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Partial differential equations driven by rough paths

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ABSTRACT

We study a class of linear first and second order partial differential equations driven by weak geometric p -rough paths, and prove the existence of a unique solution for these equations. This solution depends continuously on the driving rough path. This allows a robust approach to stochastic partial differential equations. In particular, we may replace Brownian motion by more general Gaussian and Markovian noise. Support theorems and large deviation statements all become easy corollaries of the corresponding statements of the driving process. In the case of first order equations with Gaussian noise, we discuss the existence of a density with respect to the Lebesgue measure for the solution.

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1. Introduction

The theory of rough paths can be described as an extension of the classical theory of controlled differential equations which is sufficiently robust to allow a deterministic treatment of stochastic differential equations, and equations driven by signals which are even more irregular than semimartingales. Recently various attempts have been made to extend this theory to partial differential equations (PDEs), with the aim of obtaining some form of deterministic treatment for stochastic partial differential equations (SPDEs) and at the same time allowing more general driving signals.

In [16], a non-linear evolution problem driven by a Hölder continuous path with values in a distribution space is studied. Young integration is used to obtain a mild solution for this equation. A non-linear one-dimensional wave equation driven by signals which satisfy appropriate Hölder regularity conditions is considered in [31]. The authors use a 2-dimensional Young integration theory to solve the wave equation in a mild sense. In both these papers, Hölder exponents are assumed to be

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greater than $\frac{1}{2}$ and applications to equations driven by Fractional Brownian Motion with Hurst index greater than $\frac{1}{2}$ are given.

The goal of the present paper is to deal with partial differential equations of parabolic type of the form (with summation over repeated indices)

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2} a^{ij}(t, y) \frac{\partial^2 u}{\partial y^i \partial y^j} + b^i(t, y) \frac{\partial u}{\partial y^i} - \frac{\partial u}{\partial y^k} V_l^k(y) \frac{dx_t^l}{dt} \\ &= L_t u(t, y) - \nabla u(t, y) \cdot V_l(y) \frac{dx_t^l}{dt} \end{aligned} \tag{1}$$

with given initial data $u(0, \cdot)$, subjected to a (finite-dimensional) driving signal $x_t = (x_t^1, \dots, x_t^d)$ where x_t may only possess the “rough” regularity of a typical sample path of a stochastic process; $V_l^k(\cdot)$ are sufficiently regular coefficients. By combining ideas from rough path theory and with some classical PDE estimates we are able to prove the existence of a unique solution (and then a “universal” limit theorem) for *rough partial differential equations* (RPDEs); that is, equations of type (1) where the driving signal $x(\cdot)$ is a genuine (to be precise: weak, geometric) p -rough path and where we are given initial data $u(0, \cdot) \in BUC(\mathbb{R}^e)$.³ Perhaps the easiest way to understand RPDE solutions is their identification as limit points of classical parabolic equations along the following statement (which will be implied a fortiori from Theorem 17 and the extensions discussed in Section 3.3).

Theorem 1. *Let $p \geq 1$ and assume that*

- (1) $(x^n)_{n \in \mathbb{N}}$ is a sequence of smooth paths in \mathbb{R}^d such that the sequence of step-[p] signatures $(S_{[p]}(x^n))_{n \in \mathbb{N}}$ is Cauchy in the $\frac{1}{p}$ -Hölder metric;
- (2) $V = (V_1, \dots, V_d)$ is a collection of Lip^γ vector fields on \mathbb{R}^e for $\gamma > p + 3$;
- (3) $(\phi_n)_{n \in \mathbb{N}} \subset C_b^\infty(\mathbb{R}^e)$ is Cauchy in uniform norm on \mathbb{R}^e .

Suppose that the coefficients a^{ij} and b^i are taken to be sufficiently regular such that the PDEs

$$\begin{aligned} \frac{\partial u_n}{\partial t} &= \frac{1}{2} a^{ij}(t, y) \frac{\partial^2 u_n}{\partial y^i \partial y^j} + b^i(t, y) \frac{\partial u_n}{\partial y^i} - \frac{\partial u_n}{\partial y^k} V_l^k(y) \frac{dx_t^{n,l}}{dt}, \\ u_n(0, y) &= \phi_n(y) \end{aligned}$$

have unique (bounded) classical solutions u_n . Then the sequence $(u_n)_{n \in \mathbb{N}}$ is Cauchy in the uniform norm on $[0, T] \times \mathbb{R}^e$.

The main example of such a rough path is given by (almost every realization of) *Brownian motion and Lévy’s area* and this allows for a robust treatment of the corresponding classes of SPDEs. Linear parabolic SPDEs have been studied for quite a long time. Among the early works on this subject we mention [3,22,30]. We remark that in this paper we do not make any assumptions of super-parabolicity for our coefficients. This condition stated in [22] and [30], will not come into play because the SPDEs we consider are limits of equations driven by smooth paths, and hence can be interpreted as being in Stratonovich form.

Approximation problems for SPDEs have been investigated by a number of authors using various techniques. Among these we mention Kunita (Section 6.4 in [23]), Brzeźniak and Flandoli [1] and Gyöngy [18–20], who also proved support theorems for these SPDEs. Although our results do not cover all the equations treated in these papers (our regularity assumptions on the coefficients and

³ $BUC(\mathcal{X})$ is the space of bounded uniformly continuous functions defined on \mathcal{X} . If $u \in BUC(\mathcal{X})$, then $\|u\|_{BUC(\mathcal{X})} = \sup_{x \in \mathcal{X}} |u(x)|_{\mathcal{X}}$.

initial functions tend to be stronger), the approach in this paper can be used to study equations driven by Gaussian and Markovian signals, besides Brownian motion. The use of rough path theory in the context of SPDEs has been conjectured by various people (and in particular by Lyons himself in the introduction of his '98 article [26]). The present results, together with those in the just appeared preprint [17], seem to be the first steps in this direction.

This paper is organized as follows. In Section 2 we discuss various concepts we will need from rough path theory, while in Section 3 we present our results on PDEs driven by weak geometric rough paths. Sections 4, 5 and 6 are devoted to SPDEs with multi-dimensional Brownian, Markovian and Gaussian signals (Fractional Brownian Motion, for instance, is covered for $H > \frac{1}{4}$) respectively. Using the continuity of our solution map, together with results on the support of the law and large deviation statements for Markovian and Gaussian rough paths, we get a description of the support of the law of the solution, and a generalization of the Freidlin–Wentzell theorem for these SPDEs. In the case of first order equations driven by a class of non-degenerate Gaussian signals, we also obtain the existence of a density for the solutions.

2. Preliminaries

In this section we are going to recall those notions and results from rough path theory, that will be used in the rest of this paper. For a more complete exposition of this theory, we refer the reader to [15,27,29].

By a smart limiting procedure, ordinary differential equations (ODEs) of type

$$dy_t = \sum_i V_i(y_t) dx_t^i \equiv V(y_t) dx_t$$

defined on the time interval $[0, T]$, started at $y_0 \in \mathbb{R}^e$ at time 0, with Lipschitz vector fields $V = (V_1, \dots, V_d)$ on \mathbb{R}^e give rise to so-called rough differential equations, denoted formally by

$$dy_t = V(y_t) d\mathbf{x}_t \tag{2}$$

where \mathbf{x} is weak geometric p -rough path,⁴ that is a $\frac{1}{p}$ -Hölder continuous path from $[0, T]$ to $G^{[p]}(\mathbb{R}^d)$ (the step- $[p]$ nilpotent free group over \mathbb{R}^d), i.e.

$$\|\mathbf{x}_{s,t}\| \lesssim |t - s|^{\frac{1}{p}} \quad \text{for all } s, t \in [0, T],$$

where $\|\cdot\|$ is a homogeneous norm on $G^{[p]}(\mathbb{R}^d)$. The space of weak geometric Hölder p -rough paths is denoted by $C^{\frac{1}{p}\text{-Hölder}}([0, T], G^{[p]}(\mathbb{R}^d))$, and for $\mathbf{x} \in C^{\frac{1}{p}\text{-Hölder}}([0, T], G^{[p]}(\mathbb{R}^d))$, we define

$$\|\mathbf{x}\|_{\frac{1}{p}\text{-Hölder};[0,T]} = \sup_{0 \leq s < t \leq T} \frac{\|\mathbf{x}_{s,t}\|}{|t - s|^{\frac{1}{p}}}.$$

We also set

$$d_{\frac{1}{p}\text{-Hölder};[0,T]}(\mathbf{x}, \tilde{\mathbf{x}}) = \sup_{0 \leq s < t \leq T} \frac{\|\mathbf{x}_{s,t}^{-1} \otimes \tilde{\mathbf{x}}_{s,t}\|}{|t - s|^{\frac{1}{p}}}.$$

In the next definition we explain the notion of an RDE solution for (2).

⁴ Strictly speaking we should speak of weak geometric Hölder p -rough paths.

Definition 2. Let \mathbf{x} be a weak geometric p -rough path, and suppose that $(x^n)_{n \in \mathbb{N}}$ is a sequence of smooth paths⁵ such that

$$S_{[p]}(x^n) \equiv \mathbf{x}^n \rightarrow \mathbf{x}$$

uniformly on $[0, T]$ ($S_{[p]}(x^n)$ is the step- $[p]$ signature of x^n) and $\sup_n \|x^n\|_{\frac{1}{p}\text{-Hö};[0,T]} < \infty$.⁶ We call any limit point (in uniform topology on $[0, T]$) of

$$\{\pi_{(V)}(0, y_0; x^n) : n \geq 1\}$$

an RDE solution for (2) and we denote it by $\pi_{(V)}(0, y_0; \mathbf{x})$. Here, $\pi_{(V)}(0, y_0; x^n)$ denotes the solution of the controlled differential equation

$$dy_t^n = V(y_t^n) dx_t^n$$

started at $y_0 \in \mathbb{R}^e$ at time 0.

The existence of a sequence $(x^n)_{n \in \mathbb{N}}$ with the above properties was established in [9]. In our definition, RDE solutions are genuine \mathbb{R}^e -valued paths. It is possible to define RDE solutions as proper rough paths, but this is of no significance in the present work.

The Universal Limit theorem is one of the main results in rough path theory. It gives a sufficient condition on the vector fields for the existence of a unique RDE solution, and furthermore, it states that the Itô map, which sends the driving signal to the solution, is continuous.

Theorem 3. Let \mathbf{x} be a weak geometric p -rough path and assume that the vector fields $V = (V_1, \dots, V_d)$ are $\text{Lip}^\gamma(\mathbb{R}^e)$ for $\gamma > p$. Then the RDE

$$dy_t = V(y_t) d\mathbf{x}_t$$

started at $y_0 \in \mathbb{R}^e$ at time 0, has a unique RDE solution, denoted by $\pi_{(V)}(0, y_0; \mathbf{x})$. Furthermore if $(x_n)_{n \in \mathbb{N}} \subset C^{\frac{1}{p}\text{-Hö}}([0, T], G^{[p]}(\mathbb{R}^d))$ converges to \mathbf{x} with respect to $d_{\frac{1}{p}\text{-Hö};[0,T]}$ then

$$\pi_{(V)}(0, y_0; x^n) \rightarrow \pi_{(V)}(0, y_0; \mathbf{x})$$

uniformly (in fact, in $1/p$ -Hölder norm).

Proof. Cf. [14,26,29]. \square

One of the elementary operations on rough paths described in [27], is time reversal. Given $\mathbf{x} \in C^{\frac{1}{p}\text{-Hö}}([0, T], G^{[p]}(\mathbb{R}^d))$, and a fixed $t \in (0, T]$, we can define a new weak geometric p -rough path $\widehat{\mathbf{x}}^t$ by

$$\begin{aligned} \widehat{\mathbf{x}}^t : [0, t] &\rightarrow G^{[p]}(\mathbb{R}^d), \\ s &\mapsto \widehat{\mathbf{x}}_s^t = \mathbf{x}_{t-s}. \end{aligned}$$

From [27], we know that the map which sends \mathbf{x} to $\widehat{\mathbf{x}}^t$, is continuous in $\frac{1}{p}$ -Hölder topology.

⁵ We take $C^\infty([0, T], \mathbb{R}^d)$ to be the set of paths which are restrictions on $[0, T]$ of paths in $C^\infty((-\varepsilon, T + \varepsilon), \mathbb{R}^d)$ for $\varepsilon > 0$.

⁶ Equivalently one can take x^n to be Lipschitz continuous (in the p -variation context x^n is taken to be of bounded variation).

Our interest in these time reversed paths comes from the following important fact. If we again denote the RDE solution for (2) by $\pi_{(V)}(0, y; \mathbf{x})$, we have

$$\begin{aligned} \pi_{(V)}(0, \pi_{(V)}(0, y_0; \bar{\mathbf{x}}^t); \mathbf{x})_t &= \pi_{(V)}(0, \pi_{(V)}(0, y_0; \mathbf{x})_t; \bar{\mathbf{x}}^t)_t \\ &= y_0. \end{aligned}$$

Thus,

$$\pi_{(V)}(0, \cdot; \mathbf{x})_t^{-1}(y) = \pi_{(V)}(0, y; \bar{\mathbf{x}}^t)_t,$$

i.e. for each fixed t , the inverse of the map $y \mapsto \pi_{(V)}(0, y; \mathbf{x})_t$ can be obtained by solving a rough differential equation driven by the time reversal of the original driving signal.

The inverse map $\pi_{(V)}(0, \cdot; \mathbf{x})_t^{-1}$, and thus $\pi_{(V)}(0, y; \bar{\mathbf{x}}^t)_t$, the RDE solution for

$$dy_s = V(y_s) d\bar{\mathbf{x}}_s^t$$

started at $y \in \mathbb{R}^e$ at time 0, will play a very important role in our definition of a solution for PDEs driven by weak geometric rough paths.

3. Rough partial differential equations

Consider partial differential equations of the form

$$\begin{aligned} du(t, y) &= L_t u(t, y) dt - \frac{\partial u}{\partial y^j}(t, y) V_i^j(y) dx_t^i, \\ u(0, y) &= \phi(y) \end{aligned} \tag{3}$$

on the time interval $[0, T]$, with driving signal $x : [0, T] \rightarrow \mathbb{R}^d$, d vector fields V_1, \dots, V_d on \mathbb{R}^e , initial function $\phi : \mathbb{R}^e \rightarrow \mathbb{R}$ and L_t an elliptic operator of the form

$$L_t = \frac{1}{2} a^{ij}(t, \cdot) \frac{\partial^2}{\partial y^i \partial y^j} + b^i(t, \cdot) \frac{\partial}{\partial y^i}$$

with⁷ $a : [0, T] \times \mathbb{R}^e \rightarrow S_e$ and $b : [0, T] \times \mathbb{R}^e \rightarrow \mathbb{R}^e$. In this section we are going to define a notion of a solution for the above PDE when the driving noise is a weak geometric p -rough path, and then discuss the existence and uniqueness of these solutions.

With the definition of an RDE solution (Definition 2) in mind, we give the following definition.

Definition 4. Let \mathbf{x} be a weak geometric p -rough path, and suppose that $(x^n)_{n \in \mathbb{N}}$ is a sequence of smooth paths such that

$$S_{[p]}(x^n) \equiv \mathbf{x}^n \rightarrow \mathbf{x}$$

uniformly on $[0, T]$ and $\sup_n \|x^n\|_{\frac{1}{p}\text{-Hölder}; [0, T]} < \infty$. Assume that for each $n \in \mathbb{N}$,

$$\begin{aligned} du_n(t, y) &= L_t u_n(t, y) dt - \nabla u_n(t, y) \cdot V(y) dx_t^n, \\ u_n(0, \cdot) &= \phi(\cdot) \in C_b(\mathbb{R}^e) \end{aligned}$$

⁷ S_e is the set of symmetric non-negative definite $e \times e$ real matrices.

has a unique $C_b^{1,2}$ -solution u_n . Then any limit point (in the uniform topology) of

$$\{u_n(t, y) : n \geq 1\}$$

is called a solution for the rough partial differential equation, denoted formally by

$$\begin{aligned} du(t, y) &= L_t u(t, y) dt - \nabla u(t, y) \cdot V(y) d\mathbf{x}_t, \\ u(0, y) &= \phi(y). \end{aligned} \tag{4}$$

Note that since we define a solution of an RPDE to be a limit point of approximating solutions, this definition will be justified *a posteriori* because of the uniqueness of the limit. Similarly, once uniqueness is proved, the solution will depend only on the driving rough path and not on any approximating sequence.

The rest of this section will be devoted to proving the existence and uniqueness of solutions for (4). The continuity of the map which sends the driving signal to the solution will also be proved. We will first look at the case $L_t \equiv 0$, i.e. we solve a transport equation driven by a weak geometric p -rough path. The second order equation ($L_t \neq 0$), which is treated next, can then be seen as a perturbation of the first order equation.

3.1. Linear first order RPDEs ($L_t \equiv 0$)

As a motivation for our approach, let us first recall how linear first order equations are treated in the classical and stochastic cases. Consider the PDE given in (3) with $L_t \equiv 0$. When the path $\chi : [0, T] \rightarrow \mathbb{R}^d$ is smooth and the vector fields V_i are $C^1(\mathbb{R}^e, \mathbb{R}^e)$, with an initial function $\phi \in C^1(\mathbb{R}^e, \mathbb{R})$, we can use the method of characteristics to obtain a unique solution for this equation. Indeed, let $\pi_{(V)}(0, y; \chi)$ be the unique solution of the controlled differential equation

$$dy_t = V(y_t) dx_t$$

started at $y \in \mathbb{R}^e$ at time 0. Then one can easily show that for any solution u of

$$\begin{aligned} du(t, y) + \frac{\partial u}{\partial y^j}(t, y) V_i^j(y) dx_t^i &= 0, \\ u(0, y) &= \phi(y) \end{aligned} \tag{5}$$

we have

$$u(t, \pi_{(V)}(0, y; \chi)_t) = \phi(y).$$

Thus we deduce that

$$\begin{aligned} u(t, y) &= \phi(\pi_{(V)}(0, \cdot; \chi)_t^{-1}(y)) \\ &= \phi(\pi_{(V)}(0, y; \tilde{\chi}_t^t)) \end{aligned}$$

is the unique solution of (5) with a smooth driving signal, where $\tilde{\chi}_s^t = \chi_{t-s}$ for $s \in [0, t]$.

H. Kunita studied first order SPDEs in [23] using a stochastic characteristics system, which can be thought of being a generalization of the method of characteristics to the stochastic case. For a first order linear SPDE driven by a Brownian motion $(B_t)_{t \geq 0}$ in \mathbb{R}^d ,

$$\begin{aligned} du(t, y) + V_i^j(y) \frac{\partial u}{\partial y^j}(t, y) \circ dB_t^i &= 0, \\ u(0, y) &= \phi(y) \end{aligned} \tag{6}$$

the stochastic characteristic is given by the following Stratonovich SDE,

$$dy_t = V(y_t) \circ dB_t, \quad y_0 = y \in \mathbb{R}^e. \tag{7}$$

If the vector fields V_i and the initial function ϕ are $C^{3+\varepsilon}$, then one can use the theory of stochastic flows to prove that the unique solution of (6) is given by

$$u(t, y) = \phi(\pi_{(V)}(0, \cdot; B)_t^{-1}(y))$$

where $\pi_{(V)}(0, \cdot; B)_t$ is the unique stochastic flow associated with (7).

From these brief remarks, we see that the problem of solving first order linear PDEs with smooth and Brownian signals, can be reduced to solving an ordinary and stochastic differential equation respectively. Therefore a natural question to ask is whether one can use an RDE to solve a first order linear PDE driven by a weak geometric p -rough path.

In the following theorem we give sufficient conditions on the vector fields and the initial function which guarantee the existence of a unique solution for a linear first order rough PDE driven by a weak geometric p -rough path. Moreover, we prove that the map which sends the driving signal to the solution, is continuous in the uniform topology.

Theorem 5. *Let $p \geq 1$ and let \mathbf{x} be a weak geometric p -rough path. Assume that*

- (1) $V = (V_1, \dots, V_d)$ is a collection of Lip^γ vector fields on \mathbb{R}^e for $\gamma > p$;
- (2) $\phi \in C_b^1(\mathbb{R}^e; \mathbb{R})$.

Then the RPDE

$$\begin{aligned} du(t, y) + \nabla u(t, y) \cdot V(y) d\mathbf{x}_t &= 0, \\ u(0, y) &= \phi(y) \end{aligned} \tag{8}$$

has a unique solution u , given explicitly by

$$u(t, y) = \phi(\pi_{(V)}(0, y; \tilde{\mathbf{x}}^t)_t)$$

where $\pi_{(V)}(0, y; \mathbf{x})$ was introduced in Theorem 3. We denote the solution u by $\Pi_{(V)}(0, \phi; \mathbf{x})$. Furthermore, the map

$$\mathbf{x} \mapsto u = \Pi_{(V)}(0, \phi; \mathbf{x})$$

is continuous from $C^{\frac{1}{p}-H\delta l}([0, T], G^{[p]}(\mathbb{R}^d))$ into $C([0, T] \times \mathbb{R}^e)$ when the latter is equipped with the uniform topology.

Proof. Let $(x^n)_{n \in \mathbb{N}}$ be a sequence of smooth paths such that

$$S_{[p]}(x^n) \equiv \mathbf{x}^n \rightarrow \mathbf{x} \tag{9}$$

uniformly on $[0, T]$, and

$$\sup_n \|\mathbf{x}^n\|_{\frac{1}{p}\text{-H\delta l}; [0, T]} < \infty.$$

If we consider the time reversed paths, $\overleftarrow{\mathbf{x}}^{n,t} := \mathbf{x}_{t-}^n$, we deduce from (9), that for each fixed $t \in (0, T]$,

$$\overleftarrow{\mathbf{x}}_s^{n,t} \rightarrow \overleftarrow{\mathbf{x}}_s^t$$

uniformly in $s \in [0, t]$. Furthermore, for $0 \leq s < u \leq t$,

$$\|\overleftarrow{\mathbf{x}}_{s,u}^{n,t}\| = \|\mathbf{x}_{t-u,t-s}^n\| \leq \|\mathbf{x}^n\|_{\frac{1}{p}\text{-Hö}l;[0,T]} |u - s|^{\frac{1}{p}}$$

and hence,

$$\sup_{n \in \mathbb{N}} \|\overleftarrow{\mathbf{x}}^{n,t}\|_{\frac{1}{p}\text{-Hö}l;[0,t]} \leq \sup_{n \in \mathbb{N}} \|\mathbf{x}^n\|_{\frac{1}{p}\text{-Hö}l;[0,T]} < \infty. \tag{10}$$

From the Universal Limit Theorem 3, we deduce that $\pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^{n,t})_s$ converges uniformly in $s \in [0, t]$ to $\pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^t)_s$, the unique solution of the RDE

$$dy_s = V(y_s) d\overleftarrow{\mathbf{x}}_s^t \tag{11}$$

started at $y \in \mathbb{R}^e$ at time 0. In particular we get that

$$\pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^{n,t})_t \rightarrow \pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^t)_t.$$

This can of course be done for each $t \in [0, T]$.

Our next task is to prove that the family

$$\{[0, T] \times \mathbb{R}^e \ni (t, y) \mapsto \pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^{n,t})_t \in \mathbb{R}^e\}_{n \in \mathbb{N}}$$

is equicontinuous. For $t, t' \in [0, T]$ (w.l.o.g. $t' < t$) and $y, y' \in \mathbb{R}^e$,

$$\begin{aligned} & |\pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^{n,t})_t - \pi_{(V)}(0, y'; \overleftarrow{\mathbf{x}}^{n,t'})_{t'}| \\ & \leq |\pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^{n,t})_t - \pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^{n,t'})_{t'}| \end{aligned} \tag{12}$$

$$+ |\pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^{n,t'})_{t'} - \pi_{(V)}(0, y'; \overleftarrow{\mathbf{x}}^{n,t'})_{t'}| \tag{13}$$

$$+ |\pi_{(V)}(0, y'; \overleftarrow{\mathbf{x}}^{n,t'})_{t'} - \pi_{(V)}(0, y'; \overleftarrow{\mathbf{x}}^{n,t})_{t'}|. \tag{14}$$

From the Generalized Davie Lemma in [14], we get that

$$|\pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^{n,t})_t - \pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^{n,t'})_{t'}| \leq C|t - t'|^{\frac{1}{p}}$$

where the constant C can be chosen to be independent of both n and t , but may depend on T and

$$A := \sup_{s \in [0, T]} \sup_{n \in \mathbb{N}} \|\overleftarrow{\mathbf{x}}^{n,s}\|_{\frac{1}{p}\text{-Hö}l;[0,s]} \leq \sup_{s \in [0, T]} \sup_{n \in \mathbb{N}} \|\mathbf{x}^n\|_{\frac{1}{p}\text{-Hö}l;[0,T]} < \infty. \tag{15}$$

For (13) and (14), we need the uniform continuity on $\mathbb{R}^e \times \{\mathbf{x}: \|\mathbf{x}\|_{\frac{1}{p}\text{-Hö}l;[0,T]} \leq M\}$, $M > 0$, of the Itô map $(y, \mathbf{x}) \mapsto \pi_{(V)}(0, y; \mathbf{x})$. In fact,

$$|\pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^{n,t})_{t'} - \pi_{(V)}(0, y'; \overleftarrow{\mathbf{x}}^{n,t})_{t'}| \leq |\pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^{n,t}) - \pi_{(V)}(0, y'; \overleftarrow{\mathbf{x}}^{n,t})|_{\infty;[0,t']} \rightarrow 0$$

uniformly in n , as $|y - y'| \rightarrow 0$, because the uniform bounds in (15) guarantee that we stay on a bounded set in $C^{\frac{1}{p}\text{-Hö}l}([0, T], G^{[p]}(\mathbb{R}^d))$, which does not depend on n or t .

For (14) we have

$$|\pi_{(V)}(0, y'; \overleftarrow{\mathbf{x}}^{n,t})_{t'} - \pi_{(V)}(0, y'; \overleftarrow{\mathbf{x}}^{n,t'})_{t'}| \leq |\pi_{(V)}(0, y'; \overleftarrow{\mathbf{x}}^{n,t}) - \pi_{(V)}(0, y'; \overleftarrow{\mathbf{x}}^{n,t'})|_{\infty; [0, t']}.$$

Again using the uniform continuity on $\mathbb{R}^e \times \{\mathbf{x}: \|\mathbf{x}\|_{\frac{1}{p}\text{-Hö}; [0, T]} \leq M\}$ of the Itô map $(y, \mathbf{x}) \mapsto \pi_{(V)}(0, y; \mathbf{x})$,

$$|\pi_{(V)}(0, y'; \overleftarrow{\mathbf{x}}^{n,t}) - \pi_{(V)}(0, y'; \overleftarrow{\mathbf{x}}^{n,t'})|_{\infty; [0, t']} \rightarrow 0$$

uniformly in n , as $|t - t'| \rightarrow 0$, if we can show that

$$d_{\frac{1}{p'}\text{-Hö}; [0, t']}(\overleftarrow{\mathbf{x}}^{n,t}, \overleftarrow{\mathbf{x}}^{n,t'}) \rightarrow 0 \tag{16}$$

uniformly in n , as $|t - t'| \rightarrow 0$, for some $p' > p$. From the interpolation results proved in Proposition 11 in [9], we deduce that (16) will follow if we show that

$$\sup_{s \in [0, T]} \sup_{n \in \mathbb{N}} \|\overleftarrow{\mathbf{x}}^{n,s}\|_{\frac{1}{p}\text{-Hö}; [0, s]} < \infty \tag{17}$$

and

$$d_{0; [0, t']}(\overleftarrow{\mathbf{x}}^{n,t}, \overleftarrow{\mathbf{x}}^{n,t'}) = \sup_{0 \leq s < u \leq t'} d(\overleftarrow{\mathbf{x}}^{n,t}_s, \overleftarrow{\mathbf{x}}^{n,t'}_u) \rightarrow 0 \tag{18}$$

uniformly in n as $|t - t'| \rightarrow 0$. The required uniform bounds (17) are precisely those obtained in (15). This estimate guarantees that we stay on a bounded set which does not depend on n or t . In [11], the distances d_0 and d_∞ are shown to be locally $\frac{1}{[p]}$ -Hölder equivalent, and hence (18) will follow if we can show that

$$d_{\infty; [0, t']}(\overleftarrow{\mathbf{x}}^{n,t}, \overleftarrow{\mathbf{x}}^{n,t'}) = \sup_{0 \leq s \leq t'} d(\overleftarrow{\mathbf{x}}^{n,t}_s, \overleftarrow{\mathbf{x}}^{n,t'}_s) \rightarrow 0$$

uniformly in n , as $|t - t'| \rightarrow 0$. But,

$$\begin{aligned} d_{\infty; [0, t']}(\overleftarrow{\mathbf{x}}^{n,t}, \overleftarrow{\mathbf{x}}^{n,t'}) &= \sup_{0 \leq s \leq t'} d(\overleftarrow{\mathbf{x}}^{n,t}_s, \overleftarrow{\mathbf{x}}^{n,t'}_s) \\ &= \sup_{0 \leq s \leq t'} d(\mathbf{x}^n_{t-s}, \mathbf{x}^n_{t'-s}) \\ &\leq \|\mathbf{x}^n\|_{\frac{1}{p}\text{-Hö}; [0, t']} |t - t'|^{\frac{1}{p}} \\ &\leq A |t - t'|^{\frac{1}{p}} \end{aligned}$$

and hence, the required convergence (uniform in n) is obtained. Therefore the family

$$\{(t, y) \mapsto \pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^{n,t})_t\}_{n \in \mathbb{N}}$$

is indeed equicontinuous in $t \in [0, T]$ and $y \in \mathbb{R}^e$.

From the pointwise convergence and the equicontinuity, we can conclude that

$$\pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^{n,t})_t \rightarrow \pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^t)_t$$

uniformly in $t \in [0, T]$ and $y \in \mathbb{R}^e$. The initial function ϕ is assumed to be $C_b^1(\mathbb{R}^e, \mathbb{R})$ and hence we get that

$$\phi(\pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^{n,t})_t) \rightarrow \phi(\pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^t)_t) \tag{19}$$

uniformly in $t \in [0, T]$ and $y \in \mathbb{R}^e$. Therefore if we define

$$u(t, y) = \phi(\pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^t)_t)$$

we immediately see that u is a solution of (8).

Having established the existence of a solution of (8), we now show that a Lip^γ assumption on the vector fields guarantees the uniqueness of solutions. Suppose that $v : [0, T] \times \mathbb{R}^e \rightarrow \mathbb{R}$ is another solution of (8). Then there exists a sequence of smooth paths $z^n : [0, T] \rightarrow \mathbb{R}^d$ such that

$$S_{[p]}(z^n) \equiv z^n \rightarrow \mathbf{x}$$

and $v(t, y) = \lim_{n \rightarrow \infty} v_n(t, y)$, with v_n solving

$$\begin{aligned} dv_n(t, y) + \frac{\partial v_n}{\partial y^j}(t, y) V_i^j(y) dz_t^{n,i} &= 0, \\ v_n(0, y) &= \phi(y). \end{aligned}$$

Then,

$$\begin{aligned} v(t, y) &= \lim_{n \rightarrow \infty} v_n(t, y) = \lim_{n \rightarrow \infty} \phi(\pi_{(V)}(0, y; \overleftarrow{\mathbf{z}}^{n,t})_t) \\ &= \phi(\pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^t)_t) \\ &= u(t, y) \end{aligned}$$

since $\pi_{(V)}(0, y; \overleftarrow{\mathbf{z}}^{n,t})$ converges to the unique solution $\pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^t)$ of the RDE (11). Therefore the rough solution $u(t, y) = \phi(\pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^t)_t)$ is indeed unique.

We still have to prove the continuity of the map which sends the pair the driving signal \mathbf{x} to the solution u . To this end, suppose that $(\mathbf{x}^n)_{n \in \mathbb{N}}$ is a sequence of weak geometric p -rough paths converging to \mathbf{x} in $\frac{1}{p}$ -Hölder topology, i.e. $d_{\frac{1}{p}\text{-Hölder}; [0, T]}(\mathbf{x}^n, \mathbf{x}) \rightarrow 0$. This implies a fortiori uniform convergence of \mathbf{x}^n to \mathbf{x} , with the uniform bounds $\sup_n \|\mathbf{x}^n\|_{\frac{1}{p}\text{-Hölder}; [0, T]} < \infty$. Using the same reasoning as in the existence part of the proof, we can show that

$$\pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^{n,t})_t \rightarrow \pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^t)_t$$

uniformly in $t \in [0, T]$ and $y \in \mathbb{R}^e$. Thus,

$$u_n(t, y) = \phi(\pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^{n,t})_t) \rightarrow \phi(\pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^t)_t) = u(t, y)$$

in $C([0, T] \times \mathbb{R}^e)$ equipped with the uniform topology. Therefore we conclude that the map which sends the driving signal to the solution is indeed continuous in the uniform topology. \square

Remark 6. If we take our initial function ϕ to be bounded and uniformly continuous on \mathbb{R}^e , i.e. $\phi \in BUC(\mathbb{R}^e)$, then similar reasoning as that used in the above proof, allows us to conclude that the map

$$\mathbf{x} \rightarrow \Pi_{(V)}(0, \phi; \mathbf{x}) := \phi(\pi_{(V)}(0, \cdot; \overleftarrow{\mathbf{x}} \cdot))$$

is continuous from $C^{\frac{1}{p}-H\delta l}([0, T], G^{[p]}(\mathbb{R}^d))$ into $BUC([0, T] \times \mathbb{R}^e)$. In this case however, $\phi(\pi_{(V)}(0, y; \bar{\mathbf{x}}^{n,t})_t)$ must be interpreted as a weak (e.g. viscosity) solution of

$$\begin{aligned} du_n(t, y) + \frac{\partial u_n}{\partial y^j}(t, y) V_t^j(y) dx_t^{n,i} &= 0, \\ u_n(0, y) &= \phi(y). \end{aligned}$$

Remark 7. If we take $\phi \in C^1(\mathbb{R}^e)$, but not bounded, the map $\mathbf{x} \rightarrow \Pi_{(V)}(0, \phi; \mathbf{x})$ is continuous in the compact uniform topology.

In the next corollary, we show that as in the case of classical and first order SPDEs, if we assume more regularity on the vector fields and the initial function, our solution will be smoother in y .

Corollary 8. Let $p \geq 1, k \in \{1, 2, \dots\}$ and let \mathbf{x} be a weak geometric p -rough path. Assume that

- (1) $V = (V_1, \dots, V_d)$ is a collection of Lip^γ vector fields on \mathbb{R}^e for $\gamma > p - 1 + k$;
- (2) $\phi \in C^k(\mathbb{R}^e; \mathbb{R})$.

Then the RPDE

$$\begin{aligned} du(t, y) + \nabla u(t, y) \cdot V(y) d\mathbf{x}_t &= 0, \\ u(0, y) &= \phi(y) \end{aligned}$$

has a unique solution $u \in C^{0,k}([0, T] \times \mathbb{R}^e, \mathbb{R})$.

Proof. From our assumption on the vector fields, we know that for each $t \in [0, T]$,

$$y \mapsto \pi_{(V)}(0, y; \bar{\mathbf{x}}^t)_t$$

is k times continuously differentiable (cf. [15]). Therefore we immediately deduce that for each $t \in [0, T]$,

$$u(t, \cdot) = \phi(\pi_{(V)}(0, \cdot; \bar{\mathbf{x}}^t)_t)$$

is also k times continuously differentiable. \square

3.2. Second order linear RPDEs

We study second order linear PDEs driven by weak geometric p -rough paths,

$$\begin{aligned} du(t, y) &= L_t u(t, y) dt - \nabla u(t, y) \cdot V(y) d\mathbf{x}_t, \\ u(0, y) &= \phi(y) \end{aligned} \tag{20}$$

where L_t is an elliptic operator of the form

$$L_t = \frac{1}{2} a^{ij}(t, \cdot) \frac{\partial^2}{\partial y^i \partial y^j} + b^i(t, \cdot) \frac{\partial}{\partial y^i} \tag{21}$$

with $a : [0, T] \times \mathbb{R}^e \rightarrow S_e$ and $b : [0, T] \times \mathbb{R}^e \rightarrow \mathbb{R}^e$. These equations can be regarded as perturbations of first order RPDEs. The approach we use to prove existence and uniqueness of solutions, is based on a technique for second order linear SPDEs described by Kunita in [23]. Kunita shows that solving

$$\begin{aligned}
 du(t, y) &= L_t u(t, y) dt - \nabla u(t, y) \cdot V(y) \circ dB_t, \\
 u(0, y) &= \phi(y)
 \end{aligned}
 \tag{22}$$

can be reduced to proving the existence and uniqueness of solutions for the following second order PDE,

$$\frac{\partial v}{\partial t} = \tilde{L}_t v, \quad v(0, y) = \phi(y)
 \tag{23}$$

where the coefficients of \tilde{L}_t are now random.⁸

The equivalence of problems (22) and (23) was stated under suitable assumptions by Rozovskiĭ in [32]. In [35], Tubaro used results from the theory of abstract evolution equations to solve (23), and thus obtain a unique solution for the SPDE (22).

In what follows we are going to show that these ideas can be generalized to equations driven by weak geometric p -rough paths. In our case the PDE analogue to (23) will have coefficients which depend on the flow of an RDE (we will sometimes speak of PDEs with rough coefficients).

Suppose we are given the elliptic operator L_t ,

$$L_t = \frac{1}{2} a^{ij}(t, \cdot) \frac{\partial^2}{\partial y^i \partial y^j} + b^i(t, \cdot) \frac{\partial}{\partial y^i}$$

with $a : [0, T] \times \mathbb{R}^e \rightarrow S_e$ and $b : [0, T] \times \mathbb{R}^e \rightarrow \mathbb{R}^e$. Let \mathbf{x} be a weak geometric p -rough path, $p \geq 1$, and let $V = (V_1, \dots, V_d)$ be a collection of Lip^γ ($\gamma > p + 1$) vector fields⁹ on \mathbb{R}^e . For each $t \in [0, T]$, we define the linear map $w_t^{\mathbf{x}}$ on $C^1(\mathbb{R}^e)$ by

$$\begin{aligned}
 w_t^{\mathbf{x}} : C^1(\mathbb{R}^e) &\rightarrow C^1(\mathbb{R}^e) \\
 \phi &\mapsto \Pi_{(V)}(0, \phi; \mathbf{x})_t,
 \end{aligned}$$

i.e. $w_t^{\mathbf{x}}(\phi)$ is the solution of the RPDE (20) with $L_t \equiv 0$. From Section 3.1, we deduce that $w_t^{\mathbf{x}}(\phi)(y) = \phi(\pi_{(V)}(0, y; \tilde{\mathbf{x}}^t)_t)$. This map is bijective and its inverse, $\hat{w}_t^{\mathbf{x}}$, is given by

$$\hat{w}_t^{\mathbf{x}}(\phi)(y) = \phi(\pi_{(V)}(0, y; \mathbf{x})_t).$$

We can define a new operator $L_t^{\mathbf{x}}$ by

$$L_t^{\mathbf{x}} = \hat{w}_t^{\mathbf{x}} \circ L_t \circ w_t^{\mathbf{x}}.$$

This is again a second order operator represented by

$$L_t^{\mathbf{x}} = \frac{1}{2} a_{\mathbf{x}}^{ij}(t, \cdot) \frac{\partial^2}{\partial y^i \partial y^j} + b_{\mathbf{x}}^i(t, \cdot) \frac{\partial}{\partial y^i}$$

with

$$a_{\mathbf{x}}^{ij}(t, y) = a^{kl}(t, \pi_{(V)}(0, y; \mathbf{x})_t) \partial_k \pi_{(V)}^i(0, \cdot; \tilde{\mathbf{x}}^t)_t \Big|_{\pi_{(V)}(0, y; \mathbf{x})_t} \partial_l \pi_{(V)}^j(0, \cdot; \tilde{\mathbf{x}}^t)_t \Big|_{\pi_{(V)}(0, y; \mathbf{x})_t}
 \tag{24}$$

and

⁸ These coefficients depend on the flow of the SDE $dy_t = V(y_t) \circ dB_t$.

⁹ We need this regularity condition on the vector fields, because to define $a_{\mathbf{x}}$ and $b_{\mathbf{x}}$ in (24) and (25) we need a C^2 flow for our RDE (cf. Corollary 8).

$$\begin{aligned}
 b_{\mathbf{x}}^i(t, y) &= \frac{1}{2} a^{kl}(t, \pi_{(V)}(0, y; \mathbf{x})_t) \partial_{kl} \pi_{(V)}^i(0, \cdot; \bar{\mathbf{x}}^t) \Big|_{\pi_{(V)}(0, y; \mathbf{x})_t} \\
 &\quad + b^k(t, \pi_{(V)}(0, y; \mathbf{x})_t) \partial_k \pi_{(V)}^i(0, \cdot; \bar{\mathbf{x}}^t) \Big|_{\pi_{(V)}(0, y; \mathbf{x})_t}.
 \end{aligned}
 \tag{25}$$

To better understand why we need the operator $L_t^{\mathbf{x}}$, consider the smooth path $x : [0, T] \rightarrow \mathbb{R}^d$. Suppose that u is a classical solution of

$$\begin{aligned}
 du(t, y) &= L_t u(t, y) dt - \nabla u(t, y) \cdot V(y) dx_t, \\
 u(0, y) &= \phi(y) \in C_b^2(\mathbb{R}^e),
 \end{aligned}
 \tag{26}$$

i.e. u is a $C_b^{1,2}$ function such that

$$u(t, y) = \phi(y) + \int_0^t L_s u(s, y) ds - \int_0^t \nabla u(s, y) \cdot V(y) dx_s.$$

We then have the following lemma.

Lemma 9. (Cf. [23].) u is a classical solution of (26) if and only if $v(t, y) := \hat{w}_t^{\mathbf{x}}(u(t, \cdot))(y)$ is a classical solution of

$$\frac{\partial v}{\partial t} = L_t^{\mathbf{x}} v, \quad v(0, y) = \phi(y).
 \tag{27}$$

Proof. Let u be a classical solution of (26). Then,

$$\begin{aligned}
 dv(t, y) &= du(t, \pi_{(V)}(0, y; \mathbf{x})_t) \\
 &= \partial_t u(t, \pi_{(V)}(0, y; \mathbf{x})_t) dt + \nabla u(t, \pi_{(V)}(0, y; \mathbf{x})_t) \cdot d\pi_{(V)}(0, y; \mathbf{x})_t \\
 &= \partial_t u(t, \pi_{(V)}(0, y; \mathbf{x})_t) dt + \nabla u(t, \pi_{(V)}(0, y; \mathbf{x})_t) \cdot V(\pi_{(V)}(0, y; \mathbf{x})_t) dx_t \\
 &= L_t u(t, \pi_{(V)}(0, y; \mathbf{x})_t) dt \\
 &= \hat{w}_t^{\mathbf{x}}(L_t w_t^{\mathbf{x}}(v(t, \cdot)))(y) dt
 \end{aligned}$$

and therefore $v(t, y)$ satisfies (27). \square

Conversely, we can show that if v is a $C_b^{1,2}$ -solution of (27), then

$$u(t, y) := w_t^{\mathbf{x}}(v(t, \cdot))(y)$$

is a $C_b^{1,2}$ -solution of (26).

Recall that in Definition 4, we defined a solution of the RPDE (20) to be a limit point of a sequence of solutions of equations driven by smooth paths converging to \mathbf{x} in rough path sense. Thus one of the first things that we need to do, is discuss the conditions on a, b , the vector fields $V = (V_1, \dots, V_d)$, and the initial function ϕ , which guarantee the existence of a unique $C_b^{1,2}$ -solution for the classical PDE (3). To this end, we have the following regularity hypothesis on a and b .

Hypothesis 10. $a : [0, T] \times \mathbb{R}^e \rightarrow S_e$ and $b : [0, T] \times \mathbb{R}^e \rightarrow \mathbb{R}^e$ are bounded continuous functions such that

(1) a is uniformly elliptic, i.e. there exists $\lambda > 0$ such that

$$\langle \theta, a(t, y)\theta \rangle \geq \lambda |\theta|^2$$

for all $(t, y) \in [0, T] \times \mathbb{R}^e$ and $\theta \in \mathbb{R}^e$;

(2) there exist constants $C_{a,b} > 0$ and $0 < \beta \leq 1$ such that

$$|a(t, y) - a(t', y')| + |b(t, y) - b(t', y')| \leq C_{a,b} (|t - t'|^\beta + |y - y'|^\beta)$$

for all $(t, y), (t', y') \in [0, T] \times \mathbb{R}^e$.

From Theorem 16, Chapter 1 in [5] and Theorem 3.1.1 in [34], we know that if a and b satisfy Hypothesis 10, then the PDE

$$\frac{\partial v}{\partial t} = L_t v, \quad v(0, \cdot) = \phi(\cdot) \in C_b(\mathbb{R}^e)$$

has a unique $C_b^{1,2}$ -solution.

Proposition 11. Let $V = (V_1, \dots, V_d)$ be a collection of Lip^γ vector fields, $\gamma > p + 1$, on \mathbb{R}^e , and suppose that $a : [0, T] \times \mathbb{R}^e \rightarrow S_\rho$ and $b : [0, T] \times \mathbb{R}^e \rightarrow \mathbb{R}^e$ satisfy the regularity Hypothesis 10. Then the functions a_x and b_x defined in (24) and (25) respectively, satisfy

(1) there exist constants $C_M > 0$ (uniformly on $\{\mathbf{x} : \|\mathbf{x}\|_{\frac{1}{p}\text{-Hö}} \leq M\}$) such that

$$|a_x(t, y) - a_x(t', y')| + |b_x(t, y) - b_x(t', y')| \leq C_M (|t - t'|^{\beta \wedge \frac{1}{p}} + |y - y'|^{\beta \wedge \frac{1}{p}})$$

for all $(t, y), (t', y') \in [0, T] \times \mathbb{R}^e$, where β is the Hölder exponent of a and b ;

(2) a_x is uniformly elliptic and there exists $\Lambda_M > 0$, such that

$$\inf_{\{\mathbf{x} : \|\mathbf{x}\|_{\frac{1}{p}\text{-Hö}} \leq M\}} \langle \theta, a_x(t, y)\theta \rangle \geq \Lambda_M |\theta|^2$$

for all $(t, y) \in [0, T] \times \mathbb{R}^e$ and $\theta \in \mathbb{R}^e$.

Furthermore, if we assume that a and b have two bounded, continuous spatial derivatives and $V = (V_1, \dots, V_d)$ are Lip^γ , $\gamma > p + 3$, then $a_x(t, \cdot)$ and $b_x(t, \cdot)$ are again C_b^2 functions and their C^2 -norms are uniform over $\{\mathbf{x} : \|\mathbf{x}\|_{\frac{1}{p}\text{-Hö}} \leq M\}$.

Proof. Remark that

$$\pi_{(V)}(0, y, \mathbf{x})_t = y + \pi_{(\tilde{V})}(0, \cdot, \mathbf{x})_t$$

where $\tilde{V} = V(y + \cdot)$ has the same Lip^γ -norm as V . It follows that estimates for any derivatives are automatically uniform over y . For instance (cf. [15]),

$$|D\pi_{(V)}(0, y, \tilde{\mathbf{x}}^t)_t| \leq C_1 \exp(C_1 \cdot (|V|_{\text{Lip}^\gamma} \|\mathbf{x}\|_{\frac{1}{p}\text{-Hö}; [0, T]})) \tag{28}$$

where C_1 is a constant independent of y . If we iterate this argument, we can deduce from (24) and (25), and our regularity assumption on a and b , that $a_x(t, \cdot)$ and $b_x(t, \cdot)$ are again twice differentiable

in space, with bounded derivatives. Furthermore, we can also see from (28), that the C^2 -norms¹⁰ $\|a_x(t, \cdot)\|_{C^2}$ and $\|b_x(t, \cdot)\|_{C^2}$ are bounded uniformly on $\{x: \|x\|_{\frac{1}{p}\text{-H\"{o}l}} \leq M\}$.

To prove the uniform ellipticity of a_x , we first note that by assumption, there exists $\lambda > 0$, such that

$$\langle \theta, a(t, y)\theta \rangle \geq \lambda|\theta|^2$$

for all $(t, y) \in [0, T] \times \mathbb{R}^e$ and $\theta \in \mathbb{R}^e$. Hence,

$$\begin{aligned} \langle \theta, a_x(t, y)\theta \rangle &\geq \lambda |D\pi_{(V)}(0, \cdot; \bar{x}^t)|_{\pi_{(V)}(0, y, x_t)} \cdot \theta|^2 \\ &\geq \frac{\lambda}{|(D\pi_{(V)}(0, \cdot; \bar{x}^t)|_{\pi_{(V)}(0, y, x_t)})^{-1}|} |\theta|^2 \end{aligned}$$

since

$$\begin{aligned} |\theta| &= |(D\pi_{(V)}(0, \cdot; \bar{x}^t)|_{\pi_{(V)}(0, y, x_t)})^{-1} \cdot (D\pi_{(V)}(0, \cdot; \bar{x}^t)|_{\pi_{(V)}(0, y, x_t)}) \cdot \theta| \\ &\leq |(D\pi_{(V)}(0, \cdot; \bar{x}^t)|_{\pi_{(V)}(0, y, x_t)})^{-1}| |(D\pi_{(V)}(0, \cdot; \bar{x}^t)|_{\pi_{(V)}(0, y, x_t)}) \cdot \theta|. \end{aligned}$$

To obtain the uniform ellipticity, we note that

$$(D\pi_{(V)}(0, \cdot; \bar{x}^t)|_{\pi_{(V)}(0, y, x_t)})^{-1} = D\pi_{(V)}(0, \cdot; x_t)|_{\pi_{(V)}(0, y, x_t)}.$$

Using the already discussed uniformity (with respect to the starting point) of the Jacobian and other derivatives of the flow, we see that

$$|(D\pi_{(V)}(0, \cdot; \bar{x}^t)|_{\pi_{(V)}(0, y, x_t)})^{-1}| \leq C_M$$

where the constant C_M does not depend on y . This finishes the proof of the third part of the proposition, since,

$$\inf_{\{x: \|x\|_{\frac{1}{p}\text{-H\"{o}l}} \leq M\}} \langle \theta, a_x(t, y)\theta \rangle \geq \Lambda_M |\theta|^2$$

with $\Lambda_M = \frac{\lambda}{C_M}$.

The Hölder continuity of a_x and b_x can be deduced from the Hölder continuity of a and b , and estimates on the derivatives of the flow, similar to those in (12) and (13) in Theorem 5. \square

Therefore, given a weak geometric p -rough path x , and Lip^γ , $\gamma > p + 1$, vector fields, we can again deduce from Theorem 16, Chapter 1 in [5] and Theorem 3.1.1 in [34] that the PDE with rough coefficients

$$\frac{\partial v}{\partial t} = L_t^x v, \quad v(0, \cdot) = \phi(\cdot) \in C_b(\mathbb{R}^e)$$

has a unique $C_b^{1,2}$ -solution. We have the following proposition on classical PDEs of the form (3).

¹⁰ For $f \in C^k(\mathbb{R}^e, \mathbb{R})$, we define $\|f\|_{C^k} = \sum_{0 \leq |\alpha| \leq k} \sup_x |D^\alpha f(x)|$.

Proposition 12. Let $x : [0, T] \rightarrow \mathbb{R}^d$ be a smooth path. Assume that

- (1) $V = (V_1, \dots, V_d)$ is a collection of Lip^2 vector fields on \mathbb{R}^e ;
- (2) L_t is a second order elliptic operator of the form (21), with coefficients $a : [0, T] \times \mathbb{R}^e \rightarrow S_e$ and $b : [0, T] \times \mathbb{R}^e \rightarrow \mathbb{R}^e$ satisfying the regularity Hypothesis 10;
- (3) $\phi \in C_b(\mathbb{R}^e, \mathbb{R})$.

Then the PDE

$$\begin{aligned} du(t, y) &= L_t u(t, y) dt - \nabla u(t, y) \cdot V(y) dx_t, \\ u(0, y) &= \phi(y) \end{aligned} \tag{29}$$

has a unique $C_b^{1,2}$ -solution u , which we denote by $\Pi_{(a,b,V)}(0, \phi; x)$.

Remark 13. If $x \in W^{1,1}([0, T], \mathbb{R}^d)$, we can define the solution to (29) via the unique extension of the map $C^\infty([0, T], \mathbb{R}^d) \ni x \mapsto u = \Pi_{(a,b,V)}(0, \phi; x)$ with respect to the $W^{1,1}$ -norm (cf. Remark 25).

Proof of Proposition 12. This result follows immediately from Theorem 16, Chapter 1 in [5] and Theorem 3.1.1 in [34]. \square

If we now go back to RPDEs, we see from the remarks after Proposition 11 and the result in Lemma 9, that an obvious candidate for the solution of the RPDE (20) is given by

$$u(t, y) = w_t^x(v)(y) = v(t, \pi_{(V)}(0, y; \bar{x}^t)_t) \tag{30}$$

where v is the unique $C_b^{1,2}$ -solution of

$$\frac{\partial v}{\partial t} = L_t^x v, \quad v(0, \cdot) = \phi(\cdot) \in C_b(\mathbb{R}^e, \mathbb{R}).$$

To be able to show that u is the unique solution for (20), we first have to prove two propositions, which we will use to show that u is in fact the uniform limit of solutions of classical PDEs.

Proposition 14. Let $V = (V_1, \dots, V_d)$ be a collection of Lip^γ vector fields, $\gamma > p + 1$, on \mathbb{R}^e and suppose that $a : [0, T] \times \mathbb{R}^e \rightarrow S_e$ and $b : [0, T] \times \mathbb{R}^e \rightarrow \mathbb{R}^e$ satisfy the regularity Hypothesis 10. Let $(x^n)_{n \in \mathbb{N}}$ be a sequence of weak geometric p -rough paths converging to a weak geometric p -rough path x uniformly on $[0, T]$ with uniform bounds, i.e. $\sup_n \|x^n\|_{\frac{1}{p}\text{-Hö}} < \infty$. Then,

$$a_{x^n}(t, y) \rightarrow a_x(t, y)$$

and

$$b_{x^n}(t, y) \rightarrow b_x(t, y)$$

uniformly on $[0, T] \times \mathbb{R}^e$.

Proof. To prove that $a_{x^n}(t, y) \rightarrow a_x(t, y)$ converges uniformly on $[0, T] \times \mathbb{R}^e$, we are first going to obtain pointwise convergence, and then show that the family $\{(t, y) \mapsto a_{x^n}(t, y)\}_{n \in \mathbb{N}}$ is equicontinuous. For fixed $(t, y) \in [0, T] \times \mathbb{R}^e$,¹¹

¹¹ In this proof we use the notation $\xi^{(t,y)} = \pi_{(V)}(0, y; x)_t$ and $\zeta(t, y) = \pi_{(V)}(0, y; \bar{x}^t)_t$.

$$\begin{aligned}
 & |a_{x^n}^{ij}(t, y) - a_x^{ij}(t, y)| \\
 & \leq |a^{kl}(t, \xi_n) \partial_k \zeta_n^i(t, \xi_n) \partial_l \zeta_n^j(t, \xi_n) - a^{kl}(t, \xi) \partial_k \zeta^i(t, \xi) \partial_l \zeta^j(t, \xi)| \\
 & \leq |a^{kl}(t, \xi_n) - a^{kl}(t, \xi)| |\partial_k \zeta^i(t, \xi) \partial_l \zeta^j(t, \xi)| \\
 & \quad + |a^{kl}(t, \xi_n)| |\partial_k \zeta_n^i(t, \xi_n) \partial_l \zeta_n^j(t, \xi_n) - \partial_k \zeta^i(t, \xi) \partial_l \zeta^j(t, \xi)| \\
 & \leq C_{a,b} |\xi_n(t, y) - \xi(t, y)|^\beta |D\zeta(t, \xi(t, y))|^2 \\
 & \quad + \|a\|_\infty |D\zeta_n(t, \xi_n) - D\zeta(t, \xi)| (|D\zeta_n(t, \xi_n)| + |D\zeta(t, \xi)|).
 \end{aligned}$$

If we let $n \rightarrow \infty$, we deduce from (28) and the continuity of the Itô map, that $a_{x^n}(t, y)$ converges pointwise to $a_x(t, y)$.

To prove equicontinuity, take $(t, y), (t', y') \in [0, T] \times \mathbb{R}^e$. Then,

$$\begin{aligned}
 & |a_{x^n}^{ij}(t, y) - a_{x^n}^{ij}(t', y')| \\
 & \leq |a^{kl}(t, \xi_n^{(t,y)}) \partial_k \zeta_n^i(t, \xi_n^{(t,y)}) \partial_l \zeta_n^j(t, \xi_n^{(t,y)}) - a^{kl}(t', \xi_n^{(t',y')}) \partial_k \zeta_n^i(t', \xi_n^{(t',y')}) \partial_l \zeta_n^j(t', \xi_n^{(t',y')})| \\
 & \leq |a^{kl}(t, \xi_n^{(t,y)}) - a^{kl}(t', \xi_n^{(t',y')})| |\partial_k \zeta_n^i(t, \xi_n^{(t,y)}) \partial_l \zeta_n^j(t, \xi_n^{(t,y)})| \\
 & \quad + |a^{kl}(t', \xi_n^{(t',y')})| |\partial_k \zeta_n^i(t, \xi_n^{(t,y)}) \partial_l \zeta_n^j(t, \xi_n^{(t,y)}) - \partial_k \zeta_n^i(t', \xi_n^{(t',y')}) \partial_l \zeta_n^j(t', \xi_n^{(t',y')})| \\
 & \leq C_{a,b} (|t - t'|^\beta + |\xi_n^{(t,y)} - \xi_n^{(t',y')}|^\beta) |D\zeta^n(t, \xi_n^{(t,y)})|^2 \tag{31} \\
 & \quad + \|a\|_\infty (|D\zeta^n(t, \xi_n^{(t,y)})| + |D\zeta^n(t', \xi_n^{(t',y')})|) |D\zeta^n(t, \xi_n^{(t,y)}) - D\zeta^n(t', \xi_n^{(t',y')})|. \tag{32}
 \end{aligned}$$

Since $\xi_n^{(t,y)} \rightarrow \xi^{(t,y)}$ uniformly on $[0, T] \times \mathbb{R}^e$, we deduce that $\{(t, y) \mapsto \xi_n^{(t,y)}\}_{n \in \mathbb{N}}$ is equicontinuous, and hence we can make (31) arbitrarily small by taking $|t - t'|$ and $|y - y'|$ small enough. The term (32) can also be made arbitrarily small by taking t close to t' and y close to y' , because the family

$$\{(t, y) \mapsto D\zeta_n(t, y)\}_{n \in \mathbb{N}} \tag{33}$$

is also equicontinuous. This follows because $D\zeta_n$ solves an RDE, and hence similar reasoning as that in Theorem 5 can be used to prove the equicontinuity of (33).

Therefore we can conclude that

$$a_{x^n}(t, y) \rightarrow a_x(t, y)$$

uniformly on $[0, T] \times \mathbb{R}^e$. The uniform convergence of $b_{x^n}(t, y)$ to $b_x(t, y)$ can be proved using a similar procedure. \square

Before proving the second proposition, we recall a result by Oleinik (cf. Theorem 3.2.4 in [34]).

Theorem 15 (Oleinik estimate). *Let L_t be an elliptic operator of the form (21) with $a \in C_b^{0,2}([0, T] \times \mathbb{R}^e; S_e)$ and $b \in C_b^{0,2}([0, T] \times \mathbb{R}^e; \mathbb{R}^e)$. Given $\phi \in C_b^2(\mathbb{R}^e)$ and $g \in C_b^{0,2}([0, T] \times \mathbb{R}^e)$, suppose that $f \in C_b^{1,2}([0, T] \times \mathbb{R}^e)$ satisfies*

$$\frac{\partial f}{\partial t} - L_t f = g$$

with $f(0, \cdot) = \phi$. If

$$f \in C_b^{0,2}([0, T] \times \mathbb{R}^e) \cap C^{0,4}([0, T] \times \mathbb{R}^e)$$

then $\frac{\partial f}{\partial t} \in C^{0,2}([0, T] \times \mathbb{R}^e)$ and there exist constants A and B such that

$$\sup_{0 \leq t \leq T} \|f(t, \cdot)\|_{C^2} \leq A(1 + \|\phi\|_{C^2}) + B \sup_{0 \leq t \leq T} \|g(t, \cdot)\|_{C^2}.$$

Using these estimates, we have the following result.

Proposition 16. *Suppose that for each $n \in \mathbb{N}$, $a_n : [0, T] \times \mathbb{R}^e \rightarrow S_e$ and $b_n : [0, T] \times \mathbb{R}^e \rightarrow \mathbb{R}^e$ satisfy the regularity Hypothesis 10, and furthermore assume that they have continuous bounded first and second order spatial derivatives which are bounded independently of n .*

Let $a : [0, T] \times \mathbb{R}^e \rightarrow S_e$ and $b : [0, T] \times \mathbb{R}^e \rightarrow \mathbb{R}^e$ satisfy the regularity Hypothesis 10, and suppose that they have bounded first and second order spatial derivatives. Assume that

$$a_n(t, y) \rightarrow a(t, y)$$

and

$$b_n(t, y) \rightarrow b(t, y)$$

uniformly on $[0, T] \times \mathbb{R}^e$. Set

$$L_t^n = \frac{1}{2} a_n^{ij}(t, \cdot) \frac{\partial^2}{\partial y^i \partial y^j} + b_n^i(t, \cdot) \frac{\partial}{\partial y^i}$$

and

$$L_t = \frac{1}{2} a^{ij}(t, \cdot) \frac{\partial^2}{\partial y^i \partial y^j} + b^i(t, \cdot) \frac{\partial}{\partial y^i}.$$

Then if we define $v, v_n : [0, T] \times \mathbb{R}^e \rightarrow \mathbb{R}$ to be the unique $C_b^{1,2}$ -solutions of

$$\frac{\partial v}{\partial t} = L_t v, \quad v(0, \cdot) = \phi(\cdot) \in C_b^2(\mathbb{R}^e) \tag{34}$$

and

$$\frac{\partial v_n}{\partial t} = L_t^n v_n, \quad v_n(0, \cdot) = \phi(\cdot) \in C_b^2(\mathbb{R}^e) \tag{35}$$

respectively, we have that

$$v_n(t, y) \rightarrow v(t, y)$$

uniformly on $[0, T] \times \mathbb{R}^e$.

Proof. From Theorems 12 and 16, Chapter 1 in [5], we know that (34) and (35) have unique $C_b^{1,2}$ -solutions v and v_n , given by

$$v(t, y) = \int_{\mathbb{R}^e} \Gamma(t, y; 0, z) \phi(z) dz \tag{36}$$

and

$$v_n(t, y) = \int_{\mathbb{R}^e} \Gamma_n(t, y; 0, z) \phi(z) dz \tag{37}$$

where $\Gamma(t, y; 0, z)$ and $\Gamma_n(t, y; 0, z)$ are fundamental solutions of $\frac{\partial v}{\partial t} = L_t v$ and $\frac{\partial v_n}{\partial t} = L_t^n v_n$ respectively. Furthermore, since a and b have bounded continuous first and second order spatial derivatives, we deduce from Proposition 11 and Theorem 10, Chapter 3 in [5] that $v_n, v \in C^{0,4}([0, T] \times \mathbb{R}^e)$. Thus it follows from Theorem 15 that

$$\sup_{0 \leq t \leq T} \|v_n(t, \cdot)\|_{C^2} \leq K_1(1 + \|\phi\|_{C^2})$$

where the constant K_1 can be taken to be independent of n because of the uniform boundedness assumption on the spatial derivatives of a_n and b_n . Then,

$$\begin{aligned} \left| \frac{\partial v_n}{\partial t} - L_t v_n \right| &= |L_t^n v_n - L_t v_n| \\ &\leq (|a_n(t, y) - a(t, y)| + |b_n(t, y) - b(t, y)|) \|v_n(t, \cdot)\|_{C^2} \\ &\leq K_1(1 + \|\phi\|_{C^2})(|a_n(t, y) - a(t, y)| + |b_n(t, y) - b(t, y)|) \end{aligned}$$

and hence

$$\frac{\partial v_n}{\partial t} - L_t v_n \rightarrow 0 \tag{38}$$

uniformly on $[0, T] \times \mathbb{R}^e$. Our next task is to deduce from (38) that the sequence $\{v_n\}$ converges uniformly. To do this, recall (Theorem 12 in [5]) that under a local Hölder continuity assumption on the function g ,

$$\hat{v}(t, y) = \int_{\mathbb{R}^e} \phi(y) \Gamma(t, y; 0, z) dz - \int_0^t \left(\int_{\mathbb{R}^e} g(s, z) \Gamma(t, y; s, z) dz \right) \tag{39}$$

solves the inhomogeneous PDE

$$\frac{\partial \hat{v}}{\partial t} - L_t \hat{v} = g, \quad \hat{v}(0, y) = \phi(y).$$

Trivially, for $v_{n,m} := v_n - v_m$, we have

$$\frac{\partial v_{n,m}}{\partial t} - L_t v_{n,m} = g_{n,m}$$

with $g_{n,m} = (\frac{\partial}{\partial t} - L_t)v_{n,m}(t, y)$. We can use the representation (39), together with (38) to deduce that $\{v_n\}$ converges uniformly on $[0, T] \times \mathbb{R}^e$ to some function \tilde{v} .

The last step in this proof is to show that $\tilde{v} = v$. This follows because if we repeat the above argument with $g_n = (\frac{\partial}{\partial t} - L_t)v_n$, we get that

$$\tilde{v}(t, y) = \int_{\mathbb{R}^e} \Gamma(t, y; 0, z) \phi(z) dz$$

and thus from (36) we see that $v = \tilde{v}$. Therefore we can conclude that

$$v_n(t, y) \rightarrow v(t, y)$$

uniformly on $[0, T] \times \mathbb{R}^e$. \square

In the following theorem we prove the existence of a unique bounded solution for a linear second order RPDE. Furthermore, we prove that the map which sends the driving signal to the solution is continuous in the uniform topology.

Theorem 17. *Let $p \geq 1$ and let \mathbf{x} be a weak geometric p -rough path. Assume that*

- (1) $V = (V_1, \dots, V_d)$ is a collection of Lip^γ vector fields on \mathbb{R}^e for $\gamma > p + 3$;
- (2) $a : [0, T] \times \mathbb{R}^e \rightarrow S_e$ and $b : [0, T] \times \mathbb{R}^e \rightarrow \mathbb{R}^e$ satisfy the regularity Hypothesis 10, and furthermore, have continuous bounded first and second order spatial derivatives;
- (3) $\phi \in C_b^2(\mathbb{R}^e, \mathbb{R})$.

Assume L_t is of the form (21) with coefficients a, b . Then the RPDE

$$\begin{aligned} du(t, y) &= L_t u(t, y) dt - \nabla u(t, y) \cdot V(y) d\mathbf{x}_t, \\ u(0, y) &= \phi(y) \end{aligned} \tag{40}$$

has a unique (bounded) solution u , given by

$$u(t, y) = v(t, \pi_{(V)}(0, y; \tilde{\mathbf{x}}^t)_t)$$

where v is the $C_b^{1,2}$ -solution of

$$\frac{\partial v}{\partial t} = L_t^\mathbf{x} v, \quad v(0, \cdot) = \phi(\cdot).$$

We denote this solution by $\Pi_{(a,b,V)}(0, \phi; \mathbf{x})$. Furthermore the map

$$\mathbf{x} \rightarrow u = \Pi_{(a,b,V)}(0, \phi; \mathbf{x})$$

is continuous from $C^{\frac{1}{p}\text{-Hö}}([0, T], G^{[p]}(\mathbb{R}^d))$ into $C([0, T] \times \mathbb{R}^e; \mathbb{R})$ when the latter is equipped with the uniform topology.

Proof. We note that the Lip^γ , $\gamma > p + 3$, condition on the vector fields guarantees a C^4 flow for the associated RDE, and hence the coefficients $a_\mathbf{x}$ and $b_\mathbf{x}$ will have bounded continuous first and second order spatial derivatives (cf. Proposition 11). Let $(\mathbf{x}^n)_{n \in \mathbb{N}}$ be a sequence of smooth paths such that

$$S_{[p]}(\mathbf{x}^n) \equiv \mathbf{x}^n \rightarrow \mathbf{x}$$

uniformly on $[0, T]$, and

$$\sup_n \|\mathbf{x}^n\|_{\frac{1}{p}\text{-Hö}l; [0, T]} < \infty.$$

Let u_n be the unique $C_b^{1,2}$ -solution of

$$du_n(t, y) = L_t u_n(t, y) dt - \nabla u_n(t, y) \cdot V(y) d\mathbf{X}_t^n,$$

$$u_n(0, y) = \phi(y).$$

We know that such a solution exists from Proposition 12. Then,

$$u_n(t, y) = v_n(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{X}}^{n,t})_t)$$

where v_n is the unique $C_b^{1,2}$ -solution of

$$\frac{\partial v_n}{\partial t} = L_t^{\mathbf{X}^n} v_n, \quad v_n(0, y) = \phi(y).$$

We claim that the function u defined by

$$u(t, y) = v(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{X}}^t)_t)$$

is a solution of (40), and hence we have to show that

$$u(t, y) = \lim_{n \rightarrow \infty} u_n(t, y)$$

uniformly on $[0, T] \times \mathbb{R}^e$, i.e.

$$v(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{X}}^t)_t) = \lim_{n \rightarrow \infty} v_n(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{X}}^{n,t})_t)$$

uniformly on $[0, T] \times \mathbb{R}^e$.

Our first task is to prove pointwise convergence. For fixed $(t, y) \in [0, T] \times \mathbb{R}^e$,

$$\begin{aligned} & |v_n(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{X}}^{n,t})_t) - v(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{X}}^t)_t)| \\ & \leq |v_n(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{X}}^{n,t})_t) - v(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{X}}^{n,t})_t)| \\ & \quad + |v(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{X}}^{n,t})_t) - v(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{X}}^t)_t)|. \end{aligned}$$

The second term on the right-hand side of this inequality can be made arbitrarily small by taking n large enough since $v(t, \cdot)$ is continuous, and

$$\pi_{(V)}(0, y; \overleftarrow{\mathbf{X}}^{n,t})_t \rightarrow \pi_{(V)}(0, y; \overleftarrow{\mathbf{X}}^t)_t. \tag{41}$$

For the other term in the inequality, we have that

$$\begin{aligned} & |v_n(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{X}}^{n,t})_t) - v(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{X}}^{n,t})_t)| \\ & \leq |v_n(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{X}}^{n,t})_t) - v_n(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{X}}^t)_t)| \\ & \quad + |v(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{X}}^{n,t})_t) - v(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{X}}^t)_t)| \\ & \quad + |v_n(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{X}}^t)_t) - v(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{X}}^t)_t)|. \end{aligned} \tag{42}$$

From the results in Proposition 11, we see that Oleinik’s estimates in Theorem 15 can be used for v_n and v , to get

$$\begin{aligned} & |v_n(t, \pi_{(V)}(0, y; \widehat{\mathbf{x}}^{n,t}_t) - v_n(t, \pi_{(V)}(0, y; \widehat{\mathbf{x}}^t_t)| \\ & \leq K_1(1 + \|\phi\|_{C^1})|\pi_{(V)}(0, y; \widehat{\mathbf{x}}^{n,t}_t - \pi_{(V)}(0, y; \widehat{\mathbf{x}}^t_t)| \end{aligned} \tag{43}$$

and

$$\begin{aligned} & |v(t, \pi_{(V)}(0, y; \widehat{\mathbf{x}}^{n,t}_t) - v(t, \pi_{(V)}(0, y; \widehat{\mathbf{x}}^t_t)| \\ & \leq K_2(1 + \|\phi\|_{C^1})|\pi_{(V)}(0, y; \widehat{\mathbf{x}}^{n,t}_t - \pi_{(V)}(0, y; \widehat{\mathbf{x}}^t_t)|. \end{aligned} \tag{44}$$

Hence we deduce from (41) that both (43) and (44) go to zero as $n \rightarrow \infty$.

The remaining term in (42) can also be made arbitrarily small as $n \rightarrow \infty$ because the convergence results in Propositions 14 and 16 can be used to deduce that $v_n \rightarrow v$.

To prove that the family

$$\{[0, T] \times \mathbb{R}^e \ni (t, y) \mapsto v_n(t, \pi_{(V)}(0, y; \widehat{\mathbf{x}}^{n,t}_t)\}_{n \in \mathbb{N}}$$

is equicontinuous, we take $t', t \in [0, T]$ (w.l.o.g. $t' < t$) and $y', y \in \mathbb{R}^e$, and consider

$$\begin{aligned} & |v_n(t, \pi_{(V)}(0, y; \widehat{\mathbf{x}}^{n,t}_t) - v_n(t', \pi_{(V)}(0, y'; \widehat{\mathbf{x}}^{n,t'}_{t'})| \\ & \leq |v_n(t, \pi_{(V)}(0, y; \widehat{\mathbf{x}}^{n,t}_t) - v_n(t', \pi_{(V)}(0, y; \widehat{\mathbf{x}}^{n,t}_t)| \\ & \quad + |v_n(t', \pi_{(V)}(0, y; \widehat{\mathbf{x}}^{n,t}_t) - v_n(t', \pi_{(V)}(0, y'; \widehat{\mathbf{x}}^{n,t'}_{t'})|. \end{aligned} \tag{45}$$

For the first term,

$$\begin{aligned} |v_n(t, \pi_{(V)}(0, y; \widehat{\mathbf{x}}^{n,t}_t) - v_n(t', \pi_{(V)}(0, y; \widehat{\mathbf{x}}^{n,t}_t)| & \leq \int_{t'}^t \left| \frac{\partial v_n}{\partial s}(s, \pi_{(V)}(0, y; \widehat{\mathbf{x}}^{n,t}_t) \right| ds \\ & = \int_{t'}^t |L_s^{\mathbf{x}^n} v_n(s, \pi_{(V)}(0, y; \widehat{\mathbf{x}}^{n,t}_t)| ds \\ & \leq K_3(1 + \|\phi\|_{C^2})|t - t'| \end{aligned} \tag{46}$$

where K_3 is a constant which does not depend on n . To get the last inequality we again use the estimate in Theorem 15. For the other term in (45),

$$\begin{aligned} & |v_n(t', \pi_{(V)}(0, y; \widehat{\mathbf{x}}^{n,t}_t) - v_n(t', \pi_{(V)}(0, y'; \widehat{\mathbf{x}}^{n,t'}_{t'})| \\ & \leq K_4(1 + \|\phi\|_{C^1})|\pi_{(V)}(0, y; \widehat{\mathbf{x}}^{n,t}_t - \pi_{(V)}(0, y'; \widehat{\mathbf{x}}^{n,t'}_{t'})|. \end{aligned} \tag{47}$$

In Theorem 5, we proved that the family

$$\{[0, T] \times \mathbb{R}^e \ni (t, y) \mapsto \pi_{(V)}(0, y; \widehat{\mathbf{x}}^{n,t}_t) \in \mathbb{R}^e\}_{n \in \mathbb{N}}$$

is equicontinuous and hence we deduce from (46) and (47) that

$$\{(t, y) \mapsto v_n(t, \pi_{(V)}(0, y; \widehat{\mathbf{x}}^{n,t}_t)\}_{n \in \mathbb{N}}$$

is also equicontinuous.

Therefore we can conclude that

$$u(t, y) = v(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^t))$$

is indeed a solution of (40).

Having established existence of solutions for (40), we now prove uniqueness. However, as in the case of first order equations, this follows immediately from the pointwise convergence of

$$v_n(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^{n,t})) \rightarrow v(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^t))$$

proved in the first part of the proof.

We still have to prove the continuity of the map which sends the driving signal \mathbf{x} to the solution u . To this end, suppose that $(\mathbf{x}^n)_{n \in \mathbb{N}}$ is a sequence of weak geometric p -rough paths converging to \mathbf{x} in $\frac{1}{p}$ -Hölder topology, i.e. $d_{\frac{1}{p}\text{-Hölder}; [0, T]}(\mathbf{x}^n, \mathbf{x}) \rightarrow 0$. This implies a fortiori uniform convergence with the uniform bounds $\sup_n \|\mathbf{x}^n\|_{\frac{1}{p}\text{-Hölder}; [0, T]} < \infty$. Using the same reasoning as in the existence part of the proof, we can show that

$$v_n(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^{n,t})) \rightarrow v(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^t))$$

uniformly in $t \in [0, T]$ and $y \in \mathbb{R}^e$. Thus,

$$u_n(t, y) = v_n(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^{n,t})) \rightarrow v(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^t)) = u(t, y)$$

in $C([0, T] \times \mathbb{R}^e, \mathbb{R})$ equipped with the uniform topology. Therefore we conclude that the map which sends the driving signal to the solution is indeed continuous in the uniform topology. \square

3.3. RPDEs with $BUC(\mathbb{R}^e)$ initial functions

We consider RPDEs with initial functions which are less smooth than those used to obtain the existence and uniqueness result in the previous section.

Proposition 18. *Let $p \geq 1$ and let \mathbf{x} be a weak geometric p -rough path. Assume that*

- (1) $V = (V_1, \dots, V_d)$ is a collection of Lip^γ vector fields on \mathbb{R}^e for $\gamma > p + 3$;
- (2) $a : [0, T] \times \mathbb{R}^e \rightarrow S_e$ and $b : [0, T] \times \mathbb{R}^e \rightarrow \mathbb{R}^e$ satisfy the regularity Hypothesis 10, and furthermore, have continuous bounded first and second order spatial derivatives;
- (3) $(\phi_n)_{n \in \mathbb{N}} \subset C_b^2(\mathbb{R}^e, \mathbb{R})$ such that $\|\phi_n - \phi_m\|_{BUC(\mathbb{R}^e)} \rightarrow 0$ as $n, m \rightarrow \infty$.

Then,

$$\|\Pi_{(a,b,V)}(0, \phi_n; \mathbf{x}) - \Pi_{(a,b,V)}(0, \phi_m; \mathbf{x})\|_{BUC([0, T] \times \mathbb{R}^e)} \rightarrow 0$$

as $n, m \rightarrow \infty$.

Proof. In Theorem 17, we have shown that

$$\Pi_{(a,b,V)}(0, \phi_n; \mathbf{x})(t, y) = v_n(t, \pi_{(V)}(0, y; \overleftarrow{\mathbf{x}}^t))$$

where v_n is the unique $C_b^{1,2}$ -solution of

$$\frac{\partial v_n}{\partial t} = L_t^{\mathbf{x}} v_n, \quad v_n(0, y) = \phi_n(y).$$

Using the Feynman–Kac formula for parabolic equations which are inhomogeneous in time (cf. Theorem 2.2 in [4]), we get that

$$v_n(t, y) = \mathbb{E}(\phi_n(Y_t^{t,y}))$$

where for $t > 0$, $Y^{t,y}$ is the solution of the Itô SDE,

$$\begin{aligned} dY_s^{t,y} &= \sigma^{\mathbf{x}}(t-s, Y_s^{t,y}) dW_s + b^{\mathbf{x}}(t-s, Y_s^{t,y}) ds, \\ Y_0^{t,y} &= y \end{aligned}$$

with $a^{\mathbf{x}} = \sigma^{\mathbf{x}}(\sigma^{\mathbf{x}})^T$ and a Brownian motion (W), and thus,

$$\begin{aligned} &\sup_{(t,y) \in [0,T] \times \mathbb{R}^e} |v_n(t, \pi_{(V)}(0, y; \bar{\mathbf{x}}^t)_t) - v_m(t, \pi_{(V)}(0, y; \bar{\mathbf{x}}^t)_t)| \\ &= \sup_{(t,y) \in [0,T] \times \mathbb{R}^e} |\mathbb{E}(\phi_n(Y_t^{t, \pi_{(V)}(0, y; \bar{\mathbf{x}}^t)_t}) - \phi_m(Y_t^{t, \pi_{(V)}(0, y; \bar{\mathbf{x}}^t)_t}))| \\ &\leq \sup_{z \in \mathbb{R}^e} |\phi_n(z) - \phi_m(z)|. \end{aligned}$$

The required result follows immediately from this estimate. \square

Using the above proposition and Theorem 17, we can now prove a continuity result which implies Theorem 1.

Theorem 19. *Let $p \geq 1$, and assume that*

- (1) $(\mathbf{x}^n)_{n \in \mathbb{N}}$ is a sequence of weak geometric p -rough paths such that

$$\mathbf{x}^n \rightarrow \mathbf{x}$$

uniformly on $[0, T]$ and $\sup_n \|\mathbf{x}^n\|_{\frac{1}{p}\text{-Höf}; [0, T]} < \infty$ (this holds in particular if $d_{\frac{1}{p}\text{-Höf}}(\mathbf{x}^n, \mathbf{x}) \rightarrow 0$);

- (2) $V = (V_1, \dots, V_d)$ is a collection of Lip^γ vector fields on \mathbb{R}^e for $\gamma > p + 3$;
- (3) $a : [0, T] \times \mathbb{R}^e \rightarrow S_e$ and $b : [0, T] \times \mathbb{R}^e \rightarrow \mathbb{R}^e$ satisfy the regularity Hypothesis 10, and furthermore, have continuous bounded first and second order spatial derivatives;
- (4) $(\phi_n)_{n \in \mathbb{N}} \subset BUC(\mathbb{R}^e)$ such that $\|\phi_n - \phi\|_{BUC(\mathbb{R}^e)} \rightarrow 0$ as $n \rightarrow \infty$.

Then

$$\|\Pi_{(a,b,V)}(0, \phi_n; \mathbf{x}^n) - \Pi_{(a,b,V)}(0, \phi; \mathbf{x})\|_{BUC([0, T] \times \mathbb{R}^e)} \rightarrow 0$$

as $n, m \rightarrow \infty$.

Proof. Let $\varepsilon > 0$ be given. From the proof of Proposition 18, we know that, for $n \geq n_0(\varepsilon)$ large enough

$$\|\Pi_{(a,b,V)}(0, \phi_n; \mathbf{x}_n) - \Pi_{(a,b,V)}(0, \phi; \mathbf{x}_n)\|_{BUC([0, T] \times \mathbb{R}^e)} \leq \|\phi_n - \phi\|_{BUC(\mathbb{R}^e)} < \varepsilon.$$

On the other hand, for any k and smooth function ϕ^k

$$\begin{aligned} & \left\| \Pi_{(a,b,V)}(0, \phi; \mathbf{x}_n) - \Pi_{(a,b,V)}(0, \phi; \mathbf{x}) \right\|_{BUC([0,T] \times \mathbb{R}^e)} \\ & \leq \left\| \Pi_{(a,b,V)}(0, \phi; \mathbf{x}_n) - \Pi_{(a,b,V)}(0, \phi^k; \mathbf{x}_n) \right\|_{BUC([0,T] \times \mathbb{R}^e)} \\ & \quad + \left\| \Pi_{(a,b,V)}(0, \phi^k; \mathbf{x}_n) - \Pi_{(a,b,V)}(0, \phi^k; \mathbf{x}) \right\|_{BUC([0,T] \times \mathbb{R}^e)} \\ & \quad + \left\| \Pi_{(a,b,V)}(0, \phi^k; \mathbf{x}) - \Pi_{(a,b,V)}(0, \phi; \mathbf{x}) \right\|_{BUC([0,T] \times \mathbb{R}^e)} \end{aligned}$$

and by taking $k = k(\varepsilon)$ large enough we see that

$$\begin{aligned} & \left\| \Pi_{(a,b,V)}(0, \phi; \mathbf{x}_n) - \Pi_{(a,b,V)}(0, \phi; \mathbf{x}) \right\|_{BUC([0,T] \times \mathbb{R}^e)} \\ & \leq \varepsilon + \left\| \Pi_{(a,b,V)}(0, \phi^k; \mathbf{x}_n) - \Pi_{(a,b,V)}(0, \phi^k; \mathbf{x}) \right\|_{BUC([0,T] \times \mathbb{R}^e)} + \varepsilon \end{aligned}$$

and by taken n also larger than $n_1(k, \varepsilon) = n_1(\varepsilon)$ we use Theorem 17 to see that

$$\left\| \Pi_{(a,b,V)}(0, \phi; \mathbf{x}_n) - \Pi_{(a,b,V)}(0, \phi; \mathbf{x}) \right\|_{BUC([0,T] \times \mathbb{R}^e)} \leq 3\varepsilon$$

so that, all in all,

$$\left\| \Pi_{(a,b,V)}(0, \phi_n; \mathbf{x}^n) - \Pi_{(a,b,V)}(0, \phi; \mathbf{x}) \right\|_{BUC([0,T] \times \mathbb{R}^e)} \leq 4\varepsilon$$

and for n large enough. \square

4. Application to SPDEs

As is well known [15,27,29], Brownian motion in \mathbb{R}^d , $B = (B^1, \dots, B^d)$, can be enhanced with Lévy’s area and a.s. yields a geometric p -rough path, $p \in (2, 3)$, denoted by $\mathbf{B}(\omega) \in C^{\frac{1}{p}-H\ddot{o}l}([0, T], G^2(\mathbb{R}^d))$. In the rest of this section we assume that the elliptic operator L_t is given by

$$L_t = \frac{1}{2} a^{ij}(t, \cdot) \frac{\partial^2}{\partial y^i \partial y^j} + b^i(t, \cdot) \frac{\partial}{\partial y^i}$$

with a and b satisfying Hypothesis 10, and having bounded continuous first and second order spatial derivatives.

Proposition 20. *Let $V = (V_1, \dots, V_d)$ be a collection of Lip^γ vector fields on \mathbb{R}^e , $\gamma > 5$, and suppose that $\phi \in C_b^2(\mathbb{R}^e)$. The RPDE solution $\Pi_{(a,b,V)}(0, \phi; \mathbf{B})$ to*

$$\begin{aligned} du(t, y) &= L_t u(t, y) dt - \nabla u(t, y) \cdot V(y) d\mathbf{B}_t(\omega), \\ u(0, y) &= \phi(y) \end{aligned}$$

constructed for fixed ω in a set of full measure, gives a solution $u(t, y; \omega)$ to the Stratonovich SPDE

$$\begin{aligned} du(t, y) &= L_t u(t, y) dt - \nabla u(t, y) \cdot V(y) \circ d\mathbf{B}_t, \\ u(0, y) &= \phi(y). \end{aligned} \tag{48}$$

Proof. Let $B(n)$ denote the piecewise linear approximation to B . It is clear from Section 6.4 in [23], that the solution to

$$du(t, y) = L_t u(t, y) dt - \nabla u(t, y) \cdot V(y) dB_t(n)$$

converges, at least for fixed t, y and in probability, to the Stratonovich SPDE solution (48). At the same time, $S_2(B(n)) \rightarrow \mathbf{B}$ a.s. in $C^{\frac{1}{p}-H\delta l}([0, T], G^2(\mathbb{R}^d))$. By the continuity result for RPDEs, we see that the solution to

$$\begin{aligned} du(t, y) &= L_t u(t, y) dt - \nabla u(t, y) \cdot V(y) d\mathbf{B}_t(\omega), \\ u(0, y) &= \phi(y) \end{aligned}$$

is (a version of) the solution to the Stratonovich SPDE. \square

In the case of SDEs, if we consider different approximations to Brownian motion, the solutions of the corresponding ODEs do not always converge to the solution of the Stratonovich SDE. As shown in [21], these limits solve Stratonovich SDEs with additional drift terms. All this has been studied from the rough path theory point of view in [25] and [7]. One of the examples considered in these papers is the so-called *McShane approximation*¹² to Brownian motion in \mathbb{R}^2 . From [7,25], the step-2 signature of these approximations converge in $\frac{1}{p}$ -Hölder topology, $p > 2$, to a geometric p -rough path $\tilde{\mathbf{B}}$, which is basically Brownian motion enhanced with an area which is different from the usual Lévy area, i.e.

$$\tilde{\mathbf{B}}_t = \exp(B_t + A_t + \Gamma t)$$

where A_t is the Lévy area, and

$$\Gamma = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$$

for some c which may be $\neq 0$.

Furthermore, for $\text{Lip}^{2+\varepsilon}(\mathbb{R}^2)$ vector fields $V = (V_1, V_2)$, it is shown in [7] that y_t is a solution of

$$dy_t = V(y_t) d\tilde{\mathbf{B}}_t$$

started at $y_0 \in \mathbb{R}^e$ if and only if y_t solves

$$dy_t = V(y_t) d\mathbf{B}_t + c[V_1, V_2](y_t) dt$$

started at $y_0 \in \mathbb{R}^2$. Here \mathbf{B} is the Stratonovich Enhanced Brownian motion. Thus,

$$\pi_{(V_1, V_2)}(0, y_0; \tilde{\mathbf{B}}) = \pi_{(c[V_1, V_2], V_1, V_2)}(0, y_0; (t, \mathbf{B}))$$

where (t, \mathbf{B}) is the canonical time–space rough path associated with B . With the above in mind, we prove the following result.¹³

¹² Cf. [21, p. 392] or Section 5.7 in [23].

¹³ This proposition is related to some results on approximations for SPDEs presented by I. Gyöngy at the Institute Mittag-Leffler in 2007.

Proposition 21. Let $V = (V_1, V_2)$ be Lip^γ , $\gamma > 5$, vector fields on \mathbb{R}^2 , and suppose that $\phi \in C_b^2(\mathbb{R}^2)$. Let $B(n)$ be the McShane approximation to Brownian motion. Then the $C_b^{1,2}$ -solutions to

$$\begin{aligned} du^n(t, y) &= L_t u^n(t, y) dt - \nabla u^n(t, y) \cdot V(y) dB_t(n), \\ u^n(0, y) &= \phi(y) \end{aligned} \tag{49}$$

converge to the solution of the Stratonovich SPDE

$$\begin{aligned} dv(t, y) &= (L_t v(t, y) - \nabla v(t, y) \cdot c[V_1, V_2](y)) dt - \nabla v(t, y) \cdot V(y) \circ dB_t, \\ v(0, y) &= \phi(y). \end{aligned} \tag{50}$$

Proof. From our continuity result in Theorem 17, we know that

$$u^n(t, y) \rightarrow u(t, y)$$

uniformly on $[0, T] \times \mathbb{R}^2$, where u is the unique solution of the RPDE

$$\begin{aligned} du(t, y) &= L_t u(t, y) dt - \nabla u(t, y) \cdot V(y) d\tilde{\mathbf{B}}_t(\omega), \\ u(0, y) &= \phi(y). \end{aligned}$$

Furthermore,

$$u(t, y) = v(t, \pi_{(V)}(0, y; \tilde{\mathbf{B}}^t_t)) \tag{51}$$

where v is the unique $C_b^{1,2}$ -solution of

$$\frac{\partial v}{\partial t} = L_t^{\tilde{\mathbf{B}}} v, \quad v(0, y) = \phi(y).$$

But from the results in [7], we deduce that

$$w_t^{\tilde{\mathbf{B}}} = w_t^{\mathbf{X}}$$

where $\mathbf{X} = (t, \mathbf{B}_t)$, and hence,

$$L_t^{\tilde{\mathbf{B}}} = L_t^{\mathbf{X}}.$$

Therefore v solves

$$\frac{\partial v}{\partial t} = L_t^{\mathbf{X}} v, \quad v(0, y) = \phi(y)$$

and since

$$\pi_{(V_1, V_2)}(0, y; \tilde{\mathbf{B}}) = \pi_{(c[V_1, V_2], V_1, V_2)}(0, y; (t, \mathbf{B}))$$

we deduce that u defined in (51) solves

$$dv(t, y) = (L_t v(t, y) - \nabla v(t, y) \cdot c[V_1, V_2](y)) dt - \nabla v(t, y) \cdot V(y) dB_t,$$

$$v(0, y) = \phi(y).$$
(52)

From Proposition 20, we get that u solves the Stratonovich SPDE (50). \square

In Theorem 17, we saw that $\mathbf{x} \mapsto \Pi_{(a,b,V)}(0, \phi; \mathbf{x})$ is continuous as a map from $C^{\frac{1}{p}-H\delta l}([0, T], G^{[p]}(\mathbb{R}^d))$ into $C([0, T] \times \mathbb{R}^e, \mathbb{R})$, with uniform topology, whenever $V \in \text{Lip}^\gamma(\mathbb{R}^e)$, $\gamma > p + 3$, and $\phi \in C_b^2(\mathbb{R}^e, \mathbb{R})$.

Theorem 22 (Support). Assume $h \in C^2([0, T], \mathbb{R}^d)$ and $\delta > 0$. Then¹⁴

$$\mathbb{P}(|\Pi_{(a,b,V)}(0, \phi; \mathbf{B}) - \Pi_{(a,b,V)}(0, \phi; h)|_{\infty; [0, T] \times \mathbb{R}^e} > \delta \mid |B - h|_{\infty; [0, T]} < \varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

In particular, the topological support of the solution to the Stratonovich SPDE (48) is the closure of

$$\{\Pi_{(a,b,V)}(0, \phi; h) : h \in C^2([0, T], \mathbb{R}^d)\}$$

in uniform topology.

Proof. The conditioning statement is a direct consequence of the main result of [6] and continuity of the RPDE solution map $\Pi_{(a,b,V)}(0, \phi; \cdot)$. Since $\{|B - h|_{\infty; [0, T]} < \varepsilon\}$ has positive probability this implies

$$\{\Pi_{(a,b,V)}(0, \phi; h) : h \in C^2([0, T], \mathbb{R}^d)\} \subset \text{support}(\mathbb{P}_* \Pi_{(a,b,V)}(0, \phi; \mathbf{B})).$$

The other inclusion holds since

$$\text{support}(\mathbb{P}_* \Pi_{(a,b,V)}(0, \phi; \mathbf{B})) \subset \overline{\{\Pi_{(a,b,V)}(0, \phi; h) : h \in C^\infty([0, T], \mathbb{R}^d)\}}.$$

This follows directly from continuity of $\Pi_{(a,b,V)}(0, \phi; \cdot)$, provided we can find smooth approximations B^n to B , such that $d_{\frac{1}{p}-H\delta l; [0, T]}(S_2(B^n), \mathbf{B}) \rightarrow 0$. We know that such approximations exist from the Karhunen–Loeve expansion of Brownian motion based on the sin/cos basis of L^2 , and general results of rough path convergence of the Karhunen–Loeve expansion proved in [12]. \square

Remark 23. It is easy to see that the closure of $\{\Pi_{(a,b,V)}(0, \phi; h) : h \in C^2([0, T], \mathbb{R}^d)\}$ coincides with the closure of $\{\Pi_{(a,b,V)}(0, \phi; h) : h \in W^{1,2}([0, T], \mathbb{R}^d)\}$.

Clearly, $\Pi_{(a,b,V)}(0, \phi; \mathbf{B}(\sqrt{\varepsilon} \cdot))$ converges in distribution as $\varepsilon \rightarrow 0$ to $\Pi_{(a,b,V)}(0, \phi; 0)$, the solution of the PDE $\frac{\partial v}{\partial t} = L_t v$. The following LDP principle quantifies the rate of this convergence.

Theorem 24 (Large deviations). The family $(\mathbb{P}_* \Pi_{(a,b,V)}(0, \phi; \mathbf{B}(\sqrt{\varepsilon} \cdot)))$ satisfies a large deviation principle with good rate function

$$J(u) = \inf \left\{ \frac{1}{2} \int_0^T |\dot{h}_t|^2 dt : h \in W^{1,2}([0, T], \mathbb{R}^d) \text{ and } \Pi_{(a,b,V)}(0, \phi; h) = u \right\}.$$

¹⁴ The infinity norm of $B - h$ is based on Euclidean norm on \mathbb{R}^d .

Proof. One of the results proved in [8] says that the random variables $\mathbf{B}(\sqrt{\varepsilon}\cdot)$ satisfy a large deviation principle in $\frac{1}{p}$ -Hölder topology with good rate function

$$I(\mathbf{x}) = \frac{1}{2} \int_0^T |\dot{h}_t|^2 dt \quad \text{if } S_2(h) = \mathbf{x} \text{ for some } h \in W^{1,2}([0, T], \mathbb{R}^d)$$

$$= +\infty \quad \text{otherwise.}$$

Using the continuity of $\Pi_{(a,b,V)}(0, \phi; \cdot)$ and the contraction principle, the required large deviation principle for $(\mathbb{P}_* \Pi_{(a,b,V)}(0, \phi; \mathbf{B}(\sqrt{\varepsilon}\cdot)))$ follows immediately. \square

5. SPDEs with Markovian noise

Let X be a Markov process with uniformly elliptic generator in divergence form (cf. [33]). The coefficient matrix in the generator need not have any regularity (beyond measurability), in which case X is not a semi-martingale.¹⁵ Stochastic area cannot be defined via iterated stochastic integrals but there are alternative constructions [10,24,28] that lift X to a “Markovian” rough path $\mathbf{X} \in C^{\frac{1}{p}\text{-Hölder}}([0, T], G^2(\mathbb{R}^d))$ for any $p \in (2, 3)$. With the RPDE approach, we can then give a meaning to the SPDE¹⁶

$$du(t, y) = L_t u(t, y) dt - \nabla u(t, y) \cdot V(y) d\mathbf{X}_t,$$

$$u(0, y) = \phi(y) \tag{53}$$

which generalizes Stratonovich SPDEs to “SPDEs with (uniformly elliptic) Markovian noise”. Various convergence results proved in [10] together with our RPDE continuity result, give an appealing probabilistic meaning to such SPDE solutions. For instance, if the coefficient matrix is mollified (with parameter ε) so that X^ε is a semi-martingale, one constructs without difficulties (cf. [23]) a Stratonovich solution to

$$du^\varepsilon(t, y) = L_t u^\varepsilon(t, y) dt - \nabla u^\varepsilon(t, y) \cdot V(y) \circ dX_t^\varepsilon,$$

$$u^\varepsilon(0, y) = \phi(y)$$

and as $\varepsilon \rightarrow 0$, the solution u^ε converges in distribution to the solution of (53). Similarly, if X is replaced by a piecewise linear approximation X^n , we can solve

$$du^n(t, y) = L_t u^n(t, y) dt - \nabla u^n(t, y) \cdot V(y) dX_t^n,$$

$$u^n(0, y) = \phi(y)$$

as (time-inhomogeneous) linear second order PDE and as $n \rightarrow \infty$ we have convergence (in probability) to the solution of (53). Support and large deviation properties for Markovian rough paths were established in [10] and similar reasoning as in the Brownian case leads to support and large deviation statements for these SPDEs with Markovian noise. The details are straight-forward and omitted.

¹⁵ Nonetheless, sample paths properties of X are very similar to those of Brownian motion.
¹⁶ We assume that the coefficients a and b , together with the vector field V have enough regularity (namely the assumptions made at the beginning of Section 4) for the RPDE to have unique solutions.

6. SPDEs with Gaussian noise

Let $X = (X^1, \dots, X^d)$ be a continuous centered Gaussian process with independent components started at zero, and suppose that its covariation R^X has finite ρ -variation (in 2D-sense) with $\rho \in [1, 2)$, bounded by a Hölder dominated control.¹⁷ Then from [11], we know that for $p \in (2\rho, 4)$, X lifts to a geometric Hölder p -rough path $\mathbf{X} = \mathbf{X}(\omega)$, a “Gaussian rough path”. With Lip^γ vector fields $V = (V_1, \dots, V_d)$, $\gamma > p + 3$, and $\phi \in C_b^2(\mathbb{R}^e; \mathbb{R})$, the RPDE

$$\begin{aligned} du(t, y) &= L_t u(t, y) dt - \nabla u(t, y) \cdot V(y) d\mathbf{X}_t, \\ u(0, y) &= \phi(y) \end{aligned} \tag{54}$$

can be solved for almost every ω and has the obvious interpretation of an SPDE with Gaussian noise. (The setup of [11] includes (multi-dimensional) Brownian motion with $\rho = 1$, fractional Brownian motion with $\rho = 1/(2H)$ for $H \in (1/4, 1/2)$, the case $H > 1/2$ being trivial, the Ornstein–Uhlenbeck process, and the Brownian bridge process, among many other examples.)

There is an equivalent statement for most of what has been said in Section 5: various weak and strong approximation results make the interpretation of the solution to (54) easy. Replacing X by piecewise linear approximations X^n (or mollifier approximations X^δ) reduces (54) to a (time-inhomogeneous) linear second order PDE, and as $n \rightarrow \infty$ (resp. $\delta \rightarrow 0$), these solutions converge (in probability) to the solution of (54).

There is a support result for such Gaussian rough paths (always in the appropriate $1/p$ -Hölder rough paths topology cf. [12]) and with the continuity of $\mathbf{X} \mapsto \Pi_{(a,b,V)}(0, \phi; \mathbf{X})$, the solution map to (54), we immediately get that the support of the law of $\Pi_{(a,b,V)}(0, \phi; \mathbf{X})$, in uniform topology, is the closure of all second order PDE solutions $\Pi_{(a,b,V)}(0, \phi; h)$ where $h \in \mathcal{H}$, the Cameron–Martin space associated with X .

Remark 25. Thanks to the known embedding $\mathcal{H} \hookrightarrow C^{\rho\text{-var}}([0, T], \mathbb{R}^d)$, $\rho \in [1, 2)$ (cf. [11]), we can define $S_{[p]}(h)$, $p \in (2\rho, 4)$, using Young integration. If we denote $S_{[p]}(h)$ by \mathbf{h} , then

$$\begin{aligned} \|\mathbf{h}_{s,t}\| &\leq |\mathbf{h}|_{\rho\text{-var};[s,t]} \leq K_1 |h|_{\rho\text{-var};[s,t]} \quad (\text{cf. [26, Theorem 2.2.1]}) \\ &\leq K_1 |h|_{\mathcal{H}} \sqrt{R^X_{\rho\text{-var};[s,t]^2}} \quad (\text{cf. [11, Proposition 16]}) \\ &\leq K_2 |h|_{\mathcal{H}} |t - s|^{\frac{1}{2\rho}} \\ &\leq K_3 |t - s|^{\frac{1}{p}} \end{aligned}$$

and thus $\mathbf{h} \in C^{\frac{1}{p}\text{-Höl}}([0, T], G^{[p]}(\mathbb{R}^d))$. Therefore, when we refer to $\Pi_{(a,b,V)}(0, \phi; h)$, we are basically considering $\Pi_{(a,b,V)}(0, \phi; \mathbf{h}) = \Pi_{(a,b,V)}(0, \phi; S_{[p]}(h))$, which we know exists from Theorem 17. By a basic theorem in rough path theory [26, Theorem 1] $h \rightarrow \Pi_{(a,b,V)}(0, \phi; S_{[p]}(h))$ is continuous in ρ -variation and thus here we are dealing with a Young PDE.

There is also an LDP for $(\delta_\varepsilon \mathbf{X}; \varepsilon > 0)$ where δ_ε is the dilation operator which generalizes scalar multiplication on \mathbb{R}^d to $G^{[p]}(\mathbb{R}^d)$, $p \in (2\rho, \gamma)$ (cf. [13]). Keeping $u^\varepsilon(0, y) = \phi(y)$ for all $\varepsilon > 0$, we abuse notation and write

$$du^\varepsilon(t, y) = L_t u^\varepsilon(t, y) dt - \varepsilon \nabla u^\varepsilon(t, y) \cdot V(y) d\mathbf{X}_t$$

¹⁷ A 2D control ω is Hölder dominated if there exists a constant C such that for all $0 \leq s < t \leq T$, $\omega([s, t]^2) \leq C|t - s|$. In particular, this implies that $R^X_{\rho\text{-var};[s,t]^2} \leq C|t - s|^{\frac{1}{p}}$.

rather than

$$du^\varepsilon(t, y) = L_t u^\varepsilon(t, y) dt - \nabla u^\varepsilon(t, y) \cdot V(y) d(\delta_\varepsilon \mathbf{X})_t.$$

Then the laws of $u^\varepsilon(t, y; \omega)$ satisfy an LDP (in uniform topology) with good rate function

$$J(u) = \inf \left\{ \frac{1}{2} |h|_{\mathcal{H}}^2 : h \in \mathcal{H} \text{ and } \Pi_{(V)}(0, \phi; h) = u \right\}.$$

6.1. Density result for non-degenerate first order SPDEs with Gaussian noise

We now discuss whether the solution of the first order SPDE

$$\begin{aligned} du(t, y) + \nabla u(t, y) \cdot V(y) d\mathbf{X}_t &= 0, \\ u(0, y) &= \phi(y) \end{aligned} \tag{55}$$

at some fixed point in time-space, i.e. $\Pi_{(V)}(0, \phi; \mathbf{X}(\omega))(t, y) = \phi(\pi_{(V)}(0, y; \overline{\mathbf{X}(\omega)}^t)_t)$, admits a density with respect to Lebesgue measure. The question obviously reduces to establishing a density for $\pi_{(V)}(0, y; \overline{\mathbf{X}}^t)_t$ and then imposing the necessary non-degeneracy conditions on ϕ . The existence of a density for the solution of an RDE driven by a Gaussian signal was proved in [2] under the following assumptions on the vector fields and the driving signal.

Hypothesis 26 (Ellipticity assumption on the vector fields). The vector fields V_1, \dots, V_d span the tangent space at y .

Hypothesis 27 (Non-degeneracy of the Gaussian process on $[0, T]$). Fix $T > 0$. We assume that for any smooth $f = (f_1, \dots, f_d) : [0, T] \rightarrow \mathbb{R}^d$,

$$\left(\int_0^T f dh \equiv \sum_{k=1}^d \int_0^T f_k dh^k = 0 \ \forall h \in \mathcal{H} \right) \implies f \equiv 0$$

where \mathcal{H} is the Cameron–Martin space associated with the Gaussian process.

As remarked in the same paper, non-degeneracy on $[0, T]$ implies non-degeneracy on $[0, t]$ for any $t \in (0, T]$. We then have the following theorem.

Theorem 28. (Cf. [2].) Let \mathbf{X} be a natural lift of a continuous, centered Gaussian process with independent components $X = (X_1, \dots, X_d)$, with finite $\rho \in [1, 2)$ -variation of the covariance, bounded by a Hölder dominated control, and non-degenerate in the sense of Hypothesis 27. Let $V = (V_1, \dots, V_d)$ be a collection of Lip^γ vector fields on \mathbb{R}^e , $\gamma > 2\rho$, which satisfy the ellipticity Hypothesis 26 at $y \in \mathbb{R}^e$. Then the solution of the rough differential equation

$$dY_t = V(Y_t) d\mathbf{X}_t, \quad Y_0 = y$$

admits a density at all times $t \in (0, T]$ with respect to Lebesgue measure on \mathbb{R}^e .

Using Theorem 5 and the above, we can prove the following result on the existence of a density for the solution of an RPDE.

Theorem 29. Let \mathbf{X} be a natural lift of a continuous, centered Gaussian process with independent components $X = (X_1, \dots, X_d)$, with finite $\rho \in [1, 2)$ -variation of the covariance, bounded by a Hölder dominated control, and non-degenerate in the sense of Hypothesis 27. Let $V = (V_1, \dots, V_d)$ be a collection of $\text{Lip}^\gamma(\mathbb{R}^e)$ vector fields, $\gamma > 2\rho$, and suppose¹⁸ that $\phi \in C^1(\mathbb{R}^e; \mathbb{R})$ is non-degenerate, i.e. $\nabla\phi \neq 0$ everywhere. With $\mathbf{X} = \mathbf{X}(\omega)$, the solution $u(t, y) = u(t, y; \omega)$ to the random RPDE

$$\begin{aligned} du(t, y) + \nabla u(t, y) \cdot V(y) d\mathbf{X}_t &= 0, \\ u(0, y) &= \phi(y) \end{aligned} \tag{56}$$

has a density with respect to the Lebesgue measure, for each $t \in (0, T]$ and for each $y \in \mathbb{R}^e$ for which Hypothesis 26 holds.

Proof. Fix $t \in (0, T]$, and choose $y \in \mathbb{R}^e$ such that the vector fields V_1, \dots, V_d span the tangent space at y . We have to show that

$$u(t, y) = \phi(\pi_{(V)}(0, y; \bar{\mathbf{X}}^t))$$

has a density.

We first note that $\bar{\mathbf{X}}^t$ is again a Gaussian geometric p -rough path, $p \in (2\rho, 4)$, defined on $[0, t]$. We want to prove that $\bar{\mathbf{X}}^t$ satisfies the non-degeneracy Hypothesis 27 on $[0, t]$. Let f be a smooth function and suppose that $\int f dg \equiv 0$ for all $g \in \mathcal{G}$, the Cameron–Martin space associated with $\bar{\mathbf{X}}^t$. Recall that elements of \mathcal{G} are of the form, $g_s = \mathbb{E}(\bar{X}_s^t \xi(g))$ where $\xi(g)$ is a Gaussian random variable. For $s \in [0, t]$,

$$g_s = \mathbb{E}(\bar{X}_s^t \xi(g)) = \mathbb{E}(X_{t-s} \xi(g)) = h_{t-s}$$

for some $h \in \mathcal{H}$, the Cameron–Martin space associated with X . Thus

$$\left(\int_0^t f_s dg_s \equiv 0 \ \forall g \in \mathcal{G} \right) \Leftrightarrow \left(\int_0^t f_s dh_{t-s} \equiv 0 \ \forall h \in \mathcal{H} \right).$$

Since f is smooth, the above integrals are Riemann–Stieltjes integrals, and so, using a simple change of variable, we get that

$$\int_0^t f_s dh_{t-s} = - \int_0^t f_{t-s} dh_s \equiv 0 \quad \forall h \in \mathcal{H}.$$

But $-f_{t-\cdot}$ is of course a smooth function, and hence it follows from the non-degeneracy condition on X , that $-f_{t-\cdot} \equiv 0$. This implies that $f \equiv 0$.

Therefore the Gaussian process $\bar{\mathbf{X}}^t$ also satisfies the non-degeneracy Hypothesis 27, and hence we can deduce from our choice of $y \in \mathbb{R}^e$ and Theorem 28, that the random variable $\pi_{(V)}(0, y; \bar{\mathbf{X}}^t)_t$ has a density with respect to the Lebesgue measure on \mathbb{R}^e .

From the non-degeneracy assumption on the initial function ϕ , the existence of a density for $u(t, y)$ now follows immediately. \square

¹⁸ Cf. Remark 7.

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