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Note

Bijective mapping preserving intersecting antichains for k-valued cubes

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ABSTRACT

Generalizing a result of Miyakawa, Nozaki, Pogosyan and Rosenberg, we prove that there exists a one-to-one correspondence between the set of intersecting antichains in a subset of the lower half of the k-valued n-cube and the set of intersecting antichains in the k-valued $(n-1)$-cube.

1. Introduction

Let k and n be positive integers with $k \geq 2$, and let $E = \{0, \ldots, k-1\}$. A *k-valued n-cube* is the cartesian power E^n. Write $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ for $\boldsymbol{a}, \boldsymbol{b} \in E^n$. Write $\boldsymbol{a} \preceq \boldsymbol{b}$ if $a_i \leq b_i$ for all $i \in [n]$, where $[n] = \{1, \ldots, n\}$. We call $\mathcal{A} \subseteq E^n$ an *antichain* if there exist no different elements $\boldsymbol{a}, \boldsymbol{b}$ of $\mathcal{A}$ such that $\boldsymbol{a} \preceq \boldsymbol{b}$. A family $\mathcal{A} \subseteq E^n$ is *intersecting* if for all $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{A}$ there exists $i \in [n]$ such that $a_i + b_i \geq k$. This is a natural generalization of the binary case ($k = 2$), where the elements of E^n can be interpreted as the subsets of $[n]$ and an intersecting antichain is an antichain consisting of pairwise intersecting sets. The restriction in the definition applies also when $\boldsymbol{b} = \boldsymbol{a}$, so no $\boldsymbol{a} \in E^n$ with $a_i < \frac{k}{2}$ for all $i \in [n]$ is an element of any intersecting antichain, because then $a_i + a_i < k$ for all $i \in [n]$.

In the binary case, there exists a bijective map from the "lower half" of the n-cube onto the $(n-1)$-cube that preserves intersecting antichains in both directions [4]. Answering a question of Miyakawa [3], we present a generalization to the k-valued case. The proof is slightly simpler than that of [4] for the case $k = 2$. More information on intersecting antichains can be found in [2].

The *weight* of an element $\boldsymbol{a} \in E^n$, written $w(\boldsymbol{a})$, is defined by $w(\boldsymbol{a}) = a_1 + \cdots + a_n$. For $0 \leq t \leq n(k-1)$, the tth *level* $\mathcal{B}_t$ of E^n is $\mathcal{B}_t = \{\boldsymbol{a} \in E^n : w(\boldsymbol{a}) = t\}$.

Now we define the *lower half* L_n by restricting the first entries as follows.

Let $g = \lfloor \frac{n(k-1)}{2} \rfloor$ and notice that $g = \frac{1}{2}(nk - n - 1)$ if $n(k-1)$ is odd and $g = \frac{1}{2}n(k-1)$ otherwise. Let $\mathcal{C}_i = \{(a_1, \ldots, a_n) \in E^n : a_1 = i\}$. Let

$$L_n = \begin{cases} (\mathcal{B}_0 \cup \cdots \cup \mathcal{B}_g) \cap (\mathcal{C}_0 \cup \mathcal{C}_{k-1}) & \text{if } n(k-1) \text{ is odd,} \\ ((\mathcal{B}_0 \cup \cdots \cup \mathcal{B}_{g-1}) \cap (\mathcal{C}_0 \cup \mathcal{C}_{k-1})) \cup (\mathcal{B}_g \cap \mathcal{C}_0) & \text{otherwise.} \end{cases}$$

This set can be given also as follows: Let $g' = \lfloor \frac{n(k-1)-1}{2} \rfloor$, and notice that $g' = \frac{1}{2}(nk - n - 1) = g$ if $n(k-1)$ is odd and $g' = \frac{1}{2}n(k-1) - 1 = g - 1$ otherwise. Thus

$$L_n = \begin{cases} (\mathcal{B}_0 \cup \cdots \cup \mathcal{B}_{g'}) \cap (\mathcal{C}_0 \cup \mathcal{C}_{k-1}) & \text{if } n(k-1) \text{ is odd,} \\ ((\mathcal{B}_0 \cup \cdots \cup \mathcal{B}_{g'}) \cap (\mathcal{C}_0 \cup \mathcal{C}_{k-1})) \cup (\mathcal{B}_{g'+1} \cap \mathcal{C}_0) & \text{otherwise.} \end{cases}$$

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Hence, g is the maximum weight of the elements of L_n beginning with 0. Similarly, g' is the maximum weight of the elements of L_n beginning with $k-1$. Notice that $g+1+g' = n(k-1)$.

Furthermore, note that the notation "lower half" is slightly misleading, given the facts that $|L_n| = k^{n-1} = |E^n|/k$ and that for $k > 2$ and $n > 1$, there exist elements $\boldsymbol{a} \in L_n$ and $\boldsymbol{c} \notin L_n$ satisfying $w(\boldsymbol{a}) < w(\boldsymbol{c})$. However, we stick to the notation "lower half", mainly for the following reasons. The bounds g and g' for the maximum weight of elements in L_n are both asymptotically half of the maximum possible weight of an element of E^n. For every $\boldsymbol{a} \in L_n$ and $\boldsymbol{b} \in L_n$ with $\boldsymbol{a} \preceq \boldsymbol{b}$ and $\boldsymbol{b} \in L_n$, we obtain $\boldsymbol{a} \in L_n$. Finally, both $\mathcal{C}_0 \cap L_n$ and $\mathcal{C}_{k-1} \cap L_n$ have asymptotically half the size of E^{n-1}. Here we consider n to be growing when speaking about asymptotics.

2. A map from L_n to E^{n-1}

For $a \in E$, let $\overline{a} = k-1-a$. Define a map φ from L_n into E^{n-1} by setting

$$\varphi((a_1,\dots,a_n)) = \begin{cases} (a_2,\dots,a_n) & \text{if } a_1 = 0,\\ (\overline{a}_2,\dots,\overline{a}_n) & \text{if } a_1 = k-1.\end{cases}$$

We observe that $\overline{\overline{a}} = a$, and $a = b$ if and only if $\overline{a} = \overline{b}$. Concerning the weight w, note that

$$w(\varphi(\boldsymbol{a})) = \begin{cases} w(\boldsymbol{a}) & \text{if } a_1 = 0,\\ (k-1)(n-1) - (w(\boldsymbol{a}) - (k-1)) & \text{if } a_1 = k-1.\end{cases}$$

Lemma 1. *If $\boldsymbol{a}, \boldsymbol{b} \in L_n$ with $a_1 = 0$ and $b_1 = k-1$, then*

$$w(\varphi(\boldsymbol{a})) < w(\varphi(\boldsymbol{b})).$$

Proof. We have

$$\begin{aligned} w(\varphi(\boldsymbol{b})) &= (k-1)(n-1) - (w(\boldsymbol{b}) - (k-1)) = n(k-1) - w(\boldsymbol{b})\\ &= g+1+g' - w(\boldsymbol{b}) \geq g+1 \geq w(\boldsymbol{a}) + 1 = w(\varphi(\boldsymbol{a})) + 1\\ &> w(\varphi(\boldsymbol{a})). \quad \square \end{aligned}$$

Lemma 2. *The map φ is injective.*

Proof. Consider distinct $\boldsymbol{a}, \boldsymbol{b} \in L_n$. If $a_1 = b_1$, then we obtain immediately from the definition of φ that $\varphi(\boldsymbol{a}) \neq \varphi(\boldsymbol{b})$. If $a_1 \neq b_1$, then by symmetry we may assume $a_1 = 0$ and $b_1 = k-1$. By Lemma 1, $w(\varphi(\boldsymbol{b})) > w(\varphi(\boldsymbol{a}))$, so $\varphi(\boldsymbol{a}) \neq \varphi(\boldsymbol{b})$. □

Lemma 3. *The map φ is surjective.*

Proof. We have to show that for all $\boldsymbol{b} = (b_1,\dots,b_{n-1}) \in E^{n-1}$ there exists $\boldsymbol{a} \in L_n$ such that $\varphi(\boldsymbol{a}) = \boldsymbol{b}$. We construct this $\boldsymbol{a}$ as follows: Let

$$\boldsymbol{a} = \begin{cases} (0, b_1,\dots,b_{n-1}) & \text{if } w(\boldsymbol{b}) \leq g,\\ (k-1, \overline{b}_1,\dots,\overline{b}_{n-1}) & \text{if } w(\boldsymbol{b}) > g.\end{cases}$$

If $w(\boldsymbol{b}) \leq g$, then $w(\boldsymbol{a}) = w(\boldsymbol{b}) \leq g$. If $w(\boldsymbol{b}) > g$, then $w(\boldsymbol{a}) = k-1+((k-1)(n-1) - w(\boldsymbol{b})) < n(k-1) - g = g'+1$, so $w(\boldsymbol{a}) \leq g'$. Thus in both cases $\boldsymbol{a} \in L_n$, and $\varphi(\boldsymbol{a}) = \boldsymbol{b}$. □

Corollary 1. *The map $\varphi : L_n \to E^{n-1}$ is a bijection.*

Lemma 4. *Both φ and its inverse preserve intersecting antichains.*

Proof. Due to the definition of an intersecting antichain, it is sufficient to prove the lemma for antichains $\mathcal{A}$ with $|\mathcal{A}| \in \{1, 2\}$.

Let $\boldsymbol{a}, \boldsymbol{b} \in L_n$, and let $\{\boldsymbol{a}, \boldsymbol{b}\}$ be an intersecting antichain.

If $a_1 = b_1 = 0$, then $\{\varphi(\boldsymbol{a}), \varphi(\boldsymbol{b})\}$ is an intersecting antichain.

If $a_1 = b_1 = k-1$, then

$$\begin{aligned} w(\varphi(\boldsymbol{a})) + w(\varphi(\boldsymbol{b})) &= (k-1)(n-1) - (w(\boldsymbol{a}) - (k-1)) + (k-1)(n-1) - (w(\boldsymbol{b}) - (k-1))\\ &\geq 2n(k-1) - 2\left\lfloor \frac{n(k-1)-1}{2} \right\rfloor\\ &> (k-1)(n-1). \end{aligned}$$

Thus, there exists $i \in \{2, \ldots, n\}$ such that $\bar{a}_i + \bar{b}_i \geq k$, and hence $\{\varphi(\boldsymbol{a}), \varphi(\boldsymbol{b})\}$ is intersecting. Furthermore, if $\boldsymbol{a} = \boldsymbol{b}$, then $\{\varphi(\boldsymbol{a}), \varphi(\boldsymbol{b})\} = \{\varphi(\boldsymbol{a})\}$, and hence $\{\varphi(\boldsymbol{a}), \varphi(\boldsymbol{b})\}$ is an antichain. If $\boldsymbol{a} \neq \boldsymbol{b}$, then by the antichain property, there exist $i, j \in \{2, \ldots, n\}$ with $a_i < b_i$ and $a_j > b_j$. Thus $\bar{a}_i > \bar{b}_i$ and $\bar{a}_j < \bar{b}_j$, and hence $\{\varphi(\boldsymbol{a}), \varphi(\boldsymbol{b})\}$ is an antichain.

If $a_1 \neq b_1$, then we may assume $a_1 = 0$ and $b_1 = k - 1$. Observe that $\boldsymbol{a} \neq \boldsymbol{b}$. By Lemma 1, $w(\varphi(\boldsymbol{a})) < w(\varphi(\boldsymbol{b}))$, and thus $\varphi(\boldsymbol{a}) \not\geq \varphi(\boldsymbol{b})$. Since $\{\boldsymbol{a}, \boldsymbol{b}\}$ is intersecting, there exists $i \in \{2, \ldots, n\}$ such that $a_i + b_i \geq k$. Thus $\bar{b}_i = k - 1 - b_i < a_i$, so $\varphi(\boldsymbol{a}) \not\leq \varphi(\boldsymbol{b})$. Consequently $\{\varphi(\boldsymbol{a}), \varphi(\boldsymbol{b})\}$ is an antichain. Since $\{\boldsymbol{a}, \boldsymbol{b}\}$ is an antichain, there exists $i \in \{2, \ldots, n\}$ such that $a_i > b_i$, so $a_i + \bar{b}_i = a_i + k - 1 - b_i > k - 1$, and hence $\{\varphi(\boldsymbol{a}), \varphi(\boldsymbol{b})\}$ is intersecting.

Now let $\boldsymbol{a}, \boldsymbol{b} \in E^{n-1}$, and let $\{\boldsymbol{a}, \boldsymbol{b}\}$ be an intersecting antichain. By the proof of Lemma 3, for $\boldsymbol{b} \in E^{n-1}$,

$$\varphi^{-1}(\boldsymbol{b}) = \begin{cases} (0, b_1, \ldots, b_{n-1}) & \text{if } w(\boldsymbol{b}) \leq g, \\ (k-1, \bar{b}_1, \ldots, \bar{b}_{n-1}) & \text{if } w(\boldsymbol{b}) > g. \end{cases}$$

If $w(\boldsymbol{a}) \leq g$ and $w(\boldsymbol{b}) \leq g$, then $\{\varphi^{-1}(\boldsymbol{a}), \varphi^{-1}(\boldsymbol{b})\}$ is an intersecting antichain.

If $w(\boldsymbol{a}) > g$ and $w(\boldsymbol{b}) > g$, then the first entry of both $\varphi^{-1}(\boldsymbol{a})$ and $\varphi^{-1}(\boldsymbol{b})$ is $k - 1$, so $\{\varphi^{-1}(\boldsymbol{a}), \varphi^{-1}(\boldsymbol{b})\}$ is intersecting. Furthermore, if $\boldsymbol{a} = \boldsymbol{b}$, then $\{\varphi^{-1}(\boldsymbol{a}), \varphi^{-1}(\boldsymbol{b})\} = \{\varphi^{-1}(\boldsymbol{a})\}$, and hence $\{\varphi^{-1}(\boldsymbol{a}), \varphi^{-1}(\boldsymbol{b})\}$ is an antichain. If $\boldsymbol{a} \neq \boldsymbol{b}$, then there exist $i, j \in [n-1]$ with $a_i < b_i$ and $a_j > b_j$. Thus $\bar{a}_i > \bar{b}_i$, $\bar{a}_j < \bar{b}_j$, and hence $\{\varphi^{-1}(\boldsymbol{a}), \varphi^{-1}(\boldsymbol{b})\}$ is an antichain.

In the remaining case, we may assume $w(\boldsymbol{a}) \leq g$ and $w(\boldsymbol{b}) > g$. Observe that $\boldsymbol{a} \neq \boldsymbol{b}$. The first entry of $\varphi^{-1}(\boldsymbol{a})$ is 0, and the first entry of $\varphi^{-1}(\boldsymbol{b})$ is $k - 1$, so $\varphi^{-1}(\boldsymbol{a}) \not\geq \varphi^{-1}(\boldsymbol{b})$. Since $\{\boldsymbol{a}, \boldsymbol{b}\}$ is intersecting, there exists $i \in [n-1]$ such that $a_i + b_i \geq k$. Thus $a_i \geq k - b_i = \bar{b}_i + 1 > \bar{b}_i$, and hence $\varphi^{-1}(\boldsymbol{a}) \not\leq \varphi^{-1}(\boldsymbol{b})$. Consequently $\{\varphi^{-1}(\boldsymbol{a}), \varphi^{-1}(\boldsymbol{b})\}$ is an antichain. Since $\{\boldsymbol{a}, \boldsymbol{b}\}$ is an antichain, there exists $i \in [n-1]$ such that $a_i > b_i$; thus $a_i + \bar{b}_i = a_i + k - 1 - b_i > k - 1$, and hence $\{\varphi^{-1}(\boldsymbol{a}), \varphi^{-1}(\boldsymbol{b})\}$ is intersecting. □

From Corollary 1 and Lemma 4 we immediately obtain the main result of this note.

Theorem 1. *The map φ is bijective and preserves intersecting antichains in both directions.*

3. The maximum size of an antichain and an intersecting antichain in E^n and L_n

In this section, we look at the maximum possible size of an antichain and an intersecting antichain in E^n and L_n, giving an application of Theorem 1.

By a result of de Bruijn et al. [1], E^n is a *symmetric chain order*, meaning that it can be partitioned into *chains* (totally ordered sets), the weights of each of whose elements are consecutive and symmetric about the middle level; see also [2]. Hence E^n has the Sperner property, meaning that a maximum level is a maximum antichain. For the maximum size of an antichain in L_n, we state the following:

Theorem 2. *The set $(\mathcal{B}_{\lfloor \frac{(n-1)(k-1)}{2} \rfloor} \cap \mathcal{C}_0) \cup (\mathcal{B}_{g'} \cap \mathcal{C}_{k-1})$ is a maximum antichain in L_n.*

Proof. To show that an antichain cannot be larger than this set, let $\mathcal{A}$ be a maximum antichain in L_n. Clearly, $\mathcal{A} \cap \mathcal{C}_0$ and $\mathcal{A} \cap \mathcal{C}_{k-1}$ are antichains as well. Since $\mathcal{C}_0$ is isomorphic to E^{n-1}, $\mathcal{B}_{\lfloor \frac{(n-1)(k-1)}{2} \rfloor} \cap \mathcal{C}_0$ is a maximum antichain in $\mathcal{C}_0$. Similarly, $\mathcal{C}_{k-1}$ is isomorphic to E^{n-1} and can be partitioned into symmetric chains. Since $g' - (k-1) < \lfloor \frac{(n-1)(k-1)}{2} \rfloor$, the level $\mathcal{B}_{g'}$ is below the middle level in $\mathcal{C}_{k-1}$. Hence, we can shift each antichain in $L_n \cap \mathcal{C}_{k-1}$ to the corresponding antichain in $\mathcal{B}_{g'} \cap \mathcal{C}_{k-1}$ by replacing each element by the intersection of its chain with the level $\mathcal{B}_{g'}$. Thus, $\mathcal{B}_{g'} \cap \mathcal{C}_{k-1}$ is a maximum antichain in $\mathcal{C}_{k-1}$, involving $|\mathcal{A}| \leq |(\mathcal{B}_{\lfloor \frac{(n-1)(k-1)}{2} \rfloor} \cap \mathcal{C}_0) \cup (\mathcal{B}_{g'} \cap \mathcal{C}_{k-1})|$.

To see that this set is an antichain, we only have to show that for each $\boldsymbol{a} \in \mathcal{B}_{\lfloor \frac{(n-1)(k-1)}{2} \rfloor} \cap \mathcal{C}_0$ and $\boldsymbol{b} \in \mathcal{B}_{g'} \cap \mathcal{C}_{k-1}$, the elements are incomparable. Since $a_1 = 0 < k - 1 = b_1$, we get $\boldsymbol{a} \not\geq \boldsymbol{b}$. Since

$$\sum_{i=2}^{n} a_i = w(\boldsymbol{a}) - 0 = \left\lfloor \frac{(n-1)(k-1)}{2} \right\rfloor > g' - (k-1) = w(\boldsymbol{b}) - (k-1) = \sum_{i=2}^{n} b_i,$$

we obtain $\boldsymbol{a} \not\leq \boldsymbol{b}$. Thus our chosen set is an antichain. □

For the maximum size of an intersecting antichain in E^n, we see in [2] that it equals $|\mathcal{B}_{g+1}|$ (in the notation of [2], the maximum size of a dynamically intersecting Sperner family equals $|\mathcal{B}_{\lfloor \frac{n(k-1)+2}{2} \rfloor}|$). Using Theorem 1, we observe the following corollary:

Corollary 2. *The maximum size of an intersecting antichain in L_n is $|\mathcal{B}_{\lfloor \frac{(n-1)(k-1)+2}{2} \rfloor} \cap \mathcal{C}_0|$.*

4. Remarks

In the definition of L_n, we can replace $\mathcal{C}_0$ by $\mathcal{C}_i$ and $\mathcal{C}_{k-1}$ by $\mathcal{C}_{k-1-i}$ with $0 \le i < \frac{k-1}{2}$. We obtain

$$L_{n,i} = \begin{cases} (\mathcal{B}_0 \cup \cdots \cup \mathcal{B}_g) \cap (\mathcal{C}_i \cup \mathcal{C}_{k-1-i}) & \text{if } n(k-1) \text{ is odd,} \\ ((\mathcal{B}_0 \cup \cdots \cup \mathcal{B}_{g-1}) \cap (\mathcal{C}_i \cup \mathcal{C}_{k-1-i})) \cup (\mathcal{B}_g \cap \mathcal{C}_i) & \text{otherwise.} \end{cases}$$

The analogue on $L_{n,i}$ of the map φ on L_n also is a bijection to E^{n-1} and preserves intersecting antichains. The only place where the proof is not completely identical is the case $a_1 = b_1 = k-1-i$ in the first direction of Lemma 4. In this case, we have

$$\begin{aligned} w(\varphi(\boldsymbol{a})) + w(\varphi(\boldsymbol{b})) &= (k-1)(n-1) - (w(\boldsymbol{a}) - (k-1-i)) + (k-1)(n-1) - (w(\boldsymbol{b}) - (k-1-i)) \\ &\ge 2(n-1)(k-1) - 2\left\lfloor \frac{n(k-1)-1}{2} \right\rfloor + 2(k-1-i) \\ &> 2(k-1)(n-1) - n(k-1) + (k-1) \\ &= (k-1)(n-1). \end{aligned}$$

Furthermore, g can be replaced by $g+z$ and g' by $g'-z$ with $z \in \{0, \ldots, g'\}$ such that

$$L_n^z := ((\mathcal{B}_0 \cup \cdots \cup \mathcal{B}_{g+z}) \cap \mathcal{C}_0) \cup ((\mathcal{B}_0 \cup \cdots \cup \mathcal{B}_{g'-z}) \cap \mathcal{C}_{k-1}).$$

As in the definition in Lemma 3, for $\boldsymbol{b} \in E^{n-1}$ we have

$$\varphi^{-1}(\boldsymbol{b}) = \begin{cases} (0, b_1, \ldots, b_{n-1}) & \text{if } w(\boldsymbol{b}) \le g+z, \\ (k-1, \bar{b}_1, \ldots, \bar{b}_{n-1}) & \text{if } w(\boldsymbol{b}) > g+z. \end{cases}$$

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