_b\mathrm{wal}_l(x) = {}_b\mathrm{wal}_{k \oplus l}(x)$ and ${}_b\mathrm{wal}_k(x) \cdot {}_b\mathrm{wal}_k(y) = {}_b\mathrm{wal}_k(x \oplus y)$. Furthermore, $\overline{{}_b\mathrm{wal}_k(x)} = {}_b\mathrm{wal}_{\ominus k}(x)$.

Now we turn to duality theory for nets. Let $\mathcal{K}_{b,r}^s = \{0, \ldots, b^r - 1\}^s$. We also assume there is an ordering of the elements in $\mathcal{K}_{b,r}^s$ which can be arbitrary but needs to be the same in each instance, i.e., let $\mathcal{K}_{b,r}^s = \{\boldsymbol{k}_0, \ldots, \boldsymbol{k}_{b^{sr}-1}\}$. (Note that $|\mathcal{K}_{b,r}^s| = b^{sr}$.) By this we mean that in expressions like $\sum_{\boldsymbol{k} \in \mathcal{K}_{b,r}^s}$, $(a_{\boldsymbol{k},\boldsymbol{l}})_{\boldsymbol{k},\boldsymbol{l} \in \mathcal{K}_{b,r}^s}$, and $(c_{\boldsymbol{k}})_{\boldsymbol{k} \in \mathcal{K}_{b,r}^s}$ the elements $\boldsymbol{k}$ and $\boldsymbol{l}$ run through the set $\mathcal{K}_{b,r}^s$ always in the same order.

The following $b^{sr} \times b^{sr}$ matrix plays a central role in the duality theory for higher order nets

$$ \mathbf{W}_r := ({}_b\mathrm{wal}_{\boldsymbol{k}}(b^{-r}\boldsymbol{l}))_{\boldsymbol{k},\boldsymbol{l} \in \mathcal{K}_{b,r}^s}. $$

We call $\mathbf{W}_r$ a *Walsh matrix*.

In the following we denote by A^* the conjugate transpose of a matrix A over the complex numbers $\mathbb{C}$, i.e., $A^* = \overline{A}^{\top}$.

Lemma 2.1. *The Walsh matrix* $\mathbf{W}_r$ *is invertible and its inverse is given by* $\mathbf{W}_r^{-1} = b^{-sr}\mathbf{W}_r^*$.

Proof. Let $\mathbf{A} = (a_{\boldsymbol{k},\boldsymbol{l}})_{\boldsymbol{k},\boldsymbol{l}\in\mathcal{K}_{b,r}^s} = b^{-sr}\mathbf{W}_r\mathbf{W}_r^*$. Then, using the orthogonality of the Walsh functions, we obtain

$$a_{\boldsymbol{k},\boldsymbol{l}} = \frac{1}{b^{sr}}\sum_{\boldsymbol{h}\in\mathcal{K}_{b,r}^s} {}_b\mathrm{wal}_{\boldsymbol{k}}\left(b^{-r}\boldsymbol{h}\right)\overline{{}_b\mathrm{wal}_{\boldsymbol{l}}\,(b^{-r}\boldsymbol{h})} = \frac{1}{b^{sr}}\prod_{j=1}^{s}\sum_{h=0}^{b^r-1} {}_b\mathrm{wal}_{k_j\ominus l_j}(h/b^r) = \begin{cases}1 & \text{if } \boldsymbol{k}=\boldsymbol{l},\\ 0 & \text{if } \boldsymbol{k}\neq\boldsymbol{l}\end{cases}$$

where $\boldsymbol{k} = (k_1, \ldots, k_s)$ and $\boldsymbol{l} = (l_1, \ldots, l_s)$ are in $\mathcal{K}_{b,r}^s$. □

Let $b \geq 2$ and $r, N \geq 1$ be integers. For a multiset $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ in $[0, 1)^s$ and $\boldsymbol{k} \in \mathcal{K}_{b,r}^s$ we define

$$c_{\boldsymbol{k}} = c_{\boldsymbol{k}}(\mathcal{P}) := \sum_{h=0}^{N-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h)$$

(note that $|c_{\boldsymbol{k}}| \leq N$ and $c_{\boldsymbol{0}} = N$) and the vector

$$\vec{C} = \vec{C}(\mathcal{P}) := (c_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathcal{K}_{b,r}^s}. \tag{1}$$

For $\boldsymbol{a} = (a_1, \ldots, a_s) \in \mathcal{K}_{b,r}^s$ define the elementary b-adic interval

$$E_{\boldsymbol{a}} := \prod_{j=1}^{s}\left[\frac{a_j}{b^r}, \frac{a_j+1}{b^r}\right).$$

Lemma 2.2. *We have*

$$\sum_{\boldsymbol{k}\in\mathcal{K}_{b,r}^s} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}\ominus\boldsymbol{y}) = \begin{cases}|\mathcal{K}_{b,r}^s| & \text{if } \boldsymbol{x},\boldsymbol{y}\in E_{\boldsymbol{a}} \text{ for some } \boldsymbol{a}\in\mathcal{K}_{b,r}^s,\\ 0 & \text{otherwise}.\end{cases}$$

Proof. We have $x, y \in [ab^{-r}, (a+1)b^{-r})$ for some $0 \leq a < b^r$ if and only if the b-adic digit expansions of x and y coincide at the first r digits. From this the result follows. □

Let $\boldsymbol{x} \in E_{\boldsymbol{a}}$ for some $\boldsymbol{a} \in \mathcal{K}_{b,r}^s$. Then, using Lemma 2.2, we have

$$\begin{aligned}\frac{1}{|\mathcal{K}_{b,r}^s|}\sum_{\boldsymbol{k}\in\mathcal{K}_{b,r}^s} c_{\boldsymbol{k}}\overline{{}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})} &= \frac{1}{|\mathcal{K}_{b,r}^s|}\sum_{\boldsymbol{k}\in\mathcal{K}_{b,r}^s}\sum_{h=0}^{N-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h\ominus\boldsymbol{x})\\ &= \sum_{h=0}^{N-1}\frac{1}{|\mathcal{K}_{b,r}^s|}\sum_{\boldsymbol{k}\in\mathcal{K}_{b,r}^s} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h\ominus\boldsymbol{x})\\ &= |\{h : \boldsymbol{x}_h \in E_{\boldsymbol{a}}\}| =: m_{\boldsymbol{a}}.\end{aligned}$$

Definition 2.2. Let $b \geq 2$ and $r, N \geq 1$ be integers. Let $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ be a multiset in $[0, 1)^s$ and let $\mathcal{K}_{b,r}^s = \{0, \ldots, b^r - 1\}^s$.

1. For $\boldsymbol{a} \in \mathcal{K}_{b,r}^s$ let

   $$m_{\boldsymbol{a}} = m_{\boldsymbol{a}}(\mathcal{P}) := |\{h : \boldsymbol{x}_h \in E_{\boldsymbol{a}}\}|$$

   and

   $$\vec{M} = \vec{M}(\mathcal{P}) := (m_{\boldsymbol{a}})_{\boldsymbol{a}\in\mathcal{K}_{b,r}^s}.$$

   Then we call the vector $\vec{M}$ the *point set vector* (with resolution r).
2. The vector $\vec{C} = \vec{C}(\mathcal{P})$ from (1) is called the *dual vector* (with respect to the Walsh matrix $\mathbf{W}_r$).
3. The set

   $$\mathcal{D}_r = \mathcal{D}_r(\mathcal{P}) := \{\boldsymbol{k} \in \mathcal{K}_{b,r}^s : c_{\boldsymbol{k}} \neq 0\}$$

   is called the *dual set* (with respect to the Walsh matrix $\mathbf{W}_r$).

In order to relate this definition to the classical duality theory [12] (and to [7]), we briefly consider the special case where the point set is a digital net. From [12] it follows that for a digital net $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}$ we have

$$c_{\boldsymbol{k}} = \begin{cases} b^m & \text{if } \boldsymbol{k} \text{ is in the dual net} \\ 0 & \text{otherwise,} \end{cases}$$

where the dual net (for a digital (t, m, s)-net) is as defined in [8, Definition 4.76] and coincides with $\mathcal{D}_m$ defined above (and with $\mathcal{D}_n$ in the higher order case). For non-digital nets on the other hand, $c_{\boldsymbol{k}}$ can also take on values different from 0 and b^m.

The relationship between a point set vector and its dual vector is stated in the following theorem.

Theorem 2.1. *Let $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ be a multiset in $[0, 1)^s$ and let $r \in \mathbb{N}$. Let $\vec{M}$ be the point set vector with resolution r and $\vec{C}$ be the dual vector with respect to $\mathbf{W}_r$ defined as above. Then*

$$\frac{1}{|\mathcal{K}^s_{b,r}|}\mathbf{W}_r\vec{C} = \vec{M} \quad \text{and} \quad \vec{C} = \mathbf{W}_r^*\vec{M}. \tag{2}$$

Proof. The first result follows from Lemma 2.2 and the second result follows from Lemma 2.1 and the identity $\vec{C} = |\mathcal{K}^s_{b,r}|\mathbf{W}_r^{-1}\vec{M} = \mathbf{W}_r^*\vec{M}$. □

The vector $\vec{C}$ carries sufficient information to construct a point set in the following way: Given $\vec{C}$, we can use Theorem 2.1 to determine how many points are to be placed in the interval $E_{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathcal{K}^s_{b,r}$.

Note that for the $(t, \alpha, \beta, n, m, s)$-net property it is of no importance where exactly within an interval $E_{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathcal{K}^s_{b,n}$, the points are placed. Hence we can reconstruct a net from a dual vector with respect to $\mathbf{W}_r$ provided that $r \geq \lfloor \beta n \rfloor - t$. In other words, if one knows the dual vector of a net, then one can use this dual vector to obtain the net via Theorem 2.1 provided that the resolution is greater than or equal to the strength of the net.

In analogy, the dual space of a digital net also allows us to reconstruct the original point set, see [12]. Although $\vec{C}$ is different from the dual space for digital nets, it contains the same information and can be used in a manner similar to the dual space. This will be shown below by the example of the direct product construction, the $(u, u + v)$-construction, the matrix-product construction and the double m construction for higher order nets. In case $\mathcal{P}$ is a digital $(t, \alpha, \beta, n \times m, s)$-net, the dual set $\mathcal{D}_n$ defined in Definition 2.2 coincides with the dual space defined in [7] intersected with $\mathcal{K}^s_{b,n}$, and if $\mathcal{P}$ is a digital (t, m, s)-net, it coincides with the dual space in [12] intersected with $\mathcal{K}^s_{b,m}$.

Although the above results hold for arbitrary point sets, in the following we consider point sets which are nets and show how to relate the quality of a $(t, \alpha, \beta, n, m, s)$-net to its dual set. To this end we need to introduce a function which was first introduced in [4] in the context of applying digital nets to quasi-Monte Carlo integration of smooth functions and which is related to the quality of suitable digital nets. For $k \in \mathbb{N}_0$ and $\alpha \geq 1$ let

$$\mu_\alpha(k) = \begin{cases} a_1 + \cdots + a_{\min(\nu,\alpha)} & \text{for } k > 0, \\ 0 & \text{for } k = 0, \end{cases}$$

where for $k > 0$ we assume that $k = \kappa_1 b^{a_1-1} + \cdots + \kappa_\nu b^{a_\nu-1}$ with $0 < \kappa_1, \ldots, \kappa_\nu < b$ and $1 \leq a_\nu < \cdots < a_1$. Note that for $\alpha = 1$ we obtain the well-known Niederreiter–Rosenbloom–Tsfasman (NRT) weight (see, for example, [8, Section 7.1]).

For a vector $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ we define $\mu_\alpha(\boldsymbol{k}) = \mu_\alpha(k_1) + \cdots + \mu_\alpha(k_s)$ and for a subset $\mathcal{Q}$ of $\mathcal{K}^s_{b,r}$ with $\mathcal{Q} \setminus \{\mathbf{0}\} \neq \emptyset$ and $\alpha \geq 1$ define

$$\rho_\alpha(\mathcal{Q}) := \min_{\boldsymbol{k} \in \mathcal{Q}\setminus\{\mathbf{0}\}} \mu_\alpha(\boldsymbol{k}).$$

For $\mathcal{Q} \subseteq \{\mathbf{0}\}$ we set $\rho_\alpha(\mathcal{Q}) = r + 1$.

Let $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\} \subset [0, 1)^s$. In the following we consider for which cases we have $\mathcal{D}_r(\mathcal{P}) = \{\mathbf{0}\}$ (note that $\mathbf{0} \in \mathcal{D}_r(\mathcal{P})$ for any point set $\mathcal{P}$ with at least one point). If $\mathcal{D}_r(\mathcal{P}) = \{\mathbf{0}\}$, then we have $c_{\mathbf{0}} \neq 0$ and $c_{\boldsymbol{k}} = 0$ for all $\boldsymbol{k} \in \mathcal{K}^s_{b,r} \setminus \{\mathbf{0}\}$. By Theorem 2.1 we have $\vec{M}(\mathcal{P}) = c_{\mathbf{0}} b^{-rs}(1, 1, \ldots, 1)^\top$, that is, each box $E_{\boldsymbol{a}}$ contains exactly $c_{\mathbf{0}} b^{-rs}$ points for all $\boldsymbol{a} \in \mathcal{K}^s_{b,r}$ and $\mathcal{P}$ consists of $N = c_{\mathbf{0}}$ points altogether. This is the only case for which $\mathcal{D}_r(\mathcal{P}) = \{\mathbf{0}\}$.

Conversely, since the number of points in $E_{\boldsymbol{a}}$ must be an integer, it follows that $c_{\mathbf{0}} b^{-rs} \in \mathbb{N}$, i.e., b^{rs} divides $c_{\mathbf{0}}$ and therefore b^{rs} divides N. From this we conclude that if we choose a resolution $r \in \mathbb{N}$ such that $b^{rs} > N$, i.e., $r > \frac{1}{s}\log_b N$, then $\mathcal{D}_r(\mathcal{P}) \neq \{\mathbf{0}\}$. For a net with $N = b^m$ points this means that we require $r > m/s$.

The following theorem establishes a relationship between $\rho_\alpha(\mathcal{Q})$ and the quality of a $(t, \alpha, \beta, n, m, s)$-net.

Theorem 2.2. *Let $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\} \subset [0, 1)^s$ be a multiset. Then $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b if and only if $\rho_\alpha(\mathcal{D}_{\lfloor \beta n \rfloor - t}) \geq \lfloor \beta n \rfloor - t + 1$. If $\mathcal{P}$ is a strict $(t_0, \alpha, \beta, n, m, s)$-net in base b, then $\rho_\alpha(\mathcal{D}_{\lfloor \beta n \rfloor - t_0}) = \lfloor \beta n \rfloor - t_0 + 1$.*

Proof. It was shown in [1, Theorem 1] that $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b if and only if for all $\boldsymbol{k} \in \mathbb{N}_0^s$ satisfying $0 < \mu_\alpha(\boldsymbol{k}) \leq \lfloor \beta n \rfloor - t$ we have $\sum_{h=0}^{b^m-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) = 0$ and this is equivalent to $\rho_\alpha(\mathcal{D}_{\lfloor \beta n \rfloor - t}) \geq \lfloor \beta n \rfloor - t + 1$, since for $\boldsymbol{k} \in \mathbb{N}_0^s$ with $\mu_\alpha(\boldsymbol{k}) \leq \lfloor \beta n \rfloor - t$ we have $\boldsymbol{k} \in \mathcal{K}^s_{b,\lfloor \beta n \rfloor - t}$, with equality holding if $\mathcal{P}$ is a strict $(t, \alpha, \beta, n, m, s)$-net. □

Let now $\mathcal{P}$ be a strict $(t_0, \alpha, \beta, n, m, s)$-net in base b. Let $r \geq \lfloor \beta n \rfloor - t_0$. Then $\mathcal{D}_r \supseteq \mathcal{D}_{\lfloor \beta n \rfloor - t_0}$ and $\mathcal{D}_r \setminus \mathcal{D}_{\lfloor \beta n \rfloor - t_0} \subseteq \mathcal{K}_{b,r}^s \setminus \mathcal{K}_{b,\lfloor \beta n \rfloor - t_0}^s$. For any $\boldsymbol{k} \in \mathcal{K}_{b,r}^s \setminus \mathcal{K}_{b,\lfloor \beta n \rfloor - t_0}^s$ we have $\mu_\alpha(\boldsymbol{k}) \geq \lfloor \beta n \rfloor - t_0 + 1$. Theorem 2.2 implies that $\rho_\alpha(\mathcal{D}_{\lfloor \beta n \rfloor - t_0}) = \lfloor \beta n \rfloor - t_0 + 1$ and hence $\rho_\alpha(\mathcal{D}_r) = \rho_\alpha(\mathcal{D}_{\lfloor \beta n \rfloor - t_0}) = \lfloor \beta n \rfloor - t_0 + 1$. In particular, for all $r, r' \geq \lfloor \beta n \rfloor - t_0$ we have

$$\rho_\alpha(\mathcal{D}_r) = \rho_\alpha(\mathcal{D}_{r'}) = \rho_\alpha(\mathcal{D}_n) = \lfloor \beta n \rfloor - t_0 + 1, \tag{3}$$

since $n \geq \lfloor \beta n \rfloor - t_0$.

3. Propagation rules for $(t, \alpha, \beta, n, m, s)$-nets

In this section, we introduce several propagation rules for $(t, \alpha, \beta, n, m, s)$-nets, many of which generalise the analogous results for the digital case given in [7].

3.1. The direct product of two $(t, \alpha, \beta, n, m, s)$-nets

Let $\mathcal{P}_1 = \{\boldsymbol{x}_h\}_{h=0}^{b^{m_1}-1}$ be a $(t_1, \alpha_1, \beta_1, n_1, m_1, s_1)$-net in base b and $\mathcal{P}_2 = \{\boldsymbol{y}_i\}_{i=0}^{b^{m_2}-1}$ be a $(t_2, \alpha_2, \beta_2, n_2, m_2, s_2)$-net in base b. Note that Definition 1.1 implies that we may assume that $\beta_1 n_1$ and $\beta_2 n_2$ are integers (since the $i_{j,l}$ are integers). Based on $\mathcal{P}_1$ and $\mathcal{P}_2$ a new $(t, \alpha, \beta, n, m, s)$-net in base b is formed, where $n = n_1 + n_2$, $m = m_1 + m_2$, and $s = s_1 + s_2$. The points of $\mathcal{P}$ are defined to be the direct product of the points from $\mathcal{P}_1$ and $\mathcal{P}_2$, i.e., $\mathcal{P}$ is the multiset of b^m points

$$(\boldsymbol{x}_h, \boldsymbol{y}_i), \quad \text{for } 0 \leq h \leq b^{m_1} - 1 \text{ and } 0 \leq i \leq b^{m_2} - 1 \tag{4}$$

in some order. The following theorem gives information on the t-value of the resulting $(t, \alpha, \beta, n, m, s)$-net.

Theorem 3.1 (*Propagation Rule 7*). *Let $\mathcal{P}_1$ be a $(t_1, \alpha_1, \beta_1, n_1, m_1, s_1)$-net in base b where we assume that $\beta_1 n_1$ is an integer, $\mathcal{P}_2$ is a $(t_2, \alpha_2, \beta_2, n_2, m_2, s_2)$-net in base b where we assume that $\beta_2 n_2$ is an integer, and $\mathcal{P}$ be defined as above. Then $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b, where $\alpha = \max(\alpha_1, \alpha_2)$, $\beta = \min(\beta_1, \beta_2)$, and*

$$t \leq \max(\beta_1 n_1 + t_2, \beta_2 n_2 + t_1).$$

Proof. Let $\vec{C} = (c_{\boldsymbol{k}})_{\boldsymbol{k} \in \mathcal{K}_{b,n}^s}$ be the dual vector of $\mathcal{P}$, $\vec{C}_1 = (c_{1,\boldsymbol{k}'})_{\boldsymbol{k}' \in \mathcal{K}_{b,n}^{s_1}}$ be the dual vector of $\mathcal{P}_1$ and $\vec{C}_2 = (c_{2,\boldsymbol{k}''})_{\boldsymbol{k}'' \in \mathcal{K}_{b,n}^{s_2}}$ be the dual vector of $\mathcal{P}_2$. Then, for $\boldsymbol{k} = (\boldsymbol{k}', \boldsymbol{k}'')$, where $\boldsymbol{k}' \in \mathcal{K}_{b,n}^{s_1}$, $\boldsymbol{k}'' \in \mathcal{K}_{b,n}^{s_2}$, $\boldsymbol{k} \in \mathcal{K}_{b,n}^s$, we have

$$c_{\boldsymbol{k}} = \sum_{h=0}^{b^{m_1}-1} \sum_{i=0}^{b^{m_2}-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h, \boldsymbol{y}_i) = \sum_{h=0}^{b^{m_1}-1} {}_b\mathrm{wal}_{\boldsymbol{k}'}(\boldsymbol{x}_h) \sum_{i=0}^{b^{m_2}-1} {}_b\mathrm{wal}_{\boldsymbol{k}''}(\boldsymbol{y}_i) = c_{1,\boldsymbol{k}'} c_{2,\boldsymbol{k}''}.$$

Hence $c_{\boldsymbol{k}} \neq 0$ if and only if $c_{1,\boldsymbol{k}'} \neq 0$ and $c_{2,\boldsymbol{k}''} \neq 0$. Note that $c_{1,\boldsymbol{0}}, c_{2,\boldsymbol{0}} \neq 0$ and $\boldsymbol{k} = (\boldsymbol{k}', \boldsymbol{k}'') \neq \boldsymbol{0}$ implies that either $\boldsymbol{k}' \neq \boldsymbol{0}$ or $\boldsymbol{k}'' \neq \boldsymbol{0}$ or both $\boldsymbol{k}', \boldsymbol{k}'' \neq \boldsymbol{0}$. Therefore

$$\rho_\alpha(\mathcal{D}_n) = \min(\rho_\alpha(\mathcal{D}_{n,1}), \rho_\alpha(\mathcal{D}_{n,2})) \geq \min(\rho_{\alpha_1}(\mathcal{D}_{n,1}), \rho_{\alpha_2}(\mathcal{D}_{n,2})),$$

where $\mathcal{D}_n, \mathcal{D}_{n,1}, \mathcal{D}_{n,2}$ are the dual sets of $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2$.

Let t_0 be the integer such that $\mathcal{P}$ is a strict $(t_0, \alpha, \beta, n, m, s)$-net in base b, where $\alpha = \max(\alpha_1, \alpha_2)$, $\beta = \min(\beta_1, \beta_2)$, $n = n_1 + n_2$, $m = m_1 + m_2$, and $s = s_1 + s_2$. Then, by Theorem 2.2, Eq. (3), and the assumption that $\beta_1 n_1$ and $\beta_2 n_2$ are integers, we have

$$\begin{aligned}
t_0 &= \lfloor \beta n \rfloor - \rho_\alpha(\mathcal{D}_n) + 1 \\
&\leq \lfloor \beta n \rfloor + 1 - \min(\rho_{\alpha_1}(\mathcal{D}_{n,1}), \rho_{\alpha_2}(\mathcal{D}_{n,2})) \\
&\leq \lfloor \beta n \rfloor - \min(\beta_1 n_1 - t_1, \beta_2 n_2 - t_2) \\
&\leq \lfloor \beta n \rfloor - \min(-\beta_2 n_2 + \beta n_2 + \beta_1 n_1 - t_1, -\beta_1 n_1 + \beta n_1 + \beta_2 n_2 - t_2) \\
&\leq \lfloor \beta n \rfloor - \min(\lfloor \beta n \rfloor - \beta_2 n_2 - t_1, \lfloor \beta n \rfloor - \beta_1 n_1 - t_2) \\
&= \max(\beta_2 n_2 + t_1, \beta_1 n_1 + t_2).
\end{aligned}$$

This shows that $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b for any integer t such that $t_0 \leq t \leq \max(\beta_2 n_2 + t_1, \beta_1 n_1 + t_2)$. Hence the result follows. □

3.2. The $(u, u + v)$-construction

The $(u, u + v)$-construction in the context of $(t, \alpha, \beta, n, m, s)$-nets in base b has already been discussed in the recent paper [1, Section 5]. Hence we simply recall the construction and state the result. Assume we are given a $(t_1, \alpha, \beta_1, n_1, m_1, s_1)$-net $\mathcal{P}_1$ denoted by $\{\boldsymbol{x}_h\}_{h=0}^{b^{m_1}-1}$ and a $(t_2, \alpha, \beta_2, n_2, m_2, s_2)$-net $\mathcal{P}_2$ denoted by $\{\boldsymbol{y}_i\}_{i=0}^{b^{m_2}-1}$, where we assume $s_1 \leq s_2$. Again we may assume that $\beta_1 n_1$ and $\beta_2 n_2$ are integers. Further, w.l.o.g. we may assume that $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,s_1})$ with $x_{h,j} = \xi_{h,j,1}/b + \cdots + \xi_{h,j,n_1}/b^{n_1}$ and $\boldsymbol{y}_i = (y_{i,1}, \ldots, y_{i,s_2})$ with $y_{i,j} = \eta_{i,j,1}/b + \cdots + \eta_{i,j,n_2}/b^{n_2}$ (if there are digits $\xi_{h,j,r} \neq 0$ for $r > n_1$ or $\eta_{i,j,r} \neq 0$ for $r > n_2$ we can slightly modify $\mathcal{P}_1, \mathcal{P}_2$ by setting $\xi_{h,j,r} = 0$ for $r > n_1$ and $\eta_{i,j,r} = 0$for $r > n_2$, without changing the $(t_w, \alpha, \beta_w, n_w, m_w, s_w)$-net property of $\mathcal{P}_w$, $w = 1, 2$). Set further

$\ell := \min(2\beta_1 n_1 - 2t_1 + 1, \beta_2 n_2 - t_2)$.

We define a new point set $\mathcal{P} = \{\boldsymbol{z}_h\}_{h=0}^{b^{m_1+m_2}-1}$, $\boldsymbol{z}_h = (z_{h,1}, \ldots, z_{h,s_1+s_2})$, consisting of $b^{m_1+m_2}$ points in $[0, 1)^{s_1+s_2}$ as follows:

- For $j = 1, \ldots, s_1$, $i = 0, \ldots, b^{m_2} - 1$ and $h = 0, \ldots, b^{m_1} - 1$ we set

$$z_{ib^{m_1}+h,j} = \frac{\xi_{h,j,1} \ominus \eta_{i,j,1}}{b} + \cdots + \frac{\xi_{h,j,\min(\ell,n_1)} \ominus \eta_{i,j,\min(\ell,n_1)}}{b^{\min(\ell,n_1)}} + \left(\frac{\xi_{h,j,\ell+1}}{b^{\ell+1}} + \cdots + \frac{\xi_{h,j,n_1}}{b^{n_1}}\right)\mathbf{1}_{n_1>\ell} + \left(\frac{\ominus\eta_{i,j,n_1+1}}{b^{n_1+1}} + \cdots + \frac{\ominus\eta_{i,j,\ell}}{b^{\ell}}\right)\mathbf{1}_{n_1<\ell}.$$

- For $j = s_1 + 1, \ldots, s_1 + s_2$, $i = 0, \ldots, b^{m_2} - 1$ and $h = 0, \ldots, b^{m_1} - 1$ we set

$$z_{ib^{m_1}+h,j} = y_{i,j-s_1}.$$

Then we have the following result, which was first shown in [1, Theorem 3]:

Theorem 3.2 (*Propagation Rule 8*). *Let $b \geq 2$ be an integer, let $\mathcal{P}_1$ be a $(t_1, \alpha, \beta_1, n_1, m_1, s_1)$-net in base b where we assume that $\beta_1 n_1$ is an integer, and $\mathcal{P}_2$ be a $(t_2, \alpha, \beta_2, n_2, m_2, s_2)$-net in base b where we assume that $\beta_2 n_2$ is an integer. Then $\mathcal{P}$ defined as above is a $(t, \alpha, \beta, n, m, s)$-net in base b, where $n = n_1 + n_2$, $m = m_1 + m_2$, $s = s_1 + s_2$, $\beta = \min(\beta_1, \beta_2)$, and*

$$t = \lfloor \beta n \rfloor - \ell.$$

Remark 3.1. Note that we defined the $(u, u + v)$-construction in such a way that it yields the same point set as the $(u, u + v)$-construction for digital nets as considered in [7].

3.3. The matrix-product construction

In this subsection, we will assume that b is prime. We first introduce matrices which are nonsingular by column (NSC), see [2]. Let A be an $M \times M$ matrix over $\mathbb{Z}_b$. For $1 \leq l \leq M$, let A_l denote the $l \times M$ matrix consisting of the first l rows of A. For $1 \leq k_1 < \cdots < k_l \leq M$, let $A(k_1, \ldots, k_l)$ denote the $l \times l$ matrix consisting of the columns $k_1, \ldots, k_l$ of A_l.

Definition 3.1. An $M \times M$ matrix A defined over $\mathbb{Z}_b$ is called *nonsingular by column (NSC)* if $A(k_1, \ldots, k_l)$ is nonsingular for each $1 \leq l \leq M$ and $1 \leq k_1 < \cdots < k_l \leq M$.

It is known that an $M \times M$ NSC matrix over $\mathbb{Z}_b$ exists if and only if $1 \leq M \leq b$, see [2, Section 3]. For any integer $1 \leq M \leq b$, an explicit $M \times M$ upper triangular NSC matrix over $\mathbb{Z}_b$ is given in [2, Section 5.2].

For the remainder of this section, we will assume that $A = (A_{k,l})$ is an $M \times M$ upper triangular NSC matrix over $\mathbb{Z}_b$ (upper triangular means that $A_{k,l} = 0$ for all $1 \leq l < k \leq M$).

We now describe how to construct a point set from an NSC matrix, based on the so-called matrix-product construction:

Let $1 \leq s_1 \leq \cdots \leq s_M$ be integers and define $\sigma_0 := 0$ and $\sigma_k := s_1 + \cdots + s_k$ for $1 \leq k \leq M$. Let $s := \sigma_M$. For $1 \leq k \leq M$ let $\mathcal{P}_k = \{\boldsymbol{x}_h^{(k)}\}_{h=0}^{b^{m_k}-1}$, where $\boldsymbol{x}_h^{(k)} = (x_{h,\sigma_{k-1}+1}^{(k)}, \ldots, x_{h,\sigma_k}^{(k)})$ for $0 \leq h < b^{m_k}$, be $(t_k, \alpha, \beta_k, n_k, m_k, s_k)$-nets in base b where we assume that $\beta_k n_k$ is an integer. (As with the $(u, u + v)$-construction, one can without loss of generality assume that $x_{h,j}^{(k)} = \xi_{h,j,1}^{(k)}/b + \xi_{h,j,2}^{(k)}/b^2 + \cdots$ with $\xi_{h,j,c}^{(k)} = 0$ for $c > n_k$, as setting the remaining digits to zero does not affect the quality of the net $\mathcal{P}_k$. However, this is not necessary as the results in this subsection also hold otherwise.)

We now define $V = (V_{k,l})_{k,l=1}^M := A^{-1} \in \mathbb{Z}_b^{M\times M}$ and note that V is upper triangular. For

$$h = h_1 + h_2 b^{m_1} + \cdots + h_M b^{m_1+m_2+\cdots+m_{M-1}},$$

with integers $0 \leq h_k < b^{m_k}$ (hence $0 \leq h < b^m$ where $m = m_1 + \cdots + m_M$) and for $\sigma_{k-1} < j \leq \sigma_k$, $k = 1, \ldots, M$, define

$$z_{h,j} := V_{k,k} x_{h_k,j}^{(k)} \oplus \cdots \oplus V_{k,M} x_{h_M,j}^{(M)}, \tag{5}$$

where $\oplus$ and also the multiplication are carried out digitwise modulo b, i.e., $z_{h,j} = \zeta_{h,j,1}/b + \zeta_{h,j,2}/b^2 + \cdots$ where

$$\zeta_{h,j,c} = V_{k,k}\xi_{h_k,j,c}^{(k)} + \cdots + V_{k,M}\xi_{h_M,j,c}^{(M)} \in \mathbb{Z}_b \quad \text{for all } c \geq 1,$$

with $x_{h_l,j}^{(l)} = \xi_{h_l,j,1}^{(l)}/b + \xi_{h_l,j,2}^{(l)}/b^2 + \cdots$ for $k \leq l \leq M$, where addition and multiplication are carried out in $\mathbb{Z}_b$, and where we assume that for each h and j infinitely many of the digits $\zeta_{h,j,c}$, $c = 1, 2, \ldots$ are different from $b-1$ (if this is not the case, then, for example by modifying any of the digits $\zeta_{h,j,c}$, $c > \max_{1\leq k\leq M} n_k$, will solve this problem without affecting the quality of the point set; indeed, the forthcoming Theorem 3.3 will establish that the digits $\zeta_{h,j,c}$ with $c > \min_{1\leq k\leq M}(M-k+1)(\beta_k n_k - t_k)$ can be modified arbitrarily since they do not influence the quality of the net; this way, for $M = 2$, the $(u, u+v)$-construction can be viewed as a special case of the matrix product construction).

Analogously to the notation used above, we write $\bigoplus_{l=1}^k A_{l,k}\boldsymbol{u}_l^{(k)} = A_{1,k}\boldsymbol{u}_1^{(k)} \oplus \cdots \oplus A_{k,k}\boldsymbol{u}_k^{(k)}$, where the addition and multiplication are carried out digitwise modulo b.

Now we define $\mathcal{P} = \{\boldsymbol{z}_0, \ldots, \boldsymbol{z}_{b^m-1}\}$ with $m = m_1 + \cdots + m_M$ through $\boldsymbol{z}_h := (z_{h,1}, \ldots, z_{h,s})$ for $0 \le h < b^m$.

Lemma 3.1. *Let $\boldsymbol{d} = (\boldsymbol{d}_1, \ldots, \boldsymbol{d}_M) \in \mathcal{K}_{b,n}^s$ with $\boldsymbol{d}_k \in \mathcal{K}_{b,n}^{s_k}$ and assume that $\boldsymbol{d}_k = \bigoplus_{l=1}^k A_{l,k}\boldsymbol{u}_l^{(k)}$ where for $l \le k$, $\boldsymbol{u}_l^{(k)} = (\boldsymbol{u}_l, \boldsymbol{0}) \in \mathcal{K}_{b,n}^{s_k}$ for some $\boldsymbol{u}_l \in \mathcal{K}_{b,n}^{s_l}$. Then we have*

$$\frac{1}{b^{m_1+m_2+\cdots+m_M}} \sum_{h=0}^{b^{m_1+m_2+\cdots+m_M}-1} {}_b\mathrm{wal}_{\boldsymbol{d}}(\boldsymbol{z}_h) = \prod_{r=1}^{M} \left(\frac{1}{b^{m_r}} \sum_{h_r=0}^{b^{m_r}-1} {}_b\mathrm{wal}_{\boldsymbol{u}_r}(\boldsymbol{x}_{h_r}^{(r)}) \right).$$

Proof. Let $\boldsymbol{z}_h = (\boldsymbol{z}_h^{(s_1)}, \ldots, \boldsymbol{z}_h^{(s_M)}) \in [0,1)^{s_1+\cdots+s_M}$ where $\boldsymbol{z}_h^{(s_k)} = (z_{h,\sigma_{k-1}+1}, \ldots, z_{h,\sigma_k}) \in [0,1)^{s_k}$ for $1 \le k \le M$. For $\boldsymbol{d} = (\boldsymbol{d}_1, \ldots, \boldsymbol{d}_M) \in \mathcal{K}_{b,n}^s$ with $\boldsymbol{d}_k \in \mathcal{K}_{b,n}^{s_k}$ we have

$$\sum_{h=0}^{b^{m_1+\cdots+m_M}-1} {}_b\mathrm{wal}_{\boldsymbol{d}}(\boldsymbol{z}_h) = \sum_{h=0}^{b^{m_1+\cdots+m_M}-1} \prod_{k=1}^{M} {}_b\mathrm{wal}_{\boldsymbol{d}_k}(\boldsymbol{z}_h^{(s_k)}).$$

By assumption we have $\boldsymbol{d}_k = \bigoplus_{l=1}^k A_{l,k}\boldsymbol{u}_l^{(k)}$, where for $l \le k$, $\boldsymbol{u}_l^{(k)} = (\boldsymbol{u}_l, \boldsymbol{0}) \in \mathcal{K}_{b,n}^{s_k}$ for some $\boldsymbol{u}_l \in \mathcal{K}_{b,n}^{s_l}$. Let furthermore $\overline{\boldsymbol{u}}_l = (\boldsymbol{u}_l, \boldsymbol{0}) \in \mathcal{K}_{b,n}^s$. Then for each of the above summands we have

$$\begin{aligned}
\prod_{k=1}^{M} {}_b\mathrm{wal}_{\boldsymbol{d}_k}(\boldsymbol{z}_h^{(s_k)}) &= \prod_{k=1}^{M} {}_b\mathrm{wal}_{\oplus_{l=1}^k A_{l,k}\boldsymbol{u}_l^{(k)}}(\boldsymbol{z}_h^{(s_k)}) \\
&= \prod_{k=1}^{M} {}_b\mathrm{wal}_{\oplus_{l=1}^k A_{l,k}\boldsymbol{u}_l^{(k)}}(z_{h,\sigma_{k-1}+1}, \ldots, z_{h,\sigma_k}) \\
&= \prod_{k=1}^{M} \prod_{r=k}^{M} {}_b\mathrm{wal}_{\oplus_{l=1}^k A_{l,k}\boldsymbol{u}_l^{(k)}}(V_{k,r}x_{h_r,\sigma_{k-1}+1}^{(r)}, \ldots, V_{k,r}x_{h_r,\sigma_k}^{(r)}) \\
&= \prod_{k=1}^{M} \prod_{r=k}^{M} {}_b\mathrm{wal}_{V_{k,r}\left(\oplus_{l=1}^k A_{l,k}\boldsymbol{u}_l^{(k)}\right)}(x_{h_r,\sigma_{k-1}+1}^{(r)}, \ldots, x_{h_r,\sigma_k}^{(r)}) \\
&= \prod_{k=1}^{M} \prod_{r=k}^{M} {}_b\mathrm{wal}_{V_{k,r}\left(\oplus_{l=1}^k A_{l,k}\boldsymbol{u}_l^{(k)}\right)}(\boldsymbol{x}_{h_r}^{(r)}) \\
&= \prod_{r=1}^{M} \prod_{k=1}^{r} {}_b\mathrm{wal}_{V_{k,r}\left(\oplus_{l=1}^k A_{l,k}\boldsymbol{u}_l^{(k)}\right)}(\boldsymbol{x}_{h_r}^{(r)}) \\
&= \prod_{r=1}^{M} {}_b\mathrm{wal}_{\oplus_{k=1}^r V_{k,r}\left(\oplus_{l=1}^k A_{l,k}\boldsymbol{u}_l^{(k)}\right)}(\boldsymbol{x}_{h_r}^{(r)}) \\
&= \prod_{r=1}^{M} {}_b\mathrm{wal}_{\oplus_{k=1}^r V_{k,r}\left(\oplus_{l=1}^k A_{l,k}\overline{\boldsymbol{u}}_l\right)}((\boldsymbol{x}_{h_r}^{(r)}, \boldsymbol{0})),
\end{aligned}$$

where $(\boldsymbol{x}_{h_r}^{(r)}, \boldsymbol{0}) \in [0,1)^s$ is just the concatenation of $\boldsymbol{x}_{h_r}^{(r)} \in [0,1)^{s_r}$ and the $s - s_r$ dimensional zero vector $\boldsymbol{0}$. Since $V = A^{-1}$ we now have

$$\bigoplus_{k=l}^{r} V_{k,r}A_{l,k} = \begin{cases} 1 & \text{if } r = l \\ 0 & \text{if } r \ne l. \end{cases}$$

Hence we obtain $\bigoplus_{k=1}^r V_{k,r} \bigoplus_{l=1}^k A_{l,k}\overline{\boldsymbol{u}}_l = \bigoplus_{l=1}^r \overline{\boldsymbol{u}}_l \bigoplus_{k=l}^r V_{k,r}A_{l,k} = \overline{\boldsymbol{u}}_r$ and hence

$$\prod_{r=1}^{M} {}_b\mathrm{wal}_{\oplus_{k=1}^r V_{k,r}\left(\oplus_{l=1}^k A_{l,k}\overline{\boldsymbol{u}}_l\right)}((\boldsymbol{x}_{h_r}^{(r)}, \boldsymbol{0})) = \prod_{r=1}^{M} {}_b\mathrm{wal}_{\overline{\boldsymbol{u}}_r}((\boldsymbol{x}_{h_r}^{(r)}, \boldsymbol{0})) = \prod_{r=1}^{M} {}_b\mathrm{wal}_{\boldsymbol{u}_r}(\boldsymbol{x}_{h_r}^{(r)}).$$

Hence

$$\begin{aligned}
\frac{1}{b^{m_1+\cdots+m_M}} \sum_{h=0}^{b^{m_1+\cdots+m_M}-1} {}_b\mathrm{wal}_{\boldsymbol{d}}(\boldsymbol{z}_h) &= \frac{1}{b^{m_1+\cdots+m_M}} \sum_{h=0}^{b^{m_1+\cdots+m_M}-1} \prod_{r=1}^{M} {}_b\mathrm{wal}_{\boldsymbol{u}_r}(\boldsymbol{x}_{h_r}^{(r)}) \\
&= \prod_{r=1}^{M} \left(\frac{1}{b^{m_r}} \sum_{h_r=0}^{b^{m_r}-1} {}_b\mathrm{wal}_{\boldsymbol{u}_r}(\boldsymbol{x}_{h_r}^{(r)}) \right). \quad \square
\end{aligned}$$

For the rest of the subsection, we make the convention that

$$\mu_\alpha(\boldsymbol{d}) = \sum_{k=1}^{M} \mu_\alpha(\boldsymbol{d}_k).$$

If $\mu_\alpha(\boldsymbol{d}) > 0$, then there exists at least one integer l, so that $\boldsymbol{u}_l \neq \boldsymbol{0}$; the largest integer l so that $\boldsymbol{u}_l \neq \boldsymbol{0}$ is denoted by l^*. We need the following lemma.

Lemma 3.2. *Let $\boldsymbol{d}$ be as in Lemma 3.1 with $\mu_\alpha(\boldsymbol{d}) > 0$ and let l^* denote the largest integer l so that $\boldsymbol{u}_l \neq \boldsymbol{0}$. Then we have $\mu_\alpha(\boldsymbol{d}) \geq (M - l^* + 1)\mu_\alpha(\boldsymbol{u}_{l^*})$.*

Proof. The proof follows along the same lines as the proofs of [7, Lemmas 2 and 3]. □

We can now show the main result of this subsection.

Theorem 3.3 (*Propagation Rule 9*). *Let $1 \leq s_1 \leq \cdots \leq s_M$ be integers. For $1 \leq k \leq M$ let $\mathcal{P}_k = \{\boldsymbol{x}_h^{(k)}\}_{h=0}^{b^{m_k}-1}$, where $\boldsymbol{x}_h^{(k)} = (x_{h,\sigma_{k-1}+1}^{(k)}, \ldots, x_{h,\sigma_k}^{(k)})$ for $0 \leq h < b^{m_k}$, be $(t_k, \alpha, \beta_k, n_k, m_k, s_k)$-nets in base b where we assume that $\beta_k n_k$ is an integer.*

The multiset $\mathcal{P} = \{\boldsymbol{z}_0, \ldots, \boldsymbol{z}_{b^m-1}\}$, where $\boldsymbol{z}_h := (z_{h,1}, \ldots, z_{h,s})$ and where the $z_{h,j}$ are given by (5), forms a $(t, \alpha, \beta, n, m, s)$-net, where $s = s_1 + \cdots + s_M$, $n = \max_{1 \leq k \leq M} n_k$, $m = m_1 + \cdots + m_M$, $\beta = \min(1, \alpha m/n)$ and

$$t \leq \beta n - \min_{1 \leq l \leq M} (M - l + 1)(\beta_l n_l - t_l).$$

Proof. According to [1, Theorem 1] it is enough to show that

$$\frac{1}{b^{m_1+m_2+\cdots+m_M}} \sum_{h=0}^{b^{m_1+m_2+\cdots+m_M}-1} {}_b\mathrm{wal}_{\boldsymbol{d}}(\boldsymbol{z}_h) = 0$$

for all $\boldsymbol{d} \in \mathbb{N}_0^s$ satisfying $0 < \mu_\alpha(\boldsymbol{d}) \leq \beta n - t$. As $\boldsymbol{d}$ must satisfy $\mu_\alpha(\boldsymbol{d}) \leq \beta n - t$ we may restrict ourselves to $\boldsymbol{d} \in \mathcal{K}_{b,n}^s$ satisfying $0 < \mu_\alpha(\boldsymbol{d}) \leq \beta n - t$. From Lemma 3.1, we know that

$$\frac{1}{b^{m_1+m_2+\cdots+m_M}} \sum_{h=0}^{b^{m_1+m_2+\cdots+m_M}-1} {}_b\mathrm{wal}_{\boldsymbol{d}}(\boldsymbol{z}_h) = \prod_{r=1}^{M} \left(\frac{1}{b^{m_r}} \sum_{h_r=0}^{b^{m_r}-1} {}_b\mathrm{wal}_{\boldsymbol{u}_r}(\boldsymbol{x}_{h_r}^{(r)}) \right). \tag{6}$$

Assume now that $\boldsymbol{d} \in \mathcal{K}_{b,n}^s$ is such that $0 < \mu_\alpha(\boldsymbol{d}) \leq \beta n - t$, then there exists an integer l so that $\mu_\alpha(\boldsymbol{u}_l) > 0$ and as before, we denote the largest integer l so that $\mu_\alpha(\boldsymbol{u}_l) > 0$ by l^*. We now use Lemma 3.2 to conclude that

$$\begin{aligned}(M - l^* + 1)(\beta_{l^*} n_{l^*} - t_{l^*}) &\geq \min_{1 \leq l \leq M} (M - l + 1)(\beta_l n_l - t_l) \\ &= \beta n - t \geq \mu_\alpha(\boldsymbol{d}) \geq (M - l^* + 1)\mu_\alpha(\boldsymbol{u}_{l^*}).\end{aligned}$$

Hence we have shown that $0 < \mu_\alpha(\boldsymbol{u}_{l^*}) \leq \beta_{l^*} n_{l^*} - t_{l^*}$ and therefore

$$\frac{1}{b^{m_{l^*}}} \sum_{h_{l^*}=0}^{b^{m_{l^*}}-1} {}_b\mathrm{wal}_{\boldsymbol{u}_{l^*}}(\boldsymbol{x}_{h_{l^*}}^{(l^*)}) = 0,$$

i.e., the l^*th factor in Eq. (6) is zero. □

3.4. A double m-construction

In this section, we aim to generalise a propagation rule referred to as "double m-construction" in [7, Section 3.4], which again generalises a propagation rule from [12] for digital (t, m, s)-nets.

Assume we are given a $(t_1, \alpha_1, \beta_1, n, m, s)$-net in base b, denoted by $\mathcal{P}_1 = \{\boldsymbol{x}_h\}_{h=0}^{b^m-1}$, and a $(t_2, \alpha_2, \beta_2, n, m, s)$-net in base b, denoted by $\mathcal{P}_2 = \{\boldsymbol{y}_i\}_{i=0}^{b^m-1}$. For $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,s})$, we write

$$x_{h,j} = \frac{\xi_{h,j,1}}{b} + \cdots + \frac{\xi_{h,j,n}}{b^n}$$

and for $\boldsymbol{y}_i = (y_{i,1}, \ldots, y_{i,s})$, we set

$$y_{i,j} = \frac{\eta_{i,j,1}}{b} + \cdots + \frac{\eta_{i,j,n}}{b^n}.$$

Furthermore, the dual set associated with $\mathcal{P}_1$ is denoted by $\mathcal{D}_n^{(1)}$ and the dual set associated with $\mathcal{P}_2$ by $\mathcal{D}_n^{(2)}$. We are now in a position to define a multiset $\mathcal{P} := \{\boldsymbol{z}_0, \ldots, \boldsymbol{z}_{b^{2m}-1}\}$ as follows: For $h' = hb^m + i$, $0 \le h \le b^m - 1$, $0 \le i \le b^m - 1$, we set

$$z_{h',j} = \frac{\xi_{h,j,1} \oplus \eta_{i,j,1}}{b} + \cdots + \frac{\xi_{h,j,n} \oplus \eta_{i,j,n}}{b^n} + \frac{0 \ominus \eta_{i,j,1}}{b^{n+1}} + \cdots + \frac{0 \ominus \eta_{i,j,n}}{b^{2n}}, \tag{7}$$

$h' = 0, \ldots, b^{2m} - 1$, $j = 1, \ldots, s$. We now define a set $\mathcal{N}$, which in the forthcoming Lemma 3.3 will be shown to be the dual set of $\mathcal{P}$. Let $\boldsymbol{a}_r = (a_{r,1}, \ldots, a_{r,s}) \in \mathcal{D}_n^{(r)}$, $r = 1, 2$ and define $\boldsymbol{k} = \boldsymbol{k}(\boldsymbol{a}_1, \boldsymbol{a}_2) := (k_1, \ldots, k_s)$, where

$$k_j = a_{1,j} + b^n(a_{1,j} \oplus a_{2,j}), \quad j = 1, \ldots, s,$$

then we set $\mathcal{N} = \{\boldsymbol{k}(\boldsymbol{a}_1, \boldsymbol{a}_2) \in \mathcal{K}_{2n,b}^s : \boldsymbol{a}_1 \in \mathcal{D}_n^{(1)}, \boldsymbol{a}_2 \in \mathcal{D}_n^{(2)}\}$.

Lemma 3.3. *The set $\mathcal{N} = \{\boldsymbol{k} \in \mathcal{K}_{2n,b}^s : \boldsymbol{a}_1 \in \mathcal{D}_n^{(1)}, \boldsymbol{a}_2 \in \mathcal{D}_n^{(2)}\}$ is the dual set of $\mathcal{P} = \{\boldsymbol{z}_0, \ldots, \boldsymbol{z}_{b^{2m}-1}\}$ where $\boldsymbol{z}_h := (z_{h,1}, \ldots, z_{h,s})$ and where the $z_{h,j}$ are given by Eq. (7).*

Proof. Let $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathcal{K}_{b,2n}^s$, where $k_j = a_{1,j} + b^n(a_{1,j} \oplus a_{2,j})$, $j = 1, \ldots, s$, and where $\boldsymbol{a}_r = (a_{r,1}, \ldots, a_{r,s}) \in \mathcal{K}_{b,n}^s$, $r = 1, 2$. Clearly,

$$c_{\boldsymbol{k}} = \sum_{h'=0}^{b^{2m}-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{z}_{h'}) = \sum_{h=0}^{b^m-1} \sum_{i=0}^{b^m-1} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{z}_{hb^m+i}) = \sum_{h=0}^{b^m-1} \sum_{i=0}^{b^m-1} \prod_{j=1}^{s} {}_b\mathrm{wal}_{k_j}(z_{hb^m+i,j}).$$

For brevity, we set $k_j = k_j^{(1)} + b^n k_j^{(2)}$, where $k_j^{(1)}$ and $k_j^{(2)}$ have the b-adic expansions $k_j^{(1)} = \sum_{l=1}^n k_{j,l}^{(1)} b^{l-1}$ and $k_j^{(2)} = \sum_{l=1}^n k_{j,l}^{(2)} b^{l-1}$. Hence

$$\begin{aligned}
{}_b\mathrm{wal}_{k_j}(z_{hb^m+i,j}) &= \exp\left[\frac{2\pi \mathtt{i}}{b}\left(\sum_{l=1}^{n} k_{j,l}^{(1)}(\xi_{h,j,l} \oplus \eta_{i,j,l}) + \sum_{l=n+1}^{2n} k_{j,l-n}^{(2)}(0 \ominus \eta_{i,j,l-n})\right)\right] \\
&= \exp\left[\frac{2\pi \mathtt{i}}{b} \sum_{l=1}^{n} k_{j,l}^{(1)}(\xi_{h,j,l} \oplus \eta_{i,j,l})\right] \exp\left[\frac{2\pi \mathtt{i}}{b} \sum_{l=1}^{n} k_{j,l}^{(2)}(0 \ominus \eta_{i,j,l})\right] \\
&= {}_b\mathrm{wal}_{k_j^{(1)}}(x_{h,j} \oplus y_{i,j})\, {}_b\mathrm{wal}_{k_j^{(2)}}(0 \ominus y_{i,j}) \\
&= {}_b\mathrm{wal}_{a_{1,j}}(x_{h,j})\, {}_b\mathrm{wal}_{a_{1,j}}(y_{i,j})\, {}_b\mathrm{wal}_{a_{1,j}}(0 \ominus y_{i,j})\, {}_b\mathrm{wal}_{a_{2,j}}(0 \ominus y_{i,j}) \\
&= {}_b\mathrm{wal}_{a_{1,j}}(x_{h,j})\, \overline{{}_b\mathrm{wal}_{a_{2,j}}(y_{i,j})},
\end{aligned}$$

and further

$$\begin{aligned}
c_{\boldsymbol{k}} &= \sum_{h=0}^{b^m-1} \sum_{i=0}^{b^m-1} \prod_{j=1}^{s} {}_b\mathrm{wal}_{a_{1,j}}(x_{h,j})\, \overline{{}_b\mathrm{wal}_{a_{2,j}}(y_{i,j})} \\
&= \sum_{h=0}^{b^m-1} \sum_{i=0}^{b^m-1} {}_b\mathrm{wal}_{\boldsymbol{a}_1}(\boldsymbol{x}_h)\, \overline{{}_b\mathrm{wal}_{\boldsymbol{a}_2}(\boldsymbol{y}_i)} \\
&= \sum_{h=0}^{b^m-1} {}_b\mathrm{wal}_{\boldsymbol{a}_1}(\boldsymbol{x}_h) \sum_{i=0}^{b^m-1} \overline{{}_b\mathrm{wal}_{\boldsymbol{a}_2}(\boldsymbol{y}_i)} \\
&= \sum_{h=0}^{b^m-1} {}_b\mathrm{wal}_{\boldsymbol{a}_1}(\boldsymbol{x}_h)\, \overline{\sum_{i=0}^{b^m-1} {}_b\mathrm{wal}_{\boldsymbol{a}_2}(\boldsymbol{y}_i)}.
\end{aligned}$$

If $\boldsymbol{k} \in \mathcal{N}$, then $\boldsymbol{a}_1 \in \mathcal{D}_n^{(1)}$ and $\boldsymbol{a}_2 \in \mathcal{D}_n^{(2)}$, so we have $c_{\boldsymbol{k}} \neq 0$ and hence $\boldsymbol{k}$ is in the dual set of $\mathcal{P}$. If on the other hand $\boldsymbol{k}$ is in the dual set of $\mathcal{P}$, then $c_{\boldsymbol{k}} \neq 0$ and hence $\boldsymbol{a}_1 \in \mathcal{D}_n^{(1)}$ and $\boldsymbol{a}_2 \in \mathcal{D}_n^{(2)}$, so $\boldsymbol{k} \in \mathcal{N}$. □

In order to bound the quality parameter of $\mathcal{P} = \{\boldsymbol{z}_0, \ldots, \boldsymbol{z}_{b^{2m}-1}\}$, we define

$$d = d(\mathcal{D}_n^{(1)}, \mathcal{D}_n^{(2)}) := \max_{1 \le j \le s} \max_{R_j} \max(0, \mu_\alpha(a_{1,j}) - \mu_\alpha(a_{1,j} \oplus a_{2,j})),$$

where R_j is the set of all ordered pairs $(\boldsymbol{a}_1, \boldsymbol{a}_2)$, with $\boldsymbol{a}_r = (a_{r,1}, \ldots, a_{r,s}) \in \mathcal{D}_n^{(r)} \setminus \{\boldsymbol{0}\}$, $a_{1,i} \oplus a_{2,i} = 0$ for $i \neq j$ and $a_{1,j} \oplus a_{2,j} \neq 0$. We define the max over R_j to be zero if R_j is empty. We can now prove the main result of this subsection.

Theorem 3.4 (*Propagation Rule 10*). *Let $\mathcal{P}_1$ be a $(t_1, \alpha_1, \beta_1, n, m, s)$-net in base b with dual set $\mathcal{D}_n^{(1)}$ and $\mathcal{P}_2$ be a $(t_2, \alpha_2, \beta_2, n, m, s)$-net in base b with dual set $\mathcal{D}_n^{(2)}$. Let $d = d(\mathcal{D}_n^{(1)}, \mathcal{D}_n^{(2)})$. Then the point set given by Eq. (7) is a $(t, \alpha, \beta, 2n, 2m, s)$-net in base b with $\alpha = \max(\alpha_1, \alpha_2)$, $\beta = \min(\beta_1, \beta_2)$ and*

$$t \le \max(\lfloor 2\beta n\rfloor - n - \lfloor \beta_1 n\rfloor + t_1 + d, \lfloor 2\beta n\rfloor - n - \lfloor \beta_2 n\rfloor + t_2, 0),$$

if $\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)} = \{\mathbf{0}\}$, and

$$t \le \max(\lfloor 2\beta n\rfloor - n - \lfloor \beta_1 n\rfloor + t_1 + d, \lfloor 2\beta n\rfloor - n - \lfloor \beta_2 n\rfloor + t_2, \lfloor 2\beta n\rfloor + 1 - \rho_\alpha(\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)}), 0),$$

if $\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)} \neq \{\mathbf{0}\}$.

Proof. Clearly, $0 < \beta \le 1, \alpha \ge 1$. We show a lower bound for $\mu_\alpha(\boldsymbol{k})$ for all nonzero vectors $\boldsymbol{k} \in \mathcal{N}$, which by Lemma 3.3 is the dual set of the point set given by Eq. (7). To this end we use the property that $\rho_\alpha(\mathcal{D}_n^{(r)}) \ge \rho_{\alpha_r}(\mathcal{D}_n^{(r)}) \ge \lfloor \beta_r n\rfloor - t_r + 1$, as $\alpha \ge \alpha_r$, $r = 1, 2$. For $\boldsymbol{k} \in \mathcal{N}$, $\boldsymbol{k} \neq \mathbf{0}$, we have $\boldsymbol{k} = \boldsymbol{a}_1 + b^n(\boldsymbol{a}_1 \oplus \boldsymbol{a}_2)$ with $\boldsymbol{a}_1 \in \mathcal{D}_n^{(1)}$ and $\boldsymbol{a}_2 \in \mathcal{D}_n^{(2)}$ (not both of them are zero) and hence

$$\mu_\alpha(\boldsymbol{k}) = \mu_\alpha(\boldsymbol{a}_1 + b^n(\boldsymbol{a}_1 \oplus \boldsymbol{a}_2)).$$

We consider four different cases:

1. If $\boldsymbol{a}_1 = \mathbf{0}$, then $\boldsymbol{a}_2 \neq \mathbf{0}$, and hence
$$\mu_\alpha(\boldsymbol{k}) = \mu_\alpha(b^n\boldsymbol{a}_2) \ge n + \mu_\alpha(\boldsymbol{a}_2) \ge n + \rho_\alpha(\mathcal{D}_n^{(2)}) \ge n + \lfloor \beta_2 n\rfloor - t_2 + 1.$$
2. If $\boldsymbol{a}_2 = \mathbf{0}$, then $\boldsymbol{a}_1 \neq \mathbf{0}$, and we obtain in a similar manner that
$$\mu_\alpha(\boldsymbol{k}) \ge \mu_\alpha(b^n\boldsymbol{a}_1) \ge n + \rho_\alpha(\mathcal{D}_n^{(1)}) \ge n + \lfloor \beta_1 n\rfloor - t_1 + 1.$$
3. If $\boldsymbol{a}_1, \boldsymbol{a}_2 \neq \mathbf{0}$, but $\boldsymbol{a}_1 \oplus \boldsymbol{a}_2 = \mathbf{0}$ (i.e., $\boldsymbol{a}_1 = \ominus \boldsymbol{a}_2$), then $\boldsymbol{a}_1 \in \mathcal{D}_n^{(2)}$, so $\boldsymbol{a}_1 \in \mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)}$. Consequently, if $\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)} = \{\mathbf{0}\}$, this is not possible. If $\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)} \neq \{\mathbf{0}\}$, then
$$\mu_\alpha(\boldsymbol{k}) = \mu_\alpha(\boldsymbol{a}_1) \ge \rho_\alpha(\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)}).$$
4. If $\boldsymbol{a}_1, \boldsymbol{a}_2 \neq \mathbf{0}$ and $\boldsymbol{a}_1 \oplus \boldsymbol{a}_2 \neq \mathbf{0}$, then we have
$$\begin{aligned}\mu_\alpha(\boldsymbol{k}) &= \sum_{j=1}^{s} \mu_\alpha(a_{1,j} + b^n(a_{1,j} \oplus a_{2,j})) \\ &= \sum_{\substack{j=1\\ a_{j,1}\oplus a_{j,2}\neq 0}}^{s} \mu_\alpha(a_{1,j} + b^n(a_{1,j} \oplus a_{2,j})) + \sum_{\substack{j=1\\ a_{j,1}\oplus a_{j,2}= 0}}^{s} \mu_\alpha(a_{1,j}) \\ &\ge \sum_{\substack{j=1\\ a_{j,1}\oplus a_{j,2}\neq 0}}^{s} \mu_\alpha(b^n(a_{1,j} \oplus a_{2,j})) + \sum_{\substack{j=1\\ a_{j,1}\oplus a_{j,2}= 0}}^{s} \mu_\alpha(a_{1,j}). \qquad (8)\end{aligned}$$

   We now distinguish between two subcases: Firstly, assume that the first sum in Eq. (8) has at least two terms, then $\mu_\alpha(\boldsymbol{k}) \ge 2n + 2$. Otherwise, it has exactly one term, say for $j = j_0$, which gives a smaller value than $2n + 2$. In this subcase we have
$$\begin{aligned}\mu_\alpha(\boldsymbol{k}) &= \mu_\alpha(b^n(a_{1,j_0} \oplus a_{2,j_0})) + \mu_\alpha(\boldsymbol{a}_1) - \mu_\alpha(a_{1,j_0}) \\ &\ge n + \mu_\alpha(\boldsymbol{a}_1) - (\mu_\alpha(a_{1,j_0}) - \mu_\alpha(a_{1,j_0} \oplus a_{2,j_0})) \\ &\ge n + \rho_\alpha(\mathcal{D}_n^{(1)}) - d(\mathcal{D}_n^{(1)}, \mathcal{D}_n^{(2)}) \\ &\ge n + \lfloor \beta_1 n\rfloor - t_1 + 1 - d(\mathcal{D}_n^{(1)}, \mathcal{D}_n^{(2)}).\end{aligned}$$

Hence combining the four cases we have

$$\rho_\alpha(\mathcal{N}) \ge \min(n + \lfloor \beta_1 n\rfloor - t_1 + 1 - d(\mathcal{D}_n^{(1)}, \mathcal{D}_n^{(2)}), n + \lfloor \beta_2 n\rfloor - t_2 + 1, \rho_\alpha(\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)})),$$

if $\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)} \neq \{\mathbf{0}\}$, and

$$\rho_\alpha(\mathcal{N}) \ge \min(n + \lfloor \beta_1 n\rfloor - t_1 + 1 - d(\mathcal{D}_n^{(1)}, \mathcal{D}_n^{(2)}), n + \lfloor \beta_2 n\rfloor - t_2 + 1),$$

if $\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)} = \{\mathbf{0}\}$. Now the result follows from Theorem 2.2. □

3.5. A base change propagation rule

In this subsection we show how one can obtain a net in base b from a net in base b^L. Thereby we generalise [11, Propagation Rule 7] (see also [7, Propagation Rule XI]) to $(t, \alpha, \beta, n, m, s)$-nets. The proof technique and the construction follows [11, Proposition 7] very closely.

Theorem 3.5 (*Propagation Rule 11*). *If there exists a $(t, \alpha, \beta, n, m, s)$-net in base b^L with an integer $L \geq 1$, then there exists a $(t, \alpha, \beta, n, mL, sL)$-net in base b.*

Proof. Let $\mathcal{P} = \{\boldsymbol{x}_h\}_{h=0}^{(b^L)^m-1}$ be a $(t, \alpha, \beta, n, m, s)$-net in base b^L. Without loss of generality we may assume that $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,s})$ with

$$x_{h,j} = \sum_{l=1}^{n} \xi_{h,j,l}(b^L)^{-l} \quad \text{for } 0 \leq h \leq (b^L)^m - 1,$$

where all $\xi_{h,j,l} \in \mathbb{Z}_{b^L}$. Let the expansion of $\xi_{h,j,l}$ in base b be

$$\xi_{h,j,l} = \sum_{k=1}^{L} z_{h,l,k}^{(j)} b^{k-1} \quad \text{for } 0 \leq h \leq (b^L)^m - 1, 1 \leq j \leq s, 1 \leq l \leq n,$$

where all $z_{h,l,k}^{(j)} \in \mathbb{Z}_b$. Now we define a multiset $\mathcal{Q} = \{\boldsymbol{w}_0, \ldots, \boldsymbol{w}_{b^{mL}-1}\}$ whose elements are in $[0, 1)^{sL}$. The coordinate indices range from 1 to sL, and so we can denote them by $(j-1)L + k$ with $1 \leq j \leq s$ and $1 \leq k \leq L$. Let $w_{h,(j-1)L+k}$ denote the corresponding coordinates of the point $\boldsymbol{w}_h$. To complete the definition of $\mathcal{Q}$, we put

$$w_{h,(j-1)L+k} = \sum_{l=1}^{n} z_{h,l,k}^{(j)} b^{-l} \quad \text{for } 1 \leq j \leq s, 1 \leq k \leq L, 0 \leq h \leq b^{mL} - 1.$$

We will now show that $\mathcal{Q}$ is a $(t, \alpha, \beta, n, mL, sL)$-net in base b. To this end we fix $\boldsymbol{\nu}, \boldsymbol{a}_{\boldsymbol{\nu}}, \boldsymbol{i}_{\boldsymbol{\nu}}$ so that $1 \leq i_{(j-1)L+k,\nu_{(j-1)L+k}} < \cdots < i_{(j-1)L+k,1}$, for $1 \leq k \leq L$ and $1 \leq j \leq s$, so that $\sum_{j=1}^{s} \sum_{k=1}^{L} \sum_{l=1}^{\min(\nu_{(j-1)L+k},\alpha)} i_{(j-1)L+k,l} \leq \beta n - t$.

For $\boldsymbol{w}_h$ to be in $J(\boldsymbol{a}_{\boldsymbol{\nu}}, \boldsymbol{i}_{\boldsymbol{\nu}})$, we need

$$w_{h,(j-1)L+k,l} = a_{(j-1)L+k,l} \quad \text{for all } l \in \{i_{(j-1)L+k,\nu_{(j-1)L+k}}, \ldots, i_{(j-1)L+k,1}\},$$

which is satisfied if and only if $z_{h,l,k}^{(j)} = a_{(j-1)L+k,l}$ for all l from the above mentioned range.

For $1 \leq j \leq s$ we define $\bigcup_{k=1}^{L} \{i_{(j-1)L+k,\nu_{(j-1)L+k}}, \ldots, i_{(j-1)L+k,1}\} = \{e_{j,\widetilde{\nu}_j}, \ldots, e_{j,1}\}$. For $l \in \{e_{j,\widetilde{\nu}_j}, \ldots, e_{j,1}\}$, we set $\widetilde{a}_{j,l} = \sum_{k=1}^{L} a_{(j-1)L+k,l} b^{k-1}$, where unspecified $a_{(j-1)L+k,l}$ are chosen arbitrarily. In fact, the number of $a_{(j-1)L+k,l}$ chosen arbitrarily is given by

$$\sum_{j=1}^{s} \sum_{k=1}^{L} (\widetilde{\nu}_j - \nu_{(j-1)L+k}) = L \sum_{j=1}^{s} \widetilde{\nu}_j - \sum_{j=1}^{s} \sum_{k=1}^{L} \nu_{(j-1)L+k}.$$

Hence there are $b^{L\sum_{j=1}^{s}\widetilde{\nu}_j - \sum_{j=1}^{s}\sum_{k=1}^{L}\nu_{(j-1)L+k}}$ generalised elementary intervals of the form

$$J(\widetilde{\boldsymbol{a}}, \boldsymbol{e}) = \prod_{j=1}^{s} \bigcup_{\substack{\widetilde{a}_{j,l}=0 \\ l \in \{1,\ldots,n\} \setminus \{e_{j,\widetilde{\nu}_j},\ldots,e_{j,1}\}}}^{b^L-1} \left[\frac{\widetilde{a}_{j,1}}{b^L} + \cdots + \frac{\widetilde{a}_{j,n}}{(b^L)^n}, \frac{\widetilde{a}_{j,1}}{b^L} + \cdots + \frac{\widetilde{a}_{j,n}}{(b^L)^n} + \frac{1}{(b^L)^n} \right)$$

of volume $(b^L)^{-\sum_{j=1}^{s}\widetilde{\nu}_j}$. However,

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha)} e_{j,l} \leq \sum_{j=1}^{s} \sum_{k=1}^{L} \sum_{l=1}^{\min(\nu_{(j-1)L+k},\alpha)} i_{(j-1)L+k,l} \leq \beta n - t,$$

hence by the $(t, \alpha, \beta, n, m, s)$-net property of $\mathcal{P}$, $J(\widetilde{\boldsymbol{a}}, \boldsymbol{e})$ contains $(b^L)^{m-\sum_{j=1}^{s}\widetilde{\nu}_j}$ points and hence $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains

$$b^{L\sum_{j=1}^{s}\widetilde{\nu}_j - \sum_{j=1}^{s}\sum_{k=1}^{L}\nu_{(j-1)L+k}} (b^L)^{(m-\sum_{j=1}^{s}\widetilde{\nu}_j)} = b^{Lm - \sum_{j=1}^{s}\sum_{k=1}^{L}\nu_{(j-1)L+k}}$$

points of $\mathcal{Q}$ as required. $\square$

3.6. Pirsic's base change rule

In this subsection, we present a generalisation of Pirsic's base change rule, see [14, Lemma 12], also [15]. This result shows how to interpret a $(t, \alpha, \beta, n, m, s)$-net in base b^L as a $(t', \alpha', \beta', n', m', s)$-net in base $b^{L'}$. Furthermore, we state some special cases, in particular, we show how to interpret a $(t, \alpha, \beta, n, m, s)$-net in base b as a $(t', \alpha', \beta', n', m', s)$-net in base $b^{L'}$ and how to interpret a $(t, \alpha, \beta, n, m, s)$-net in base b^L as a $(t', \alpha', \beta', n', m', s)$-net in base b.

Theorem 3.6 (*Propagation Rule 12*)**.** *Let $n, n', m, m', s, \alpha, L$ and $L' \in \mathbb{N}$, where $\gcd(L, L') = 1, mL = m'L', nL = n'L'$, let $0 < \beta \le 1$ be a real number and let $0 \le t \le \beta n$ and βn be integers. Then a $(t, \alpha L', \beta, n, m, s)$-net in base b^L is a $(t', \alpha, \frac{\beta}{L'}, n', m', s)$-net in base $b^{L'}$, where*

$$t' = \min\left(\left\lceil \frac{tL + s\alpha(L-1)L' - \frac{(L'-1)L'}{2} + (-L' \pmod L)\beta n'}{L'(L' + (-L' \pmod L))} \right\rceil, \left\lceil \frac{tL + (s\alpha L' - 1)(L-1) - \frac{(L'-1)L'}{2}}{L'^2} \right\rceil\right).$$

Proof. The proof proceeds as follows: We start with a generalised elementary interval for the point set in base $b^{L'}$, then change this into a generalised elementary interval in base b and consequently rewrite the latter as a union of intervals in base b^L.

Assume we are given an arbitrary generalised elementary interval $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ in base $b^{L'}$ for some given values of $\boldsymbol{\nu}, \boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}$, such that $\nu_j \ge 0$, $1 \le i_{j,\nu_j} < \cdots < i_{j,1}$, $j = 1, \ldots, s$, and such that for a nonnegative integer t''

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \le \frac{\beta}{L'} n' - t''. \tag{9}$$

Without loss of generality, we assume that there exists at least one ν_j satisfying $\nu_j > 0$, then $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ admits the following representation:

$$J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}) = \prod_{j=1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n'\} \setminus \left\{i_{j,\nu_j},\ldots,i_{j,1}\right\}}}^{b^{L'}-1} \left[\frac{a_{j,1}}{b^{L'}} + \cdots + \frac{a_{j,n'}}{(b^{L'})^{n'}}, \frac{a_{j,1}}{b^{L'}} + \cdots + \frac{a_{j,n'}}{(b^{L'})^{n'}} + \frac{1}{(b^{L'})^{n'}}\right).$$

As $a_{j,l} \in \{0, \ldots, b^{L'} - 1\}$ it has a b-adic representation of the form $a_{j,l} = a_{j,l,1} + a_{j,l,2} b + \cdots + a_{j,l,L'} b^{L'-1}$, and hence

$$\frac{a_{j,l}}{(b^{L'})^l} = \frac{a_{j,l,L'}}{b^{(l-1)L'+1}} + \cdots + \frac{a_{j,l,2}}{b^{lL'-1}} + \frac{a_{j,l,1}}{b^{lL'}},$$

for $1 \le l \le n'$ where $a_{j,l,g} \in \{0, \ldots, b-1\}$. We now set

$$\frac{a_{j,l}}{(b^{L'})^l} = \sum_{k=(l-1)L'+1}^{lL'} \frac{\widetilde{a}_{j,k}}{b^k},$$

i.e. $\widetilde{a}_{j,lL'-g+1} = a_{j,l,g}$, $1 \le l \le n'$, $1 \le g \le L'$ and $1 \le j \le s$. We can now rewrite the above interval as a generalised elementary interval in base b,

$$J(\widetilde{\boldsymbol{i}}_{\boldsymbol{\nu}}, \widetilde{\boldsymbol{a}}_{\boldsymbol{\nu}}) = \prod_{j=1}^{s} \bigcup_{\substack{\widetilde{a}_{j,l}=0 \\ l \in \{1,\ldots,n'L'\} \setminus \left\{\widetilde{i}_{j,\nu_j L'}, \widetilde{i}_{j,\nu_j L'-1},\ldots,\widetilde{i}_{j,1}\right\}}}^{b-1} \left[\frac{\widetilde{a}_{j,1}}{b} + \cdots + \frac{\widetilde{a}_{j,L'}}{b^{L'}} + \cdots + \frac{\widetilde{a}_{j,n'L'}}{b^{n'L'}}, \right.$$
$$\left. \frac{\widetilde{a}_{j,1}}{b} + \cdots + \frac{\widetilde{a}_{j,L'}}{b^{L'}} + \cdots + \frac{\widetilde{a}_{j,n'L'}}{b^{n'L'}} + \frac{1}{b^{n'L'}}\right),$$

where

$$\widetilde{i}_{j,(k-1)L'+g} = i_{j,k} L' + 1 - g$$

for $1 \le g \le L'$ and $1 \le k \le \nu_j$. Clearly,

$$J(\widetilde{\boldsymbol{i}}_{\boldsymbol{\nu}}, \widetilde{\boldsymbol{a}}_{\boldsymbol{\nu}}) = \prod_{j=1}^{s} \bigcup_{\substack{\widetilde{a}_{j,l}=0 \\ l \in \{1,\ldots,nL\} \setminus \left\{\widetilde{i}_{j,\nu_j L'},\ldots,\widetilde{i}_{j,1}\right\}}}^{b-1} \left[\frac{\widetilde{a}_{j,1}}{b} + \cdots + \frac{\widetilde{a}_{j,nL}}{b^{nL}}, \frac{\widetilde{a}_{j,1}}{b} + \cdots + \frac{\widetilde{a}_{j,nL}}{b^{nL}} + \frac{1}{b^{nL}}\right).$$

Now for $1 \le j \le s$ and $1 \le k \le \nu_j L'$ we define integers $r_{j,k}$ and $e_{j,k}$ such that $0 \le r_{j,k} < L$ and

$$\widetilde{i}_{j,k} = e_{j,k}L - r_{j,k}.$$

Note that it is possible that $e_{j,k} = e_{j,k'}$ for $k \ne k'$. Let now $\{\widetilde{e}_{j,\widetilde{\nu}_j}, \ldots, \widetilde{e}_{j,1}\}$ be the set of distinct elements of $\{e_{j,\nu_j L'}, \ldots, e_{j,1}\}$. Then $\widetilde{\nu}_j \le \nu_j L'$ and $\{e_{j,\nu_j L'}, \ldots, e_{j,1}\} = \{\widetilde{e}_{j,\widetilde{\nu}_j}, \ldots, \widetilde{e}_{j,1}\}$.

Let $\widetilde{\boldsymbol{\nu}} = (\widetilde{\nu}_1, \ldots, \widetilde{\nu}_s)$. For $1 \le j \le s$ for fixed $\widetilde{a}_{j,l}$ and $\widetilde{e}_{j,k}L - (L-1) \le l \le \widetilde{e}_{j,k}L$, where $1 \le k \le \widetilde{\nu}_j$, we set

$$\widetilde{\widetilde{a}}_{j,\widetilde{e}_{j,k}} = b^{L-1}\widetilde{a}_{j,\widetilde{e}_{j,k}L-(L-1)} + b^{L-2}\widetilde{a}_{j,\widetilde{e}_{j,k}L-(L-2)} + \cdots + \widetilde{a}_{j,\widetilde{e}_{j,k}L}.$$

Furthermore, for fixed j, only $\nu_j L'$ of the $\widetilde{a}_{j,l}$, where $\widetilde{e}_{j,k}L - (L-1) \le l \le \widetilde{e}_{j,k}L$ and $1 \le k \le \widetilde{\nu}_j$, are specified in $\widetilde{\boldsymbol{a}}_{\boldsymbol{\nu}}$. Hence $J(\widetilde{\boldsymbol{i}}_{\boldsymbol{\nu}}, \widetilde{\boldsymbol{a}}_{\boldsymbol{\nu}})$, and therefore also $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$, is the union of $b^{L\sum_{j=1}^s \widetilde{\nu}_j - L'\sum_{j=1}^s \nu_j}$ disjoint intervals of the form

$$\prod_{j=1}^{s} \bigcup_{\substack{\widetilde{\widetilde{a}}_{j,l}=0 \\ l\in\{1,\ldots,n\}\setminus\left\{\widetilde{e}_{j,\widetilde{\nu}_j},\ldots,\widetilde{e}_{j,1}\right\}}}^{b^L-1} \left[\frac{\widetilde{\widetilde{a}}_{j,1}}{(b^L)} + \frac{\widetilde{\widetilde{a}}_{j,2}}{(b^L)^2} + \cdots + \frac{\widetilde{\widetilde{a}}_{j,n}}{(b^L)^n}, \frac{\widetilde{\widetilde{a}}_{j,1}}{(b^L)} + \frac{\widetilde{\widetilde{a}}_{j,2}}{(b^L)^2} + \cdots + \frac{\widetilde{\widetilde{a}}_{j,n}}{(b^L)^n} + \frac{1}{(b^L)^n}\right).$$

If we can show that

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha L')} \widetilde{e}_{j,l} \le \beta n - t, \tag{10}$$

then each interval contains $(b^L)^{m-|\widetilde{\boldsymbol{\nu}}|_1}$ points, and consequently $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains

$$(b^L)^{m-|\widetilde{\boldsymbol{\nu}}|_1} b^{|\widetilde{\boldsymbol{\nu}}|_1 L - |\boldsymbol{\nu}|_1 L'} = b^{mL-|\boldsymbol{\nu}|_1 L'} = b^{m'L'-|\boldsymbol{\nu}|_1 L'} = (b^{L'})^{m'-|\boldsymbol{\nu}|_1}$$

points and the proof is complete. Hence $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains the right number of points if Eq. (10) is satisfied, or equivalently, if

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha L')} \widetilde{e}_{j,l}L \le L(\beta n - t).$$

So $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ still contains the right number of points if

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L',\alpha L')} \widetilde{i}_{j,l} + \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L',\alpha L')} r_{j,l} \le L(\beta n - t). \tag{11}$$

We now find a bound for $\sum_{j=1}^s \sum_{l=1}^{\min(\nu_j L',\alpha L')} \widetilde{i}_{j,l}$:

$$\begin{aligned}
\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L',\alpha L')} \widetilde{i}_{j,l} &= \sum_{j=1}^{s} \sum_{l=1}^{L'\min(\nu_j,\alpha)} \widetilde{i}_{j,l} \\
&= \sum_{j=1}^{s} \sum_{k=1}^{\min(\nu_j,\alpha)} \sum_{g=1}^{L'} \widetilde{i}_{j,(k-1)L'+g} = \sum_{j=1}^{s} \sum_{k=1}^{\min(\nu_j,\alpha)} \sum_{g=1}^{L'} (i_{j,k}L' + 1 - g) \\
&= \sum_{j=1}^{s} \sum_{k=1}^{\min(\nu_j,\alpha)} \left[\sum_{g=1}^{L'} i_{j,k}L' - \sum_{g=1}^{L'-1} g\right] \\
&= \sum_{j=1}^{s} \sum_{k=1}^{\min(\nu_j,\alpha)} \left[i_{j,k}L'^2 - \frac{(L'-1)L'}{2}\right] \\
&\le \sum_{j=1}^{s} \sum_{k=1}^{\min(\nu_j,\alpha)} \left[i_{j,k}L'^2\right] - \frac{(L'-1)L'}{2} \\
&\le \beta n' L' - t'' L'^2 - \frac{(L'-1)L'}{2},
\end{aligned} \tag{12}$$

where we used Eq. (9). Combining Eqs. (11) and (12) we find that $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains the right number of points if

$$t''L'^2 + \frac{(L'-1)L'}{2} - \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha L')} r_{j,l} \geq tL.$$

That is, we can set

$$t' = \min\left\{ t'' : t''L'^2 + \frac{(L'-1)L'}{2} - M(t'') \geq tL \right\}, \tag{13}$$

where

$$M(t'') = \max\left\{ \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha L')} (-\widetilde{i}_{j,l} (\mathrm{mod}\ L)) : i_{j,l} \geq 0 \text{ and } \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j, \alpha)} i_{j,l} \leq \frac{\beta}{L'} n' - t'' \right\},$$

where we recall $\widetilde{i}_{j,l} = \widetilde{e}_{j,l} L - r_{j,l}$ for $1 \leq l \leq \nu_j L'$ and $1 \leq j \leq s$, and $\widetilde{i}_{j,(k-1)L'+g} = i_{j,k} L' + 1 - g$ for $1 \leq g \leq L'$ and $1 \leq k \leq \nu_j$. We now aim to find an upper bound for $\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha L')} (-\widetilde{i}_{j,l} (\mathrm{mod}\ L))$. We have

$$\begin{aligned}
\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha L')} (-\widetilde{i}_{j,l} (\mathrm{mod}\ L)) &= \sum_{j=1}^{s} \sum_{k=1}^{\min(\nu_j, \alpha)} \sum_{g=1}^{L'} (-i_{j,k} L' - 1 + g (\mathrm{mod}\ L)) \\
&\leq \sum_{j=1}^{s} \sum_{k=1}^{\min(\nu_j, \alpha)} \sum_{g=1}^{L'} (-i_{j,k} L' (\mathrm{mod}\ L)) + \sum_{j=1}^{s} \sum_{k=1}^{\min(\nu_j, \alpha)} \sum_{g=1}^{L'} (g - 1 (\mathrm{mod}\ L)) \\
&\leq \sum_{j=1}^{s} \sum_{k=1}^{\min(\nu_j, \alpha)} \sum_{g=1}^{L'} (-L' (\mathrm{mod}\ L)) i_{j,k} + s\alpha(L-1)L' \\
&\leq (-L' (\mathrm{mod}\ L)) L' \left(\frac{\beta}{L'} n' - t'' \right) + s\alpha(L-1)L' \\
&= (-L' (\mathrm{mod}\ L))(\beta n' - t''L') + s\alpha(L-1)L'.
\end{aligned}$$

From Eq. (13) it follows that

$$t' \leq \min\left\{ t'' : t''L'^2 + \frac{(L'-1)L'}{2} - \left((-L' (\mathrm{mod}\ L))(\beta n' - t''L') + (L-1)L'\alpha s \right) \geq tL \right\}.$$

This condition is satisfied for all t'' with

$$t'' \geq \left\lceil \frac{tL + s\alpha(L-1)L' - \frac{(L'-1)L'}{2} + (-L' (\mathrm{mod}\ L))\beta n'}{L'(L' + (-L' (\mathrm{mod}\ L)))} \right\rceil,$$

which gives the first bound. For the second bound, let

$$t'' = \left\lceil \frac{tL + (s\alpha - 1)(L-1) - \frac{(L'-1)L'}{2}}{L'^2} \right\rceil,$$

then, using Eq. (12), we have

$$\begin{aligned}
\sum_{j=1}^{s} \sum_{l=1}^{\min(\widetilde{\nu}_j, \alpha L')} \widetilde{e}_{j,l} &\leq \frac{1}{L} \left(\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha L')} \widetilde{i}_{j,l} + \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha L')} r_{j,l} \right) \\
&\leq \frac{1}{L} \left(\beta n' L' - t''L'^2 - \frac{(L'-1)L'}{2} + \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha L')} r_{j,l} \right) \\
&\leq \frac{1}{L} \left(\beta n' L' - t''L'^2 - \frac{(L'-1)L'}{2} + s\alpha(L-1)L' \right) \\
&\leq \frac{1}{L} \left(\beta n' L' - Lt - (s\alpha L' - 1)(L-1) + \frac{(L'-1)L'}{2} - \frac{(L'-1)L'}{2} + s\alpha(L-1)L' \right) \\
&= \beta n - t + \frac{L-1}{L}.
\end{aligned}$$

By assumption, βn is an integer, $\sum_{j=1}^{s}\sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha L')}\widetilde{e}_{j,l}$ is an integer, hence

$$\sum_{j=1}^{s}\sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha L')}\widetilde{e}_{j,l} \le \beta n - t,$$

which completes the proof. □

We point out that $\alpha L'$ changes to α in Theorem 3.6. Using Propagation Rule (2), we can establish the following corollary to Theorem 3.6, which avoids a change in the parameter α.

Corollary 3.1. *Let $n, n', m, m', s, \alpha, L$ and $L' \in \mathbb{N}$, where $\gcd(L, L') = 1$, $mL = m'L'$, $nL = n'L'$, let $0 < \beta \le 1$ be a real number and let $0 \le t \le \beta n$ and βn be integers. Then a $(t, \alpha, \beta, n, m, s)$-net in base b^L is a $(t', \alpha, \frac{\beta}{L'}, n', m', s)$-net in base $b^{L'}$, where*

$$t' = \min\left(\left\lceil \frac{tL + s\alpha(L-1)L' - \frac{(L'-1)L'}{2} + (-L'(\mathrm{mod}\ L))\beta n'}{L'(L' + (-L'(\mathrm{mod}\ L)))}\right\rceil, \left\lceil \frac{tL + (s\alpha L' - 1)(L-1) - \frac{(L'-1)L'}{2}}{L'^2}\right\rceil\right).$$

However, in some cases it is possible to improve on Corollary 3.1.

Theorem 3.7 (*Propagation Rule 14*)**.** *Let $n, n', m, m', s, \alpha, L$ and $L' \in \mathbb{N}$, $L' \ge \alpha$ where $\gcd(L, L') = 1$, $mL = m'L'$, $nL = n'L'$, let $0 < \beta \le 1$ be a real number and let $0 \le t \le \beta n$ and βn be integers. Then a $(t, \alpha, \beta, n, m, s)$-net in base b^L is a $(t', \alpha, \frac{\beta}{\alpha}, n', m', s)$-net in base $b^{L'}$, where*

$$t' = \min\left(\left\lceil \frac{tL + sf(\alpha, L) - \frac{(\alpha-1)\alpha}{2} + (-L'(\mathrm{mod}\ L))\beta n'}{\alpha(L' + (-L'(\mathrm{mod}\ L)))}\right\rceil, \left\lceil \frac{tL + (s\alpha - 1)(L-1) - \frac{(\alpha-1)\alpha}{2}}{\alpha L'}\right\rceil\right),$$

and where

$$\begin{aligned} f(\alpha, L) &= \sum_{l=1}^{\alpha}(l - 1(\mathrm{mod}\ L)) \\ &= \frac{1}{2}\left(L(L-1)\left\lfloor\frac{\alpha-1}{L}\right\rfloor + \left(\alpha - L\left\lfloor\frac{\alpha-1}{L}\right\rfloor\right)\left(\alpha - L\left\lfloor\frac{\alpha-1}{L}\right\rfloor - 1\right)\right). \end{aligned}$$

Proof. Using the same definitions as in the proof of Theorem 3.6, we aim to establish that the assumption

$$\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \le \frac{\beta}{\alpha}n' - t''$$

where t'' is a nonnegative integer, implies that

$$\sum_{j=1}^{s}\sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha)}\widetilde{e}_{j,l} \le \beta n - t. \tag{14}$$

We proceed in a manner similar to the proof of Theorem 3.6, i.e. $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains the right number of points if Eq. (14) is satisfied which in turn is equivalent to

$$\sum_{j=1}^{s}\sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha)}\widetilde{e}_{j,l}L \le \beta nL - tL,$$

and hence $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ still contains the right number of points if

$$\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j L',\alpha)}\widetilde{i}_{j,l} + \sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j L',\alpha)} r_{j,l} \le \beta nL - tL. \tag{15}$$

We now find a bound for $\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j L',\alpha)}\widetilde{i}_{j,l}$. We have

$$\begin{aligned}\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_jL',\alpha)}\widetilde{i}_{j,l} &= \sum_{\substack{j=1\\ \nu_j>0}}^{s}\sum_{l=1}^{\alpha}\widetilde{i}_{j,l}\\ &= \sum_{\substack{j=1\\ \nu_j>0}}^{s}\sum_{l=1}^{\alpha}\left[i_{j,1}L'+1-l\right]\\ &= L'\sum_{\substack{j=1\\ \nu_j>0}}^{s}\sum_{l=1}^{\alpha}i_{j,1}+\sum_{\substack{j=1\\ \nu_j>0}}^{s}\sum_{l=1}^{\alpha}(1-l)\\ &= \alpha L'\sum_{\substack{j=1\\ \nu_j>0}}^{s}i_{j,1}-\sum_{\substack{j=1\\ \nu_j>0}}^{s}\frac{(\alpha-1)\alpha}{2}\\ &\le \alpha L'\left(\frac{\beta}{\alpha}n'-t''\right)-\frac{(\alpha-1)\alpha}{2}. \end{aligned} \tag{16}$$

Hence, combining Eqs. (15) and (16), we find that $J(\boldsymbol{i}_\nu, \boldsymbol{a}_\nu)$ contains the right number of points if

$$t''\alpha L'+\frac{(\alpha-1)\alpha}{2}-\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_jL',\alpha)}r_{j,l}\ge tL.$$

We set

$$t'=\min\left\{t'':t''\alpha L'+\frac{(\alpha-1)\alpha}{2}-M(t'')\ge tL\right\},$$

where

$$M(t'')=\max\left\{\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_jL',\alpha)}(-\widetilde{i}_{j,l}(\mathrm{mod}\ L)):i_{j,1}\ge 0,\ \sum_{\substack{j=1\\ \nu_j>0}}^{s}i_{j,1}\le\frac{\beta}{\alpha}n'-t''\right\}.$$

We now establish a bound for $\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_jL',\alpha)}(-\widetilde{i}_{j,l}(\mathrm{mod}\ L))$, where we set $f(\alpha,L)=\sum_{l=1}^{\alpha}(l-1(\mathrm{mod}\ L))$. We have

$$\begin{aligned}\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_jL',\alpha)}(-\widetilde{i}_{j,l}(\mathrm{mod}\ L)) &= \sum_{\substack{j=1\\ \nu_j>0}}^{s}\sum_{l=1}^{\alpha}\left(-i_{j,1}L'-1+l(\mathrm{mod}\ L)\right)\\ &\le \sum_{\substack{j=1\\ \nu_j>0}}^{s}\sum_{l=1}^{\alpha}(-i_{j,1}L'(\mathrm{mod}\ L))+\sum_{\substack{j=1\\ \nu_j>0}}^{s}\sum_{l=1}^{\alpha}(l-1(\mathrm{mod}\ L))\\ &\le (-L'(\mathrm{mod}\ L))\alpha\left(\frac{\beta}{\alpha}n'-t''\right)+sf(\alpha,L).\end{aligned}$$

Hence

$$t'\le\min\left\{t'':t''\alpha L'+\frac{(\alpha-1)\alpha}{2}-\left((-L'(\mathrm{mod}\ L))(\beta n'-t''\alpha)+sf(\alpha,L)\right)\ge tL\right\},$$

which is satisfied for all t'' with

$$t''\ge\left\lceil\frac{tL+sf(\alpha,L)+(-L'(\mathrm{mod}\ L))\beta n'-\frac{(\alpha-1)\alpha}{2}}{\alpha(L'+(-L'(\mathrm{mod}\ L)))}\right\rceil.$$

To obtain the second bound, we set

$$t''=\left\lceil\frac{tL+(s\alpha-1)(L-1)-\frac{(\alpha-1)\alpha}{2}}{\alpha L'}\right\rceil.$$

Consequently,

$$\begin{aligned}\sum_{j=1}^{s}\sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha)}\widetilde{e}_{j,l} &\le \frac{1}{L}\left(\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_jL',\alpha)}\widetilde{i}_{j,l}+\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_jL',\alpha)}r_{j,l}\right)\\ &\le \frac{\alpha L'}{L}\left(\frac{\beta}{\alpha}n'-t''\right)-\frac{(\alpha-1)\alpha}{2L}+\frac{s\alpha(L-1)}{L}\\ &\le \beta n - t + \frac{L-1}{L},\end{aligned}$$

hence $\sum_{j=1}^{s}\sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha)}\widetilde{e}_{j,l} \le \beta n - t$ and the proof is complete. □

In the following corollary, we recover the result due to Pirsic.

Corollary 3.2. *Let m, m', L and $L' \in \mathbb{N}$, $\gcd(L, L') = 1$, $mL = m'L'$ and let $0 \le t \le m$ be an integer. Then a (t, m, s)-net in base b^L is a (t', m', s)-net in base $b^{L'}$, with*

$$t' = \min\left(\left\lceil \frac{tL + (-L'(\mathrm{mod}\ L))m'}{L' + (-L'(\mathrm{mod}\ L))}\right\rceil, \left\lceil \frac{tL + (s-1)(L-1)}{L'}\right\rceil\right).$$

Proof. The proof follows immediately from Theorem 3.7, where we set $\alpha = \beta = 1$, $n = m$ and $n' = m'$ and notice that $f(1, L) = 0$. □

We again remark that in Theorem 3.6, $\alpha L'$ changes to α. However, when considering a base change from b^L to b, there is no need to change α, as the following theorem shows, which can be regarded as a generalisation of [7, Theorem 9] and [13, Lemma 9].

Theorem 3.8. *For $\alpha, n, m, s, L \in \mathbb{N}$, $0 < \beta \le 1$ a real number and $0 \le t' \le \beta n$ an integer, a $(t', \alpha, \beta, n, m, s)$-net in base b^L is a $(t, \alpha, \beta, nL, mL, s)$-net in base b, where*

$$t \le (t' + e)L + (s\alpha - 1)(L - 1)$$

and $e = 0$ if βn is an integer and $e = 1$ otherwise.

Proof. The proof is similar to the proof of Theorem 3.6. □

Finally, we consider a base change from b to $b^{L'}$, which can be considered to be a generalisation of [9, Lemma 2.9].

Theorem 3.9. *Let $n, m, s, \alpha, L' \in \mathbb{N}$, let $0 < \beta \le 1$ be a real number and let $0 \le t \le \beta n/L'$ be an integer. Then a $(tL'^2 + \frac{(L'-1)L'}{2}, \alpha L', \beta, nL', mL', s)$-net in base b is a $(t, \alpha, \frac{\beta}{L'}, n, m, s)$-net in base $b^{L'}$.*

Proof. The proof is similar to the proof of Theorem 3.6. □

Furthermore, we point out that $\alpha L'$ changes to α in Theorem 3.9. Using Propagation Rule (2), we can establish the following corollary to Theorem 3.9, which avoids a change in the parameter α.

Corollary 3.3. *Let $n, m, s, \alpha, L' \in \mathbb{N}$, let $0 < \beta \le 1$ be a real number and let $0 \le t \le \beta n/L'$ be an integer. Then a $(tL'^2 + \frac{(L'-1)L'}{2}, \alpha, \beta, nL', mL', s)$-net in base b is a $(t, \alpha, \frac{\beta}{L'}, n, m, s)$-net in base $b^{L'}$.*

However, in some cases it is possible to improve on Corollary 3.3.

Theorem 3.10. *Let $n, m, s, \alpha, L' \in \mathbb{N}$, $L' \ge \alpha$, then a $(t\alpha L' + \frac{(\alpha-1)\alpha}{2}, \alpha, \beta, nL', mL', s)$-net in base b is a $(t, \alpha, \frac{\beta}{\alpha}, n, m, s)$-net in base $b^{L'}$.*

Proof. The proof proceeds along the same lines as the proof of Theorem 3.7. □

3.7. A higher order to higher order construction

Next we consider a propagation rule which was referred to as "A higher order to higher order construction" in [7]. In [4], it was shown how to construct digital $(t, \alpha, \beta, n \times m, s)$-nets from digital (t, m, sd)-nets. Essentially, the "higher order to higher order construction" in [7] replaces the digital (t, m, sd)-net with a digital $(t, \alpha, \beta, n \times m, sd)$-net, but makes use of the same construction algorithm. We now show that the same idea can be used for $(t, \alpha, \beta, n, m, s)$-nets. Assume we are given a multiset $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{b^m-1}\}$ forming a $(t', \alpha', \beta', n, m, sd)$-net in base b. We write $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,sd})$ and $x_{h,j} = \xi_{h,j,1}/b + \xi_{h,j,2}/b^2 + \cdots$ for all $0 \le h \le b^m - 1$ and $1 \le j \le sd$.

Then we construct a multiset $\{\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{b^m-1}\}$ as follows: For $0 \le h < b^m$ we set $\boldsymbol{y}_h = (y_{h,1}, \ldots, y_{h,s})$ in $[0, 1)^s$ where for $1 \le j \le s$,

$$y_{h,j} = \sum_{l=1}^{n} \sum_{k=1}^{d} \xi_{h,(j-1)d+k,l} b^{-k-(l-1)d}. \tag{17}$$

Theorem 3.11 (*Propagation Rule 15*). *Let $d \in \mathbb{N}$ and let the multiset $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ be a $(t', \alpha', \beta', n, m, sd)$-net in base b, where we assume that $\beta' n$ is an integer.*

Then for any $\alpha \ge 1$, the multiset $\{\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{b^m-1}\}$, defined by Eq. (17), forms a $(t, \alpha, \beta' \min(1, \alpha/(\alpha' d)), dn, m, s)$-net in base b with

$$t = \left\lceil \min\left(d, \frac{\alpha}{\alpha'}\right) \min\left(\beta' n, t' + \left\lfloor \frac{\alpha' s(d-1)}{2} \right\rfloor\right)\right\rceil.$$

Proof. The case where $\beta' n \le t' + \lfloor \alpha' s(d-1)/2 \rfloor$ is trivial. Hence we assume from now on that $\beta' n > t' + \lfloor \alpha' s(d-1)/2 \rfloor$ and that we deal with an arbitrary generalised elementary interval $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$, for some given values of $\boldsymbol{\nu}, \boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}$, such that $1 \le i_{j,\nu_j} < \cdots < i_{j,1}$, $\nu_j \ge 0$, for $1 \le j \le s$ and

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \le \beta' \min\left(1, \frac{\alpha}{\alpha' d}\right) dn - t.$$

We need to show that $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains $b^{m-|\boldsymbol{\nu}|_1}$ points. For $\boldsymbol{y}_h$, $0 \le h \le b^m - 1$, to be in $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$, we need for $0 \le h \le b^m - 1$, $1 \le j \le s$, $1 \le l \le n$ and $1 \le k \le d$,

$$\eta_{h,j,(l-1)d+k} = a_{j,(l-1)d+k} \quad \text{whenever } (l-1)d + k \in \{i_{j,\nu_j}, \ldots, i_{j,1}\},$$

where $y_{h,j} := \eta_{h,j,1}/b + \cdots + \eta_{h,j,dn}/b^{dn}$. But from the construction method we find that the condition $\eta_{h,j,(l-1)d+k} = a_{j,(l-1)d+k}$ is equivalent to $\xi_{h,(j-1)d+k,l} = a_{j,(l-1)d+k}$. As $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{b^m-1}\}$ forms a $(t', \alpha', \beta', n, m, sd)$-net, we translate the above condition into a condition on a generalised elementary interval of dimension sd. In particular we set

$$a'_{(j-1)d+k,l} = a_{j,(l-1)d+k}, \quad \text{if } (l-1)d + k \in \{i_{j,\nu_j}, \ldots, i_{j,1}\}.$$

Also, for each choice of $1 \le j \le s$ and $1 \le k \le d$ we let $w_{(j-1)d+k}$ denote the largest integer such that there are $e_{(j-1)d+k,1} > \cdots > e_{(j-1)d+k,w_{(j-1)d+k}} > 0$ for which

$$\{(e_{(j-1)d+k,u} - 1)d + k : u = 1, \ldots, w_{(j-1)d+k}\} \subseteq \{i_{j,\nu_j}, \ldots, i_{j,1}\}.$$

If no such $w_{(j-1)d+k}$ exists we set $w_{(j-1)d+k} = 0$ and hence $\{(e_{(j-1)d+k,u} - 1)d + k : u = 1, \ldots, w_{(j-1)d+k}\} = \emptyset$. Consequently, for dimension $(j-1)d + k$ with $1 \le j \le s$ and $1 \le k \le d$, the digits $a'_{(j-1)d+k,1}, \ldots, a'_{(j-1)d+k,w_{(j-1)d+k}}$ are specified whenever $w_{(j-1)d+k} > 0$. In particular, $w_{(j-1)d+k}$ gives the number of digits in dimension $(j-1)d + k$ that the generalised elementary interval corresponding to the $(t', \alpha', \beta', n, m, sd)$-net contributes to dimension j of the generalised elementary interval corresponding to the $(t, \alpha, \beta' \min(1, \alpha/(\alpha' d)), dn, m, s)$-net. We hence note that

$$\sum_{k=1}^{d} w_{(j-1)d+k} = \nu_j \quad \text{for } 1 \le j \le s \tag{18}$$

and obtain the following generalised elementary interval $J(\boldsymbol{e}_{\boldsymbol{w}}, \boldsymbol{a}'_{\boldsymbol{w}})$ of dimension sd, where $\boldsymbol{e}_{\boldsymbol{w}} = (e_{1,w_1}, \ldots, e_{1,1}, \ldots, e_{sd,w_{sd}}, \ldots, e_{sd,1})$ and $\boldsymbol{a}'_{\boldsymbol{w}} = (a'_{1,w_1}, \ldots, a'_{1,1}, \ldots, a'_{sd,w_{sd}}, \ldots, a'_{sd,1})$. By the property of the $(t', \alpha', \beta', n, m, sd)$-net, if

$$\sum_{j=1}^{sd} \sum_{l=1}^{\min(w_j,\alpha')} e_{j,l} \le \beta' n - t', \tag{19}$$

then $J(\boldsymbol{e}_{\boldsymbol{w}}, \boldsymbol{a}'_{\boldsymbol{w}})$ contains $b^{m-\sum_{j=1}^{sd} w_j} = b^{m-\sum_{j=1}^{s} \nu_j}$ points, where we used Eq. (18), as required. By distinguishing the cases $\alpha' d \le \alpha$ and $\alpha' d > \alpha$, it was shown in [7] that Eq. (19) holds, which completes the proof. $\square$

Remark 3.2. Similar to [7, Example 1] one can employ a $(0, m, 2)$-net in base b to show that Theorem 3.11 cannot be improved on in general.

Corollary 3.4. *Let $d \in \mathbb{N}$ and let $\{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{b^m-1}\}$ be a (t', m, sd)-net in base b. Then for every $\alpha \geq 1$, the multiset $\{\boldsymbol{y}_0, \dots, \boldsymbol{y}_{b^m-1}\}$ defined by Eq. (17) forms a $(t, \alpha, \min(1, \frac{\alpha}{d}), dm, m, s)$-net in base b with*

$$t = \min(d, \alpha) \min\left(m, t' + \left\lfloor \frac{s(d-1)}{2} \right\rfloor\right).$$

Proof. The proof follows immediately from Remark 1.1 and by setting $\alpha' = \beta' = 1$ and $n = m$ in Theorem 3.11. □

Theorem 3.11 can be improved when $\alpha = \alpha'$, which we show in the following.

Proposition 3.1 (*Propagation Rule 16*)**.** *Let $\alpha, d \in \mathbb{N}$ and let $\{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{b^m-1}\}$ form a $(t, \alpha, \beta, n, m, sd)$-net in base b. Then the multiset $\{\boldsymbol{y}_0, \dots, \boldsymbol{y}_{b^m-1}\}$ defined by Eq. (17) forms a $(t, \alpha, \beta, n, m, s)$-net in base b.*

Proof. Let $\boldsymbol{\nu} = (\nu_1, \dots, \nu_s) \in \{0, \dots, nd\}^s$ be given and for $j = 1, \dots, s$ let $dn > i_{j,1} > \cdots > i_{j,\nu_j} > 0$ be such that

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\alpha,\nu_j)} i_{j,l} \leq \beta n - t.$$

Let $\boldsymbol{i}_{\boldsymbol{\nu}} = (i_{1,1}, \dots, i_{1,\nu_1}, \dots, i_{s,1}, \dots, i_{s,\nu_s})$, $\boldsymbol{a}_{\boldsymbol{\nu}} = (a_{1,i_{1,1}}, \dots, a_{1,i_{1,\nu_1}}, \dots, a_{s,i_{s,1}}, \dots, a_{s,i_{s,\nu_s}}) \in \{1, \dots, nd\}^{|\nu|_1}$, and a generalised elementary interval

$$J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}) = \prod_{j=1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l\in\{1,\dots,nd\}\setminus\{i_{j,1},\dots,i_{j,\nu_j}\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,nd}}{b^{nd}}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,nd}}{b^{nd}} + \frac{1}{b^{nd}}\right),$$

where $\{i_{j,1}, \dots, i_{j,\nu_j}\} = \emptyset$ in case $\nu_j = 0$ for $1 \leq j \leq s$, be given.

Let $\boldsymbol{y}_h = (y_{h,1}, \dots, y_{h,s})$ with $y_{h,s} = \eta_{h,j,1}/b + \eta_{h,j,2}/b^2 + \cdots$. Then $\boldsymbol{y}_h \in J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ if and only if $\eta_{h,j,l} = a_{j,l}$ for all $l \in \{i_{j,1}, \dots, i_{j,\nu_j}\}$ and $1 \leq j \leq s$.

We define now a new generalised elementary interval J' in dimension sd such that $\boldsymbol{y}_h \in J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ if and only if $\boldsymbol{x}_h \in J'$. To this end, for $j = 1, \dots, s$, let $a'_{(j-1)d+k,l} = a_{j,(l-1)d+k}$ where $1 \leq k \leq d$ and $1 \leq l \leq n$ are such that $(l-1)d+k \in \{i_{j,1}, \dots, i_{j,\nu_j}\}$. For $j' = 1, \dots, sd$ we have now specified $a'_{j',i'}$ for certain values of $i' \in \{1, \dots, n\}$. Let $U_{j'}$ be the set of i' for which $a'_{j',i'}$ is specified, i.e.,

$$U_{j'} = \{1 \leq i' \leq n : (i'-1)d + j' - (j-1)d \in \{i_{j,1}, \dots, i_{j,\nu_j}\} \text{ for } j = \lceil j'/d \rceil\}.$$

We set $U_{j'} = \{i'_{j',1}, \dots, i'_{j',\nu'_{j'}}\}$, where we assume that the elements are ordered such that $n \geq i'_{j',1} > \cdots > i'_{j',\nu'_{j'}} > 0$. Define now $\nu' = (\nu'_1, \dots, \nu'_{sd}) \in \{0, \dots, n\}^{sd}$, $\boldsymbol{i}'_{\nu'} = (i'_{1,1}, \dots, i'_{1,\nu'_1}, \dots, i'_{sd,1}, \dots, i'_{sd,\nu'_{sd}})$, and $\boldsymbol{a}' = (a'_{1,i'_{1,1}}, \dots, a'_{1,i'_{1,\nu'_1}}, \dots, a'_{sd,i'_{sd,1}}, \dots, a'_{sd,i'_{sd,\nu'_{sd}}})$. Then $J' = J(\boldsymbol{i}'_{\nu'}, \boldsymbol{a}'_{\nu'})$ has the property that $\boldsymbol{y}_h \in J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ if and only if $\boldsymbol{x}_h \in J(\boldsymbol{i}'_{\nu'}, \boldsymbol{a}'_{\nu'})$.

Note that $\nu'_{(j-1)d+1} + \cdots + \nu'_{(j-1)d+d} = \nu_j$ for $1 \leq j \leq s$ and therefore $|\boldsymbol{\nu}|_1 = |\boldsymbol{\nu}'|_1$. Thus if J' contains $b^{m-|\boldsymbol{\nu}'|_1}$ points, then $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains $b^{m-|\boldsymbol{\nu}|_1}$ points. The former will be the case if $\sum_{j=1}^{sd} \sum_{l=1}^{\min(\alpha,\nu'_j)} i'_{j,l} \leq \beta n - t$, which we show in the following.

If $\nu_j < \alpha$, then

$$\begin{aligned}\sum_{j'=(j-1)d+1}^{jd} \sum_{l=1}^{\nu'_{j'}} i'_{j',l} &\leq \left\lceil \frac{i_{j,1}}{\alpha} \right\rceil + \cdots + \left\lceil \frac{i_{j,\nu_j}}{\alpha} \right\rceil \\ &\leq \frac{i_{j,1} + \cdots + i_{j,\nu_j} + \nu_j(\alpha-1)}{\alpha} \\ &\leq i_{j,1} + \cdots + i_{j,\nu_j}\end{aligned}$$

since $i_1 + \cdots i_{\nu_j} \geq \frac{\nu_j(\nu_j+1)}{2}$.

If $\nu_j \geq \alpha$, then

$$\begin{aligned}\sum_{j'=(j-1)d+1}^{jd} \sum_{l=1}^{\min(\nu'_{j'},\alpha)} i'_{j',l} &\leq \left\lceil \frac{i_{j,1}}{\alpha} \right\rceil + \cdots + \left\lceil \frac{i_{j,\alpha}}{\alpha} \right\rceil + \left\lceil \frac{i_{j,\alpha}-1}{\alpha} \right\rceil + \cdots + \left\lceil \frac{i_{j,\alpha}-\alpha(d-1)}{\alpha} \right\rceil \\ &\leq i_{j,1} + \cdots + i_{j,\alpha}.\end{aligned}$$

Therefore we have

$$\sum_{j=1}^{sd}\sum_{l=1}^{\min(\alpha,\nu'_j)} i'_{j,l} \le \sum_{j=1}^{s}\sum_{l=1}^{\min(\alpha,\nu_j)} i_{j,l} \le \beta n - t.$$

Hence the result follows since $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ is a $(t, \alpha, \beta, n, m, sd)$-net and therefore J' contains $b^{m-|\boldsymbol{\nu}'|_1}$ points. □

4. Propagation rules for $(t, \alpha, \beta, \sigma, s)$-sequences and an application

Based on results from Section 3 we deduce properties of $(t, \alpha, \beta, \sigma, s)$-sequences in base b.

4.1. A higher order to higher order construction for $(t, \alpha, \beta, \sigma, s)$-sequences

We use the higher order to higher order construction from Section 3.7 to construct $(t, \alpha, \beta, \sigma, s)$-sequences in base b.

Assume we are given an infinite sequence $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots\}$ forming a $(t', \alpha', \beta', \sigma, sd)$-sequence in base b. We write $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,sd})$ and $x_{h,j} = \xi_{h,j,1}/b + \xi_{h,j,2}/b^2 + \cdots$ for all $h \ge 0$ and $1 \le j \le sd$.

Then we construct an infinite sequence $\{\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots\}$ as follows: For $h \ge 0$ we set $\boldsymbol{y}_h = (y_{h,1}, \ldots, y_{h,s})$ in $[0, 1)^s$ where

$$y_{h,j} = \sum_{l=1}^{\infty}\sum_{k=1}^{d} \xi_{h,(j-1)d+k,l}\, b^{-k-(l-1)d}. \tag{20}$$

Theorem 4.1. *Let $\alpha', d, s, \sigma \in \mathbb{N}, 0 < \beta' \le 1$ be such that $\beta'\sigma$ is an integer, and $t' \ge 0$ be an integer. Let $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots\}$ be a $(t', \alpha', \beta', \sigma, sd)$-sequence in base b. Then for any $\alpha \ge 1$, the infinite sequence $\{\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots\}$ defined by Eq. (20) forms a $(t, \alpha, \beta' \min(1, \alpha/(\alpha' d)), d\sigma, s)$-sequence in base b with*

$$t = \left\lceil \min\left(d, \frac{\alpha}{\alpha'}\right)\left(t' + \left\lfloor \frac{\alpha' s(d-1)}{2} \right\rfloor\right)\right\rceil.$$

Proof. We need to show that for all $k \ge 0$ and all $m > \frac{t}{\beta' \min(1, \frac{\alpha}{\alpha' d}) d\sigma}$ the multiset $\{\boldsymbol{y}_{kb^m}, \ldots, \boldsymbol{y}_{(k+1)b^m-1}\}$ forms a $(t, \alpha, \beta' \min(1, \frac{\alpha}{\alpha' d}), d\sigma m, m, s)$-net in base b. It is clear that $m > \frac{t'}{\beta'\sigma}$ and hence $\{\boldsymbol{x}_{kb^m}, \ldots, \boldsymbol{x}_{(k+1)b^m-1}\}$ forms a $(t', \alpha', \beta', \sigma m, m, sd)$-net in base b. But $\beta'\sigma m$ is an integer, hence $\{\boldsymbol{y}_{kb^m}, \ldots, \boldsymbol{y}_{(k+1)b^m-1}\}$ forms a $(t, \alpha, \beta' \min(1, \frac{\alpha}{\alpha' d}), d\sigma m, m, s)$-net in base b, by Theorem 3.11, where $t \le \lceil \min(d, \frac{\alpha}{\alpha'})(t' + \lfloor \frac{\alpha' s(d-1)}{2} \rfloor)\rceil$. Hence Eq. (20) defines a $(t, \alpha, \beta' \min(1, \alpha/(\alpha' d)), d\sigma, s)$-sequence. □

Remark 4.1. As in Remark 3.2 and [7, Example 1] one can employ a $(0, 2)$-sequence in base b to show that Theorem 4.1 cannot be improved on in general.

Similar to Corollary 3.4 in Section 3.7, we consider the following special case.

Corollary 4.1. *Let $\alpha', d, s, \sigma \in \mathbb{N}, 0 < \beta' \le 1$ be such that $\beta'\sigma$ is an integer, and $t' \ge 0$ be an integer. Let $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots\}$ be a (t', sd)-sequence in base b. Then for any $\alpha \ge 1$, the infinite sequence $\{\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots\}$ defined by Eq. (20) forms a $(t, \alpha, \min(1, \frac{\alpha}{d}), d, s)$-sequence in base b with*

$$t = \min(d, \alpha)\left(t' + \left\lfloor \frac{s(d-1)}{2} \right\rfloor\right).$$

The following result is analogous to Proposition 3.1.

Proposition 4.1. *Let $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots\}$ be a $(t, \alpha, \beta, \sigma, sd)$-sequence in base b. Then the infinite sequence $\{\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots\}$ defined by Eq. (20) forms a $(t, \alpha, \beta, \sigma, s)$-sequence in base b.*

Proof. We need to show that for $k \ge 0, m > t/(\beta\sigma)$, the multiset $\{\boldsymbol{y}_{kb^m}, \ldots, \boldsymbol{y}_{(k+1)b^m-1}\}$ forms a $(t, \alpha, \beta, \sigma m, m, s)$-net in base b. But for $k \ge 0, m > t/(\beta\sigma)$, $\{\boldsymbol{x}_{kb^m}, \ldots, \boldsymbol{x}_{(k+1)b^m-1}\}$ forms a $(t, \alpha, \beta, \sigma m, m, sd)$-net in base b, hence, by Proposition 3.1, $\{\boldsymbol{y}_{kb^m}, \ldots, \boldsymbol{y}_{(k+1)b^m-1}\}$ forms a $(t, \alpha, \beta, \sigma m, m, s)$-net in base b. □

4.2. A base reduction for $(t, \alpha, \beta, \sigma, s)$-sequences

We show that a $(t', \alpha, \beta, \sigma, s)$-sequence in base b^L can be considered as a $(t, \alpha, \beta, \sigma, s)$-sequence in base b with some quality parameter t. The following theorem generalises [13, Proposition 4].

Theorem 4.2. *Let $\sigma, s, \alpha, L' \in \mathbb{N}$, let $0 < \beta \leq 1$ be a real number and $t' \geq 0$ be an integer. A $(t', \alpha, \beta, \sigma, s)$-sequence in base b^L is a $(t, \alpha, \beta, \sigma, s)$-sequence in base b with*

$$t = (t' + e)L + (s\alpha - 1 + \beta\sigma)(L - 1),$$

where $e = 0$ if $\beta\sigma$ is an integer and $e = 1$ otherwise.

Proof. Let $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots\}$ be a $(t', \alpha, \beta, \sigma, s)$-sequence in base b^L, t as above and fix $m > \frac{t}{\beta\sigma}$ and write it in the form $m = pL + r$ with integers p and r such that $0 \leq r < L$. Note that $p > \frac{t'}{\beta\sigma}$. For a fixed integer $k \geq 0$, we consider the multiset $\mathcal{P} = \{\boldsymbol{x}_{kb^m}, \ldots, \boldsymbol{x}_{(k+1)b^m-1}\}$. Then $\mathcal{P}$ can be split up into b^r multisets $\{\boldsymbol{x}_{lb^{pL}}, \ldots, \boldsymbol{x}_{(l+1)b^{pL}-1}\}$ where $kb^r \leq l < (k+1)b^r$. As $p > \frac{t'}{\beta\sigma}$, each of these subsequences forms a $(t', \alpha, \beta, \sigma p, p, s)$-net in base b^L, which by Theorem 3.8 is a $((t'+e)L + (s\alpha - 1)(L-1), \alpha, \beta, \sigma pL, pL, s)$-net in base b. A $((t'+e)L + (s\alpha - 1)(L-1), \alpha, \beta, \sigma pL, pL, s)$-net in base b is also a $((t'+e)L + (s\alpha - 1 + \beta\sigma)(L-1), \alpha, \beta, \sigma m, pL, s)$-net in base b, as the strength of the latter is smaller than the strength of the former. An application of Propagation Rule (6) shows that $\mathcal{P}$ is a $((t'+e)L + (s\alpha - 1 + \beta\sigma)(L-1), \alpha, \beta, \sigma m, pL + r, s)$-net in base b, and hence a $(t, \alpha, \beta, \sigma m, m, s)$-net in base b. □

4.3. A base expansion for $(t, \alpha, \beta, \sigma, s)$-sequences

Here we consider a base change in the opposite direction: We show that a $(t, \alpha, \beta, \sigma, s)$-sequence in base b can be interpreted as a $(t', \alpha', \beta', \sigma, s)$-sequence in base $b^{L'}$. The following theorem generalises Theorem 3.9 from Section 3.6 to $(t, \alpha, \beta, \sigma, s)$-sequences (see also [13, Proposition 5]).

Theorem 4.3. *Let $\sigma, s, \alpha, L' \in \mathbb{N}$, let $0 < \beta \leq 1$ be a real number and $u \geq 0$ be an integer. Then a $(u, \alpha L', \beta, \sigma, s)$-sequence in base b is a $(t, \alpha, \frac{\beta}{L'}, \sigma, s)$-sequence in base $b^{L'}$, with*

$$t = \left\lceil \frac{u}{L'^2} - \frac{(L'-1)}{2L'} \right\rceil.$$

Proof. Denote the $(u, \alpha L', \beta, \sigma, s)$-sequence in base b by $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots\}$, which is of course also a $(tL'^2 + \frac{(L'-1)L'}{2}, \alpha L', \beta, \sigma, s)$-sequence in base b. By Definition 1.2 for all integers $k \geq 0$ and $m \geq 1$ the finite subsequence

$$\{\boldsymbol{x}_{kb^{mL'}}, \ldots, \boldsymbol{x}_{(k+1)b^{mL'}-1}\} \tag{21}$$

forms a $(\min(tL'^2 + \frac{(L'-1)L'}{2}, \beta\sigma mL'), \alpha L', \beta, \sigma mL', mL', s)$-net in base b. We consider two cases:

1. Assume first that m is such that $tL'^2 + \frac{(L'-1)L'}{2} \leq \beta\sigma mL'$, then by Theorem 3.9, the multiset given by Eq. (21) forms a $(t, \alpha, \frac{\beta}{L'}, \sigma m, m, s)$-net in base $b^{L'}$. Furthermore, $tL'^2 + \frac{(L'-1)L'}{2} \leq \beta\sigma mL'$ implies that $t \leq \lfloor \frac{\beta}{L'}\sigma m \rfloor$.
2. Now assume $\beta\sigma mL' < tL'^2 + \frac{(L'-1)L'}{2}$. According to Remark 1.2, the multiset given by Eq. (21) forms a $(\lfloor \frac{\beta}{L'}\sigma m \rfloor, \alpha, \frac{\beta}{L'}, \sigma m, m, s)$-net in base $b^{L'}$. Furthermore, $\beta\sigma mL' < tL'^2 + \frac{(L'-1)L'}{2}$ implies that $\lfloor \frac{\beta}{L'}\sigma m \rfloor \leq t$.

Hence the multiset given in Eq. (21) is a $(\min(t, \lfloor \frac{\beta}{L'}\sigma m \rfloor), \alpha, \frac{\beta}{L'}, \sigma m, m, s)$-net in base $b^{L'}$. We conclude that for all m such that $\frac{\beta}{L'}\sigma m > t$ we obtain a $(t, \alpha, \frac{\beta}{L'}, \sigma m, m, s)$-net in base $b^{L'}$ and therefore a $(t, \alpha, \frac{\beta}{L'}, \sigma, s)$-sequence in base $b^{L'}$. □

We also consider a special case based on Theorem 3.10.

Theorem 4.4. *Let $\sigma, s, \alpha, L' \in \mathbb{N}$, $L' \geq \alpha$, let $0 < \beta \leq 1$ be a real number and $t \geq 0$ be an integer. Then a $(t\alpha L' + \frac{(\alpha-1)\alpha}{2}, \alpha, \beta, \sigma, s)$-sequence in base b is a $(t, \alpha, \frac{\beta}{\alpha}, \sigma, s)$-sequence in base $b^{L'}$.*

Proof. We denote the $(t\alpha L' + \frac{(\alpha-1)\alpha}{2}, \alpha, \beta, \sigma, s)$-sequence in base b by $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots\}$. Then by Definition 1.2 for all integers $k \geq 0$ and $m \geq 1$ the finite subsequence

$$\{\boldsymbol{x}_{kb^{mL'}}, \ldots, \boldsymbol{x}_{(k+1)b^{mL'}-1}\} \tag{22}$$

forms a $(\min(t\alpha L' + \frac{(\alpha-1)\alpha}{2}, \beta\sigma mL'), \alpha, \beta, \sigma mL', mL', s)$-net in base b. We consider two cases:

1. Assume that $t\alpha L' + \frac{(\alpha-1)\alpha}{2} \leq \beta\sigma mL'$. Then by Theorem 3.10 the multiset given in Eq. (22) is a $(t, \alpha, \frac{\beta}{\alpha}, \sigma m, m, s)$-net in base $b^{L'}$. Furthermore, $t\alpha L' + \frac{(\alpha-1)\alpha}{2} \leq \beta\sigma mL'$ implies that $t \leq \lfloor \frac{\beta}{\alpha}\sigma m \rfloor$.
2. Assume that $\beta\sigma mL' < t\alpha L' + \frac{(\alpha-1)\alpha}{2}$. According to Remark 1.2, the multiset given in Eq. (22) forms a $(\lfloor \frac{\beta}{\alpha}\sigma m \rfloor, \alpha, \frac{\beta}{\alpha}, \sigma m, m, s)$-net in base $b^{L'}$. Furthermore, $\beta\sigma mL' < t\alpha L' + \frac{(\alpha-1)\alpha}{2}$ implies that $\lfloor \frac{\beta}{\alpha}\sigma m \rfloor \leq t$.

Hence the multiset given in Eq. (22) is a $(\min(t, \lfloor \frac{\beta}{\alpha}\sigma m \rfloor), \alpha, \frac{\beta}{\alpha}, \sigma m, m, s)$-net in base $b^{L'}$. We conclude that for all m such that $\frac{\beta}{\alpha}\sigma m > t$ we obtain a $(t, \alpha, \frac{\beta}{\alpha}, \sigma m, m, s)$-net in base $b^{L'}$ and therefore a $(t, \alpha, \frac{\beta}{\alpha}, \sigma, s)$-sequence in base $b^{L'}$. □

4.4. An explicit bound for $t_b(\alpha, s)$ for prime powers b

In this subsection the least value t such that there exists a $(t, \alpha, \beta, \sigma, s)$-sequence in base b is studied.

Definition 4.1. For integers $b \geq 2, s \geq 1, \alpha \geq 1$, let $t_b(\alpha, s)$ denote the least value t such that there exists a $(t, \alpha, \beta, \sigma, s)$-sequence in base b with $\alpha = \beta\sigma$.

Remark 4.2. In [6, Definition 6] the analogous quantity for the digital case has been introduced: Let b be a prime power, then $d_b(\alpha, s)$ denotes the smallest value of t such that there exists a digital $(t, \alpha, \beta, \sigma, s)$-sequence over the finite field $\mathbb{F}_b$ with $\alpha = \beta\sigma$.

In this case it is known (see [6, Theorem 7]) that for all $s \geq 1$ and $\alpha \geq 2$ we have

$$s\frac{\alpha(\alpha-1)}{2} - \alpha < d_q(\alpha, s) \leq s\alpha^2 \frac{c}{\log q} + \alpha + \alpha \left\lfloor \frac{s(\alpha-1)}{2} \right\rfloor,$$

where $c > 0$ is an absolute constant. Note that these bounds also apply to (nondigital) $(t, \alpha, \beta, \sigma, s)$-sequences where $\alpha = \beta\sigma$.

The following corollary follows from Theorems 4.2 and 4.3. Setting $\alpha = \beta = \sigma = 1$ and making use of Theorems 4.2 and 4.4, we could even recover [13, Corollary 4].

Corollary 4.2. *For all integers $b \geq 2, s \geq 1, \alpha \geq 1, \alpha = \beta\sigma$, we have*

$$\frac{t_b(\alpha, s) - (s\alpha - 1 + \beta\sigma)(L-1)}{L} \leq t_{b^L}(\alpha, s) \leq \left\lceil \frac{t_b(\alpha L, s) - \frac{(L-1)L}{2}}{L^2} \right\rceil.$$

The next theorem provides an explicit bound for $t_b(\alpha, s)$ for prime powers b. Setting $\alpha = \beta = \sigma = 1$, this result recovers [13, Proposition 6].

Theorem 4.5. *For every prime power b, we have*

$$t_b(\alpha, s) \leq \frac{2bs\alpha^2}{b-1} - 2\frac{b\alpha^{3/2}s^{1/2}}{(b^2-1)^{1/2}} + 2\alpha \left\lfloor \frac{s(\alpha-1)}{2} \right\rfloor + s\alpha - 1 + \alpha.$$

Proof. We use Theorem 4.2 with $L = 2$ to obtain

$$t_b(\alpha, s) \leq 2t_{b^2}(\alpha, s) + (s\alpha - 1 + \alpha).$$

By Corollary 4.1, where we set $d = \alpha$,

$$t_{b^2}(\alpha, s) \leq \alpha t_{b^2}(1, s\alpha) + \alpha \left\lfloor \frac{s(\alpha-1)}{2} \right\rfloor,$$

where $t_{b^2}(1, s\alpha)$ corresponds to the least value t such that there exists a $(t, s\alpha)$-sequence in base b^2. From [13, Theorem 5] we obtain

$$t_{b^2}(1, s\alpha) \leq \frac{bs\alpha}{b-1} - \frac{b(s\alpha)^{1/2}}{(b^2-1)^{1/2}}$$

and the result follows. □

Acknowledgements

The first and second authors gratefully acknowledge the support of the Australian Research Council under its Centre of Excellence Program. The second author is also supported by an Australian Research Council Queen Elizabeth II Research Fellowship. FP is supported by the Austrian Science Foundation (FWF), Project S9609, that is part of the Austrian National Research Network "Analytic Combinatorics and Probabilistic Number Theory".

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