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Fixed point theorems for generalized contractions in ordered metric spaces

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Abstract

The purpose of this paper is to present some fixed point results for self-generalized contractions in ordered metric spaces. Our results generalize and extend some recent results of A.C.M. Ran, M.C. Reurings [A.C.M. Ran, M.C. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2004) 1435–1443], J.J. Nieto, R. Rodríguez-López [J.J. Nieto, R. Rodríguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order 22 (2005) 223–239; J.J. Nieto, R. Rodríguez-López, Existence and uniqueness of fixed points in partially ordered sets and applications to ordinary differential equations, Acta Math. Sin. (Engl. Ser.) 23 (2007) 2205–2212], J.J. Nieto, R.L. Pouso, R. Rodríguez-López [J.J. Nieto, R.L. Pouso, R. Rodríguez-López, Fixed point theorem theorems in ordered abstract sets, Proc. Amer. Math. Soc. 135 (2007) 2505–2517], A. Petruşel, I.A. Rus [A. Petruşel, I.A. Rus, Fixed point theorems in ordered L-spaces, Proc. Amer. Math. Soc. 134 (2006) 411–418] and R.P. Agarwal, M.A. El-Gebeily, D. O'Regan [R.P. Agarwal, M.A. El-Gebeily, D. O'Regan, Generalized contractions in partially ordered metric spaces, Appl. Anal., in press]. As applications, existence and uniqueness results for Fredholm and Volterra type integral equations are given.

Keywords: Fixed point; Monotone operator; Ordered metric space; Generalized contraction; Integral equation

1. Introduction

Recently, Ran and Reurings [11] proved the following Banach–Caccioppoli type principle in ordered metric spaces.

Theorem 1.1. *(See Ran and Reurings [11].) Let X be a partially ordered set such that every pair $x, y \in X$ has a lower and an upper bound. Let d be a metric on X such that the metric space (X, d) is complete. Let $f : X \to X$ be a continuous and monotone (i.e., either decreasing or increasing) operator. Suppose that the following two assertions hold*:

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(1) *there exists* $a \in]0,1[$ *such that* $d(f(x), f(y)) \leqslant a \cdot d(x,y)$, *for each* $x, y \in X$ *with* $x \geqslant y$;
(2) *there exists* $x_0 \in X$ *such that* $x_0 \leqslant f(x_0)$ *or* $x_0 \geqslant f(x_0)$.

Then f *has an unique fixed point* $x^* \in X$, *i.e.* $f(x^*) = x^*$, *and for each* $x \in X$ *the sequence* $(f^n(x))_{n\in\mathbb{N}}$ *of successive approximations of* f *starting from* x *converges to* $x^* \in X$.

Since then several authors considered the problem of existence (and uniqueness) of a fixed point for contraction-type operators on partially ordered sets.

In 2005 J.J. Nieto and R. Rodríguez-López proved a modified variant of Theorem 1.1, by removing the continuity of f. Their result (see [7, Theorem 2.3]) is the following.

Theorem 1.2. *(See Nieto and Rodríguez-López [7].) Let* X *be a partially ordered set such that every pair* $x, y \in X$ *has a lower or an upper bound. Let* d *be a metric on* X *such that the metric space* (X, d) *is complete. Let* $f : X \to X$ *be an increasing operator. Suppose that the following three assertions hold*:

(1) *there exists* $a \in]0,1[$ *such that* $d(f(x), f(y)) \leqslant a \cdot d(x,y)$, *for each* $x, y \in X$ *with* $x \geqslant y$;
(2) *there exists* $x_0 \in X$ *such that* $x_0 \leqslant f(x_0)$;
(3) *if an increasing sequence* (x_n) *converges to* x *in* X, *then* $x_n \leqslant x$ *for all* $n \in \mathbb{N}$.

Then f *has a unique fixed point* $x^* \in X$ *and for each* $x \in X$ *the sequence* $(f^n(x))_{n\in\mathbb{N}}$ *of successive approximations of* f *starting from* x *converges to* $x^* \in X$.

Notice that, the case of decreasing operators is treated in J.J. Nieto and R. Rodríguez-López [9], where some interesting applications to ordinary differential equations with periodic boundary conditions are also given.

Also, J.J. Nieto, R.L. Pouso and R. Rodríguez-López, in a very recent paper, improve some results given by A. Petruşel and I.A. Rus in [10] in the setting of abstract L-spaces in the sense of Fréchet, see for example Theorems 3.3 and 3.5 in [8].

On the other hand, very recently, R.P. Agarwal, M.A. El-Gebeily and D. O'Regan in [1] extended Ran and Reurings result for the case of generalized φ-contractions. The main result in [1] is the following theorem.

Theorem 1.3. *(See Agarwal, El-Gebeily and O'Regan [1].) Let* X *be a partially ordered set and* d *be a metric on* X *such that the metric space* (X, d) *is complete. Let* $f : X \to X$ *be an increasing operator such that the following three assertions hold*:

(1) *there exists an increasing mapping* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\lim_{n\to+\infty} \varphi^n(t) = 0$ *for each* $t > 0$, *such that for each* $x, y \in X$ *with* $x \geqslant y$ *we have*
$$d\big(f(x), f(y)\big) \leqslant \varphi\left(\max\left\{d(x,y), d\big(x, f(x)\big), d\big(y, f(y)\big), \frac{1}{2}\big[d\big(x, f(y)\big) + d\big(y, f(x)\big)\big]\right\}\right);$$
(2) *there exists* $x_0 \in X$ *such that* $x_0 \leqslant f(x_0)$;
(3) [f *is continuous*] *or* [*if an increasing sequence* $(x_n) \subset X$ *converges to* x *in* X, *then* $x_n \leqslant x$ *for all* $n \in \mathbb{N}$].

Then f *has at least one fixed point in* X.

Finally, let us notice that, if X is a nonempty set endowed with a partial order $\leqslant$ and a metric d, some fixed point results for operators $f : (C[a,b], X) \to X$ are given in Z. Drici, F.A. McRae, J. Vasundhara Devi [2].

The purpose of this paper is to generalize and extend Theorems 1.1–1.3. Some applications to integral equations are also given.

2. Notations and basic concepts

Let $f : X \to X$ be an operator. Then $f^0 := 1_X$, $f^1 := f, \ldots, f^{n+1} = f \circ f^n$, $n \in \mathbb{N}$, denote the iterate operators of f. By $I(f)$ we will denote the set of all nonempty invariant subsets of f, i.e. $I(f) := \{Y \subset X \mid f(Y) \subseteq Y\}$.

Also, by $F_f := \{x \in X \mid x = f(x)\}$ we will denote the fixed point set of the operator f, while $A_f(x^*) := \{x \in X \mid f^n(x) \to x^*, \text{ as } n \to +\infty\}$ denotes the attractor basin of f with respect to $x^* \in X$.

Let X be a nonempty set. Denote by $\Delta(X)$ the diagonal of $X \times X$. Also, let $s(X) := \{(x_n)_{n\in N} \mid x_n \in X,\ n \in N\}$.

Let $c(X) \subset s(X)$ a subset of $s(X)$ and $\mathrm{Lim} : c(X) \to X$ an operator. By definition the triple $(X, c(X), \mathrm{Lim})$ is called an L-space (Fréchet [3]) if the following conditions are satisfied:

(i) If $x_n = x$, for all $n \in N$, then $(x_n)_{n\in N} \in c(X)$ and $\mathrm{Lim}(x_n)_{n\in N} = x$.
(ii) If $(x_n)_{n\in N} \in c(X)$ and $\mathrm{Lim}(x_n)_{n\in N} = x$, then for all subsequences, $(x_{n_i})_{i\in N}$, of $(x_n)_{n\in N}$ we have that $(x_{n_i})_{i\in N} \in c(X)$ and $\mathrm{Lim}(x_{n_i})_{i\in N} = x$.

By definition an element of $c(X)$ is a convergent sequence, $x := \mathrm{Lim}(x_n)_{n\in N}$ is the limit of this sequence and we also write $x_n \to x$ as $n \to +\infty$.

In what follow we denote an L-space by $(X, \to)$.

In this setting, if $U \subset X \times X$, then an operator $f : X \to X$ is called orbitally U-continuous (see [8]) if: $[x \in X$ and $f^{n(i)}(x) \to a \in X$, as $i \to +\infty$ and $(f^{n(i)}(x), a) \in U$ for any $i \in \mathbb{N}]$ imply $[f^{n(i)+1}(x) \to f(a)$, as $i \to +\infty]$.

Let $(X, \leqslant)$ be a partially ordered set, i.e. X is a nonempty set and $\leqslant$ is a reflexive, transitive and anti-symmetric relation on X. Denote

$$X_{\leqslant} := \big\{(x, y) \in X \times X \,\big|\, x \leqslant y \text{ or } y \leqslant x\big\}.$$

Also, if $x, y \in X$, with $x \leqslant y$, then by $[x, y]_{\leqslant}$ we will denote the ordered segment joining x and y, i.e. $[x, y]_{\leqslant} := \{z \in X \mid x \leqslant z \leqslant y\}$. In the same setting, consider $f : X \to X$. Then, $(LF)_f := \{x \in X \mid x \leqslant f(x)\}$ is the lower fixed point set of f, while $(UF)_f := \{x \in X \mid x \geqslant f(x)\}$ is the upper fixed point set of f. Also, if $f : X \to X$ and $g : Y \to Y$, then the Cartesian product of f and g is denoted by $f \times g$ and it is defined in the following way: $f \times g : X \times Y \to X \times Y$, $(f \times g)(x, y) := (f(x), g(y))$.

Definition 2.1. Let X be a nonempty set. Then, by definition $(X, \to, \leqslant)$ is an ordered L-space if and only if:

(i) $(X, \to)$ is an L-space;
(ii) $(X, \leqslant)$ is a partially ordered set;
(iii) $(x_n)_{n\in\mathbb{N}} \to x$, $(y_n)_{n\in\mathbb{N}} \to y$ and $x_n \leqslant y_n$, for each $n \in \mathbb{N} \Rightarrow x \leqslant y$.

Throughout this paper we suppose that $(X, \to, \leqslant)$ is an ordered L-space. If (X, d) is a metric space, then the convergence structure is given by the metric and the triple $(X, d, \leqslant)$ will be called an ordered metric space.

We will also consider in this paper the following assertions:

$(*)$ if $(x_n)_{n\in\mathbb{N}} \to x$, $(z_n)_{n\in\mathbb{N}} \to x$ and $x_n \leqslant y_n \leqslant z_n$, for each $n \in \mathbb{N}$, then $y_n \to x$.
$(**)$ if $(y_i)_{i\in N}$ and $(z_i)_{i\in N}$ are subsequences of $(x_n)_{n\in N}$ such that $\{y_i\colon i \in \mathbb{N}\} \cup \{z_i\colon i \in \mathbb{N}\} = \{x_n\colon n \in \mathbb{N}\}$ and $(y_i)_{i\in N}, (z_i)_{i\in N} \in c(X)$ with $\mathrm{Lim}(y_i)_{i\in N} = x$ and $\mathrm{Lim}(z_i)_{i\in N} = x$, then $(x_n)_{n\in N} \in c(X)$ and $\mathrm{Lim}(x_n)_{n\in N} = x$.

Recall now the following important abstract concept.

Definition 2.2. (See Rus [13].) Let $(X, \to)$ be an L-space. An operator $f : X \to X$ is, by definition, a Picard operator (briefly PO) if:

(i) $F_f = \{x^*\}$;
(ii) $(f^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$, for all $x \in X$.

Several classical results in fixed point theory can be easily transcribed in terms of the Picard operators, see [10,12, 14]. In I.A. Rus [13] the basic theory of Picard operators is presented.

3. Fixed point results

Our starting result is a slight modified version of the main abstract result in [8] (see Theorem 3.5) and in [10] (see Lemma 4.1). For the sake of completeness we present it here.

Lemma 3.1. *Let $(X, \to)$ be an L-space and U a symmetric subset of $X \times X$ such that $\Delta(X) \subset U$. Let $f : X \to X$ be an operator. Suppose that:*

(i) *for each $x, y \in X$ with $(x, y) \notin U$ there exists $z \in X$ such that $(x, z) \in U$ and $(y, z) \in U$;*
(ii) *there exist $x_0, x^* \in X$ such that $x_0 \in A_f(x^*)$;*
(iii) *$(x, y) \in U$ and $x \in A_f(x^*)$ implies $y \in A_f(x^*)$.*

Then $A_f(x^*) = X$.

Moreover, if

(a) *f is orbitally continuous*

or

(b) *f is orbitally U-continuous and there exists a subsequence $(f^{n_k}(x_0))_{k\in\mathbb{N}}$ of $(f^n(x_0))_{n\in\mathbb{N}}$ such that $(f^{n_k}(x_0), x^*) \in U$ for each $k \in \mathbb{N}$,*

then $F_f = \{x^*\}$ and thus f is a PO.

A natural consequence of the above result follows by choosing $U := X_{\leqslant}$.

Lemma 3.2. *(See [8, Theorem 3.3].) Let $(X, \to, \leqslant)$ be an ordered L-space and $f : X \to X$ be an operator. Suppose that*:

(i) *for each $x, y \in X$ with $(x, y) \notin X_{\leqslant}$ there exists $z \in X$ such that $(x, z) \in X_{\leqslant}$ and $(y, z) \in X_{\leqslant}$;*
(ii) *there exist $x_0, x^* \in X$ such that $x_0 \in A_f(x^*)$;*
(iii) *$(x, y) \in X_{\leqslant}$ and $x \in A_f(x^*)$ implies $y \in A_f(x^*)$;*
(iv)$_{\mathrm{a}}$ *f is orbitally continuous*

or

(iv)$_{\mathrm{b}}$ *f is orbitally $X_{\leqslant}$-continuous and there exists a subsequence $(f^{n_k}(x_0))_{k\in\mathbb{N}}$ of $(f^n(x_0))_{n\in\mathbb{N}}$ such that $(f^{n_k}(x_0), x^*) \in X_{\leqslant}$, for each $k \in \mathbb{N}$.*

Then f is a PO.

Recall that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison function if it is increasing and $\varphi^k(t) \to 0$, as $k \to +\infty$. As a consequence, we also have $\varphi(t) < t$, for each $t > 0$, $\varphi(0) = 0$ and φ is right continuous at 0. For example, $\varphi(t) = at$ (where $a \in [0, 1[$), $\varphi(t) = \frac{t}{1+t}$ and $\varphi(t) = \ln(1 + t)$, $t \in \mathbb{R}_+$, are comparison functions.

If (X, d) is a metric space, then an operator $f : X \to X$ is said to be a φ-contraction if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function and $d(f(x), f(y)) \leqslant \varphi(d(x, y))$, for all $x, y \in X$. We refer to Jachymski and Jóźwik [6] and I.A. Rus [12] for a detailed study of φ-contractions.

The first main result of this section is a fixed point theorem for a φ-contraction on an ordered complete metric space.

Theorem 3.3. *Let $(X, d, \leqslant)$ be an ordered metric space and $f : X \to X$ be an operator. We suppose that*:

(i) *for each $x, y \in X$ with $(x, y) \notin X_{\leqslant}$ there exists $c(x, y) \in X$ such that $(x, c(x, y)) \in X_{\leqslant}$ and $(y, c(x, y)) \in X_{\leqslant}$;*
(ii) *$X_{\leqslant} \in I(f \times f)$;*
(iii) *if $(x, y) \in X_{\leqslant}$ and $(y, z) \in X_{\leqslant}$, then $(x, z) \in X_{\leqslant}$;*

(iv) *there exists* $x_0 \in X$ *such that* $(x_0, f(x_0)) \in X_{\leqslant}$;
(v)$_{\mathrm{a}}$ f *is orbitally continuous*

or

(v)$_{\mathrm{b}}$ f *is orbitally* $X_{\leqslant}$*-continuous and there exists a subsequence* $(f^{n_k}(x_0))_{k\in\mathbb{N}}$ *of* $(f^n(x_0))_{n\in\mathbb{N}}$ *such that* $(f^{n_k}(x_0), x^*) \in X_{\leqslant}$ *for each* $k \in \mathbb{N}$;
(vi) *there exists a comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $d(f(x), f(y)) \leqslant \varphi(d(x, y))$*, for each* $(x, y) \in X_{\leqslant}$;
(vii) *the metric* d *is complete.*

Then f *is a PO.*

Proof. Let $x_0 \in X$ be such that $(x_0, f(x_0)) \in X_{\leqslant}$. Suppose first that $x_0 \neq f(x_0)$. Then, from (ii) we obtain

$$\big(f(x_0), f^2(x_0)\big), \big(f^2(x_0), f^3(x_0)\big), \ldots, \big(f^n(x_0), f^{n+1}(x_0)\big), \ldots \in X_{\leqslant}.$$

From (vi) we get, by induction, that $d(f^n(x_0), f^{n+1}(x_0)) \leqslant \varphi^n(d(x_0, f(x_0))$, for each $n \in \mathbb{N}$. Since $\varphi^n(d(x_0, f(x_0)) \to 0$ as $n \to +\infty$, for an arbitrary $\varepsilon > 0$ we can choose $N \in \mathbb{N}^*$ such that $d(f^n(x_0), f^{n+1}(x_0)) < \varepsilon - \varphi(\varepsilon)$, for each $n \geqslant N$. Since $(f^n(x_0), f^{n+1}(x_0)) \in X_{\leqslant}$ for all $n \in \mathbb{N}$, we have for all $n \geqslant N$ that

$$\begin{aligned} d\big(f^n(x_0), f^{n+2}(x_0)\big) &\leqslant d\big(f^n(x_0), f^{n+1}(x_0)\big) + d\big(f^{n+1}(x_0), f^{n+2}(x_0)\big) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi\big(d\big(f^n(x_0), f^{n+1}(x_0)\big)\big) \leqslant \varepsilon. \end{aligned}$$

Now since $(f^n(x_0), f^{n+2}(x_0)) \in X_{\leqslant}$ (see (iii)) we have for any $n \geqslant N$ that

$$\begin{aligned} d\big(f^n(x_0), f^{n+3}(x_0)\big) &\leqslant d\big(f^n(x_0), f^{n+1}(x_0)\big) + d\big(f^{n+1}(x_0), f^{n+3}(x_0)\big) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi\big(d\big(f^n(x_0), f^{n+2}(x_0)\big)\big) \leqslant \varepsilon. \end{aligned}$$

By induction we have

$$d\big(f^n(x_0), f^{n+k}(x_0)\big) < \varepsilon, \quad \text{for any } k \in \mathbb{N}^* \text{ and } n \geqslant N.$$

Hence $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d). From (vii) we have $(f^n(x_0))_{n\in\mathbb{N}} \to x^*$, as $n \to +\infty$.

Let $x \in X$ be arbitrarily chosen. Then:

(1) If $(x, x_0) \in X_{\leqslant}$, then $(f^n(x), f^n(x_0)) \in X_{\leqslant}$ and thus $d(f^n(x), f^n(x_0)) \leqslant \varphi^n(d(x, x_0))$, for each $n \in \mathbb{N}$. Letting $n \to +\infty$ we obtain that $(f^n(x))_{n\in\mathbb{N}} \to x^*$.

(2) If $(x, x_0) \notin X_{\leqslant}$, then, from (i), there exists $c(x, x_0) \in X$ such that $(x, c(x, x_0)) \in X_{\leqslant}$ and $(x_0, c(x, x_0)) \in X_{\leqslant}$. From the second relation, as before, we get $d(f^n(x_0), f^n(c(x, x_0))) \leqslant \varphi^n(d(x_0, c(x, x_0)))$, for each $n \in \mathbb{N}$ and hence $(f^n(c(x, x_0)))_{n\in\mathbb{N}} \to x^*$, as $n \to +\infty$. Then, using the first relation we infer that $d(f^n(x), f^n(c(x, x_0))) \leqslant \varphi^n(d(x, c(x, x_0)))$, for each $n \in \mathbb{N}$ and so, by letting again $n \to +\infty$, we conclude $(f^n(x))_{n\in\mathbb{N}} \to x^*$.

Now we will prove that $x^* \in F_f$. If (v)$_{\mathrm{a}}$ holds, then clearly $x^* \in F_f$. If we suppose that (v)$_{\mathrm{b}}$ takes place, then since $(f^{n_k}(x_0))_{k\in\mathbb{N}} \to x^*$ and $(f^{n_k}(x_0), x^*) \in X_{\leqslant}$ for all $k \in \mathbb{N}$ we obtain, from the orbitally $X_{\leqslant}$-continuity of f, that $f^{n_k+1}(x_0) \to f(x^*)$ as $k \to +\infty$. Thus $x^* = f(x^*)$. If we have $f(y) = y$ for some $y \in X$, then from above, we must have $f^n(y) \to x^*$, so $y = x^*$.

If $f(x_0) = x_0$, then x_0 plays the role of x^*. □

Remark 3.4. Equivalent representation of condition (iv) are

(iv)′ there exists $x_0 \in X$ such that $x_0 \leqslant f(x_0)$ or $x_0 \geqslant f(x_0)$;
(iv)″ $(LF)_f \cup (UF)_f \neq \emptyset$.

Remark 3.5. Condition (ii) can be replaced by each of the following assertions:

(ii)′ $f : (X, \leqslant) \to (X, \leqslant)$ is increasing;
(ii)″ $f : (X, \leqslant) \to (X, \leqslant)$ is decreasing.

However, it is easy to see that assertion (ii) in Theorem 3.3 is more general, see [10] for example.

Notice that with the above remarks and with the φ-contraction condition, Theorem 3.3 generalizes Theorem 2.1 in [1], Theorems 2.2–2.3 in [7] and Theorem 2.1 in [11].

In certain situations, the condition:

(iii) if $(x, y) \in X_{\leqslant}$ and $(y, z) \in X_{\leqslant}$ then $(x, z) \in X_{\leqslant}$,

can be removed.

For example, as a consequence of Theorem 3.3, we have the following result. For the sake of completeness, we will sketch here a direct proof of it.

Theorem 3.6. *Let $(X, d, \leqslant)$ be an ordered metric space and $f : X \to X$ be an operator. We suppose that*:

(i) *for each $x, y \in X$ with $(x, y) \notin X_{\leqslant}$ there exists $c(x, y) \in X$ such that $(x, c(x, y)) \in X_{\leqslant}$ and $(y, c(x, y)) \in X_{\leqslant}$;*
(ii) *$f : (X, \leqslant) \to (X, \leqslant)$ is increasing*;
(iii) *there exists $x_0 \in X$ such that $x_0 \leqslant f(x_0)$*;
(iv)$_{\mathrm{a}}$ *f is orbitally continuous*

or

(iv)$_{\mathrm{b}}$ *f is orbitally $X_{\leqslant}$-continuous and there exists a subsequence $(f^{n_k}(x_0))_{k\in\mathbb{N}}$ of $(f^n(x_0))_{n\in\mathbb{N}}$ such that $(f^{n_k}(x_0), x^*) \in X_{\leqslant}$ for each $k \in \mathbb{N}$,*

or

(iv)$_{\mathrm{c}}$ *if an increasing sequence (x_n) converges to x in X, then $x_n \leqslant x$ for all $n \in \mathbb{N}$*;
(v) *there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $d(f(x), f(y)) \leqslant \varphi(d(x, y))$, for each $(x, y) \in X_{\leqslant}$*;
(vi) *the metric d is complete.*

Then f is a PO.

Proof. Since $f : (X, \leqslant) \to (X, \leqslant)$ is increasing and $x_0 \leqslant f(x_0)$ we immediately have $x_0 \leqslant f(x_0) \leqslant f^2(x_0) \leqslant \cdots \leqslant f^n(x_0) \leqslant \cdots$. Hence from (v) we obtain $d(f^n(x_0), f^{n+1}(x_0)) \leqslant \varphi^n(d(x_0, f(x_0)))$, for each $n \in \mathbb{N}$. By a similar approach as in the proof of Theorem 3.3 we obtain

$$d\big(f^n(x_0), f^{n+k}(x_0)\big) < \varepsilon, \quad \text{for any } k \in \mathbb{N}^* \text{ and } n \geqslant N.$$

Hence $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d). From (vi) we have $(f^n(x_0))_{n\in\mathbb{N}} \to x^*$, as $n \to +\infty$.

Now we will prove that $x^* \in F_f$. For the cases (iii)$_{\mathrm{a}}$ and (iii)$_{\mathrm{b}}$ the conclusion follows in a similar way to Theorem 3.3. If (iii)$_{\mathrm{c}}$ takes place, then, since $(f^n(x_0))_{n\in\mathbb{N}} \to x^*$, given any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}^*$ such that for each $n \geqslant N_\epsilon$ we have $d(f^n(x_0), x^*) < \epsilon$. On the other hand, for each $n \geqslant N_\epsilon$, since $f^n(x_0) \leqslant x^*$, we get

$$d\big(x^*, f\big(x^*\big)\big) \leqslant d\big(x^*, f^{n+1}(x_0)\big) + d\big(f\big(f^n(x_0)\big), f\big(x^*\big)\big) \leqslant d\big(x^*, f^{n+1}(x_0)\big) + \varphi\big(d\big(f^n(x_0), x^*\big)\big) < 2\epsilon.$$

Thus $x^* \in F_f$.

The uniqueness of the fixed point follows by contradiction. Suppose there exists $y^* \in F_f$, with $x^* \neq y^*$. There are two possible cases:

(a) if $(x^*, y^*) \in X_{\leqslant}$, then $0 < d(y^*, x^*) = d(f^n(y^*), f^n(x^*)) \leqslant \varphi^n(d(y^*, x^*)) \to 0$ as $n \to +\infty$, which is a contradiction. Hence $x^* = y^*$;

(b) if $(x^*, y^*) \notin X_{\leqslant}$, then there exists $c^* \in X$ such that $(x^*, c^*) \in X_{\leqslant}$ and $(y^*, c^*) \in X_{\leqslant}$. The monotonicity condition implies that $f^n(x^*)$ and $f^n(c^*)$ are comparable, as well as, $f^n(c^*)$ and $f^n(y^*)$. Hence

$$\begin{aligned} 0 < d\big(y^*, x^*\big) &= d\big(f^n\big(y^*\big), f^n\big(x^*\big)\big) \leqslant d\big(f^n\big(y^*\big), f^n\big(c^*\big)\big) + d\big(f^n\big(c^*\big), f^n\big(x^*\big)\big) \\ &\leqslant \varphi^n\big(d\big(y^*, c^*\big)\big) + \varphi^n\big(d\big(c^*, x^*\big)\big) \to 0 \end{aligned}$$

as $n \to +\infty$, which is again a contradiction. Thus $x^* = y^*$. □

Remark 3.7. It is easy to see that a dual result to Theorem 3.6 can be proved. More precisely, Theorem 3.6 holds if we replace condition (iii) by

(iii)′ there exists $x_0 \in X$ such that $x_0 \geqslant f(x_0)$;

and condition $(\mathrm{iv})_{\mathrm{c}}$ by

$(\mathrm{iv})'_{\mathrm{c}}$ if a decreasing sequence (x_n) converges to x in X, then $x_n \geqslant x$ for all $n \in \mathbb{N}$.

Remark 3.8. Other results of the above type can be obtained by putting instead of a complete ordered metric space one of the following ordered L-structures (see also [4,5,8,10,14]):

(a) $(X, d, \leqslant)$ an ordered complete generalized metric space (i.e., $d(x, y) \in \mathbb{R}^n_+$);
(b) $(X, \mathcal{F}, T)$ a complete Menger space.

Another result of this type is:

Theorem 3.9. *Let $(X, \to, \leqslant)$ be an ordered L-space such that $(X, \to, \leqslant)$ satisfy the condition $(*)$ in Section 2 and $f : X \to X$ be an operator. We suppose that*:

(i) *for each $x, y \in X$ with $(x, y) \notin X_{\leqslant}$ there exist $m(x, y), M(x, y) \in X$ such that $x, y \in [m(x, y), M(x, y)]_{\leqslant}$;*
(ii) *[f is increasing] or [f is decreasing and $(X, \to, \leqslant)$ has the property $(**)$ in Section 2];*
(iii) *there exist $x_0, x^* \in X$ such that $x_0 \in A_f(x^*)$;*
$(\mathrm{iv})_{\mathrm{a}}$ *f is orbitally continuous*

or

$(\mathrm{iv})_{\mathrm{b}}$ *f is orbitally $X_{\leqslant}$-continuous and there exists a subsequence $(f^{n_k}(x_0))_{k\in\mathbb{N}}$ of $(f^n(x_0))_{n\in\mathbb{N}}$ such that $(f^{n_k}(x_0), x^*) \in X_{\leqslant}$ for each $k \in \mathbb{N}$;*
(v) *if $(x, x_0) \in X_{\leqslant}$, then $x \in A_f(x^*)$.*

Then f is a PO.

Proof. From (iii) and (iv) we have that $x^* \in F_f$.

Let $x \in X$ be arbitrarily chosen.

(1) If $(x, x_0) \in X_{\leqslant}$, then from (v) we obtain $(f^n(x))_{n\in\mathbb{N}} \to x^*$, as $n \to +\infty$.

(2) If $(x, x_0) \notin X_{\leqslant}$, then by (i) we have that $x, x_0 \in [m(x, x_0), M(x, x_0)]_{\leqslant}$. Since $x_0 \in [m(x, x_0), M(x, x_0)]_{\leqslant}$ and taking into account (v) it follows that

$$\big(f^n\big(m(x, x_0)\big)\big)_{n\in\mathbb{N}} \to x^* \quad \text{and} \quad \big(f^n\big(M(x, x_0)\big)\big)_{n\in\mathbb{N}} \to x^*, \quad \text{as } n \to +\infty.$$

If f is increasing, then from $m(x, x_0) \leqslant x \leqslant M(x, x_0)$ and hypothesis $(*)$ we obtain $(f^n(x))_{n\in\mathbb{N}} \to x^*$, as $n \to +\infty$. If f is decreasing, then $m(x, x_0) \leqslant x \leqslant M(x, x_0)$ implies that $f^{2k}(x) \to x^*$ and $f^{2k+1}(x) \to x^*$, as $k \to +\infty$. From $(**)$ we get that $f^n(x) \to x^*$, as $n \to +\infty$. Hence, f is a PO. □

A consequence of the above theorem is:

Theorem 3.10. *Let $(X, d, \leqslant)$ be an ordered metric space satisfying the condition $(*)$ in Section 2 and $f : X \to X$ be an operator. We suppose that*:

(i) *for each $x, y \in X$ with $(x, y) \notin X_{\leqslant}$ there exist $m(x, y), M(x, y) \in X$ such that $x, y \in [m(x, y), M(x, y)]_{\leqslant}$;*
(ii) *if $(x, y) \in X_{\leqslant}$ and $(y, z) \in X_{\leqslant}$, then $(x, z) \in X_{\leqslant}$;*
(iii) *f is increasing or decreasing;*
(iv) *there exists $x_0 \in X$ such that $(x_0, f(x_0)) \in X_{\leqslant}$;*
$(\mathrm{v})_{\mathrm{a}}$ *f is orbitally continuous*

or

(v)$_b$ *f is orbitally $X_{\leqslant}$-continuous and there exists a subsequence $(f^{n_k}(x_0))_{k\in\mathbb{N}}$ of $(f^n(x_0))_{n\in\mathbb{N}}$ such that, if $f^{n_k}(x_0) \to x^*$ as $k \to \infty$, then $(f^{n_k}(x_0), x^*) \in X_{\leqslant}$ for each $k \in \mathbb{N}$;*

(vi) *the metric d is complete;*

(vii) *there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $d(f(x), f(y)) \leqslant \varphi(d(x, y))$, for each $(x, y) \in X_{\leqslant}$.*

Then $f : (X, d) \to (X, d)$ is a PO.

Proof. Let $x_0 \in X$ be such that $(x_0, f(x_0)) \in X_{\leqslant}$. Then from (iii) it follows $(f(x_0), f^2(x_0)), (f^2(x_0), f^3(x_0)), \dots, (f^n(x_0), f^{n+1}(x_0)), \dots \in X_{\leqslant}$. From (vii) we get that $d(f^n(x_0), f^{n+1}(x_0)) \leqslant \varphi^n(d(x_0, f(x_0)))$, for each $n \in \mathbb{N}$. As in the proof of Theorem 3.3, we obtain that $(f^n(x_0))_{n\in\mathbb{N}} \to x^*$, as $n \to +\infty$.

Let $x \in X$ be arbitrary. Then:

(1) If $(x, x_0) \in X_{\leqslant}$, then $(f^n(x), f^n(x_0)) \in X_{\leqslant}$ and so $d(f^n(x), f^n(x_0)) \leqslant \varphi^n(d(x, x_0))$, for each $n \in \mathbb{N}$. Letting $n \to +\infty$ we obtain that $(f^n(x))_{n\in\mathbb{N}} \to x^*$.

(2) If $(x, x_0) \notin X_{\leqslant}$, then, from (i), there exist $m(x, x_0), M(x, x_0) \in X$ such that $x, x_0 \in [m(x, x_0), M(x, x_0)]_{\leqslant}$. From $m(x, x_0) \leqslant x_0 \leqslant M(x, x_0)$ we get that $(f^n(m(x, x_0)))_{n\in\mathbb{N}} \to x^*$ and $(f^n(M(x, x_0)))_{n\in\mathbb{N}} \to x^*$, as $n \to +\infty$. From the relation $m(x, x_0) \leqslant x \leqslant M(x, x_0)$, condition (iii) and the above convergence we infer that $(f^n(x))_{n\in\mathbb{N}} \to x^*$. The rest of the proof, namely the fact $x^* \in F_f$, runs identically as before. □

For the case of a generalized φ-contraction an existence result for the fixed point can also be established.

Theorem 3.11. *Let $(X, d, \leqslant)$ be an ordered metric space and $f : X \to X$ be an operator. We suppose that*:

(i) *$X_{\leqslant} \in I(f \times f)$;*

(ii) *if $(x, y) \in X_{\leqslant}$ and $(y, z) \in X_{\leqslant}$, then $(x, z) \in X_{\leqslant}$;*

(iii) *there exists $x_0 \in X$ such that $(x_0, f(x_0)) \in X_{\leqslant}$;*

(iv)$_a$ *f is orbitally continuous*

or

(iv)$_b$ *f is orbitally $X_{\leqslant}$-continuous and there exists a subsequence $(f^{n_k}(x_0))_{k\in\mathbb{N}}$ of $(f^n(x_0))_{n\in\mathbb{N}}$ such that $(f^{n_k}(x_0), x^*) \in X_{\leqslant}$ for each $k \in \mathbb{N}$;*

(v) *there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$d\big(f(x), f(y)\big) \leqslant \varphi\left(\max\left\{d(x, y), d\big(x, f(x)\big), d\big(y, f(y)\big), \frac{1}{2}\big[d\big(x, f(y)\big) + d\big(y, f(x)\big)\big]\right\}\right),$$

for each $(x, y) \in X_{\leqslant}$;

(vi) *the metric d is complete.*

Then $F_f \neq \emptyset$.

Proof. Let $x_0 \in X$ be such that $(x_0, f(x_0)) \in X_{\leqslant}$. Suppose first that $x_0 \neq f(x_0)$. Then, from (i) we obtain

$$\big(f(x_0), f^2(x_0)\big), \big(f^2(x_0), f^3(x_0)\big), \dots, \big(f^n(x_0), f^{n+1}(x_0)\big), \dots \in X_{\leqslant}.$$

We claim that

$$(***) \qquad d\big(f^n(x_0), f^{n+1}(x_0)\big) \leqslant \varphi\big(d\big(f^{n-1}(x_0), f^n(x_0)\big)\big), \quad \text{for each } n \in \mathbb{N}.$$

To see $(***)$ we consider

$$d\big(f^n(x_0), f^{n+1}(x_0)\big) \leqslant \varphi\left(\max\left\{d\big(f^{n-1}(x_0), f^n(x_0)\big), d\big(f^n(x_0), f^{n+1}(x_0)\big), d\big(f^n(x_0), f^{n-1}(x_0)\big),\right.\right.$$
$$\left.\left.\frac{1}{2}\big[d\big(f^n(x_0), f^n(x_0)\big) + d\big(f^{n-1}(x_0), f^{n+1}(x_0)\big)\big]\right\}\right) \leqslant \varphi(M_n),$$

where

$$M_n := \max\left\{d\big(f^{n-1}(x_0), f^n(x_0)\big), d\big(f^n(x_0), f^{n+1}(x_0)\big), \frac{1}{2}\big[d\big(f^{n-1}(x_0), f^n(x_0)\big) + d\big(f^n(x_0), f^{n+1}(x_0)\big)\big]\right\}.$$

(1) If $M_n = d(f^{n-1}(x_0), f^n(x_0))$ we are done.

(2) If $M_n = d(f^n(x_0), f^{n+1}(x_0))$, then $d(f^n(x_0), f^{n+1}(x_0)) = 0$. Since if not, then

$$d\big(f^n(x_0), f^{n+1}(x_0)\big) \leqslant \varphi\big(d\big(f^n(x_0), f^{n+1}(x_0)\big)\big) < d\big(f^n(x_0), f^{n+1}(x_0)\big),$$

which is a contradiction. Thus $(**)$ follows again.

(3) If $M_n = \frac{1}{2}[d(f^{n-1}(x_0), f^n(x_0)) + d(f^n(x_0), f^{n+1}(x_0))]$, then if $M_n = 0$ we have that $d(f^n(x_0), f^{n+1}(x_0)) = 0$ and $(**)$ holds. If $M_n \neq 0$, then

$$\begin{aligned} d\big(f^n(x_0), f^{n+1}(x_0)\big) &\leqslant \varphi\left(\frac{1}{2}\big[d\big(f^{n-1}(x_0), f^n(x_0)\big) + d\big(f^n(x_0), f^{n+1}(x_0)\big)\big]\right) \\ &< \frac{1}{2}\big[d\big(f^{n-1}(x_0), f^n(x_0)\big) + d\big(f^n(x_0), f^{n+1}(x_0)\big)\big]. \end{aligned}$$

Hence $d(f^n(x_0), f^{n+1}(x_0)) < d(f^{n-1}(x_0), f^n(x_0))$. In this case

$$M_n = \frac{1}{2}\big[d\big(f^{n-1}(x_0), f^n(x_0)\big) + d\big(f^n(x_0), f^{n+1}(x_0)\big)\big] < d\big(f^{n-1}(x_0), f^n(x_0)\big),$$

which contradicts the definition of M_n.

Thus in all cases $(***)$ holds.

From $(***)$ we immediately have

$$d\big(f^n(x_0), f^{n+1}(x_0)\big) \leqslant \varphi^n\big(d\big(x_0, f(x_0)\big)\big), \quad \text{for each } n \in \mathbb{N}.$$

Since $\varphi^n(d(x_0, f(x_0)) \to 0$ as $n \to +\infty$, for an arbitrary $\varepsilon > 0$ we can choose $N \in \mathbb{N}^*$ such that

$$d\big(f^n(x_0), f^{n+1}(x_0)\big) < \varepsilon - \varphi(\varepsilon), \quad \text{for each } n \geqslant N.$$

As in the proof of Theorem 3.3 we have first that

$$\begin{aligned} d\big(f^n(x_0), f^{n+2}(x_0)\big) &\leqslant d\big(f^n(x_0), f^{n+1}(x_0)\big) + d\big(f^{n+1}(x_0), f^{n+2}(x_0)\big) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi\big(d\big(f^n(x_0), f^{n+1}(x_0)\big)\big) \leqslant \varepsilon. \end{aligned}$$

Now since $(f^n(x_0), f^{n+2}(x_0)) \in X_{\leqslant}$ (see (ii)) we have for any $n \geqslant N$ that

$$\begin{aligned} &d\big(f^n(x_0), f^{n+3}(x_0)\big) \\ &\quad \leqslant d\big(f^n(x_0), f^{n+1}(x_0)\big) + d\big(f^{n+1}(x_0), f^{n+3}(x_0)\big) \\ &\quad < \varepsilon - \varphi(\varepsilon) + \varphi\Bigg(\max\Bigg\{d\big(f^n(x_0), f^{n+2}(x_0)\big), d\big(f^n(x_0), f^{n+1}(x_0)\big), d\big(f^{n+2}(x_0), f^{n+3}(x_0)\big), \\ &\qquad \frac{1}{2}\big[d\big(f^{n+1}(x_0), f^{n+2}(x_0)\big) + d\big(f^n(x_0), f^{n+3}(x_0)\big)\big]\Bigg\}\Bigg) \\ &\quad < \varepsilon - \varphi(\varepsilon) + \varphi\left(\max\left\{\varepsilon, \varepsilon - \varphi(\varepsilon), \varphi^2(\varepsilon), \frac{1}{2}\big[\varphi(\varepsilon) + d\big(f^n(x_0), f^{n+3}(x_0)\big)\big]\right\}\right) \leqslant \varepsilon - \varphi(\varepsilon) + \varphi(S_n), \end{aligned}$$

where

$$S_n := \max\left\{\varepsilon, \frac{1}{2}\big[\varphi(\varepsilon) + d\big(f^n(x_0), f^{n+3}(x_0)\big)\big]\right\}.$$

We will prove that $S_n = \varepsilon$. If not, then $S_n = \frac{1}{2}[\varphi(\varepsilon) + d(f^n(x_0), f^{n+3}(x_0))]$. Since $S_n > 0$ we have

$$d\big(f^n(x_0), f^{n+3}(x_0)\big) < \varepsilon - \varphi(\varepsilon) + \frac{1}{2}\big[\varphi(\varepsilon) + d\big(f^n(x_0), f^{n+3}(x_0)\big)\big]$$

and thus

$$d\big(f^n(x_0), f^{n+3}(x_0)\big) < 2\big[\varepsilon - \varphi(\varepsilon)\big] + \varphi(\varepsilon).$$

As a consequence $S_n < \frac{1}{2}\varphi(\varepsilon) + [\varepsilon - \varphi(\varepsilon)] + \frac{1}{2}\varphi(\varepsilon) = \varepsilon$, which contradicts the definition of S_n.

Hence $S_n = \varepsilon$ and thus $d(f^n(x_0), f^{n+3}(x_0)) < \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) = \varepsilon$.

Next, by induction, we obtain that $d(f^n(x_0), f^{n+k}(x_0)) < \varepsilon$, for any $k \in \mathbb{N}^*$ and $n \geqslant N$.

Hence $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d). From (vi) we have $(f^n(x_0))_{n\in\mathbb{N}} \to x^*$, as $n \to +\infty$.

Now we prove that $x^* \in F_f$. If $(\mathrm{iv})_\mathrm{a}$ holds, then clearly $x^* \in F_f$. If we suppose that $(\mathrm{iv})_\mathrm{b}$ takes place, then since $(f^{n_k}(x_0))_{n\in\mathbb{N}} \to x^*$ and $(f^{n_k}(x_0), x^*) \in X_\leqslant$ for all $k \in \mathbb{N}$ we obtain, from the orbitally $X_\leqslant$-continuity of f, that $f^{n_k+1}(x_0) \to f(x^*)$ as $k \to +\infty$. Thus $x^* = f(x^*)$.

If $f(x_0) = x_0$, then x_0 is a fixed point. □

4. Applications

Consider the integral equations

$$x(t) = \int_a^b K\big(t, s, x(s)\big)\, ds + g(t), \quad t \in [a, b], \tag{1}$$

and

$$x(t) = \int_a^t K\big(t, s, x(s)\big)\, ds + g(t), \quad t \in [a, b]. \tag{2}$$

The purpose of this section is to give existence results for Eqs. (1) and (2) using Theorem 3.6.

Theorem 4.1. *Consider Eq.* (1). *Suppose*

(i) $K : [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $g : [a, b] \to \mathbb{R}^n$ *are continuous*;
(ii) $K(t, s, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ *is increasing for each* $t, s \in [a, b]$;
(iii) *there exist a continuous function* $p : [a, b] \times [a, b] \to \mathbb{R}_+$ *and a comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, *such that*

$$\big|K(t, s, u) - K(t, s, v)\big| \leqslant p(t, s)\varphi\big(|u - v|\big), \quad \textit{for each } t, s \in [a, b],\ u, v \in \mathbb{R}^n,\ u \leqslant v;$$

(iv) $\sup_{t\in[a,b]} \int_a^b p(t, s)\, ds \leqslant 1$;
(v) *there exists* $x_0 \in C([a, b], \mathbb{R}^n)$ *such that* $x_0(t) \leqslant \int_a^b K(t, s, x_0(s))\, ds + g(t)$, *for any* $t \in [a, b]$.

Then the integral equation (1) *has a unique solution* x^* *in* $C([a, b], \mathbb{R}^n)$.

Proof. Let $X := C([a, b], \mathbb{R}^n)$ with the usual supremum norm, i.e., $\|x\| := \max_{t\in[a,b]} |x(t)|$, for $x \in C([a, b], \mathbb{R}^n)$. Consider on X the partial order defined by

$$x, y \in C\big([a, b], \mathbb{R}^n\big), \quad x \leqslant y \quad \text{if and only if} \quad x(t) \leqslant y(t) \quad \text{for any } t \in [a, b].$$

Then $(X, \|\cdot\|, \leqslant)$ is an ordered and complete metric space. Moreover for any increasing sequence $(x_n)_{n\in\mathbb{N}}$ in X converging to a certain $x^* \in X$ we have $x_n(t) \leqslant x^*(t)$, for any $t \in [a, b]$. Also, for every $x, y \in X$ there exists $c(x, y) \in X$ which is comparable to x and y.

Define $A : C([a, b], \mathbb{R}^n) \to C([a, b], \mathbb{R}^n)$, by the formula

$$Ax(t) := \int_a^b K\big(t, s, x(s)\big)\, ds + g(t), \quad t \in [a, b].$$

First observe that from (ii) A is increasing. Also, for each $x, y \in X$ with $x \leqslant y$ we have

$$\begin{aligned}\big|Ax(t) - Ay(t)\big| &\leqslant \int_a^b \big|K\big(t,s,x(s)\big) - K\big(t,s,y(s)\big)\big|\,ds \leqslant \int_a^b p(t,s)\varphi\big(\big|x(s)-y(s)\big|\big)\,ds \\ &\leqslant \varphi\big(\|x-y\|\big)\cdot\int_a^b p(t,s)\,ds \leqslant \varphi\big(\|x-y\|\big), \quad \text{for any } t \in [a,b].\end{aligned}$$

Hence $\|Ax - Ay\| \leqslant \varphi(\|x-y\|)$, for each $x, y \in X$ with $x \leqslant y$.

From (v) we have that $x_0 \leqslant Ax_0$.

The conclusion follows now from Theorem 3.6. □

Theorem 4.2. *Consider Eq.* (2). *Suppose*

(i) $K : [a,b] \times [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $g : [a,b] \to \mathbb{R}^n$ *are continuous*;
(ii) $K(t,s,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ *is increasing for each* $t, s \in [a,b]$;
(iii) *there exists a comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\varphi(\lambda t) \leqslant \lambda\varphi(t)$, *for each* $t \in \mathbb{R}_+$ *and each* $\lambda \geqslant 1$, *such that*

$$\big|K(t,s,u) - K(t,s,v)\big| \leqslant \varphi\big(|u-v|\big), \quad \textit{for each } t,s \in [a,b],\ u,v \in \mathbb{R}^n,\ u \leqslant v;$$

(iv) *there exists* $x_0 \in C([a,b],\mathbb{R}^n)$ *such that* $x_0(t) \leqslant \int_a^t K(t,s,x_0(s))\,ds + g(t)$, *for any* $t \in [a,b]$.

Then the integral equation (2) *has a unique solution* x^* *in* $C([a,b],\mathbb{R}^n)$.

Proof. Let $X := C([a,b],\mathbb{R}^n)$ be endowed with a Bielecki-type norm, i.e., $\|x\|_B := \max_{t\in[a,b]}(|x(t)| \cdot e^{-\tau(t-a)})$, for $x \in C([a,b],\mathbb{R}^n)$ (where $\tau > 0$ is arbitrarily chosen). Consider on X the same partial order defined before (see the proof of Theorem 4.1).

Then $(X, \|\cdot\|_B, \leqslant)$ is an ordered and complete metric space. Moreover for any increasing sequence $(x_n)_{n\in\mathbb{N}}$ in X converging to a certain $x^* \in X$ we have $x_n(t) \leqslant x^*(t)$, for any $t \in [a,b]$. Also, for every $x, y \in X$ there exists $c(x,y) \in X$ which is comparable to x and y.

Define $A : C([a,b],\mathbb{R}^n) \to C([a,b],\mathbb{R}^n)$, by the formula

$$Ax(t) := \int_a^t K\big(t,s,x(s)\big)\,ds + g(t), \quad t \in [a,b].$$

From (ii) we have that A is increasing. Also, for each $x, y \in X$ with $x \leqslant y$ we have

$$\begin{aligned}\big|Ax(t) - Ay(t)\big| &\leqslant \int_a^t \big|K\big(t,s,x(s)\big) - K\big(t,s,y(s)\big)\big|\,ds \leqslant \int_a^t \varphi\big(\big|x(s)-y(s)\big|\big)\,ds \\ &= \int_a^t \varphi\big(\big|x(s)-y(s)\big|e^{-\tau(s-a)}e^{\tau(s-a)}\big)\,ds \leqslant \int_a^t e^{\tau(s-a)}\varphi\big(\big|x(s)-y(s)\big|e^{-\tau(s-a)}\big)\,ds \\ &\leqslant \varphi\big(\|x-y\|_B\big)\int_a^t e^{\tau(s-a)}\,ds \leqslant \frac{1}{\tau}\varphi\big(\|x-y\|_B\big)e^{\tau(t-a)}, \quad \text{for any } t \in [a,b].\end{aligned}$$

Hence, for $\tau \geqslant 1$ we obtain $\|Ax - Ay\|_B \leqslant \varphi(\|x-y\|_B)$, for each $x, y \in X$ with $x \leqslant y$.

From (iv) we have that $x_0 \leqslant Ax_0$.

The conclusion follows now from Theorem 3.6. □

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References

[1] R.P. Agarwal, M.A. El-Gebeily, D. O'Regan, Generalized contractions in partially ordered metric spaces, Appl. Anal., in press.
[2] Z. Drici, F.A. McRae, J. Vasundhara Devi, Fixed point theorems in partially ordered metric spaces for operators with PPF dependence, Nonlinear Anal. 7 (2007) 641–647.
[3] M. Fréchet, Les espaces abstraits, Gauthier–Villars, Paris, 1928.
[4] O. Hadžić, E. Pap, V. Radu, Generalized contraction mapping principles in probabilistic metric spaces, Acta Math. Hungar. 101 (2003) 131–138.
[5] O. Hadžić, E. Pap, Fixed Point Theory in Probabilistic Metric Spaces, Kluwer Acad. Publ., Dordrecht, 2001.
[6] J. Jachymski, I. Jóźwik, Nonlinear contractive conditions: A comparison and related problems, Banach Center Publ. 77 (2007) 123–146.
[7] J.J. Nieto, R. Rodríguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order 22 (2005) 223–239.
[8] J.J. Nieto, R.L. Pouso, R. Rodríguez-López, Fixed point theorem theorems in ordered abstract sets, Proc. Amer. Math. Soc. 135 (2007) 2505–2517.
[9] J.J. Nieto, R. Rodríguez-López, Existence and uniqueness of fixed points in partially ordered sets and applications to ordinary differential equations, Acta Math. Sin. (Engl. Ser.) 23 (2007) 2205–2212.
[10] A. Petruşel, I.A. Rus, Fixed point theorems in ordered L-spaces, Proc. Amer. Math. Soc. 134 (2006) 411–418.
[11] A.C.M. Ran, M.C. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2004) 1435–1443.
[12] I.A. Rus, Generalized Contractions and Applications, Cluj Univ. Press, 2001.
[13] I.A. Rus, Picard operators and applications, Sci. Math. Jpn. 58 (2003) 191–219.
[14] I.A. Rus, A. Petruşel, G. Petruşel, Fixed Point Theory 1950–2000: Romanian Contributions, House of the Book of Science, Cluj-Napoca, 2002.