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A Construction Which Can Be Used to Produce Finitely Presented Infinite Simple Groups

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In this series of three papers we discuss finitely presented infinite simple groups. It is known (see [3]) that every finitely generated group with solvable word problem can be embedded in a finitely generated simple subgroup of a finitely presented group. Since all finitely generated subgroups of finitely presented simple groups have solvable word problem, it is natural to ask which classes of finitely generated groups with solvable word problem can be embedded in finitely presented simple groups. The first paper contains a method for constructing finitely presented infinite simple groups. In the second paper we show that certain finitely presented Abelian groups and all linear groups over the integers can be embedded in finitely presented simple groups. Finally, in the third paper we show that a particular finitely presented group with unsolvable conjugacy problem can be embedded in a finitely presented group and that this gives a finitely presented simple group with unsolvable conjugacy problem.

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INTRODUCTION

This paper extends the work done by Higman [1], which is in turn based on work done by Thompson [3]. The formulation here resembles that of [1] rather than [3].

The aim is to describe a procedure for constructing finitely presented infinite simple groups. We take a finitely presented simple group, $G_{n,1}$, constructed in [1] and a large group, $\mathscr{G}_{n,1}$, which contains it. We shall prove

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the result, essentially due to Thompson, that if K is a group satisfying $G_{n,1} \leqslant K \leqslant \mathscr{G}_{n,1}$ then the derived subgroup, K', is simple. This results leads us to construct subgroups of $\mathscr{G}_{n,1}$ which are finitely presented, contain $G_{n,1}$ and have derived subgroups of finite index. The resulting derived subgroups are finitely presented simple groups.

We take certain types of subgroup, H, of $\mathscr{G}_{n,1}$ and produce a set, χ, of defining relations for the group $\langle G_{n,1}, H \rangle$. We show that if H is a finitely presented subgroup of an inverse limit of wreath products, then the set of relations, χ, is finitely based. Thus $\langle G_{n,1}, H \rangle$ is a finitely presented group.

Using this procedure we can construct a finitely presented simple group containing $GL(3, \mathbb{Z})$ (see [2]), which is not contained in any of the previously known finitely presented simple groups (see [1]). So, using this method, we do indeed get some new examples of such groups.

Description of $\mathscr{G}_{n,1}$ and $G_{n,1}$

Let W be a free semigroup, with 1, freely generated by $\{a_1, \ldots, a_n\}$. We will always assume that $n \geqslant 2$; (if $n = 1$ the construction gives nothing new).

A subset of W is called a *subspace* if it is closed under right multiplication by elements of W.

A subspace, Y, of W is called *inescapable* if given any $u \in W$ there exists some $w \in W$ such that $uw \in Y$. For example, the set $Y = \{w \in W \mid w$ contains at least one $a_1\}$ is an inescapable subspace of W.

A subspace, X, is called *cofinite* if $|W \backslash X| < \infty$. Since there is a finite bound on the length of words not belonging to X, a cofinite subspace is inescapable.

A *homomorphism*, θ, between subspaces of W, is a map satisfying $(uw)\theta = (u\theta)w$, for all $w \in W$, whenever $u\theta$ is defined. An *isomorphism* is a bijective homomorphism and if the domain and range of an isomorphism are inescapable (cofinite), then it is called an inescapable (cofinite) isomorphism.

If $u, v \in W$, then u is said to be an *initial segment* of v if there exists some $w \in W$ such that $uw = v$. We call u a proper initial segment of v if $w \neq 1$.

If Z is a subspace, the set $\{y \in Z \mid$ no proper initial segment of y belongs to $Z\}$ is called the *basis* for Z. A subset of W is a basis if it is the basis of some subspace. An *inescapable* (*cofinite*) *basis* is the basis of some inescapable (cofinite) space. Bases are precisely the subsets U, of W, such that no element of U is a proper initial segment of any other element of U, and U is a basis for Z if and only if it is the unique maximal basis such that every element of Z is uw, for some $u \in U$, $w \in W$. A basis is inescapable if and only if it is a maximal basis, and is a cofinite basis if and only if it is maximal and finite. Thus every basis is contained in some inescapable basis, and it is not hard to see that every finite basis is contained in a cofinite basis (see [1]).

We are particularly interested in cofinite bases, so we note some basic results. Proofs, explanations and a detailed discussion of cofinite bases and $G_{n,1}$ in general, can be found in [1].

If $U = \{u_1, ..., u_s\}$ is a cofinite basis, then so is $U' = \{u_1, ..., u_{i-1}, u_i a_1, ..., u_i a_n, u_{i+1}, ..., u_s\}$, U' is called an *elementary expansion* of U. Any basis which can be obtained from U by a finite series of elementary expansions is called an *expansion* of U. Every cofinite basis is an expansion of the basis $\{a_1, ..., a_n\}$, and if U, V are cofinite bases we can find a cofinite basis which is an expansion of them both. Since any finite basis is part of a cofinite basis, given $z \in W$, we can find a cofinite basis, and hence a cofinite subspace, containing z.

By an *extension of an inescapable isomorphism*, θ, we mean an inescapable isomorphism, θ', such that $u\theta' = u\theta$, whenever $u\theta$ is defined. Also, θ is maximal if it has no non-trivial extensions.

LEMMA 1. *Every inescapable isomorphism, θ, has a unique maximal extension, θ^*.*

Proof. If θ has domain Y, define

$$Y^* = \{z \in W \mid \exists y \in W((zw)\theta = yw, \text{ for all } w \in W \text{ such that } zw \in Y)\}.$$

Then Y^* is an inescapable subspace, because Y is and θ is a homomorphism. Define a map, θ^*, on Y^* by the rule $z\theta^* = y$, where $(zw)\theta = yw$ whenever $zw \in Y$. If $z \in Y^*$, then $(zu)\,\theta^*$ is defined, for all $u \in W$. Choose any $w \in W$ such that $zuw \in Y$; then $(zuw)\theta = yuw$, by definition of θ^*. So $(zu)\,\theta^* = yu = (z\theta^*)u$, and θ^* is a homomorphism. If $z\theta^* = y = v\theta^*$, choose w such that $zw, vw \in Y$. Then $(zw)\theta = yw = (vw)\theta$, and so $zw = vw$. Thus $z = v$ and θ^* is injective. The range, Z^*, of θ^* is a subspace, since θ^* is an isomorphism, which contains the range of θ. Thus Z^* is an inescapable subspace and θ^* is an inescapable isomorphism which clearly extends θ.

Suppose that ϕ is an inescapable isomorphism which also extends θ. If $x\phi = v$, then, for any w such that $xw \in Y$, $(xw)\theta = vw$. So $x \in Y^*$ and $x\theta^* = v$, i.e., θ^* extends ϕ. Thus θ^* is the required maximal extension of θ.

If θ and ϕ are inescapable isomorphisms, we define the composition, $\theta \circ \phi$, by the rule $u(\theta \circ \phi) = (u\theta)\phi$, whenever $(u\theta)\phi$ is defined. It is easy to see that $\theta \circ \phi$ is an inescapable isomorphism and straightforward checking of the groups axioms yields:

LEMMA 2. *The set of maximal inescapable isomorphisms forms a group under the operation $\theta\phi = (\theta \circ \phi)^*$.*

We call this group $\mathscr{G}_{n,1}$ and it is essentially the group constructed by Thompson [3].

The maximal extension of a cofinite isomorphism is also cofinite and it is again straightforward to obtain:

LEMMA 3. *The set of maximal cofinite isomorphisms forms a subgroup $G_{n,1}$ of $\mathscr{G}_{n,1}$.*

The group $G_{n,1}$ is the same as the group so named in [1]. We note here that if $X = \{x_1, ..., x_r\}$, we can define cofinite subspaces of the set of words $XW = \{x_i w \mid 1 \leqslant i \leqslant r,\ w \in W\}$. Again we have that the set of maximal cofinite isomorphisms forms a group, $G_{n,r}$. This group is also finitely presented, and is constructed in detail in [1].

We now construct and define the elements and subgroups of $\mathscr{G}_{n,1}$ in which we are interested.

SYMBOLS AND H-EXPANSIBLE GROUPS

A *column* is an object of the form

$$\begin{pmatrix} u \\ g \\ v \end{pmatrix},$$

where u and v are words in W and $g \in \mathscr{G}_{n,1}$. We say that $k \in \mathscr{G}_{n,1}$ *has* or *contains* the above column if, for all $w \in W$ such that wg is defined, $(uw)k$ is defined and $(uw)k = v(wg)$. In this case, k is said to be *almost defined* on u.

LEMMA 4. *If k has the columns*

$$\begin{pmatrix} u \\ g \\ v \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} u \\ h \\ y \end{pmatrix},$$

then $g = h$ and $v = y$.

Proof. There exists an inescapable subspace, Y, on which both g and h are defined. By definition, for all $w \in Y$, $(uw)k = v(wg) = y(wh)$. Thus we can suppose that $v = yx$, for some $x \in W$. Then $wg = x(wh)$, for all $w \in Y$. This implies that every word in the image of Y under g has initial segment x. But Yg is an inescapable subspace and so we must have $x = 1$. Thus $v = y$ and $wh = wg$, for all $w \in Y$. Since h and g are equal on an inescapable subspace, Lemma 1 gives $h = g$.

We say that $g \in \mathscr{G}_{n,1}$ is *expansible* if there exists a cofinite basis on which g is almost defined.

Let $g \in \mathscr{G}_{n,1}$ be expansible and almost defined on the cofinite basis $\{u_1, \ldots, u_s\}$. If the columns

$$\begin{pmatrix} u_i \\ g_i \\ v_i \end{pmatrix}$$

belong to g, we call the set of these columns a *symbol* for g and write it

$$\begin{pmatrix} u_1 & \cdots & u_s \\ g_1 & \cdots & g_s \\ v_1 & \cdots & v_s \end{pmatrix}. \tag{Γ}$$

It is easy to see that $\{v_1, \ldots, v_s\}$ is a basis; to show that it is a cofinite basis we need to show that it is maximal. If $\{v_1, \ldots, v_s\}$ is not maximal, then there exists some v, which can be taken in the range of g, which is independent of this set (i.e., for no v_i does there exist $w, y \in W$ such that $vw = v_i y$). Let $xg = v$. The set $\{u_1, \ldots, u_s\}$ is a cofinite basis, so there exist w, y such that $xw = u_i y$, for some u_i. Choose z such that $(yz) g_i$ is defined and then $(xwz) g = vwz$ and $(u_i yz) g = v_i (yz) g_i$. So v and v_i are not independent, which is a contradiction. Thus $\{v_1, \ldots, v_s\}$ is also cofinite basis.

If g_i has symbol

$$\begin{pmatrix} x_1 & \cdots & x_r \\ h_1 & \cdots & h_r \\ y_1 & \cdots & y_r \end{pmatrix},$$

then the symbol

$$\begin{pmatrix} u_1 & \cdots & u_{i-1} & u_i x_1 & \cdots & u_i x_r & u_{i+1} & \cdots & u_s \\ g_1 & \cdots & g_{i-1} & h_1 & \cdots & h_r & g_{i+1} & \cdots & g_s \\ v_1 & \cdots & v_{i-1} & v_i y_1 & \cdots & v_i y_r & v_{i+1} & \cdots & v_s \end{pmatrix}$$

is also a symbol for g and we call it an *expansion* of Γ. Any symbol which can be obtained from Γ by a finite series of expansions is also called an expansion of Γ. Also, Γ is called a *contraction* of any of its expansions.

LEMMA 5. *If g and k have symbols respectively*

$$\begin{pmatrix} u_1 & \cdots & u_s \\ g_1 & \cdots & g_s \\ v_1 & \cdots & v_s \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} v_1 & \cdots & v_s \\ k_1 & \cdots & k_s \\ y_1 & \cdots & y_s \end{pmatrix},$$

then gk has symbol

$$\begin{pmatrix} u_1 & \cdots & u_s \\ g_1 k_1 & \cdots & g_s k_s \\ y_1 & \cdots & y_s \end{pmatrix}.$$

Proof. Define a map, τ, by the rule $(u_i w)\tau = y_i(wg_i k_i)$, whenever $w(g_i k_i)$ is defined. Since $\{u_1,..., u_s\}$ and $\{y_1,..., y_s\}$ are cofinite bases and g_i and k_i are inescapable isomorphisms, τ is an inescapable isomorphism.

The set $Y = \{u_i w \mid 1 \leqslant i \leqslant s,\ wg_i \text{ and } (wg_i)\,k_i \text{ defined}\}$ is an inescapable subspace and, for all $y \in Y$, $y\tau = y(gk)$. Thus, by Lemma 1, gk is the unique maximal extension of τ. If $w(g_i k_i)$ is defined, then $(u_i w)\,gk = (u_i w)\tau = y_i(wg_i k_i)$, so gk has the above symbol as required.

We call the third symbol above a *combination* of the first two.

Let $H \leqslant K \leqslant \mathscr{G}_{n,1}$. A symbol for $g \in \mathscr{G}_{n,1}$ is called an *H-symbol* if all the elements of the middle row of the symbol belong to H.

A group K (containing H, as above) is said to be *H-expansible* if it satisfies the following two conditions:

(i) for all $g \in K$, there exists a cofinite basis $\{u_1,..., u_s\}$ such that g contains the columns

$$\begin{pmatrix} u_i \\ h_i \\ v_i \end{pmatrix},$$

for some $h_i \in H$, $v_i \in W$,

(ii) there exists a set of generators, $(g_i)_{i\in I}$, for K such that every element $g_{i_1} g_{i_2} \cdots g_{i_m}$, of K, has an H-symbol which is the combination of some H-symbols for $g_{i_1},..., g_{i_m}$.

The first condition ensures that every element of K has an H-symbol and the second ensures that every relation in K has an H-symbol. We need both of these conditions when we discuss a defining set of relations for K, below.

LEMMA 6. *If K is H-expansible, then any set of generators for K satisfies condition* (ii) *above.*

Proof. Suppose that $(k_j)_{j\in J}$ also generate K. Every element $k_{j_1} \cdots k_{j_r}$ can be written in the form $g_{i_{1,1}} \cdots g_{i_{1,p}} \cdots g_{i_{r,1}} \cdots g_{i_{r,q}}$. By assumption there are H-symbols for each $g_{i_{t,s}}$ whose combination is an H-symbol for this word. But the combination of the subset of these symbols for $g_{i_{1,1}},..., g_{i_{1,p}}$ is an H-symbol for k_{j_1}. In this way we get H-symbols for each k_{j_s}, whose combination is an H-symbol for the original word considered. (Note, since $\mathscr{G}_{n,1}$ is a group, and therefore associative, the combination of symbols is an associative operation.)

Let H be a subgroup of $\mathscr{G}_{n,1}$. We are interested in groups of the form $\langle G_{n,1}, H\rangle$, the group generated by $G_{n,1}$ and H. We shall reserve script notation for subgroups of $\mathscr{G}_{n,1}$ containing $G_{n,1}$ and $\mathscr{H}$ will always be the group $\langle G_{n,1}, H\rangle$.

At this point we introduce notation which will be used throughout this and the following papers. For $h \in \mathscr{G}_{n,1}$, it is easy to see that the map given by the rules

$$\begin{aligned} a_1 w &\to a_1(wh), && \text{whenever } wh \text{ defined} \\ a_i y &\to a_i y, && 2 \leqslant i \leqslant n, \text{ for all } y \in W \end{aligned}$$

is an inescapable isomorphism. By σ_h we will always mean the unique maximal extension of this map.

LEMMA 7. *The groups H and $H^* = \{\sigma_h \mid h \in H\}$ are isomorphic.*

Proof. We define a map $\theta: H \to H^*$ by the rule $h\theta = \sigma_h$. Given $k, h \in \mathscr{G}_{n,1}$, let Y be an inescapable subspace such that, for all $y \in Y$, $(yk)h$ is defined. Let Z be the inescapable subspace consisting of all elements of the form $a_1 y$ or $a_i w$, for $y \in Y$, $w \in W$ and $2 \leqslant i \leqslant n$. For any $z \in Z$, $z\sigma_k\sigma_h = z\sigma_{kh}$, and so, by Lemma 1, $\sigma_k\sigma_h = \sigma_{kh}$. Thus θ is a homomorphism.

If $\sigma_h = \sigma_k$, then $wh = wk$, whenever both are defined. There exists an inescapable subspace on which both h and k are defined and so, again by Lemma 1, $h = k$. Thus θ is an isomorphism, as required.

We will often want to talk about elements of $\mathscr{G}_{n,1}$ in terms of their symbols. Any object of the form

$$\begin{pmatrix} u_1 & \cdots & u_s \\ g_1 & \cdots & g_s \\ v_1 & \cdots & v_s \end{pmatrix},$$

where $\{u_1, ..., u_s\}$ and $\{v_1, ..., v_s\}$ are cofinite bases and $g_i \in \mathscr{G}_{n,1}$, will be called a symbol. By *the element*, $\tau \in \mathscr{G}_{n,1}$, *with this symbol* we mean the unique maximal extension of the map τ^*, given by $(u_i w)\tau^* = v_i(wg_i)$, for all w such that wg_i is defined $(1 \leqslant i \leqslant s)$. It is easy to see that τ^* is an inescapable isomorphism so, by Lemma 1, in this way every symbol defines a unique element of $\mathscr{G}_{n,1}$.

In particular, if α is any element of $G_{n,1}$, choose a cofinite basis $\{u_1, ..., u_s\}$ in the domain of α. Then α has the symbol

$$\begin{pmatrix} u_1 & \cdots & u_s \\ 1 & \cdots & 1 \\ u_1\alpha & \cdots & u_s\alpha \end{pmatrix}.$$

Furthermore, if $\{u_1,..., u_s\}$ and $\{v_1,..., v_s\}$ are any two cofinite bases, then

$$\begin{pmatrix} u_1 & \cdots & u_s \\ 1 & \cdots & 1 \\ v_1 & \cdots & v_s \end{pmatrix}$$

is a symbol for some element of $G_{n,1}$.

Conversely, if Γ is a symbol for g in the sense defined prior to Lemma 5 and Γ defines the element ε in the manner just indicated, then we see that $(u_i w)\varepsilon = (u_i w)\,\varepsilon^* = v_i(wg_i) = (u_i w)\,g$, for all w such that wg_i is defined. Thus g and ε are equal on an inescapable subspace and so, by Lemma 1, $g = \varepsilon$. Thus a symbol for $g \in \mathscr{G}_{n,1}$ defines g and the above definitions are reasonable.

It is worth noting, and is easily seen be a similar argument to that above, that if a symbol, Δ, defines $\tau \in \mathscr{G}_{n,1}$, then any expansion of Δ also defines τ.

From now on we shall assume that the group $\mathscr{H}$ is H-expansible.

LEMMA 8. *If $\mathscr{H}$ is H-expansible, then $\mathscr{H}$ and ${}_*\mathscr{H} = \langle G_{n,1}, H^* \rangle$ are equal.*

Proof. We need to show that $H^* \subseteq \mathscr{H}$ and $H \subseteq {}_*\mathscr{H}$.

Since $\mathscr{H}$ is H-expansible, for given $h \in H$ we can find an H-symbol,

$$\begin{pmatrix} u_1 & \cdots & u_s \\ h_1 & \cdots & h_s \\ v_1 & \cdots & v_s \end{pmatrix},$$

for h. Choose a cofinite basis $\{a_1, w_2,..., w_s\}$ and let α, γ_i and β be the elements of $G_{n,1}$ with symbols respectively

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_s \\ 1 & 1 & \cdots & 1 \\ a_1 & w_2 & \cdots & w_s \end{pmatrix},$$

$$\begin{pmatrix} a_1 & w_2 & \cdots & w_{i-1} & w_i & w_{i+1} & \cdots & w_s \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ w_i & w_2 & \cdots & w_{i-1} & a_1 & w_{i+1} & \cdots & w_s \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_1 & w_2 & \cdots & w_s \\ 1 & 1 & \cdots & 1 \\ v_1 & v_2 & \cdots & v_s \end{pmatrix},$$

for $2 \leqslant i \leqslant S$. By definition, for $k \in H$, σ_k has symbols

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ k & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_1 & w_2 & \cdots & w_s \\ k & 1 & \cdots & 1 \\ a_1 & w_2 & \cdots & w_s \end{pmatrix}.$$

Then, by Lemma 5, the product $\alpha\sigma_{h_1}\gamma_2\sigma_{h_2}\cdots\gamma_s\sigma_{h_s}\gamma_s\cdots\gamma_3\gamma_2\beta$ has symbol

$$\begin{pmatrix} u_1 \cdots u_s \\ h_1 \cdots h_s \\ v_1 \cdots v_s \end{pmatrix}.$$

But a symbol for h defines the unique element h and so $h = \alpha\sigma_{h_1}\cdots\gamma_s\sigma_{h_s}\gamma_s\cdots\gamma_2\beta \in \langle G_{n,1}, H^*\rangle$, for any $h \in H$.

Since $h \in \mathscr{G}_{n,1}$, there exists some y such that h is defined on y. Expand the column

$$\begin{pmatrix} u_1 \\ h_1 \\ v_1 \end{pmatrix}$$

repeatedly until we obtain a set of columns

$$\begin{pmatrix} u_1 z_1 & \cdots & u_1 z_r \\ k_1 & \cdots & k_r \\ v_1 x_1 & \cdots & v_1 x_r \end{pmatrix},$$

where each z_i is longer, as a word, than y. The set $\{z_1, \ldots, z_r\}$ is a cofinite basis and so, for some z_j, there exists x such that $yx = z_j$. But h is defined on $u_1 yx$ so h contains the column

$$\begin{pmatrix} u_1 yx \\ 1 \\ v \end{pmatrix},$$

for some $v \in W$. Thus, by Lemma 4, $k_j = 1$ and so we can expand the above symbol for h to a symbol of the form

$$\begin{pmatrix} x_1 & \cdots & x_t & w \\ g_1 & \cdots & g_t & 1 \\ y_1 & \cdots & y_t & z \end{pmatrix},$$

for h. We can now expand this to the symbol

$$\begin{pmatrix} x_1 & \cdots & x_t & wa_1 & \cdots & wa_n \\ g_1 & \cdots & g_t & 1 & \cdots & 1 \\ y_1 & \cdots & y_t & za_1 & \cdots & za_n \end{pmatrix}.$$

By definition of σ_h, the symbol

$$\begin{pmatrix} a_1x_1 & \cdots & a_1x_t & a_1w & a_2 & \cdots & a_n \\ g_1 & \cdots & g_t & 1 & 1 & \cdots & 1 \\ a_1y_1 & \cdots & a_1y_t & a_1z & a_2 & \cdots & a_n \end{pmatrix}$$

is a symbol for σ_h. If $\delta, \rho \in G_{n,1}$ are the elements with symbols respectively

$$\begin{pmatrix} a_1x_1 & \cdots & a_1x_t & a_1w & a_2 & \cdots & a_n \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ x_1 & \cdots & x_t & wa_1 & wa_2 & \cdots & wa_n \end{pmatrix}$$

and

$$\begin{pmatrix} y_1 & \cdots & y_t & za_1 & za_2 & \cdots & za_n \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ a_1y_1 & \cdots & a_1y_t & a_1z & a_2 & \cdots & a_n \end{pmatrix}$$

then, by Lemma 5, σ_h and $\delta h\rho$ have the same symbol. Thus $\sigma_h = \delta h\rho \in \mathscr{H}$ and $\mathscr{H} = {}_*\mathscr{H}$, as required.

From the above lemma we see that we can, and we will, take $\mathscr{H}$ to be generated by the generators of $G_{n,1}$ and the set $\{\sigma_h \mid h$ belongs to some fixed set of generators of $H\}$.

LEMMA 9. *If $g \in \mathscr{H}$ and $\mathscr{H}$ is H-expansible, there exists a unique H-symbol for g which has every other H-symbol for g as an expansion.*

Proof. Suppose that

$$\Gamma = \begin{pmatrix} u_1 & \cdots & u_s \\ h_1 & \cdots & h_s \\ v_1 & \cdots & v_s \end{pmatrix} \quad \text{and} \quad \Delta = \begin{pmatrix} x_1 & \cdots & x_r \\ k_1 & \cdots & k_r \\ y_1 & \cdots & y_r \end{pmatrix}$$

are two H-symbols for g, neither of which is a non-trivial expansion of any H-symbol. Since $\{u_1,..., u_s\}$ and $\{x_1,..., x_r\}$ are cofinite bases we can assume, by swapping Γ and Δ and reordering their columns if necessary, that $u_1 = x_1z_1,..., u_t = x_1z_t$, for some cofinite basis $\{z_1,..., z_t\}$.

The set, Y_i, of all $w \in W$ such that both wh_i and $(z_iw)\,k_1$ are defined is an inescapable subspace. For all $w \in Y_i$, $v_i(wh_i) = (x_1z_iw)\,g = y_1(z_iw)\,k_1$. If $v_i p = y_1$, for some $p \in W$, then $wh_i = p(z_iw)\,k_1$. Thus every word in the inescapable subspace y_ih_i has initial segment p, so we must have $p = 1$. Hence we can assume that $v_i = y_1w_i$, for some $w_i \in W$. Since $\{v_1,..., v_s\}$ is a cofinite basis, so is $\{w_1,..., w_t\}$.

Let h be the element with symbol

$$\begin{pmatrix} z_1 & \cdots & z_t \\ h_1 & \cdots & h_t \\ w_1 & \cdots & w_t \end{pmatrix}.$$

Then Γ contains the columns

$$\begin{pmatrix} x_1 z_1 & \cdots & x_1 z_t \\ h_1 & \cdots & h_t \\ y_1 w_1 & \cdots & y_1 w_t \end{pmatrix}$$

and hence can be contracted to the symbol

$$\begin{pmatrix} x_1 & u_{t+1} & \cdots & u_s \\ h & h_{t+1} & \cdots & h_s \\ y_1 & v_{t+1} & \cdots & v_s \end{pmatrix}.$$

By Lemma 4, $h = k_1 \in H$, so this is an H-symbol for g which is a contraction of Γ. By assumption this contraction cannot be non-trivial, so we must have $z_1 = \cdots = z_t = 1$ and $u_1 = x_1$. Then, from Lemma 4, $h_1 = k_1$ and $v_1 = y_1$.

Applying this argument to each of the columns of the above two symbols in turn, we see that these symbols must be identical. Hence every H-symbol for g can be obtained by expanding a unique uncontractable H-symbol for g.

We call the unique uncontractable H-symbol for g the *shortest H-symbol* for g.

Defining Relations for $\mathscr{H}$

Throughout this section we still assume that $\mathscr{H}$ is H-expansible. The element, δ, will always be the element of order 2 with symbol

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n a_1 & a_n a_2 & \cdots & a_n a_n \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ a_n a_1 & a_2 & \cdots & a_{n-1} & a_1 & a_n a_2 & \cdots & a_n a_n \end{pmatrix}.$$

We say that a set, $\{\eta_2,\ldots,\eta_s\}$, of elements of $G_{n,1}$ is of *type s* if there exists a cofinite basis $\{a_1, w_2,\ldots, w_s\}$ such that η_i has the symbol

$$\begin{pmatrix} a_1 & w_2 & \cdots & w_{i-1} & w_i & w_{i+1} & \cdots & w_s \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ w_i & w_2 & \cdots & w_{i-1} & a_1 & w_{i+1} & \cdots & w_s \end{pmatrix}, \qquad 1 \leqslant i \leqslant s.$$

We will take as generators of $\mathscr{H}$ the generators of $G_{n,1}$ and a set $\{\sigma_h \mid h$ belongs to some fixed set of generators for $H\}$. With this set of generators, the following sets of relations will be shown to define $\mathscr{H}$.

(A) The defining relations of $G_{n,1}$ and $H^* = \{\sigma_h \mid h \in H\}$.

The relations in the following sets are written in terms of general elements of $G_{n,1}$ and H^*, not just on elements from the chosen generating set. The elements are considered as a "shorthand" notation for the words on the chosen generating set to which they correspond. Since we have relations A, it does not matter which of the words a particular element of $G_{n,1}$ or H^* is taken to represent.

(B) $\{\alpha\sigma_h = \sigma_h\alpha \mid h \in H,\ \alpha \in G_{n,1},\ \alpha$ fixes all words of the form $a_1 w\}$.

(C) $\{\delta\sigma_h \delta\sigma_k = \sigma_k \delta\sigma_h \delta \mid h, k \in H\}$.

(D) $\{\sigma_h = \tau\sigma_{h_1}\eta_2\sigma_{h_2} \cdots \eta_s\sigma_{h_s}\eta_s \cdots \eta_3\eta_2\varepsilon \mid h \in H\}$, where

$$\begin{pmatrix} u_1 & \cdots & u_s \\ h_1 & \cdots & h_s \\ v_1 & \cdots & v_s \end{pmatrix}$$

is any symbol for σ_h, $\{\eta_2,\ldots,\eta_s\}$ is any set of type s and if $\{a_1, w_2,\ldots, w_s\}$ is the basis of the domains of the η_i, then τ and ε are the elements with symbols respectively

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_s \\ 1 & 1 & \cdots & 1 \\ a_1 & w_2 & \cdots & w_s \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_1 & w_2 & \cdots & w_s \\ 1 & 1 & \cdots & 1 \\ v_1 & v_2 & \cdots & v_s \end{pmatrix}.$$

The element σ_h has unique shortest symbol

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ h & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

and considering symbols for all the elements above, we see that the relations $\mathrm{A} \cup \mathrm{B} \cup \mathrm{C} \cup \mathrm{D} = \chi$ hold in $\mathscr{G}_{n,1}$, by Lemma 5.

LEMMA 10. *If ρ is any element with a symbol of the form*

$$\begin{pmatrix} a_1 & w_2 & \cdots & w_{r-1} & w_r & w_{r+1} & \cdots & w_m \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ w_2 & w_3 & \cdots & w_r & a_1 & w_{r+1} & \cdots & w_m \end{pmatrix},$$

the relation $\rho\sigma_h\rho^{-1}\sigma_k = \sigma_k\rho\sigma_h\rho^{-1}$ is a consequence of $\mathrm{A} \cup \mathrm{B} \cup \mathrm{C}$.

Proof. Let α, β be the elements with symbols respectively

$$\begin{pmatrix} a_1 & w_2 & \cdots & w_{r-1} & w_r & w_{r+1} & \cdots & w_m \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ a_1 & x_3 & \cdots & x_r & a_n a_1 & x_{r+1} & \cdots & x_m \end{pmatrix}$$

and

$$\begin{pmatrix} a_1 & x_3 & \cdots & x_r & a_n a_1 & x_{r+1} & \cdots & x_m \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ a_1 & w_3 & \cdots & w_r & w_2 & w_{r+1} & \cdots & w_m \end{pmatrix}.$$

The relation $\alpha\delta\beta = \rho$ is in A and α, β fix a_1, so the relations

$$\begin{aligned} \rho\sigma_h\rho^{-1}\sigma_k &= \alpha\delta\beta\sigma_h\beta^{-1}\,\delta\alpha^{-1}\sigma_k && \text{(A)} \\ &= \alpha\delta\sigma_h\,\delta\sigma_k\alpha^{-1} && (\mathrm{A} \cup \mathrm{B}) \\ &= \alpha\sigma_k\,\delta\sigma_h\,\delta\alpha^{-1} && \text{(C)} \\ &= \sigma_k\alpha\delta\beta\sigma_h\beta^{-1}\,\delta\alpha^{-1} && (\mathrm{A} \cup \mathrm{B}) \\ &= \sigma_k\rho\sigma_h\rho^{-1} && \text{(A)} \end{aligned}$$

are consequences of $\mathrm{A} \cup \mathrm{B} \cup \mathrm{C}$, as required.

LEMMA 11. *If $\{\eta_2, \ldots, \eta_s\}$ is a set of type s, the relation* $\sigma_{h_i}\eta_{i+1}\sigma_{h_{i+1}} \cdots \eta_j\sigma_{h_j}\eta_j\,\eta_{j-1} \cdots \eta_{i+1}\sigma_{k_i}\eta_{i+1}\sigma_{k_{i+1}} \cdots \eta_j\sigma_{k_j} = \sigma_{h_i k_i}\eta_{i+1}\sigma_{h_{i+1}k_{i+1}} \cdots \eta_j\sigma_{h_j k_j}$ is a consequence of $\mathrm{A} \cup \mathrm{B} \cup \mathrm{C}$.

Proof. From Lemma 10 we have $\eta_r\sigma_h\eta_r\sigma_k = \sigma_k\eta_r\sigma_h\eta_r$.

Since $\eta_r\eta_m\eta_r$ fixes a_1 $(m \neq r)$, $\eta_m\eta_r\sigma_h\eta_r = \eta_r\sigma_h\eta_r\eta_m$ is a consequence of $\mathrm{A} \cup \mathrm{B}$.

Thus the relations

$$\begin{aligned} &\sigma_{h_m}\eta_{m+1}\sigma_{h_{m+1}} \cdots \eta_r\sigma_{h_r}\eta_r \cdots \eta_{m+1}\sigma_{k_m}\eta_{m+1} \\ &\quad = \sigma_{h_m} \cdots \eta_r\sigma_{h_r}\eta_{m+1}\sigma_{k_m}\eta_{m+1}\eta_r \cdots \eta_{m+2} \\ &\quad = \sigma_{h_m} \cdots \eta_r\eta_{m+1}\sigma_{k_m}\eta_{m+1}\sigma_{k_r}\eta_r \cdots \eta_{m+2} \\ &\quad = \sigma_{h_m} \cdots \eta_{r-1}\sigma_{h_{r-1}}\eta_{m+1}\sigma_{k_m}\eta_{m+1}\eta_r\sigma_{h_r}\eta_r \cdots \eta_{m+2} \\ &\quad = \sigma_{h_m}\eta_{m+1}\eta_{m+1}\sigma_{k_m}\eta_{m+1}\sigma_{h_{m+1}} \cdots \eta_r\sigma_{h_r}\eta_r \cdots \eta_{m+2} \\ &\quad = \sigma_{h_m k_m}\eta_{m+1}\sigma_{h_{m+1}} \cdots \eta_r\sigma_{h_r}\eta_r \cdots \eta_{m+2} \end{aligned}$$

are consequences of $\mathrm{A} \cup \mathrm{B} \cup \mathrm{C}$.

Applying this result to the left-hand side of the relation in the statement of the lemma, for $r=j$ and $m=i,\ i+1,\dots,j-1$, gives the desired right-hand side.

The next lemma allows us to ignore the order of the columns in a symbol when obtaining a relation from the symbol.

LEMMA 12. *If $\{\eta_2,\dots,\eta_s\}$ is a set of type s and μ is a permutation of the set $\{m,\dots,r\}$, $1\leqslant m\leqslant r\leqslant s$, then the relation $\sigma_{h_m}\eta_{m+1}\sigma_{h_{m+1}}\cdots\eta_r\sigma_{h_r}=\alpha_\mu\sigma_{h_{m\mu}}\eta_{m+1}\sigma_{h_{(m+1)\mu}}\cdots\eta_r\sigma_{h_{r\mu}}\beta_\mu$ is a consequence of $\mathrm{A}\cup\mathrm{B}\cup\mathrm{C}$, where α_μ and β_μ have symbols respectively*

$$\begin{pmatrix} w_2 & \cdots & w_m & x_m & x_{m+1} & \cdots & x_r & w_{r+1} & \cdots & w_s \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ w_2 & \cdots & w_m & a_1 & w_{m+1} & \cdots & w_r & w_{r+1} & \cdots & w_s \end{pmatrix}$$

and

$$\begin{pmatrix} w_2 & \cdots & w_m & w_{m+1} & \cdots & w_r & a_1 & w_{r+1} & \cdots & w_s \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ w_2 & \cdots & w_m & y_{m+1} & \cdots & y_r & y_{r+1} & w_{r+1} & \cdots & w_s \end{pmatrix}$$

$(x_{m\mu^{-1}}=a_1,\ x_j=w_{j\mu}\ (j\mu\neq m),\ y_{r\mu^{-1}+1}=a_1,\ y_{i+1}=w_{i\mu+1}\ (i\mu\neq r))$.

Proof. If is sufficient to show the result for any $\mu=(ij)$, $m\leqslant i<j\leqslant r$. The relations

$$\begin{aligned}
&\sigma_{h_i}\eta_{i+1}\cdots\sigma_{h_j}\\
&\quad=\eta_{i+1}\sigma_{h_{i+1}}\eta_{i+1}\sigma_{h_i}\eta_{i+1}\eta_{i+2}\sigma_{h_{i+2}}\cdots\eta_j\sigma_{h_j}\\
&\quad=\eta_{i+1}\sigma_{h_{i+1}}\eta_{i+2}\eta_{i+1}\sigma_{h_i}\eta_{i+1}\sigma_{h_{i+2}}\cdots\eta_j\sigma_{h_j}\\
&\quad=\eta_{i+1}\sigma_{h_{i+1}}\eta_{i+2}\sigma_{h_{i+2}}\cdots\eta_j\sigma_{h_j}\eta_{i+1}\sigma_{h_i}\eta_{i+1}\\
&\quad=\eta_{i+1}\sigma_{h_{i+1}}\cdots\eta_{j-1}\eta_j\sigma_{h_j}\eta_j\sigma_{h_{j-1}}\eta_j\eta_{i+1}\sigma_{h_i}\eta_{i+1}\\
&\quad=\eta_j\sigma_{h_j}\eta_j n_{i+1}\sigma_{h_{i+1}}\cdots\eta_{j-1}\sigma_{h_{j-1}}\eta_j\eta_{i+1}\sigma_{h_i}\eta_{i+1}\\
&\quad=\eta_j\sigma_{h_j}\eta_{i+1}\sigma_{h_{i+1}}\eta_{i+1}\eta_j\eta_{i+1}\eta_{i+2}\sigma_{h_{i+2}}\cdots\eta_{j-1}\sigma_{h_{j-1}}\eta_j\eta_{i+1}\sigma_{h_i}\eta_{i+1}\\
&\quad=\eta_j\sigma_{h_j}\eta_{i+1}\sigma_{h_{i+1}}\cdots\eta_{j-1}\sigma_{h_{j-1}}\eta_{j-1}\cdots\eta_{i+1}\eta_j\eta_{i+1}\cdots\eta_j\eta_{i+1}\sigma_{h_i}\eta_{i+1}\\
&\quad=\eta_j\sigma_{h_j}\eta_{i+1}\sigma_{h_{i+1}}\cdots\eta_{j-1}\sigma_{h_{j-1}}\eta_j\sigma_{h_i}\eta_{i+1}
\end{aligned}$$

are consequences of $\mathrm{A}\cup\mathrm{B}\cup\mathrm{C}$, as are the relations

$$\sigma_{h_m}\eta_{m+1}\cdots\sigma_{h_{i-1}}\eta_i\eta_j=\eta_{m+1}\cdots\eta_i\eta_j\eta_i\cdots\eta_{m+1}\sigma_{h_m}\eta_{m+1}\cdots\sigma_{h_{i-1}}\eta_i$$

and

$$\eta_{i+1}\eta_{j+1}\sigma_{h_{j+1}}\eta_{j+2}\cdots\sigma_{h_r}=\eta_{j+1}\sigma_{h_{j+1}}\cdots\eta_r\sigma_{h_r}\eta_r\cdots\eta_{j+1}\eta_{i+1}\eta_{j+1}\cdots\eta_r$$

(using the same techniques). The relations $\eta_{m+1} \cdots \eta_i \eta_j \eta_i \cdots \eta_{m+1} = \alpha_\mu$ and $\eta_r \cdots \eta_{j+1} \eta_{i+1} \eta_{j+1} \cdots \eta_r = \beta_\mu$ $(\mu = (i\,j))$ belong to A, so the relation $\sigma_{h_m} \eta_{m+1} \sigma_{h_{m+1}} \cdots \eta_r \sigma_{h_r} = \alpha \mu \sigma_{h_{m\mu}} \eta_{m+1} \sigma_{h_{(m+1)\mu}} \cdots \eta_r \sigma_{h_{r\mu}} \beta \mu$, where $\mu = (i\,j)$, is a consequence of $\mathrm{A} \cup \mathrm{B} \cup \mathrm{C}$, as required.

THEOREM 1. *If the group $\mathscr{H} = \langle G_{n,1}, H \rangle$, $H \in \mathscr{G}_{n,1}$, is H-expansible, then the relations χ, as described above are a set of defining relation for $\mathscr{H}$.*

Proof. Suppose that $\alpha_1 \sigma_{h_1} \alpha_2 \sigma_{h_2} \cdots \alpha_m \sigma_{h_m} = 1$ is a relation in $\mathscr{H}$, where $\alpha_i \in G_{n,1}$ and $h_i \in H$.

$\mathscr{H}$ is H-expansible and so, in view of Lemma 6, we can find H-symbols

$$\begin{pmatrix} u_{i-1,1} & \cdots & u_{i-1,s} \\ 1 & \cdots & 1 \\ v_{i,1} & \cdots & v_{i,s} \end{pmatrix} \quad \text{and} \quad \Gamma_i = \begin{pmatrix} v_{i,1} & \cdots & v_{i,s} \\ k_{i,1} & \cdots & k_{i,s} \\ u_{i,1} & \cdots & i_{i,s} \end{pmatrix},$$

for α_i and σ_{h_i} respectively, such that

$$\begin{pmatrix} u_{0,1} & \cdots & u_{0,s} \\ g_1 & \cdots & g_s \\ u_{m,1} & \cdots & u_{m,s} \end{pmatrix},$$

where $g_j = k_{1,j} k_{2,j} \cdots k_{m,j}$, is a symbol for the identity. By Lemma 4 we must have $g_j = 1$ and $u_{0,j} = u_{m,j}$, for $1 \leqslant j \leqslant s$. Thus the relations $\sigma_{g_j} = \sigma_{k_{1,j}} \cdots \sigma_{k_{m,j}} = 1$ are consequences of A.

For some set, $\{\eta_2, \ldots, \eta_s\}$ of type s, the relations $\sigma_{h_i} = \tau_i \sigma_{k_{i,1}} \eta_2 \sigma_{k_{i,2}} \cdots \eta_s \sigma_{k_{i,s}} \eta_s \cdots \eta_2 \varepsilon_i$ belong to D, where τ_i and ε_i have symbols

$$\begin{pmatrix} v_{i,1} & v_{i,2} & \cdots & v_{i,s} \\ 1 & 1 & \cdots & 1 \\ a_1 & w_1 & \cdots & w_s \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_1 & w_2 & \cdots & w_s \\ 1 & 1 & \cdots & 1 \\ u_{i,1} & u_{i,2} & \cdots & u_{i,s} \end{pmatrix},$$

respectively. The relations $\varepsilon_i \alpha_{i+1} \tau_{i+1} = 1$, $1 \leqslant i \leqslant m-1$, are consequences of A and so the relation

$$\alpha_1 \sigma_{h_1} \cdots \alpha_m \sigma_{h_m} = \alpha_1 \tau_1 \sigma_{k_{1,1}} \eta_2 \cdots \sigma_{k_{1,s}} \eta_s \cdots \eta_2 \sigma_{k_{2,1}} \eta_2 \cdots \sigma_{k_{2,s}} \eta_s \cdots \eta_2$$
$$\cdots \sigma_{k_{m,1}} \eta_2 \cdots \sigma_{k_{m,s}} \eta_s \cdots \eta_2 \varepsilon_m$$

is a consequence of χ. Then, by Lemma 11, the relation $\alpha_1 \sigma_{h_1} \cdots \alpha_m \sigma_{h_m} = \alpha_1 \tau_1 \sigma_{g_1} \eta_2 \sigma_{g_2} \cdots \eta_s \sigma_{g_s} \eta_s \cdots \eta_2 \varepsilon_m$ is a consequence of χ. We have already noted that the relations $\sigma_{g_j} = 1$ are consequences of χ, as are the relations $\eta_j \eta_j = \alpha_1 \tau_1 \varepsilon_m = 1$. So the relation $\alpha_1 \sigma_{h_1} \cdots \alpha_m \alpha_{h_m} = 1$ is a consequence of χ.

We wish to construct finitely presented groups so eventually we will consider cases where the relations χ are finitely based. With this in mind we prove the following lemma.

LEMMA 13. *The set of relations* $\mathrm{A} \cup \mathrm{B} \cup \mathrm{C}$ *is finitely based if* H *is finitely presented.*

Proof. $G_{n,1}$ is finitely presented, see [1], so by assumption the relations A are finitely based.

The group $G_{n,r}$ is the set of all maximal cofinite isomorphisms between subspaces of the set of all words of the form $x_j w$, for some finite set $\{x_1,..., x_r\}$ which is disjoint from W. This group is also finitely presented, see [1]. Given an element $\beta \in G_{n,n-1}$ define an element βf by the rules

$$(a_j u)(\beta f) = a_i v,$$
$$(a_1 u)(\beta f) = a_1 u,$$

where $2 \leqslant j \leqslant n$ and $(x_{j-1} u)\beta = x_{i-1} v$. It is not hard to see that $\beta f \in G_{n,1}$ and in fact that f is an isomorphism between $G_{n,n-1}$ and the subgroup, K, of elements of $G_{n,1}$ which fix all words of the form $a_1 w$. Thus K is finitely generated by, say, $\{\alpha_1,..., \alpha_m\}$. Then $\mathrm{A} \cup \mathrm{B}$ is generated by A and the finite set $\{\alpha_i \sigma_h = \sigma_h \alpha_i \mid 1 \leqslant i \leqslant m$, h belongs to a fixed finite set of generators of $H\}$.

The set $\mathrm{A} \cup \mathrm{B} \cup \mathrm{C}$ is generated by A, B and the finite set $\{\delta\sigma_h\, \delta\sigma_k = \sigma_k\, \delta\sigma_h \delta \mid h, k$ belong to the fixed set of generators of $H\}$. Thus $\mathrm{A} \cup \mathrm{B} \cup \mathrm{C}$ is finitely based.

SUBGROUPS OF WREATH PRODUCTS

We now describe a class of subgroups of $\mathscr{G}_{n,1}$ with the property that, if H belongs to this class, $\mathscr{H}$ is H-expansible, and, if H is finitely presented, so is $\mathscr{H}$.

Let A and B be permutation groups of the sets Γ and Δ, respectively, and suppose that Δ is finite. We can form the wreath product $A \operatorname{wr} B$ as a permutation group on the set $\Gamma \times \Delta$. However, it will be convenient for us to regard $A \operatorname{wr} B$ as a permutation group on $\Delta \times \Gamma$; we can do this since there is a natural bijection between $\Gamma \times \Delta$ and $\Delta \times \Gamma$. Thus, if g is an element of $A \operatorname{wr} B$, the image of $(\delta, \gamma) \in \Delta \times \Gamma$ under g is of the form $(\delta b, \gamma a_\delta)$, where $b \in B$, $a_\delta \in A$ and δb is independent of γ. We can associate g with the set $\{b, a_\delta \mid \delta \in \Delta\}$.

There is a natural projection $p: A \operatorname{wr} B \to B$ given by $gp = b$. Also, if A and

C are permutation groups and θ is a homomorphism $\theta: A \to C$, θ induces a natural homomorphism $\theta': A \text{ wr } B \to C \text{ wr } B$, given by

$$\{b, a_\delta \mid \delta \in \Delta\} \to \{b, (a_\delta)\theta \mid \delta \in \Delta\}.$$

We also note that the permutation wreath product is associative and so

$$(A \text{ wr } B) \text{ wr } C = A \text{ wr } (B \text{ wr } C).$$

Suppose that F is a free group acting faithfully on itself by right multiplication and let S_n be the symmetric group on $N = \{a_1, ..., a_n\}$, in its natural representation. We will define, inductively, $F(\text{wr } S_n)^i$ to be $(F(\text{wr } S_n)^{i-1}) \text{ wr } S_n$ and $S_n(\text{wr } S_n)^i$ to be $(S_n(\text{wr } S_n)^{i-1}) \text{ wr } S_n$. Since we have associative wreath products, we can think of $S_n(\text{wr } S_n)^{i-1}$ as acting on N^i, the cartesian product of i copies of N.

If $\theta: F \to F \text{ wr } S_n$ is any homomorphism, we define $\theta_i: F(\text{wr } S_n)^i \to F(\text{wr } S_n)^{i+1}$ to be the *natural homomorphism* induced by the homomorphism $\theta_{i-1}: F(\text{wr } S_n)^{i-1} \to F(\text{wr } S_n)^i$, for $i = 1, 2, 3, ...$ and setting $\theta_0 = \theta$. We let p_i and ϕ_i be the natural projections $p_i: F(\text{wr } S_n)^i \to S_n(\text{wr } S_n)^{i-1}$ and $\phi_i: S_n(\text{wr } S_n)^i \to S_n(\text{wr } S_n)^{i-1}$. Then the diagram

$$\begin{tikzcd}
F \arrow[r, "\theta"'] & F \text{ wr } S_n \arrow[r, "\theta_1"'] \arrow[d, "p_1"'] & F(\text{wr } S_n)^2 \arrow[r, "\theta_2"'] \arrow[d, "p_2"'] & \cdots \arrow[r] & F(\text{wr } S_n)^i \arrow[r, "\theta_i"'] \arrow[d, "p_i"'] & F(\text{wr } S_n)^{i+1} \arrow[r, "\theta_{i+1}"'] \arrow[d, "p_{i+1}"'] & \cdots \\
& S_n & S_n \text{ wr } S_n \arrow[l, "\phi_1"] & \cdots \arrow[l, "\phi_2"] & S_n(\text{wr } S_n)^{i-1} \arrow[l] & S_n(\text{wr } S_n)^i \arrow[l, "\phi_i"] & \cdots \arrow[l, "\phi_{i+1}"]
\end{tikzcd}$$

is commutative in the sense that $p_i = \theta_i p_{i+1} \phi_i$. We can read off, from the diagram, unique homomorphisms $\psi_i: F \to S_n(\text{wr } S_n)^{i-1}$ such that $\ker \psi_{i+1} \subseteq \ker \psi_i$.

Let $\Psi = \bigcap_i \ker \psi_i$ and consider the group F/Ψ. For convenience of notation we will now suppose that θ is a homomorphism $F \to F \text{ wr } S_{n-1}$, where S_{n-1} is the permutation group on $\{a_1, ..., a_{n-1}\}$.

LEMMA 14. *If we have a homomorphism $\theta: F \to F \text{ wr } S_{n-1}$ and F/Ψ is the group defined by θ as above, F/Ψ can be embedded in $\mathscr{G}_{n,1}$.*

Proof. Define a map, τ_L, by the rules

$$(ua_n w)\, \tau_L = (u(L\psi_r))\, a_n w$$

$$(a_n w)\, \tau_L = a_n w,$$

where $L \in F$, u is a word of length r which does not contain a_n and $w \in W$. Clearly, τ_L is a well-defined homomorphism between subspaces of W.

If $(a_i x)\tau_L = (a_j y)\tau_L$, then $i = j$, and, if $i = n$, $x = y$. If $u(L\psi_r)\,a_n w = v(L\psi_s)\,a_n z$, where neither v nor u contains a_n, neither $v(L\psi_s)$ nor $u(L\psi_r)$ contains a_n, so $u(L\psi_r) = v(L\psi_s)$ and $w = z$. Each of $L\psi_r$ and $L\psi_s$ is a permutation which preserves lengths of words, so we must have $r = s$ and $u = v$. Thus τ_L is an isomorphism.

Given any $z \in W$, $(za_n)\tau_L$ and $(za_n)\tau_L^{-1}$ are both defined. So za_n belongs to both the range and domain of τ_L which are thus inescapable.

Let $\varepsilon_L \in \mathscr{G}_{n,1}$ be the maximal extension of τ_L. Define $\Sigma\colon F \to \mathscr{G}_{n,1}$ by the rule $L \to \varepsilon_L$. If $\varepsilon_L = \varepsilon_K$, for some L, $K \in F$, then for *any* $u = b_1 \cdots b_r$ $(b_i \neq a_n)$ we have $u(L\psi_r)\,a_n = (ua_n)\,\varepsilon_L = (ua_n)\,\varepsilon_K = u(K\psi_r)\,a_n$. Thus $u(L\psi_r) = u(K\psi_r)$ for all words u, of length r, not containing a_n. Hence $L\psi_r = K\psi_r$, for all r, and so $LK^{-1} \in \Psi$. If $L \in \Psi$, then $L\psi_r = 1$, for all r, and so $\varepsilon_L = 1$. Thus $\ker \Sigma = \Psi$ and F/Ψ is isomorphic to a subgroup of $\mathscr{G}_{n,1}$.

Let H_θ be the subgroup of $\mathscr{G}_{n,1}$ isomorphic to F/Ψ as above. Then $H_\theta = \{\varepsilon_L \mid L \in F\}$ and is completely determined by the original homomorphism $\theta\colon F \to F \operatorname{wr} S_{n-1}$. We call H_θ *the subgroup of* $\mathscr{G}_{n,1}$ *defined by* θ.

LEMMA 15. *If $h \in H_\theta$, then h has an H_θ-symbol of the form*

$$\begin{pmatrix} a_1 & \cdots & a_{n-1} & a_n \\ h_1 & \cdots & h_{n-1} & 1 \\ a_1\pi & \cdots & a_{n-1}\pi & a_n \end{pmatrix},$$

where $\pi \in S_{n-1}$.

Proof. Let $L \in F$ be an element such that $h = \varepsilon_L$. Since $\theta\colon F \to F \operatorname{wr} S_{n-1}$, we can think of $L\theta$ as an ordered n-tuple $(L_1, \ldots, L_{n-1}, \pi)$, where $L_i \in F$ and $\pi \in S_{n-1}$. Let $h_i = \varepsilon_{L_i} \in H_\theta$ and let τ be the element with symbol

$$\begin{pmatrix} a_1 & \cdots & a_{n-1} & a_n \\ h_1 & \cdots & h_{n-1} & 1 \\ a_1\pi & \cdots & a_{n-1}\pi & a_n \end{pmatrix}.$$

We show that $h = \tau$.

The set $\{a_i u a_n w, a_n z \mid 1 \leqslant i \leqslant n-1, z, u, w \in W,\ u$ does not contain a_n and $(ua_n w)\,h_i$ defined$\}$ is an inescapable subspace. We have that $(a_n z)h = a_n z = (a_n z)\tau$ and $(a_i u a_n w)h = (a_i u)(L\psi_{r+1})\,a_n w = a_i\pi(u(L_i\psi_r))\,a_n w = a_i\pi(ua_n w)\,h_i = (a_i u a_n w)\tau$. So h and τ are equal on an inescapable subspace, and, by Lemma 1, $h = \tau$.

LEMMA 16. *If Γ is an H_θ-symbol which is the combination of H_θ-symbols $\Gamma_1,\dots,\Gamma_m$ and Δ is an expansion of Γ, there exist expansions $\Delta_1,\dots,\Delta_m$ of $\Gamma_1,\dots,\Gamma_m$, respectively, such that Δ is a combination of $\Delta_1,\dots,\Delta_m$.*

Proof. Any expansion of Γ is obtained by a finite sequence of expansions which involve replacing one column of the form

$$\begin{pmatrix} u \\ h \\ v \end{pmatrix}$$

by the columns

$$\begin{pmatrix} ua_1 & \cdots & ua_{n-1} & ua_n \\ h_1 & \cdots & h_{n-1} & 1 \\ va_1\pi & \cdots & va_{n-1}\pi & va_n \end{pmatrix},$$

where

$$\begin{pmatrix} a_1 & \cdots & a_{n-1} & a_n \\ h_1 & \cdots & h_{n-1} & 1 \\ a_1\pi & \cdots & a_{n-1}\pi & a_n \end{pmatrix}$$

is an H_θ-symbol for h. Thus, if

$$\Gamma = \begin{pmatrix} u_1 & u_2 & \cdots & u_s \\ k_2 & k_2 & \cdots & k_s \\ v_1 & v_2 & \cdots & v_s \end{pmatrix},$$

we may inductively assume that

$$\Delta = \begin{pmatrix} u_1a_1 & \cdots & u_1a_{n-1} & ua_n & u_2 & \cdots & u_s \\ h_1 & \cdots & h_{n-1} & 1 & k_2 & \cdots & k_s \\ v_1a_1\pi & \cdots & v_1a_{n-1}\pi & va_n & v_2 & \cdots & v_s \end{pmatrix}.$$

Let

$$\Gamma_i = \begin{pmatrix} u_{i-1,1} & \cdots & u_{i-1,s} \\ g_{i,1} & \cdots & g_{i,s} \\ u_{i,1} & \cdots & u_{i,s} \end{pmatrix},$$

where $u_{0,j} = u_j$, $v_{m,j} = v_j$ and $k_j = g_{1,j}g_{2,j}\cdots g_{m,j}$. If

$$\begin{pmatrix} a_1 & \cdots & a_{n-1} & a_n \\ l_{i,1} & \cdots & l_{i,n-1} & 1 \\ a_1\pi_i & \cdots & a_{n-1}\pi_i & a_n \end{pmatrix}$$

is an H_θ-symbol for $g_{i,1}$, by Lemmas 5 and 4 we must have $\pi = \pi_1 \pi_2 \cdots \pi_m$ and $h_j = l_{1,j} l_{2,j} \cdots l_{m,j}$ (since $k_1 = g_{1,1} \cdots g_{m,1}$). Thus, if we let

$$\Delta_i = \begin{pmatrix} u_{i-1,1}a_1 & \cdots & u_{i-1,1}a_{n-1} & u_{i-1,1}a_n & u_{i-1,2} & \cdots & u_{i-1,s} \\ l_{i,1} & \cdots & l_{i,n-1} & 1 & g_{i,2} & \cdots & g_{i,s} \\ u_{i,1}a_1\pi_i & \cdots & u_{i,1}a_{n-1}\pi_i & u_{i,1}a_n & u_{i,2} & \cdots & u_{i,s} \end{pmatrix},$$

Δ_i is an expansion of Γ_i and Δ is a combination of $\Delta_1, \ldots, \Delta_m$.

LEMMA 17. *If H_θ is the subgroup of $\mathscr{G}_{n,1}$ defined by θ, the group $\mathscr{H}_\theta = \langle G_{n,1}, H_\theta \rangle$ is H_θ-expansible.*

Proof. We take as generators for $\mathscr{H}_\theta$ all the elements of $G_{n,1}$ and all the elements of H_θ. These elements all have H_θ-symbols so in proving that $\mathscr{H}_\theta$ satisfies condition (ii) for H_θ-expansible groups we automatically prove that it satisfies condition (i).

We show that $\mathscr{H}_\theta$ satisfies condition (ii) by induction on the length of an element of $\mathscr{H}_\theta$ as a word in the chosen generators.

Since every generator has an H_θ-symbol, we have the first step of the induction.

Suppose that $\tau \in \mathscr{H}_\theta$ has H_θ-symbol

$$\begin{pmatrix} x_1 & \cdots & x_r \\ k_1 & \cdots & k_r \\ y_1 & \cdots & y_r \end{pmatrix}$$

and that $\alpha \in G_{n,1}$ has symbol

$$\begin{pmatrix} u_1 & \cdots & u_s \\ 1 & \cdots & 1 \\ v_1 & \cdots & v_s \end{pmatrix}.$$

Each $h \in H_\theta$ has a symbol of the form

$$\begin{pmatrix} a_1 & \cdots & a_{n-1} & a_n \\ h_1 & \cdots & h_{n-1} & 1 \\ a_1\pi & \cdots & a_{n-1}\pi & a_n \end{pmatrix},$$

so we can expand the above symbol for τ repeatedly until its bottom row is an expansion of the cofinite basis $\{u_1, \ldots, u_s\}$. Suppose the resulting symbol is

$$\begin{pmatrix} z_1 & \cdots & z_t \\ g_1 & \cdots & g_t \\ w_1 & \cdots & w_t \end{pmatrix}.$$

The identity has symbol

$$\begin{pmatrix} a_1 & \cdots & a_n \\ 1 & \cdots & 1 \\ a_1 & \cdots & a_n \end{pmatrix}$$

so we can expand the symbol above, for α, to a symbol of the form

$$\begin{pmatrix} w_1 & \cdots & w_t \\ 1 & \cdots & 1 \\ q_1 & \cdots & q_t \end{pmatrix}.$$

By Lemma 5, we can now combine these symbols to get a symbol for $\tau\alpha$. By induction hypothesis, we can assume that the original symbol for τ was a combination of H_θ-symbols for each of the generators which make up τ. Thus, by Lemma 16, we have the result for $\tau\alpha$.

If $h \in H_\theta$ has H_θ-symbol

$$\begin{pmatrix} a_1 & \cdots & a_{n-1} & a_n \\ h_1 & \cdots & h_{n-1} & 1 \\ a_1\pi & \cdots & a_{n-1}\pi & a_n \end{pmatrix},$$

we can repeatedly expand this until we obtain a symbol of the form

$$\begin{pmatrix} y_1 & \cdots & y_r \\ l_1 & \cdots & l_r \\ w_1 & \cdots & w_r \end{pmatrix},$$

for h. By Lemma 5, the combination of this symbol and the symbol above for τ is a symbol for τh and is also a combination of symbols for the generators which make up τh. This completes the induction step.

By δ_i and δ_π, $1 \leqslant i \leqslant n$, $\pi \in S_{n-1}$, we shall always mean the elements of $G_{n,1}$ with symbols respectively

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n a_1 a_1 & \cdots & a_n a_1 a_{i-1} & a_n a_1 a_i & a_n a_1 a_{i+1} & \cdots & a_n a_1 a_n & a_n a_2 & \cdots & a_n a_n \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ a_n a_1 a_i & a_2 & \cdots & a_{n-1} & a_n a_1 a_1 & \cdots & a_n a_1 a_{i-1} & a_1 & a_n a_1 a_{i+1} & \cdots & a_n a_1 a_n & a_n a_2 & \cdots & a_n a_n \end{pmatrix}$$

and

$$\begin{pmatrix} a_1a_1 & \cdots & a_1a_{n-1} & a_1a_n & a_2 & \cdots & a_n \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ a_1a_1\pi & \cdots & a_1a_{n-1}\pi & a_1a_n & a_2 & \cdots & a_n \end{pmatrix}.$$

Let E be a set of generators for H_θ and their inverses. Let $D' = \{\sigma_h = \delta\delta_1\theta_{h_1}\delta_2\sigma_{h_2}\cdots\delta_{n-1}\sigma_{h_{n-1}}\delta_n\cdots\delta_{3n-2}\delta_{3n-2}\cdots\delta_2\delta_1\delta\delta_\pi \mid h \in E\}$, where h has H_θ-symbol

$$\begin{pmatrix} a_1 & \cdots & a_{n-1} & a_n \\ h_1 & \cdots & h_{n-1} & 1 \\ a_1\pi & \cdots & a_{n-1}\pi & a_n \end{pmatrix}.$$

Let $\chi' = A \cup B \cup C \cup D' \subseteq \chi$.

LEMMA 18. *If $g \in H_\theta$ has symbol*

$$\begin{pmatrix} a_1 & \cdots & a_{n-1} & a_n \\ g_1 & \cdots & g_{n-1} & 1 \\ a_1\mu & \cdots & a_{n-1}\mu & a_n \end{pmatrix}$$

the relation $\sigma_g = \delta\delta_1\sigma_{g_1}\cdots\delta_{n-1}\sigma_{g_{n-1}}\delta_{n-1}\cdots\delta_1\delta\delta_\mu$ is a consequence of χ'.

Proof. By induction on the length of g as a word in the elements of E. If $g \in E$, then the result follows by definition of D'.
Suppose that the result holds for g and let $h \in E$ have symbol

$$\begin{pmatrix} a_1 & \cdots & a_{n-1} & a_n \\ h_1 & \cdots & h_{n-1} & 1 \\ a_1\pi & \cdots & a_{n-1}\pi & a_n \end{pmatrix}.$$

Then gh has symbol

$$\begin{pmatrix} a_1 & \cdots & a_{n-1} & a_n \\ g_1h_{1\mu} & \cdots & g_{n-1}h_{(n-1)\mu} & 1 \\ a_1\mu\pi & \cdots & a_{n-1}\mu\pi & a_n \end{pmatrix}.$$

By assumption the relation $\sigma_g = \delta\delta_1\sigma_{g_1}\cdots\delta_{n-1}\sigma_{g_{n-1}}\delta_{n-1}\cdots\delta_1\delta\delta_\mu$ is a consequence of χ' and so, using Lemma 12 with the induction hypothesis, the relation

$$\sigma_{gh} = \delta\delta_1\theta_{g_1}\cdots\delta_{n-1}\sigma_{g_{n-1}}\delta_{n-1}\cdots\delta_1\delta\delta_\mu\alpha\delta_1\sigma_{h_{1\mu}} \\ \cdots\delta_{n-1}\sigma_{h_{(n-1)\mu}}\delta_{n-1}\cdots\delta_1\beta\delta\delta_\pi$$

is a consequence of χ', where $\alpha = \delta_1 \alpha_\mu \delta_1$, $\beta = \delta_1 \cdots \delta_{n-1} \beta_\mu \delta_{n-1} \cdots \delta_1$ (α_μ, β_μ from Lemma 12). The relations $\delta\delta_\mu \delta\alpha\delta_i = \delta_i \delta\delta_\mu \delta\alpha$, $\alpha\beta = 1$ and $\delta_\mu \delta_\pi = \delta_{\mu\pi}$ are consequences of A and $\delta\delta_\mu \delta\alpha$ fixes a_1. So the relation $\sigma_{gh} = \delta\delta_1 \sigma_{g_1} \cdots \delta_{n-1} \sigma_{g_{n-1}} \delta_{n-1} \cdots \delta_2 \sigma_{h_{1\mu}} \delta_2 \cdots \sigma_{h_{(n-1)\mu}} \delta_{n-1} \cdots \delta_1 \delta\delta_{\mu\pi}$ is a consequence of χ'. We note that the δ_i belong to a set of type $3n-2$ and so we have the result for σ_{gh} by Lemma 11.

Lemma 19. *If*

$$\Gamma = \begin{pmatrix} u_1 & \cdots & u_s \\ k_1 & \cdots & k_s \\ v_1 & \cdots & v_s \end{pmatrix}$$

is any H_θ-symbol for σ_g, $g \in H_\theta$, and $\{\eta_2, \ldots, \eta_s\}$ is any set of type s, the relation $\sigma_g = \tau\sigma_{k_1}\eta_2\sigma_{k_2} \cdots \eta_s\sigma_{k_s}\eta_s \cdots \eta_2\varepsilon$ is a consequence of χ', where τ and ε have symbols

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_s \\ 1 & 1 & \cdots & 1 \\ a_1 & w_2 & \cdots & w_s \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_1 & w_2 & \cdots & w_s \\ 1 & 1 & \cdots & 1 \\ v_1 & v_2 & \cdots & v_s \end{pmatrix},$$

respectively.

Proof. By Lemma 9, Γ is an expansion of

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ g & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

and by Lemma 12, we can, and will, suppose that the columns of Γ are in any order which suits us. The proof is by induction on the number, m, of columns of the form

$$\begin{pmatrix} a_1 u \\ l \\ a_1 v \end{pmatrix}$$

in Γ.

If $m = 1$,

$$\Gamma = \begin{pmatrix} a_1 & x_2 & \cdots & x_s \\ g & 1 & \cdots & 1 \\ a_1 & x_2 & \cdots & x_s \end{pmatrix}$$

and the relation is $\sigma_g = \tau\sigma_g\varepsilon$. But $\tau\varepsilon = 1$ is a consequence of A, in this case, and τ fixes a_1, so the result is a consequence of $A \cup B$.

If $m = n$, then

$$\Gamma = \begin{pmatrix} a_1a_1 & \cdots & a_1a_{n-1} & a_1a_n & y_{n+1} & \cdots & y_s \\ g_1 & \cdots & g_{n-1} & 1 & 1 & \cdots & 1 \\ a_1a_1\mu & \cdots & a_1a_{n-1}\mu & a_1a_n & y_{n+1} & \cdots & y_s \end{pmatrix}$$

and the relation is $\sigma_g = \tau\sigma_{g_1}\eta_2\sigma_{g_2}\eta_3 \cdots \sigma_{g_{n-1}}\eta_{n-1} \cdots \eta_2\varepsilon$. Let α be the element with symbol

$$\begin{pmatrix} a_1 & w_2 & \cdots & w_n & w_{n+1} & \cdots & w_{s-1} & w_s \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_na_1a_2 & \cdots & a_na_1a_n & z_{n+1} & \cdots & z_{s-1} & a_na_1a_1 \end{pmatrix},$$

some $z_{n+1},\dots, z_s$. The relations $\alpha^{-1}\eta_j\alpha = \delta_j$, $2 \leqslant j \leqslant n$, are consequences of A and α fixes a_1 so the relation $\sigma_g = \delta\delta_1\alpha^{-1}\sigma_{g_1}\eta_2 \cdots \sigma_{g_{n-1}}\eta_{n-1} \cdots \eta_2\alpha\delta_1\delta\delta_\mu$ is a consequence of χ', by Lemma 18. The element $\tau\alpha\delta_1\delta$ fixes a_1 and so the relation $\tau\alpha\delta_1\delta\sigma_g\delta\delta_1\alpha^{-1}\tau^{-1} = \sigma_g$ is a consequence of χ'. The relation $\alpha\delta_1\delta\delta_\mu\delta\delta_1\alpha^{-1}\tau^{-1} = \varepsilon$ is a consequence of A and so we have the required relation for σ_g as a consequence of χ'.

We now consider the expansion

$$\begin{pmatrix} u_1 & \cdots & u_{s-1} & u_sa_1 & \cdots & u_sa_{n-1} & u_sa_n \\ k_1 & \cdots & k_{s-1} & h_1 & \cdots & h_{n-1} & 1 \\ v_1 & \cdots & v_{s-1} & v_sa_1\pi & \cdots & v_sa_{n-1}\pi & v_sa_n \end{pmatrix}$$

of Γ and assume the result for any set $\{v_2,\dots, v_s\}$ of type s. For a given set $\{\eta_2,\dots, \eta_r\}$ of type $r = s + n - 1$, we want to show that the relation $\theta_g = \tau\sigma_{k_1}\eta_2 \cdots \sigma_{k_{s-1}}\eta_s\sigma_{h_1} \cdots \eta_{r-1}\sigma_{h_{n-1}}\eta_{r-1} \cdots \eta_2\varepsilon$ is a consequence of χ', and we can assume that the relation $\sigma_g = \tau'\sigma_{k_1}v_2 \cdots \sigma_{k_s}v_s \cdots v_2\varepsilon'$ is a consequence of χ', for any set $\{v_2,\dots, v_s\}$ of type s.

Let β be the element with symbol

$$\begin{pmatrix} a_1 & w_2 & \cdots & w_{s-1} & w_s & w_{s+1} & \cdots & w_r \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ a_1 & z_2 & \cdots & z_{s-1} & z_sa_1 & z_sa_2 & \cdots & z_sa_n \end{pmatrix},$$

for some $z_2,\dots, z_s$. Let $v_i = \beta^{-1}\eta_i\beta$, $2 \leqslant i \leqslant s - 1$; then there is a v_s such that $\{v_2,\dots, v_s\}$ is a set of type s. Taking the symbol

$$\begin{pmatrix} z_2 & \cdots & z_s & a_1a_1 & \cdots & a_1a_{n-1} & a_1a_n \\ 1 & \cdots & 1 & h_1 & \cdots & h_{n-1} & 1 \\ z_2 & \cdots & z_s & a_1a_1\pi & \cdots & a_1a_{n-1}\pi & a_1a_n \end{pmatrix},$$

for σ_{k_s}, we have, from the case $m=n$, that the relation $\sigma_{k_s} = \tau''\eta_2 \cdots \eta_s\sigma_{h_1}\eta_{s+1} \cdots \sigma_{h_n-1}\eta_{n-1} \cdots \eta_2\varepsilon''$ is a consequence of χ'. Now, $\tau''\eta_2 \cdots \eta_{s-1} = \nu_s\beta^{-1}$, $\varepsilon''\nu_s \cdots \nu_2\varepsilon' = \varepsilon$ and $\tau'\beta^{-1} = \tau$ are consequences of A, and β fixes a_1, so the relations

$$
\begin{aligned}
\sigma_g &= \tau'\sigma_{k_1}\nu_2 \cdots \sigma_{k_s}\nu_s \cdots \nu_2\varepsilon' \\
&= \tau'\beta^{-1}\sigma_{k_1}\eta_2 \cdots \sigma_{k_{s-1}}\beta\nu_s\sigma_{k_s}\nu_s \cdots \nu_2\varepsilon' \\
&= \tau\sigma_{k_1}\eta_2 \cdots \sigma_{k_{s-1}}\beta\nu_s\tau''\eta_2 \cdots \eta_s\sigma_{h_1}\eta_{s+1} \cdots \sigma_{h_{n-1}}\eta_{n-1} \cdots \eta_2\varepsilon''\nu_s \cdots \nu_2\varepsilon' \\
&= \tau\sigma_{k_1}\eta_2 \cdots \sigma_{k_{s-1}}\eta_s\sigma_{h_1}\eta_{s+1} \cdots \sigma_{h_{n-1}}\eta_{n-1} \cdots \eta_2\varepsilon
\end{aligned}
$$

are consequences of χ', completing the induction step.

Clearly, $D' \subseteq D$. The above lemma shows that every relation in D is a consequence of χ', thus χ' is a set of defining relations for $\mathscr{H}_\theta$. We have already shown, in Lemma 13, that if H_θ is finitely presented the relations, $A \cup B \cup C$ are finitely based. If H_θ is finitely generated, we can take the set $E \subseteq H_\theta$, above, to be finite, so we can suppose that the set $D'$is finite and hence that the relations χ' are finitely based. Thus we have the following theorem.

THEOREM 2. *If θ is any homomorphism from a free group, F, to the wreath product, $F \operatorname{wr} S_{n-1}$, and if the subgroup, H_θ, of $\mathscr{G}_{n,1}$ defined by θ is finitely presented, so is the group $\langle G_{n,1}, H_\theta\rangle$.*

A USEFUL LEMMA

The constructions above have been made in such a way that we can apply the following lemma, which is essentially a result of Thompson [3]. We give a proof of the lemma since Thompson's formulation is very different from that of this work.

LEMMA 20. *If K is a subgroup of $\mathscr{G}_{n,1}$ containing $G'_{n,1}$, then the derived subgroup, K', is simple.*

Proof. We show that any non-trivial normal subgroup, N, of K contains K'. Then, since $G'_{n,1}$ is simple and nonabelian (see [1]), we have $G''_{n,1} = G'_{n,1}$ and so $G'_{n,1} \subseteq K'$. Thus K' satisfies the initial conditions on K and so any non-trivial normal subgroup of K' contains K''. But $K'' \lhd K$ and $K'' \neq 1$ since $G'_{n,1} \subseteq K''$; thus $K' \leqslant K''$ and so K' is contained in all its non-trivial normal subgroups.

Suppose that N is a non-trivial normal subgroup of K and that $\tau \in N$ is non-trivial. If $u\tau \neq u$ and $u = yw$, then $y\tau \neq y$, so choose $u \in W$ such that

$u\tau \neq u$ and suppose that τ is not defined on any proper initial segment of u. Let $u\tau = v$; then $u \neq v$.

We use u to construct a non-trivial element in $G'_{n,1} \cap N$ and then, since $G'_{n,1}$ is simple, we have $G'_{n,1} \subseteq N$. We then use this result to show that K/N is abelian.

Let $\{u, u_2, \ldots, u_s\}$ and $\{v, v_2, \ldots, v_s\}$ be cofinite bases and let α and β be the elements with symbols respectively

$$\begin{pmatrix} ua_1a_1 & ua_1a_2 & ua_1a_3 & \cdots & ua_1a_n & ua_2 & ua_3 & \cdots & ua_n & u_2 & \cdots & u_s \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ ua_1a_2 & ua_2 & ua_1a_3 & \cdots & ua_1a_n & ua_1a_1 & ua_3 & \cdots & ua_n & u_2 & \cdots & u_s \end{pmatrix}$$

and

$$\begin{pmatrix} va_1a_1 & va_1a_2 & va_1a_3 & \cdots & va_1a_n & va_2 & va_3 & \cdots & va_n & v_2 & \cdots & v_s \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ va_1a_2 & va_2 & va_1a_3 & \cdots & va_1a_n & va_1a_1 & va_3 & \cdots & va_n & v_2 & \cdots & v_s \end{pmatrix}.$$

Then α and β are even and thus belong to $G'_{n,1}$, which has index at most 2 in $G_{n,1}$ (see [1]).

Choose y such that $y\tau^{-1}$ and $y\beta$ are both defined. Since $y\beta$ is defined, y is not an initial segment of va_1 so $y\tau^{-1}$ is not an initial segment of ua_1 and $y\tau^{-1}\alpha$ is defined. If v is not an initial segment of y, then u is not an initial segment of $y\tau^{-1}$ and so $y\tau^{-1}\alpha\tau = y\tau^{-1}\tau = y = y\beta$. If $y = va_1a_1w$, some w, then $y\tau^{-1}\alpha\tau = va_1a_2w = y\beta$. Similarly, if y is of the form va_1a_iw, va_iw or v_iw (for $i \geqslant 2$), then $y\tau^{-1}\alpha\tau = y\beta$. Thus $\tau^{-1}\alpha\tau$ is equal to β on an inescapable subspace, and so $\beta = \tau^{-1}\alpha\tau$.

Then $\alpha^{-1}\tau^{-1}\alpha\tau \in G'_{n,1} \cap N$, and is non-trivial since $(ua_2a_1^i)\,\alpha^{-1}\tau^{-1}\alpha\tau \neq ua_2a_1^i$, for any i for which the left-hand side is defined.

Take η and ζ contained in K. We want to show that $N\eta\zeta = N\zeta\eta$. Choose $w \in W$ such that $(uw)\,\eta^{-1}$ and $(vw)\,\zeta^{-1}$, u, v as above, are defined. Choose $\mu, \nu \in G_{n,1}$ such that $(uw)\mu = (uw)\,\eta^{-1}$ and $(vw)\nu = (vw)\,\zeta^{-1}$. Since $G'_{n,1}$ has index at most 2 in $G_{n,1}$, it is not hard to see that we can in fact choose μ and ν to be in $G'_{n,1}$. (If a_i is not an initial segment of u, let π be the element of $G_{n,1}$ which swaps a_ia_1 and a_ia_n and fixes everything else. Then if $G_{n,1} \neq G'_{n,1}$, $\pi \notin G'_{n,1}$ and π fixes u, so $(uw)\,\pi\mu = (uw)\,\eta^{-1}$. Either μ or $\pi\mu$ belongs to $G'_{n,1}$.) Since $G'_{n,1} \subseteq N$, it is sufficient to show that $N\mu\eta\nu\zeta = N\nu\zeta\mu\eta$.

If μ and ν have symbols respectively

$$\begin{pmatrix} uw & u_2 & \cdots & u_s \\ 1 & 1 & \cdots & 1 \\ (uw)\mu & y_2 & \cdots & y_s \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} vw & v_2 & \cdots & v_s \\ 1 & 1 & \cdots & 1 \\ (vw)\nu & z_2 & \cdots & z_s \end{pmatrix},$$

let γ and ρ have symbols

$$\begin{pmatrix} a_1 & \cdots & a_{n-1} & a_n a_1 & a_n x_2 & \cdots & a_n x_s \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ uwa_1 & \cdots & uwa_{n-1} & uwa_n & u_2 & \cdots & u_s \end{pmatrix}$$

and

$$\begin{pmatrix} a_2 & \cdots & a_n & a_1 a_1 & a_1 w_2 & \cdots & a_1 w_s \\ 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ vwa_2 & \cdots & vwa_n & vwa_1 & z_2 & \cdots & z_s \end{pmatrix}.$$

Using the same techniques as above, we can choose γ and ρ to be in $G'_{n,1}$. We show that $\eta' = \gamma\mu\eta\gamma^{-1}$ and $\zeta' = \rho v\zeta\rho^{-1}$ commute, which gives the result.

If $y = a_i w'$, for any $w' \in W$, $2 \leqslant i \leqslant n-1$, then $y\eta' = y = y\zeta'$. If $y = a_1 w'$, then $y\rho$ does not have initial segment vwa_j, $j \neq 1$. So $y(\rho v\zeta)$ does not have initial segment vwa_j, $j \neq 1$, and hence $y\zeta'$ $(= y(\rho v\zeta\rho^{-1}))$ is of the form $a_1 z$, whenever it is defined. Since η' fixes all words of the form $a_1 z$, $y\eta'\zeta' = y\zeta'\eta'$. Similarly, ζ' fixes all words of the form $a_n z$ and if $y = a_n w'$, $y\eta'\zeta' = y\zeta'\eta'$, whenever $y\eta'$ is defined. Thus $\eta'\zeta'$ and $\zeta'\eta'$ are equal on an inescapable subspace and so $\eta'\zeta' = \zeta'\eta'$.

Derived Subgroups

Finally, we want finitely presented simple groups; so we are interested in the cases when $\mathscr{H}'_\theta$ is of finite index in $\mathscr{H}_\theta$. Although we cannot give necessary and sufficient conditions for this, the theorem below shows that if H_θ is a finitely presented subgroup of $\mathscr{G}_{n,1}$ defined by θ, then H_θ can be embedded in a finitely presented group with a simple derived subgroup of finite index. When H_θ is finitely presented, the lemmas and theorems above give that derived subgroup is a finitely presented simple group.

An element of $F \operatorname{wr} S_{n-1}$ is an n-tuple $(L_1,..., L_{n-1}, \pi)$, where $L_1,..., L_{n-1} \in F$, $\pi \in S_{n-1}$ and $(P, a_i)(L_1,..., L_{n-1}, \pi) = (PL_i, a_i\pi)$, for all $P \in F$. Let θ be a homomorphism from F to $F \operatorname{wr} S_{n-1}$. If $L\theta = (L_1,..., L_{n-1}, \pi)$, we call $L_1,..., L_{n-1}$ *the first θ-components of L*. The kth *θ-components* of L are defined, inductively, to be all the $(k-1)$st θ-components of $L_1,..., L_{n-1}$.

Let H_θ be the group defined by θ and let ε_L be the image of L in H_θ (see Lemma 14). Then $\varepsilon_L = 1$ if and only if $L \in \Psi$, and refering to the diagram above Lemma 14, we see that this is so if and only if all the θ-components of L lie in $\ker \psi_1$.

Let ι be the natural embedding of S_{n-1} in S_n, which identifies S_{n-1} with the set of all elements fixing a_n. We have a homomorphism $\theta^* : F \to F \operatorname{wr} S_n$, given by $L\theta^* = (L_1, \ldots, L_{n-1}, L, \pi\iota)$. Let p_i^* and ϕ_i^* be the natural maps corresponding to p_i and ϕ_i, on the diagram. Since ι is a monomorphism, we have $\ker \psi_1 = \ker \psi_1^*$ and clearly the set of θ^*-components of L is just L itself and the set of θ-components of L. Let ε_L^* be the image of L in H_{θ^*} and then $\varepsilon_L^* = 1$ if and only if $\varepsilon_L = 1$. In other words, H_θ is isomorphic to H_{θ^*}.

THEOREM 3. *Let θ be a homomorphism from a free group F to the wreath product $F \operatorname{wr} S_{n-1}$ and let H_θ be the subgroup of $\mathscr{G}_{n,1}$ defined by θ. There exists an integer m and a homomorphism $\phi : F \to F \operatorname{wr} S_{m-1}$ such that H_θ is isomorphic to H_ϕ and $\langle G_{m,1}, H_\phi \rangle'$ has finite index in $\langle G_{m,1}, H_\phi \rangle$, if H_θ is finitely generated.*

Proof. If $L\theta = (L_1, \ldots, L_{n-1}, \pi)$, then define $\phi : F \to F \operatorname{wr} S_{m-1}$ by the rule $L\phi = (L_1, \ldots, L_{n-1}, L, \ldots, L, \pi\iota)$, where ι is the natural embedding of S_{n-1} in S_{m-1} which identifies S_{n-1} with the set of all elements which fix $a_n, a_{n+1}, \ldots, a_{m-1}$. From the discussion above, we see that H_θ is isomorphic to H_ϕ.

From relations D for $\mathscr{H}_\phi$ we have

$$\sigma_h = \delta\delta_1\sigma_{h_1}\delta_2\sigma_{h_2}\cdots\delta_{n-1}\sigma_{h_{n-1}}\delta_n\sigma_h\delta_{n+1}\sigma_h\cdots\delta_{m-1}\sigma_h\delta_{m-1}\cdots\delta_1\delta\delta_{\pi\iota}.$$

If H_θ is finitely generated we can choose a finite set of generators $\{k_1, \ldots, k_r\}$, for H_ϕ. Let $G_{m,1} = G'_{m,1} \cup G'_{m,1}\alpha$. From the relation in D, above, for k_j we get $\sigma_{k_j} = \tau\sigma_{k_1}^{q_{j,1}}\sigma_{k_2}^{q_{j,2}}\cdots\sigma_{k_r}^{q_{j,r}}$, for some $\tau \in \mathscr{H}'_\phi \cup \mathscr{H}'_\phi\alpha$ and integers $q_{j1}, \ldots, q_{jr}$.

The q_{jj}, $1 \leqslant j \leqslant r$, increase with m but q_{ji} $(i \neq j)$ is independent of m. Thus we can choose m large enough so that, if

$$M = \begin{pmatrix} q_{11} - 1 & q_{12} & \cdots & q_{1r} \\ q_{21} & q_{22} - 1 & \cdots & q_{2r} \\ \vdots & \vdots & & \vdots \\ q_{r1} & q_{r2} & \cdots & q_{rr} - 1 \end{pmatrix},$$

then $\det M > 0$. Thus, adding non-zero multiples of the rows of M together, we can reduce M to the form

$$\begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b_r \end{pmatrix}.$$

Carrying out the same operations on the set of words $\{\sigma_{k_1}^{q_{11}-1}\sigma_{k_2}^{q_{12}}\cdots\sigma_{k_r}^{q_{1r}}, \ldots, \sigma_{k_1}^{q_{r1}}\sigma_{k_2}^{q_{r2}}\cdots\sigma_{k_r}^{q_{rr}-1}\}$, we see that $\sigma_{k_j}^{b_j} \in \mathscr{H}'_\phi \cup \mathscr{H}'_\phi\alpha$. Thus $\mathscr{H}'_\phi$ has index at most $2b_1b_2\cdots b_r$, in $\mathscr{H}_\phi$.

If H_θ is finitely presented we have, from Theorem 2 and Lemma 17, that $\mathscr{H}'_\phi$ is a finitely presented infinite simple group.

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