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Note

On sources in comparability graphs, with applications

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Abstract

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We characterize sources in comparability graphs and show that our result provides a unifying look at two recent results about interval graphs.

An orientation Θ of a graph G is obtained by assigning unique directions to its edges. To simplify notation, we write $xy \in \Theta$, whenever the edge xy receives the direction from x to y. A vertex w is called a *source* whenever $vw \in \Theta$ for no vertex v in G. An orientation Θ is termed *transitive* if for every vertices x, y, z, $xy \in \Theta$ and $yz \in \Theta$ implies $xz \in \Theta$. It is well known that a graph G that admits a transitive orientation is a *comparability* graph.

In this context, it makes sense to ask the following natural question:

for what vertices of a comparability graph can we find a transitive orientation that makes them into a source? (1)

The purpose of this note is to provide an answer to (1). As it turns out, our result provides a unifying look at two recent results concerning interval graphs [4, 6].

All the graphs in this work are simple with no self-loops nor multiple edges.

Familiarity with standard graph theoretical terminology compatible with Golumbic [5] is assumed. To specify our results, however, we need to define some new terms. For an arbitrary vertex w of G, the graph G^w is obtained from G by adding a new vertex w' and by making w' adjacent to w only. We claim that

w is a source in some transitive orientation of G if, and only if, the graph G^w is a comparability graph. (2)

To justify (2), we note first that if G^w is a comparability graph then, by reversing the orientation on all the edges if necessary, we guarantee that w is a source in some transitive orientation Θ of G^w. In particular, w is a source in the restriction of Θ to G.

Conversely, given a transitive orientation Θ of G that makes w into a source, the orientation $\Theta^w = \Theta \cup \{ww'\}$ of G^w is transitive, and so G^w is a comparability graph, as claimed.

A vertex w of a graph G is called *special* if w coincides with one of the highlighted verteces in some graph F_i, $(1 \leqslant i \leqslant 4)$ or in the complement $\bar{F}_5$ of the graph F_5 featured in Fig. 1. A vertex that is not special is referred to as *regular*. As it turns out, regular vertices play an important role in the answer to (1). More precisely, we state the following result.

Theorem 1. *A vertex w of a comparability graph G is a source in some transitive orientation Θ of G if, and only if, w is regular.*

Proof. First, let w be a regular vertex of G. If w fails to be a source in any transitive orientation of G then, by (2), the graph G^w is not a comparability graph. Hence G^w must contain an induced subgraph H isomorphic to one of Gallai's forbidden graphs (for a list see Gallai [2] or Duchet [1]). Since, by

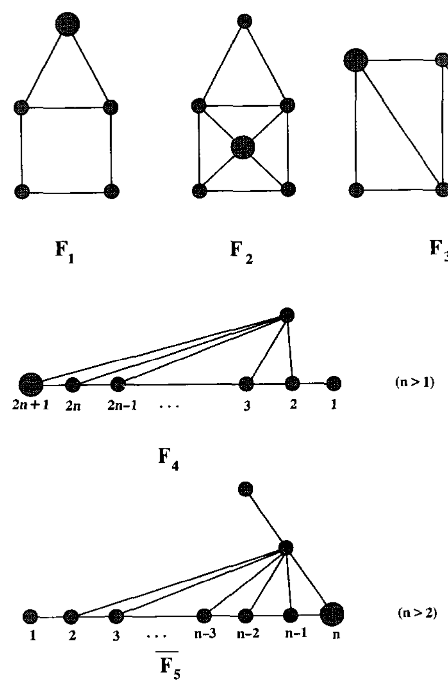

Fig. 1.

assumption, G is a comparability graph, w' must belong to H. But now, it is a straightforward observation that w must be a special vertex, a contradiction.

Conversely, let w be a source in some transitive orientation Θ of G. Again, it is routine to check that if w is special, then G^w contains one of the forbidden graphs in Gallai's catalog, contradicting (2). This completes the proof of Theorem 1. □

A graph $G=(V, E)$ is termed an *interval graph* if there exists a family $\{I_v\}_{v\in V}$ of intervals such that for distinct vertices u, v in G

$$uv \in E \text{ if, and only if, } I_u \cap I_v \neq \emptyset.$$

Such a family $\{I_v\}_{v\in V}$ of intervals is commonly referred to as an *interval representation* of G. An interval $I_x=[a_x, b_x]$, is called an *end interval* if $a_x \leqslant a_y$ for every $y \in V$; the vertex x itself is termed an *end vertex*.

An early characterization of interval graphs was proposed by Gilmore and Hoffman [3]: they showed that a graph G is an interval graph if, and only if, G itself is triangulated and its complement $\bar{G}$ is a comparability graph.

Let Θ be a transitive orientation of a graph G. A linear order $<$ on the vertex-set of G is said to be *consistent* with Θ if

$$u < v \text{ whenever } uv \in \Theta.$$

(Note that such a linear order is readily available: we only need apply a topological sort to G. Furthermore, every source in Θ can be placed first in $<$.)

We are now in a position to show that Theorem 1 implies the following result.

Corollary 1.1 (Gimbel [4]). *A vertex w in an interval graph G is an end vertex if, and only if, G contains an induced subgraphs none of the graphs $\bar{F}_1$, $\bar{F}_2$, or F_5 featured in Fig. 1, with w one of the highlighted vertices.*

Proof. The 'only if' implication is easy: we only need observe that if G contains an induced subgraph isomorphic to one of the graphs, $\bar{F}_1$, $\bar{F}_2$, or F_5 featured in Fig. 1, then the highlighted vertices cannot correspond to an end interval in any interval representation of G.

To prove the 'if' implication, assume that G contains no induced subgraph isomorphic to one of the graphs $\bar{F}_1$, $\bar{F}_2$, or F_5 featured in Fig. 1. Since G must be a triangulated graph, G cannot contain an induced subgraph isomorphic to the complement of the graph F_3 or F_4 of Fig. 1. Consequently, w is a regular vertex in $\bar{G}$. By Theorem 1, w is a source in some transitive orientation Θ of $\bar{G}$. Now a result of Gilmore and Hoffman [3] guarantees that

> *for every transitive orientation Θ of the edges of $\bar{G}$, there exists a linear order $<$ on the set of the maximal cliques of G such that $<$ is consistent with Θ and such that for every vertex x of G the maximal cliques containing x occur consecutively in $<$.*

(For a proof the interested reader is referred to Golumbic [5, pp. 172–173].)

Note, furthermore, that by virtue of (3), the set I_x of all the maximal clique containing x is an *interval.* Since w is a source in Θ, I_w becomes an end interval in $\{I_v\}_{v\in V}$, as claimed. □

Recently, Skrien and Gimbel [6] proposed to call an interval graph G *homogeneously representable* if for every vertex v of G, there exists an interval representation of G in which I_v is an end interval.

Theorem 1 implies the following characterization of the homogeneously representable interval graphs.

Corollary 1.2 (Skrien and Gimbel [6]). *An interval graph G is homogeneously representable if, and only if, G contains no induced subgraph isomorphic to one of the graphs $\bar{F}_1$ and F_5 with $n = 3$.*

Proof. The 'only if' implication is immediate; to settle the 'if' implication, we only need show that every vertex of $\bar{G}$ can be a source in some transitive orientation of $\bar{G}$, for then the conclusion follows by (3). Since G is triangulated, $\bar{G}$ cannot have an induced subgraph iomorphic to one of the graphs F_3 and F_4; further, F_1 is an induced subgraph of F_2. Consequently, every vertex of $\bar{G}$ must be regular, and Theorem 1 implies that every vertex of $\bar{G}$ is a source in some transitive orientation of $\bar{G}$. □

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