Infinitely many solutions to perturbed elliptic equations

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Abstract

A new version of perturbation theory is developed which produces infinitely many sign-changing critical points for uneven functionals. The abstract result is applied to the following elliptic equations with a Hardy potential and a perturbation from symmetry:

\[-\Delta u - \frac{\mu}{|x|^2} u = f(x, u) + p(x, u) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega\]

and

\[-\Delta u = \frac{|u|^q - 2}{|x|^q} u + p(x, u) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega,\]

where $0 < s < 2$, $\Omega$ is a smooth bounded domain of $\mathbb{R}^n$, and $p(x, u)$ is not odd in $u$. Infinitely many sign-changing solutions are obtained.

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1. Introduction

Let $E$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $I \in C^1(E, \mathbb{R})$. It is well known that if $I$ is even, i.e., $I(-u) = I(u)$, the Symmetric Mountain Pass Theorem of Ambrosetti–Rabinowitz (cf., e.g., [R1,R2,S1]) (and its variants) provides a powerful tool for studying the existence of infinitely many critical points of $I$. In this theory, since the “genus” and “degree” of an odd operator can be applied, the evenness of $I$ (i.e., oddness of $\nabla I$) plays a crucial role. A long standing question is whether the symmetry (evenness) of the functional is necessary for the existence of infinitely many critical points. Since 1980s, some mathematicians had been working on this problem for elliptic equations. They include Bahri–Brezis [BB], Bahri–Lions [BL], Struwe [S1], Bahri [Ba], Rabinowitz [R1,R2], Tanaka [Ta], Tehrani [T], etc. (More details will be mentioned in Section 3.) These papers considered the equations which have an odd principal term, and a non-odd perturbation of the principal term. In such cases, the existence of infinitely many solutions to the equation was obtained. Therefore, some partial answers to the open question were obtained under appropriate assumptions.

An open problem is when will these critical points be sign-changing with respect to the positive (negative) cone $P$ ($-P$) of $E$? In other words, how about the nodal structures of those critical points? Equivalently, for elliptic problems, how can we get infinitely many sign-changing solutions for the perturbation problems considered by those pioneering papers mentioned above.

In the present paper, we shall develop a theory which answers this open problem. Although some technical details will be formulated in the next section, we state the following theorem loosely now; it will be proved in Section 2.

**Theorem A.** Let $I \in C^1(E, \mathbb{R})$ be of the form $I' = \text{id} - K_I$ and satisfy the Palais–Smale condition, where $K_I$ is a continuous operator. Assume that $K_I(\pm D_0) \subset \pm D_0$ holds, where $D_0$ is a open convex neighborhood of the cone $P$. Let $N, M$ be two closed subspaces of $E$ with $\dim N < \infty$, $\dim N - \text{codim} M \geq 1$. Suppose that

$$Q(\rho) := \{u \in M : \|u\| = \rho\} \subset S := E \setminus (-D_0 \cup D_0).$$

Define

$$N^* = N \oplus \text{span}\{u^*\}, \quad u^* \in E \setminus N; \quad N^*_+ = \{u + tu^* : u \in N, t \geq 0\}. \quad \text{Assume that}

(i) \ I(0) = 0;
(ii) \ there \ exists \ an \ R_1 > \rho \ such \ that \ I(u) \leq 0 \ for \ all \ u \in N \ with \ \|u\| \geq R_1;
(iii) \ there \ exists \ an \ R_2 \geq R_1 \ such \ that \ I(u) \leq 0 \ for \ all \ u \in N^* \ with \ \|u\| \geq R_2.

Let $\Gamma := \{\phi \in C(E, E) : \phi \ is \ odd, \ \phi(-D_0 \cup D_0) \subset (-D_0 \cup D_0)\};$

$$\phi(u) = u \ if \ \max\{I(u), I(-u)\} \leq 0.$$
If

$$\gamma^* = \inf_{\phi \in \Gamma} \sup_{\phi(N^*) \cap S} I > \gamma^{**} = \inf_{\phi \in \Gamma} \sup_{\phi(N) \cap S} I > 0,$$

then \(K[\gamma^{**}, m_0 + 1] \cap (E \setminus (-P \cup P)) \neq \emptyset\), that is, there is a sign-changing critical point, where \(m_0 := \sup I < \infty\) and \(K[\gamma^{**}, m_0 + 1]\) denotes the set of critical points with critical values in \([\gamma^{**}, m_0 + 1]\).

This theorem gives a positive answer by showing that the critical point obtained in Theorem A is sign-changing. We also get the upper bound of the critical value. In practice, by estimating the critical values, we may get infinitely many sign-changing solutions. To prove this theorem, we will re-construct the critical value by a minmax procedure with respect to a set of mappings satisfying an “invariance” condition. By carefully analyzing the gradient flow and the invariant set of the flow, we show that the critical value corresponds to a sign-changing critical point.

The abstract theorem will be applied to the following elliptic equations with Hardy singular terms and perturbations from symmetry.

• \(-\Delta u - \mu \frac{u}{|x|^2} = f(x, u) + p(x, u), \quad \text{in } \Omega; \quad u = 0 \quad \text{on } \partial \Omega, \quad (1.1)\)

• \(-\Delta u = \mu \frac{|u|^{q-2}}{|x|^s} u + p(x, u), \quad \text{in } \Omega; \quad u = 0 \quad \text{on } \partial \Omega, \quad (1.2)\)

where \(0 < s < 2\), \(\Omega\) is a smooth bounded domain of \(\mathbb{R}^n\), \(f\) (odd) and \(p\) (non-odd) are Carathéodory functions. We will obtain infinitely many sign-changing solutions. To the best of our knowledge, the existence of infinitely many sign-changing solutions to (1.1) and (1.2) has not been studied by variational methods.

The existence of sign-changing solutions have attracted much attention in recent years. For example, in [B], the author established an abstract critical theory in partially ordered Hilbert spaces by virtue of critical groups. He studied superlinear problems. In [LW], a Ljusternik–Schnirelmann theory was established for the study of sign-changing solutions of even functionals. Some linking type theorems were also obtained in partially ordered Hilbert spaces. The methods and abstract critical point theory of [B,LW] (and [BW1]) were applied to a dense Banach space of continuous functions of a Hilbert space \(E\), where the cone has nonempty interior. This plays a crucial role. To fit that framework, the nonlinearities needed to satisfy a one-sided Lipschitz condition. A very recent paper [BLW] obtains some results for the subcritical growth case by working directly on the cone of the Sobolev space. This idea plays an important role in this paper. We refer the readers to other papers [BWe,BWe1,W,W,SWZ,WZ] and the references cited therein for sign-changing problems. Related papers and results will be mentioned later in this paper.
The paper is organized as follows: In Section 2, we establish the abstract result. Section 3 will be devoted to Eq. (1.1), and in Section 4, we deal with Eq. (1.2).

2. Abstract theorems

Let $E$ be a Hilbert space with inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$. Let $I \in C^1(E, \mathbb{R})$, $\mathcal{K} := \{ u \in E : I'(u) = 0 \}$ and $\tilde{E} := E \setminus \mathcal{K}$.

A locally Lipschitz continuous map $V : \tilde{E} \to E$ is called a pseudo-gradient vector field of $I$ if

(i) $\langle I'(u), V(u) \rangle \geq \frac{1}{2} \| I'(u) \|^2$ for all $u \in \tilde{E}$.

(ii) $\| V(u) \| \leq 2 \| I'(u) \|$ for all $u \in \tilde{E}$.

It is well known that the following Cauchy problem

$$
\frac{d \sigma(t, u)}{dt} = -V(\sigma(t, u)), \quad \sigma(0, u) = u,
$$

has a unique solution (flow) $\sigma : [0, T(u)) \times \tilde{E} \to E$, where $T(u) \in (0, \infty]$ is the largest time of the existence of $\sigma$ with initial value $u$.

Let $P$ ($-P$) denote the closed convex positive (negative) cone of $E$. For $\mu_0 > 0$, define

$$
\mathcal{D}_0 := \{ u \in E : \text{dist}(u, P) < \mu_0 \}; \quad \mathcal{D} := \mathcal{D}_0 \cup (-\mathcal{D}_0); \quad S = E \setminus \mathcal{D}.
$$

Then $\mathcal{D}_0$ is open and convex; $\mathcal{D}$ is open; $\pm P \subset \pm \mathcal{D}_0$; $S$ is closed.

First, we consider two lemmas.

Lemma 2.1. Consider the functional $I \in C^1(E, \mathbb{R})$ with gradient $I'(u) = i(u)u - K_1u$, where $i : E \to [1/2, 1]$ is a locally Lipschitz continuous function. Assume $K_1(\pm \mathcal{D}_0) \subset \pm \mathcal{D}_0$. Then there exists a locally Lipschitz continuous map $B_0 : \tilde{E} \to E$ such that $B_0(\pm \mathcal{D}_0 \cap \tilde{E}) \subset \pm \mathcal{D}_0$ and $V(u) := i(u)u - B_0(u)$ is a pseudo-gradient vector field of $I$. Moreover, if $I$ and $i$ are even functionals, $B_0$ (and hence $V$) can be chosen to be odd.

The proof of first part of the lemma can be found in [SZ] which improves a result of [S] and [LS]. The proof of the second part is standard (cf. [BLW, R2]): Let $\tilde{B}_0(u) = \frac{1}{2}(B_0 u - B_0(-u))$, then $\tilde{B}_0 : \tilde{E} \to E$ is odd and locally Lipschitz continuous. Define $\tilde{V}(u) := i(u)u - \tilde{B}_0 u$. Then $\tilde{V} : \tilde{E} \to E$ is also odd and locally Lipschitz continuous. It is easy to check that $\tilde{V}$ is a pseudo-gradient vector field of $I$. Further, for any $u \in \pm \mathcal{D}_0 \cap \tilde{E}$, we have $-u \in \mp \mathcal{D}_0 \cap \tilde{E}$, and hence $B_0(\pm u) \in \pm \mathcal{D}_0$, $-B_0(-u) \in \pm \mathcal{D}_0$. Therefore, $\tilde{B}_0 u = \frac{1}{2}B_0 u + \frac{1}{2}(-B_0(-u)) \in \pm \mathcal{D}_0$ since $\pm \mathcal{D}_0$ is convex. That is, $B_0(\pm \mathcal{D}_0 \cap \tilde{E}) \subset \pm \mathcal{D}_0$. 
We also need the following lemma which can be found in [De, Theorem 4.1] (see also [Br, Theorem 1, C]).

**Lemma 2.2.** Assume that $E$ is a Banach space, $\mathcal{M}$ is a closed convex subset of $E$, $H: \mathcal{M} \to E$ is locally Lipschitz continuous and

$$
\lim_{\lambda \to 0^+} \frac{\text{dist}(u + \lambda H(u), \mathcal{M})}{\lambda} = 0, \quad \forall u \in \mathcal{M}.
$$

Then for any given $u_0 \in \mathcal{M}$, there exists a $\delta > 0$ such that the initial value problem

$$
\frac{d\eta(t, u_0)}{dt} = H(\eta(t, u_0)), \quad \eta(0, u_0) = u_0,
$$

has a unique solution $\eta(t, u_0)$ defined on $[0, \delta)$. Moreover, $\eta(t, u_0) \in \mathcal{M}$ for all $t \in [0, \delta)$.

Consider the following vector field:

$$
W(u) := \frac{(1 + \|u\|^2 V(u)}{(1 + \|u\|^2 \|V(u)\|^2 + 1). \tag{2.1}
$$

Then $W$ is a locally Lipschitz continuous vector field over $\tilde{E}$. Obviously, $\|W(u)\| \leq \|u\| + 1$ for all $u \in \tilde{E}$. For simplicity, rewrite $W(u)$ as $W(u) = \pi(u) V(u)$, where

$$
\pi(u) := \frac{(1 + \|u\|^2}{(1 + \|u\|^2 \|V(u)\|^2 + 1} \tag{2.2}
$$

is locally Lipschitz continuous.

Recall the Cerami condition (sometimes called the weak Palais–Smale (PS) condition) [Si]): if for any sequence $\{u_n\}$ such that $\sup_n I(u_n)$ is bounded and

$$(1 + \|u_n\|) I'(u_n) \to 0,$$

then $\{u_n\}$ has a convergent subsequence. The usual PS condition implies the Cerami. Although the classical (PS) condition is enough for our applications, we still like to use the Cerami condition in the abstract result, since we believe it has many more applications to other problems.

We define

$$
K[a, b] := \{u \in E : I'(u) = 0, a \leq I(u) \leq b\},
$$

$$
I^c := \{u \in E : I(u) \leq c\}, B_R := \{u \in E : \|u\| \leq R\}.
$$
Theorem 2.1. Let $I \in C^1(E, \mathbb{R})$ be of the form $I' = \text{id} - K_I$ and satisfy the Cerami condition. Assume that $K_I(\pm D_0) \subset \pm D_0$ holds. Let $N, M$ be two subspaces of $E$ with $\dim N < \infty$, $\dim N - \text{codim} M = 1$. Suppose that

$$Q(\rho) := \{ u \in M : \| u \| = \rho \} \subset S.$$ 

Define

$$N^* = N \oplus \text{span}\{u^*\}, \quad u^* \in E \setminus N; \quad N^*_+ = \{ u + tu^* : u \in N, t \geq 0 \}.$$ 

Assume that

(i) $I(0) = 0$;
(ii) there exists a $R_1 > \rho$ such that $I(u) \leq 0$ for all $u \in N$ with $\| u \| \geq R_1$;
(iii) there exists a $R_2 \geq R_1$ such that $I(u) \leq 0$ for all $u \in N^*$ with $\| u \| \geq R_2$.

Let

$$\Gamma = \{ \phi \in C(E, E) : \phi \text{ is odd}, \phi(D) \subset D; \phi(u) = u \text{ if } \max\{I(u), I(-u)\} \leq 0 \}.$$ 

If

$$\gamma^* = \inf_{\phi \in \Gamma} \sup_{\phi(N^*_+) \cap S} I > \gamma^{**} = \inf_{\phi \in \Gamma} \sup_{\phi(N) \cap S} I > 0,$$

then $K[\gamma^{**}, m_0 + 1] \cap (E \setminus (-P \cup P)) \neq \emptyset$, where $m_0 := \sup_{N^*} I < \infty$.

Proof. By the Intersection Theorem (cf. Proposition 9.23 of [R2], Lemma 6.4 of [Str]), we see that $\phi(N \cap B_{R_1}) \cap Q(\rho) \neq \emptyset$ and that $\phi(N^*_+ \cap B_{R_1}) \cap Q(\rho) \neq \emptyset$ for each $\phi \in \Gamma$. Therefore, $\phi(N \cap B_{R_1}) \cap S \neq \emptyset$ and $\phi(N^*_+ \cap B_{R_1}) \cap S \neq \emptyset$. It follows that $\gamma^*$ and $\gamma^{**}$ are well defined. Let $\tilde{\gamma} \in (\gamma^{**}, \gamma^*)$ and

$$\Gamma_1 = \Gamma_2 \cup \Gamma_3,$$

where

$$\Gamma_2 := \{ \phi \in \Gamma : I(u) \leq \tilde{\gamma} \text{ for all } u \in \phi(N) \cap S \},$$ 

$$\Gamma_3 := \{ \phi \in \Gamma : I(\phi(u)) \leq I(u) \}.$$
Then $\Gamma_2 \neq \emptyset$; $\mathrm{id} \in \Gamma_3$. We define

$$
\gamma_0 := \inf_{\phi \in \Gamma_1} \sup_{\phi(N_1^+ \cap S)} I.
$$

(2.3)

Then $\gamma_0 \geq \gamma^* > 0$. Particularly, by assumption (iii), we may write $\gamma_0$ as

$$
\gamma_0 = \inf_{\phi \in \Gamma_1} \sup_{\phi(N_1^+ \cap B_{R_2} \cap S)} I.
$$

(2.4)

Since $\mathrm{id} \in \Gamma_3$,

$$
\gamma_0 \leq \sup_{\mathrm{id}(N_1^+ \cap B_{R_2} \cap S)} I \leq \sup_{(N_1^+ \cap B_{R_2})^*} I \leq \sup_{(N^*)} I = m_0 < \infty.
$$

Choose $\bar{\epsilon} > 0$ such that $\bar{\epsilon} < \min\{\gamma^* - \bar{\gamma}, 1, \bar{\gamma} - \gamma^{**}\}$ and assume that $K[\gamma_0 - \bar{\epsilon}, \gamma_0 + \bar{\epsilon}] \subset (-P \cup P)$ (otherwise, we are done). Assume $K[\gamma_0 - \bar{\epsilon}, \gamma_0 + \bar{\epsilon}] \neq \emptyset$. Since $K[\gamma_0 - \bar{\epsilon}, \gamma_0 + \bar{\epsilon}]$ is compact, we have that

$$
\text{dist}\left(K[\gamma_0 - \bar{\epsilon}, \gamma_0 + \bar{\epsilon}], S\right) := \delta_1 > 0.
$$

(2.5)

By the Cerami condition, there is an $\omega \in (0, \bar{\epsilon}/3)$ such that

$$
\frac{(1 + \|u\|)^2 \|I'(u)\|^2}{1 + (1 + \|u\|)^2 \|I'(u)\|^2} \geq \omega
$$

(2.6)

for

$$
u \in I^{-1}[\gamma_0 - \omega, \gamma_0 + \omega]\setminus (K[\gamma_0 - \bar{\epsilon}, \gamma_0 + \bar{\epsilon})_{\delta_1/2}.
$$

If $K[\gamma_0 - \bar{\epsilon}, \gamma_0 + \bar{\epsilon}] = \emptyset$, we let $(K[\gamma_0 - \bar{\epsilon}, \gamma_0 + \bar{\epsilon})_c = \emptyset$ for all $c > 0$. By decreasing $\omega$ if necessary, we may assume that $3\omega < \gamma_0$. Let

$$Q_1 = \{u \in E : |I(u) - \gamma_0| > 3\omega\}, \quad Q_2 = \{u \in E : |I(u) - \gamma_0| \leq 2\omega\}.
$$

Let $y(u) : E \rightarrow [0, 1]$ be locally Lipschitz continuous such that $y(u) = 1$ for all $u \in E \setminus (K[\gamma_0 - \bar{\epsilon}, \gamma_0 + \bar{\epsilon})_{\delta_1/2}$ and $y(u) = 0$ for all $u \in (K[\gamma_0 - \bar{\epsilon}, \gamma_0 + \bar{\epsilon})_{\delta_1/3}$. If $K[\gamma_0 - \bar{\epsilon}, \gamma_0 + \bar{\epsilon}] = \emptyset$, we let $y(u) \equiv 1$. Consider

$$h(u) = \frac{\text{dist}(u, Q_1)}{\text{dist}(u, Q_1) + \text{dist}(u, Q_2)}.$$
Let $\vec{W}(u) = y(u)h(u)W(u)$ if $u \in \tilde{E}$ and $\vec{W}(u) = 0$ otherwise. Here $W(u)$ is given by (2.1), where $V(u)$ corresponds to $I$ and is not necessarily odd since $I$ is not assumed odd. Then $\vec{W}$ is a locally Lipschitz vector field over $E$. We consider the following Cauchy initial value problem:

$$\frac{d\sigma(t, u)}{dt} = -\vec{W}(\sigma(t, u)), \quad \sigma(0, u) = u \in E,$$

which has a unique continuous solution $\sigma(t, u)$ in $E$ for $t \geq 0$. Evidently,

$$\frac{dI(\sigma(t, u))}{dt} \leq 0.$$

Now we claim that,

$$\sigma([0, +\infty), \tilde{D}) \subset \tilde{D}, \quad \sigma([0, +\infty), D) \subset D. \quad (2.7)$$

We first observe that $B_0(\pm D_0 \cap \tilde{E}) \subset (\pm \tilde{D}_0)$ implies that $B_0(\pm \tilde{D}_0 \cap \tilde{E}) \subset (\pm \tilde{D}_0)$. Obviously, $\sigma(t, u) = u$ for all $t \geq 0$, and $u \in \tilde{D} \cap K$. Next, we assume that $u \in \tilde{D}_0 \cap \tilde{K}$. If there were a $t_0 > 0$ such that $\sigma(t_0, u) \notin \tilde{D}_0$, then there would be a number $s_0 \in [0, t_0)$ such that $\sigma(s_0, u) \in \tilde{D}_0$ and $\sigma(t, u) \notin \tilde{D}_0$ for $s_0 < t \leq t_0$. This means that the trajectory $\sigma(t, u)$ flows out of $\tilde{D}_0$ during the time period $(s_0, t_0]$. Consider the following initial value problem

$$\frac{d\sigma(t, \sigma(s_0, u))}{dt} = -\vec{W}(\sigma(t, \sigma(s_0, u))), \quad \sigma(0, \sigma(s_0, u)) = \sigma(s_0, u) \in E.$$

It has a unique solution $\sigma(t, \sigma(s_0, u))$ for $t \geq 0$. Next, we are going to apply Lemma 2.2 to show that $\sigma(t, u)$ will flow back to $\tilde{D}_0$ for a short period of $t > s_0$ and to get a contradiction. For any $v \in \tilde{D}_0$, if $v \in K$, then $\vec{W}(v) = 0$. Hence $v + \lambda(-\vec{W}(v)) = v \in \tilde{D}_0$. If $v \notin K$, then for $\lambda \in (0, 1)$ small enough, by Lemma 2.1 and the convexity of $\tilde{D}_0$, we have that

$$v + \lambda(-\vec{W}(v)) = v + \lambda \left( -y(v)h(v)\pi(v)V(v) \right)$$

$$= v + \lambda \left( -y(v)h(v)\pi(v)(v - B_0(v)) \right)$$

$$= \left( 1 - \lambda y(v)h(v)\pi(v) \right)v + \lambda y(v)h(v)\pi(v)B_0(v) \in \tilde{D}_0.$$

This means that

$$\lim_{\lambda \to 0^+} \frac{\text{dist}(v + \lambda(-\vec{W}(v)), \tilde{D}_0)}{\lambda} = 0, \quad \forall v \in \tilde{D}_0.$$
By Lemma 2.2, there exists a $\delta > 0$ such that $\sigma(t, \sigma(s_0, u)) \in \tilde{D}_0$ for all $t \in [0, \delta)$. By the semigroup property, we see that $\sigma(t, u) \in \tilde{D}_0$ for all $t \in [s_0, s_0 + \delta)$, which contradicts the definition of $s_0$. Therefore, $\sigma([0, +\infty), \tilde{D}_0) \subset \tilde{D}_0$. Similarly, $\sigma([0, +\infty), -\tilde{D}_0) \subset -\tilde{D}_0$. That is, $\sigma([0, +\infty), D) \subset D$. To prove $\sigma([0, +\infty), \tilde{D}) \subset D$, we just show that $\sigma([0, +\infty), D_0) \subset D_0$ by negation. Assume there exist $u^* \in D_0, t_0 > 0$ such that $\sigma(t_0, u^*) \notin D_0$. Choose a neighborhood $U_{u^*}$ of $u^*$ such that $U_{u^*} \subset \tilde{D}_0$.

Then by the theory of ordinary differential equations in Banach space, we may find a neighborhood $U_{t_0}$ of $\sigma(t_0, u^*)$ such that $\sigma(t_0, \cdot) : U_{u^*} \rightarrow U_{t_0}$ is a homeomorphism. Since $\sigma(t_0, u^*) \notin D_0$, we take a $w \in U_{t_0} \setminus D_0$. Correspondingly, we find a $v \in U_{u^*}$ such that $\sigma(t_0, v) = w$, which contradicts the fact that $\sigma([0, +\infty), \tilde{D}) \subset D$, which was previously proved. Hence, (2.7) follows.

By the definition of $\gamma_0$ in (2.4), there exists a $\phi \in \Gamma_1$ such that $\phi(N_+^* \cap B_{R_2}) \cap S \subset E_0^{\gamma_0 + \omega}$. Therefore, $\phi(N_+^* \cap B_{R_2})$ is a subset of $E_0^{\gamma_0 + \omega} \cup D$.

We claim that there exists a $T_2 > 0$ such that $\sigma(T_2, \phi(N_+^* \cap B_{R_2})) \subset E_0^{\gamma_0 - \omega/2} \cup D$.

If $u \in \phi(N_+^* \cap B_{R_2}) \cap D$, we have $\sigma(t, u) \notin D$ for all $t \geq 0$ by (2.6).

If $u \in \phi(N_+^* \cap B_{R_2}), u \notin D$, then we see that $I(u) \leq \gamma_0 + \omega$. If $I(u) \leq \gamma_0 - \omega$, then $I(\sigma(t, u)) \leq I(u) \leq \gamma_0 - \omega$ for all $t$, and we are done. If $I(u) > \gamma_0 - \omega$, then $u \in I^{-1}[\gamma_0 - \omega, \gamma_0 + \omega]$.

If $\text{dist}(\sigma([0, \infty), u), K[\gamma_0 - \varepsilon, \gamma_0 + \varepsilon]) \leq \delta_1/2$, then by the definition of $\delta_1$ in (2.5), there exists a $t_m$ such that $\sigma(t_m, u) \notin S$. Moreover, we may choose a $t_m$ so that $\text{dist}(\sigma(t_m, u), S) \geq \frac{1}{2} \delta_1$.

Assume $\text{dist}(\sigma([0, \infty), u), K[\gamma_0 - \varepsilon, \gamma_0 + \varepsilon]) > \delta_1/2$. Similarly, we assume that $I(\sigma(t, u)) > \gamma_0 - \omega$ for all $t$ (otherwise, we are done). Then $\sigma(t, u) \in I^{-1}[\gamma_0 - \omega, \gamma_0 + \omega] \setminus (K[\gamma_0 - \varepsilon, \gamma_0 + \varepsilon])\delta_1$. Hence, by (2.6),

$$\frac{(1 + \|\sigma(t, u)\|^2)\|I'(\sigma(t, u))\|^2}{1 + (1 + \|\sigma(t, u)\|^2)\|I'(\sigma(t, u))\|^2} \geq \omega, \quad \forall t \geq 0$$

(2.8)

and $h(\sigma(t, u)) = 1, y(\sigma(t, u)) = 1$ for all $t \geq 0$. Therefore,

$$I(\sigma(24, u))$$

$$= I(u) + \int_0^{24} dI(\sigma(t, u))$$

$$= I(u) - \int_0^{24} \left( \frac{(1 + \|\sigma(t, u)\|^2)\|I'(\sigma(t, u), V(\sigma(t, u)))\|^2}{(1 + \|\sigma(t, u)\|^2)\|V(\sigma(t, u))\|^2 + 1} \right) dt$$

$$\leq I(u) - \int_0^{24} \left( \frac{(1 + \|\sigma(t, u)\|^2)\|I'(\sigma(t, u))\|^2}{8(1 + \|\sigma(t, u)\|^2)\|I'(\sigma(t, u))\|^2 + 2} \right) dt$$

$$\leq I(u) - 3\omega$$

$$\leq \gamma_0 - 2\omega.$$
For the case of $K[\gamma_0 - \bar{\varepsilon}, \gamma_0 + \bar{\varepsilon}] = \emptyset$, if there is a $t_0 > 0$ such that $I(\sigma(t_0, u)) \leq \gamma_0 - \omega$, then $I(\sigma(t, u)) \leq \gamma_0 - \omega$ for all $t \geq t_0$, and we are done. Otherwise, $\gamma_0 - \omega < I(\sigma(t, u)) \leq \gamma_0 + \omega$ for all $t \geq 0$. Hence, $y(\sigma(t, u)) \equiv 0$, $h(\sigma(t, u)) = 1$ for all $t \geq 0$. We still have (2.8)–(2.9).

By combining the above arguments, for any $u \in \phi(N_+^* \cap B_{R_2}) \setminus D$, there exists a $T_u > 0$ such that either $\sigma(T_u, u) \in E^{\gamma_0 - \omega / 2}$ or $\text{dist}(\sigma(T_u, u), S) \geq \frac{1}{2} \delta_1$ (hence, $\sigma(T_u, u) \in D$).

By continuity, there exists a neighborhood $U_u$ such that either $\sigma(T_u, U_u) \subset E^{\gamma_0 - \omega / 3}$ or $\text{dist}(\sigma(T_u, U_u), S) \geq \frac{1}{2} \delta_1$. Both case imply that $\sigma(T_u, U_u) \subset E^{\gamma_0 - \omega / 3} \cup (E \setminus S)$. Since $\phi(N_+^* \cap B_{R_2}) \setminus D$ is compact in $E$, we get a $T_2 > 0$ such that $\sigma(T_2, \phi(N_+^* \cap B_{R_2}) \setminus D) \subset E^{\gamma_0 - \omega / 4} \cup (E \setminus S)$. Therefore, $\sigma(T_2, \phi(N_+^* \cap B_{R_2})) \subset E^{\gamma_0 - \omega / 4} \cup (E \setminus S)$.

Since the set

$$
\{u \in E : \max[I(u), I(-u)] \leq 0\}
$$

is closed and symmetric, we denote it by $O \cup (-O)$, where $O$ is closed. Now we define

$$
h^*(u) = \begin{cases} 
\sigma(T_2, \phi(u)), & u \in N_+^* \cup \tilde{D} \cup O, \\
-\sigma(T_2, \phi(-u)), & u \in -N_+^* \cup (-\tilde{D}) \cup (-O).
\end{cases}
$$

By the definition of $\Gamma$, we have $\phi(\tilde{D}) \subset D$. Thus, $\phi(\tilde{D}) \subset \tilde{D}$. By combining this with (2.7), we see that $h^*(\tilde{D}) \subset \tilde{D}$, and $h^*(D) \subset D$. If $u \in D$, then $-u \in -\tilde{D}$ and $h^*(u) = \sigma(T_2, \phi(u)) = -h^*(-u)$. If $u \in O$, then $\max[I(u), I(-u)] \leq 0$, $\phi(u) = u$, and $\phi(-u) = -u$. In particular, $I(\pm u) \leq 0 < \gamma_0 - 3\omega$, which implies $u \in Q_1$. Hence, $h(u) = 0$ and $\sigma(T_2, u) = u$. It follows that $h^*(-u) = -h^*(u) = -u$. Finally, if $u \in N_+^*$, then $h^*(-u) = -\sigma(T_2, \phi(u)) = -h^*(u)$. Therefore, $h^*$ is odd on $(N_+^* \cup \tilde{D} \cup O) \cup (-N_+^* \cup (-\tilde{D}) \cup (-O))$. Furthermore, $h^*$ may be extended to an odd map in $C(E, E)$. That is, $h^* \in \Gamma$.

Next we show that $h^* \in \Gamma_1$. Recall that $\phi \in \Gamma_1 = \Gamma_2 \cup \Gamma_3$. If $\phi \in \Gamma_2$, then for any $u \in h^*(N) \cap S$, we have $u = h^*(w)$ for some $w \in N \subset N_+^*$. Therefore, $u = \sigma(T_2, \phi(w))$, $\phi(w) \in S$ (otherwise, $u \in \tilde{D}$), $\phi(w) \in \phi(N) \cap S$ and $I(u) \leq I(\phi(w)) \leq \gamma$. Therefore, $h^* \in \Gamma_2$. If $\phi \in \Gamma_3$, then

$$
I(h^*(u)) = I(\sigma(T_2, \phi(u)) \leq I(\phi(u)) \leq I(u)
$$

for all $u \in N \subset N_+^*$. That is, $h^* \in \Gamma_3$. Both cases imply that $h^* \in \Gamma_1$. But

$$
\sup_{h^*(N_+^* \cap B_{R_1}) \cap S} I \leq \gamma_0 - \omega / 4,
$$

and we get the desired contradiction. \(\square\)
3. Application (I)

Consider the following equation

$$-\Delta u - \mu \frac{u}{|x|^2} = f(x, u) + p(x, u) \quad \text{in } \Omega; \quad u = 0 \quad \text{on } \partial\Omega,$$

(3.1)

where $\Omega \subset \mathbb{R}^n (n \geq 3)$ is an open bounded domain with smooth boundary and containing the origin $0$; $0 \leq \mu < \bar{\mu} := (n - 2)^2/4$. In this paper, we use the letter $c$ indiscriminately to denote various positive constants when the exact values are irrelevant. Assume

$(H_1)$ $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with subcritical growth:

$$|f(x,u)| \leq c(1 + |u|^{s-1}) \quad \text{for all } u \in \mathbb{R} \text{ and a.e. } x \in \Omega,$$

where $s \in (2, 2^*)$, $2^* = 2n/(n - 2)$. Moreover, $f(x,u) = o(|u|)$ as $|u| \to 0$ uniformly in $x \in \Omega$; $f(x,u)u \geq 0$ for all $u \in \mathbb{R}$ and a.e. $x \in \Omega$.

$(H_2)$ There exists an $\omega > 2$ such that $0 < \omega F(x,u) \leq uf(x,u)$ for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$ with $|u|$ large enough.

$(H_3)$ $f(x,u)$ is odd in $u \in \mathbb{R}$.

$(H_4)$ $p : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. There is a $\sigma < \omega/2$ such that

$$|p(x,u)| \leq c(1 + |u|^{\sigma}) \quad \text{for all } u \in \mathbb{R} \text{ and a.e. } x \in \Omega.$$

Moreover, $p(x,u) = o(|u|)$ as $|u| \to 0$ uniformly in $x \in \Omega$; $p(x,u)u > 0$ for a.e. $x \in \Omega$ and all $u \in \mathbb{R} \setminus \{0\}$.

The main results of this section are the following theorems and corollary.

**Theorem 3.1.** Assume $(H_1)-(H_4)$. Then Eq. (3.1) has an infinite sequence of sign-changing solutions provided that

$$\frac{2s}{n(s-2)} - 1 > \frac{\omega}{\omega - (1 + \sigma)}.$$  

(3.2)

If the perturbation term $p$ disappears and the oddness $(H_3)$ of $f$ is cancelled completely, we have the following theorem.

**Theorem 3.2** (Without any symmetry). Assume $(H_1)-(H_2)$. Suppose that there is a $\sigma < \omega/2$ such that

$$|f(x,u) - f(x,-u)| \leq c(1 + |u|^{\sigma}) \quad \text{for all } u \in \mathbb{R} \text{ and a.e. } x \in \Omega.$$  

(3.3)
Then the following equation
\[ -\Delta u - \mu \frac{u}{|x|^2} = f(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega \] (3.4)

has an infinite sequence of sign-changing solutions provided that (3.2) holds.

The geometric meaning of (3.3): If \( f(x, u) \) is not odd in \( u \), then there is a gap between the curves of \( f(x, u) \) and \( -f(x, -u) \). Condition (3.3) implies that the height of the gap is bounded by the function \( c + c|u| \) everywhere. The height may become larger and larger as \( |u| \to \infty \) and it vanishes if and only if \( f(x, u) \) is odd in \( u \).

An example: Let \( f(x, u) = -|u|^{(3n+1)/(3n-1)} \) for \( u < 0 \); \( f(x, u) = u^{(3n+1)/(3n-1)} + u^2 \) for \( u \in [0, 1] \); \( f(x, u) = u^{(3n+1)/(3n-1)} + u^\sigma \) for \( u \geq 1 \), where \( \sigma \in [0, 9/15] \). Then \( f \) is not odd and satisfies all the assumptions of Theorem 3.2.

Although it is an immediate consequence of the above theorem, we still like to state the following corollary where only oddness is assumed in the neighborhood of infinity. It should be noted that, even if the nonlinearity is not odd only around zero, the symmetry of the energy functional is destroyed completely. The classical methods seem to be invalid.

**Corollary 3.1.** Assume \((H_1)-(H_2)\). If there exists a \( R > 0 \) such that
\[ f(x, -u) = -f(x, u) \quad \text{for a.e. } x \in \Omega, \quad |u| \geq R, \]
then Eq. (3.4) has an infinite sequence of sign-changing solutions provided that
\[ \frac{2s}{n(s-2)} - 1 > \frac{\omega}{\omega - 1}. \]

We emphasize that the above results are new even for the case of \( \mu = 0 \), which was studied by several authors concerning existence only. For instance, the special case
\[ \begin{cases} -\Delta u = |u|^{s-2}u + p(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases} \] (3.5)

was first studied by Bahri–Brestycki [BB] and Struwe [S1, S2] independently. In [Ba], Bahri considered (3.5) and proved that there is an open dense set of \( p \) in \( W^{-1,2}(\Omega) \) such that (3.5) has infinitely many solutions if \( s < 2n/(n-1) \). In [R1] (see also [R2]), Rabinowitz considered a general case of (3.5) under the assumption (3.2). The value of \( s \) in (3.2) was improved to \( s < (2n-2)/(n-2) \) by Bahri and Lions in [BL]. In [Ta], Takana studied (3.5) by Morse index methods and weaken (3.2) for \( p \) a \( C^1 \) function. In [T], Tehrani considered the case of a sign-changing potential. All the papers mentioned
above only concern the existence of infinitely many solutions. No information about the signs of the solutions was obtained.

Now we proceed to prove Theorems 3.1 and 3.2 and Corollary 3.1.

For a fixed $\mu \in [0, \bar{\mu})$, consider the Hilbert space $E$ endowed with the inner product

$$
\langle u, v \rangle_E = \int_{\Omega} \nabla u \nabla v \, dx - \mu \int_{\Omega} \frac{uv}{|x|^2} \, dx, \quad \forall u, v \in H^1_0(\Omega)
$$

and norm $\|u\|_E := \langle u, u \rangle_E^{1/2}$. This norm is equivalent to the Dirichlet norm $\|u\| = (\int_{\Omega} |\nabla u|^2 \, dx)^{1/2}$ in $H^1_0(\Omega)$ by Hardy’s inequality.

The eigenvalue problem

$$
-\Delta u - \frac{u}{|x|^2} = \lambda u
$$

has a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \to \infty$ with finite multiplicity for each $\lambda_k$. The first eigenvalue is simple with positive eigenfunction $\phi_1$; the eigenfunction $\phi_k$ corresponding to $\lambda_k$ ($k \geq 2$) is sign-changing (cf. [FG,E]). Moreover, $\phi_k \in L^p(\Omega)$ for $p < \frac{2\mu^{1/2}}{\mu - \mu^{1/2}}$, and $H^1_0(\Omega)$ can be exhausted by the eigenfunctions $\{\phi_k\}$ (cf. [Se]). We first prove the following asymptotic formula for $\{\lambda_k\}$ which shows that the eigenvalues of (3.6) have the same growth properties as the eigenvalues of $(-\Delta, 0)$. The technique being employed is motivated by the paper [LY].

**Lemma 3.1.** The following estimates hold:

$$
\lambda_k \geq (M_1 B_n)^{-\frac{2}{n}} \left( 1 + \frac{(n-2)^2}{4} \frac{4}{(n-2)^2} - \mu \right)^{-1} \left( \frac{n+2}{n} \right)^{-1} k^{\frac{2}{n}} := C_0 k^{\frac{2}{n}},
$$

for all $k \geq 1$, where $M_1 = (2\pi)^{-n} |\Omega|$, and $|\Omega|$, $B_n$ are the volumes of $\Omega$ and the unit ball of $\mathbb{R}^n$, respectively.

Before proving it, we state the following Lemma which can be found in [LY, Lemma 1].

**Lemma 3.2.** If $f$ is a real-valued function defined on $\mathbb{R}^n$ with $0 \leq f \leq M_1$ and $\int_{\mathbb{R}^n} |z|^2 f(z) \, dz \leq M_2$, then

$$
\int_{\mathbb{R}^n} f(z) \, dz \leq (M_1 B_n)^{\frac{2}{n+2}} M_2^{\frac{n}{n+2}} \left( \frac{n+2}{n} \right)^{\frac{n}{n+2}},
$$

where $B_n = \text{volume of the unit ball of } \mathbb{R}^n$. 
Proof of Lemma 3.1. Since \( \phi_i \) is an eigenfunction of \(-\Delta - \mu / |x|^2\) corresponding to \( \lambda_i \), the \( \{\phi_i\} \) are orthogonal in both \( E \) and \( L^2(\Omega) \). Assume that \( \|\phi_i\|_2 = 1 \). Define

\[
\Phi(x, y) = \sum_{i=1}^{k} \phi_i(x)\phi_i(y), \quad x, y \in \Omega
\]

and consider the \( x \)-Fourier transform of \( \Phi \):

\[
\hat{\Phi}(z, y) = \frac{1}{(2\pi)^{n/2}} \int_{\Omega} \Phi(x, y) e^{ix \cdot z} \, dx.
\] (3.7)

Then

\[
\int_{\Omega} \Phi^2(x, y) \, dx = \int_{\mathbb{R}^n} |\hat{\Phi}(z, y)|^2 \, dz, \quad y \in \Omega.
\]

It follows that

\[
\int_{\mathbb{R}^n} \int_{\Omega} |\hat{\Phi}(z, y)|^2 \, dz \, dy = \int_{\Omega} \int_{\Omega} \Phi^2(x, y) \, dx \, dy = k. \tag{3.8}
\]

Furthermore,

\[
\int_{\Omega} |\hat{\Phi}(z, y)|^2 \, dy = \frac{1}{(2\pi)^n} \int_{\Omega} \left| \int_{\Omega} \Phi(x, y) e^{ix \cdot z} \, dx \right|^2 \, dy
\]

\[
\leq \frac{1}{(2\pi)^n} \int_{\Omega} |e^{ix \cdot z}|^2 \, dx \leq \frac{1}{(2\pi)^n} |\Omega|. \tag{3.9}
\]

This follows from the fact that

\[
\int_{\Omega} \left| \int_{\Omega} \sum_{i=1}^{k} \phi_i(x)\phi_i(y)h(x) \, dx \right|^2 \, dy
\]

\[
\leq \int_{\Omega} \sum_{i=1}^{k} \left| \int_{\Omega} \phi_i(x)h(x) \, dx \right|^2 |\phi_i(y)|^2 \, dy \leq \|h\|^2, \quad h \in L^2(\Omega),
\]
since the $\phi_i$ are orthonormal. On the other hand,

$$z_j \hat{\Phi}(z, y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \Phi(x, y) z_j e^{ix \cdot z} \, dx$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\Omega} \Phi(x, y) \left(-i\right) \frac{\partial}{\partial x_j} e^{ix \cdot z} \, dx$$

$$= i \frac{1}{(2\pi)^{n/2}} \int_{\Omega} \left( \frac{\partial}{\partial x_j} \Phi(x, y) \right) e^{ix \cdot z} \, dx$$

$$= i \frac{\partial}{\partial x_j} \Phi(z, y), \quad j = 1, \ldots, n. \quad (3.10)$$

Note that $\tilde{\mu} \int_{\Omega} \frac{|u|^2}{|x|^2} \, dx \leq \|u\|^2$ by Hardy’s inequality. Hence,

$$\|u\|^2 \leq \frac{(n-2)^2}{4} \frac{\|u\|^2}{(n-2)^2 - \mu} \|E\|.$$  \quad (3.11)

Therefore, by (3.10)–(3.11),

$$\int_{\mathbb{R}^n} \int_{\Omega} |z|^2 |\hat{\Phi}(z, y)|^2 \, dy \, dz$$

$$= \int_{\mathbb{R}^n} \int_{\Omega} \left| \nabla_x \Phi(x, y) \right|^2 \, dy \, dx$$

$$= \int_{\mathbb{R}^n} \int_{\Omega} \left| \nabla_x \Phi(x, y) \right|^2 \, dy \, dx$$

$$= - \int_{\Omega} \int_{\Omega} \Phi(x, y) \Delta_x \Phi(x, y) \, dy \, dx$$

$$= - \int_{\Omega} \int_{\Omega} \left( \sum_{i=1}^{k} \phi_i(x) \phi_i(y) \right) \left( \sum_{i=1}^{k} \Delta_x \phi_i(x) \phi_i(y) \right) \, dy \, dx$$

$$= - \int_{\Omega} \int_{\Omega} \left( \sum_{i=1}^{k} \phi_i(x) \phi_i(y) \right) \sum_{i=1}^{k} \left( \mu \frac{\phi_i(x)}{|x|^2} - \lambda_i \phi_i(x) \right) \phi_i(y) \, dy \, dx$$

$$= \int_{\Omega} \int_{\Omega} \left( \sum_{i=1}^{k} \phi_i(x) \phi_i(y) \right) \left( \sum_{i=1}^{k} \lambda_i \phi_i(x) \phi_i(y) \right) \, dy \, dx$$

$$+ \mu \int_{\Omega} \int_{\Omega} \left( \sum_{i=1}^{k} \phi_i(x) \phi_i(y) \right) \sum_{i=1}^{k} \phi_i(x) \phi_i(y) \, dy \, dx$$
\[
= \sum_{i=1}^{k} \lambda_i + \mu \sum_{i=1}^{k} \int_{\Omega} \int_{\Omega} \frac{\phi_i^2(x) \phi_i^2(y)}{|x|^2} \, dx \, dy + \mu \sum_{i \neq j}^{k} \int_{\Omega} \int_{\Omega} \frac{\phi_i(x) \phi_j(y) \phi_j(x) \phi_j(y)}{|x|^2} \, dx \, dy
\]

\[
= \sum_{i=1}^{k} \lambda_i + \mu \sum_{i=1}^{k} \int_{\Omega} \int_{\Omega} \frac{\phi_i^2(x) \phi_i^2(y)}{|x|^2} \, dx \, dy \leq \sum_{i=1}^{k} \lambda_i + \sum_{i=1}^{k} \|\phi_i(x)\|^2
\]

\[
\leq \sum_{i=1}^{k} \lambda_i + \frac{(n-2)^2}{4} \sum_{i=1}^{k} \|\phi_i(x)\|^2_E
\]

\[
\leq \left(1 + \frac{(n-2)^2}{4} - \frac{4}{(n-2)^2} - \mu \right) \sum_{i=1}^{k} \lambda_i.
\]

Now we let

\[
f(z) = \int_{\Omega} |\hat{\Phi}(z_i)|^2 \, dy,
\]

\[
M_1 = (2\pi)^{-n} |\Omega|
\]

and

\[
M_2 = \left(1 + \frac{(n-2)^2}{4} - \frac{4}{(n-2)^2} - \mu \right) \sum_{i=1}^{k} \lambda_i.
\]

By Lemma 3.2 and (3.8)–(3.9), we have that

\[
k \leq (M_1 B_n)^{\frac{2}{\pi+2}} \left(\left(1 + \frac{(n-2)^2}{4} - \frac{4}{(n-2)^2} - \mu \right) \sum_{i=1}^{k} \lambda_i\right)^{\frac{n}{\pi+2}} \left(\frac{n+2}{n}\right)^{\frac{n}{\pi+2}}.
\]
Therefore,

\[
\sum_{i=1}^{k} \lambda_i \geq (M_1 B_n)^{-\frac{2}{n}} \left( 1 + \frac{(n - 2)^2}{4} \right)^{-1} \left( \frac{n + 2}{n} \right)^{-1} k \frac{n + 2}{n}.
\]

Hence,

\[
\lambda_k \geq (M_1 B_n)^{-\frac{2}{n}} \left( 1 + \frac{(n - 2)^2}{4} \right)^{-1} \left( \frac{n + 2}{n} \right)^{-1} k \frac{n + 2}{n}.
\]

\( \square \)

To prove the (PS) condition, we first recall the following lemma whose proof is standard by using Brezis–Lieb’s Lemma and Vitali’s Theorem (cf. e.g. Lemma 2.3 of [CP])

**Lemma 3.3.** Let \( \{u_m\} \subset H_0^1(\Omega) \) be such that \( u_m \rightharpoonup u \) weakly in \( H_0^1(\Omega) \). Then

(i) \( \int_\Omega |\nabla u_m|^2 \, dx = \int_\Omega |\nabla (u_m - u)|^2 \, dx + \int_\Omega |\nabla u|^2 \, dx + o(1) \);

(ii) \( \int_\Omega \frac{u_m^2}{|x|^2} \, dx = \int_\Omega \frac{(u_m - u)^2}{|x|^2} \, dx + \int_\Omega \frac{u^2}{|x|^2} \, dx + o(1) \).

Let \( N_k \) be the eigenspace of the operator \(-\Delta - \frac{\mu}{|x|^2}\) corresponding to \( \lambda_k \) and \( E_k := N_1 \oplus N_2 \oplus \cdots \oplus N_k \). Let

\[
P := \{ u \in E : u(x) \geq 0 \text{ for a.e. } x \in \Omega \}.
\]

Then \( P \) (\(-P\)) is the positive (negative) cone of \( E \).

Define

\[
G(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega \frac{u^2}{|x|^2} \, dx - \int_\Omega F(x, u) \, dx = \frac{1}{2} \| u \|^2_E - \int_\Omega F(x, u) \, dx
\]

and

\[
I(u) := \frac{1}{2} \| u \|^2_E - \int_\Omega F(x, u) \, dx - \int_\Omega P(x, u) \, dx, \quad u \in E.
\]

(3.12)

(3.13)
Then $G$ and $I$ are in $C^1(E, \mathbb{R})$. In applications of Theorem 2.1, usually the assumption “$Q(\rho) \subset S$” is not true as long as $\dim M = \infty$. Therefore, we are going to consider an approximation of $E$: $E_1 \subset E_2 \subset \cdots \subset E_k \subset \cdots \subset E_m \subset \cdots$, where the $E_k$ are described above. Note that $\dim E_k < \infty$ for each $k$. For each $k > 2$, define

$$G_k := G|_{E_k}, \quad I_k := I|_{E_k}.$$ 

Then $G_k, I_k \in C^1(E_k, \mathbb{R})$.

**Lemma 3.4.** Assume $(H_1)$–$(H_4)$. Then $I_k$ (and hence $G_k$) satisfies the (PS) condition.

**Proof.** Assume that $\{u_m\} \subset E_k$ is a (PS) sequence: $\sup_{m \geq 1} |I_k(u_m)|$ is bounded and $I'_k(u_m) \to 0$ as $m \to \infty$. We first prove that $\{u_m\}$ is $\|\cdot\|_E$-bounded. Assume that $\|u_m\|_E \to \infty$. Then we have

$$\int_{\Omega} \frac{2F(x, u_m) + 2P(x, u_m)}{\|u_m\|_E^2} dx \to 1. \quad (3.14)$$

Let $w_m = \frac{u_m}{\|u_m\|_E}$. Then $w_m \rightharpoonup w^*$ weakly in $E$, strongly in $L^2(\Omega)$ and a.e. in $\Omega$. Define $\Omega_1 = \{x \in \Omega : w^*(x) \neq 0\}$. Then $\frac{2F(x, u_m)}{u_m^2} w_m^2 \to \infty$ for $x \in \Omega_1$ by $(H_2)$. If $\Omega_1$ has positive measure, then

$$\int_{\Omega} \frac{2F(x, u_m)}{\|u_m\|_E^2} dx = \int_{\Omega} \frac{2F(x, u_m)}{u_m^2} w_m^2 dx \geq \int_{\Omega_1} \frac{2F(x, u_m)}{u_m^2} w_m^2 dx \to \infty.$$ 

This contradicts (3.14). Thus, the measure of $\Omega_1$ must be zero, i.e., $w^* \equiv 0$ a.e. in $\Omega$. On the other hand, if we choose $2 < \omega_0 < \omega$, we have

$$\int_{\Omega} \left( \frac{\omega_0 F(x, u_m) - u_m f(x, u_m)}{u_m^2} + \frac{\omega_0 P(x, u_m) - u_m p(x, u_m)}{u_m^2} \right) w_m^2 dx \to \frac{\omega_0}{2} - 1. \quad (3.15)$$

However, by $(H_2)$ and $(H_4)$,

$$\omega_0 F(x, u) - f(x, u)u \leq -c|u|^{\omega} + c,$$

$$\omega_0 P(x, u) - p(x, u)u \leq c + c|u|^\sigma + 1$$
for all $u \in \mathbb{R}$ and a.e. $x \in \Omega$. Note that $\sigma + 1 < \omega$ and

$$\limsup_{m \to \infty} \left( \frac{\omega_0 F(x, u_m) - u_m f(x, u_m)}{u_m^2} u_m^2 + \frac{\omega_0 P(x, u_m) - u_m p(x, u_m)}{u_m^2} u_m^2 \right) \leq \limsup_{m \to \infty} \left( -c |u_m|^{\omega_0} + c \right) \frac{u_m^2}{u_m^2} \leq 0.$$  \hfill (3.16)

We observe that (3.15)–(3.16) imply that $\omega_0 - 2 \leq 0$, a contradiction. Therefore, $\{\|u_m\|_E\}$ and $\{\|u_m\|\}$ are both bounded, since both norms are equivalent on $E_k$. We may assume that

$$u_m \to u \text{ weakly in } E_k \text{ in both the } \| \cdot \| \text{ and } \| \cdot \|_E \text{ topologies;}$$

$$u_m \to u \text{ strongly in } L^2(\Omega); \ u_m(x) \to u(x) \text{ for a.e. } x \in \Omega.$$

On the other hand, by Lemma 3.3,

$$\|u\|^2 + \|u_m - u\|^2 - \mu \int_\Omega \frac{u^2}{|x|^2} \, dx - \mu \int_\Omega \frac{(u_m - u)^2}{|x|^2} \, dx$$

$$- \int_\Omega (f(x, u_m)u_m - p(x, u_m)u_m) \, dx$$

$$= \|u_m\|^2 + o(1) - \mu \int_\Omega \frac{u_m^2}{|x|^2} \, dx + \mu \int_\Omega \frac{u_m^2}{|x|^2} \, dx - \mu \int_\Omega \frac{u^2}{|x|^2} \, dx$$

$$- \mu \int_\Omega \frac{(u_m - u)^2}{|x|^2} \, dx - \int_\Omega (f(x, u_m)u_m - p(x, u_m)u_m) \, dx$$

$$= \|u_m\|^2 - \mu \int_\Omega \frac{u_m^2}{|x|^2} \, dx - \int_\Omega (f(x, u_m)u_m - p(x, u_m)u_m) \, dx + o(1)$$

$$= \langle I'(u_m), u_m \rangle_E + o(1)$$

$$\to 0.$$

Note that $I_k'(u) = 0$ because of the weak continuity of $I_k'$. In addition to this, we note that $\{f(x, u_m)u_m + p(x, u_m)u_m\}$ is uniformly integrable due to the boundedness of $\{u_m\}$. This implies

$$\|u_m - u\|^2_E$$

$$= \int_\Omega \left( f(x, u_m)u_m + p(x, u_m)u_m - f(x, u)u - p(x, u)u \right) \, dx + o(1)$$

$$= o(1).$$
Hence,

\[ \|u_m - u\|^2_2 \to 0 \]

as \( m \to \infty \). \( \square \)

**Lemma 3.5.** Under the assumptions of Theorem 3.1, there exist \( \rho_k > 0 \) and \( C_1 > 0 \) such that

\[ I(u) \geq C_1 \lambda_k^{s(1-2)/(s-2)} := \delta_k, \quad u \in Q(\rho_k) := \{ u \in E_{k-1}^+ : \|u\|_E := \rho_k \}, \]

where \( \alpha := n(1/2 - 1/s) \) and \( C_1 \) is independent of \( k \). Moreover, \( \rho_k \to \infty \) as \( k \to \infty \).

**Proof.** By (H1)–(H4), for any \( \varepsilon > 0 \) small enough, there exists a \( C_\varepsilon > 0 \) such that

\[ F(x, u) + P(x, u) \leq \varepsilon |u|^2 + C_\varepsilon |u|^s \quad \text{for all} \quad u \in \mathbb{R} \quad \text{and a.e.} \ x \in \Omega. \]

Recall the Gagliardo–Nirenberg inequality [N]:

\[ \|u\|_s \leq c_0 \|\nabla u\|^2 \|u\|_2^{1-\alpha} \quad \text{for all} \quad u \in H^1_0(\Omega), \]

where \( \alpha = n(1/2 - 1/s) \), and \( c_0 \) is a constant depending on \( s \) and \( n \). Note that

\[ \lambda_k \|u\|^2_2 \leq \|u\|^2_E \quad \text{for all} \quad u \in E_{k-1}^+. \]

For \( \varepsilon \) small enough, we have the following estimates:

\[ I(u) \geq \frac{1}{2} \|u\|^2_E - \int_{\Omega} (\varepsilon |u|^2 + C_\varepsilon |u|^s) \, dx \]

\[ \geq \frac{1}{4} \|u\|^2_E - c_1 \|\nabla u\|^2 \|u\|^2_2 \]

\[ \geq \frac{1}{4} \|u\|^2_E - c_2 \|u\|_E \lambda_k^{-s(1-2)/2} \]

\[ \geq \frac{1}{8} \rho_k^2 \]

for \( u \in E_{k-1}^+ \) with \( \|u\|_E := \rho_k \). \( \square \)

Given \( m > k + 2 \), let \( P_m = P \cap E_m \) be the positive cone in \( E_m \) and \( Q(\rho_k, m) := \{ u \in E_{k-1}^+ \cap E_m : \|u\|_E := \rho_k \} \). Since \( Q(\rho_k, m) \) is compact in \( E_m \) and includes only sign-changing elements, it is easy to check that

\[ \text{dist}(Q(\rho_k, m), \pm P_m) := d_m > 0. \quad (3.17) \]
For any $\mu_m \in (0, d_m/4)$, define
\begin{equation}
\mathcal{D}_0(m, \mu_m) := \{ u \in E_m : \text{dist}(u, P_m) < \mu_m \}. \tag{3.18}
\end{equation}

Then $\mathcal{D}_0(m, \mu_m)$ is open and convex in $E_m$, $\pm P_m \subset \pm \mathcal{D}_0(m, \mu_m)$ and
\begin{equation}
Q(p_k, m) \subset S_m := E_m \setminus \mathcal{D}_m \quad \text{where} \quad \mathcal{D}_m := -\mathcal{D}_0(m, \mu_m) \cup \mathcal{D}_0(m, \mu_m). \tag{3.19}
\end{equation}

Evidently, the gradient of $I_m := I|_{E_m}$ can be expressed as $I'_m = \text{id} - \text{Proj}_m K_I$, where $K_I : E \to E$ is given by $K_I u = (-\Delta - \mu/|x|^2)^{-1}(f(\cdot, u(\cdot)) + p(\cdot, u(\cdot)))$ for all $u \in E$, $\text{Proj}_m$ is the projection on $E_m$ from $E$; $\langle K_I u, w \rangle_E := \int_{\mathbb{R}^n} (f(x, u) + p(x, u)) w \, dx$ for all $u, w \in E$. If $p \equiv 0$, we denote $K_I$ by $K_G$.

**Lemma 3.6.** Under the assumptions of the $(H_1), (H_2)$ and $(H_4)$, there exists a $\mu_m \in (0, d_m/4)$ such that $\text{Proj}_m K_I(\pm \mathcal{D}_0(m, \mu_m)) \subset \pm \mathcal{D}_0(m, \mu_m)$ and $\text{Proj}_m K_G(\pm \mathcal{D}_0(m, \mu_m)) \subset \pm \mathcal{D}_0(m, \mu_m)$.

**Proof.** We modify an argument of [BLW]. Write $u^+ = \max\{u, 0\}, u^- = \min\{u, 0\}$. For any $u \in E_m$,
\begin{equation}
\|u^\pm\|_t = \min_{w \in (\mp P_m)} \|u - w\|_t \leq C_t \min_{w \in (\mp P_m)} \|u - w\|_E = C_t \text{dist}_E(u, \mp P_m), \tag{3.20}
\end{equation}

where $t \in [2, 2^*]$ and $C_t > 0$ is a constant. By assumptions $(H_1), (H_2)$, for each $\varepsilon' > 0$, there exists a $C_{\varepsilon'} > 0$ such that
\begin{equation}
f(x, u)u + p(x, u)u \leq \varepsilon' u^2 + C_{\varepsilon'}|u|^s \tag{3.21}
\end{equation}

for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$. Let $v = \text{Proj}_m K_I(u)$. In view of (3.20), (3.21) and the fact that $f(x, t)t \geq 0, p(x, t)t \geq 0$ for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$, we have for $\varepsilon' > 0$ small enough
\begin{align*}
\text{dist}_E(v, \mp P_m) \|v^\pm\|_E &\leq \|v^\pm\|_E^2 \\
&= \langle v, v^\pm \rangle_E \\
&= \int_{\mathbb{R}^n} (|f(x, u^\pm)| + |p(x, u^\pm)|)|v^\pm| \, dx \\
&\leq \int_{\mathbb{R}^n} (\varepsilon'|u^\pm| + C_{\varepsilon'}|u^\pm|^{s-1})|v^\pm| \\
&\leq \left( \frac{2}{5} \text{dist}_E(u, \mp P_m) + C \text{dist}_E(u, \mp P_m)^{s-1} \right) \|v^\pm\|_E;
\end{align*}
that is,
\[ \text{dist}_E \left( \text{Proj}_m K_I(u), \mp P_m \right) \leq \left( \frac{2}{5} \right) \text{dist}_E (u, \mp P_m) + C \text{dist}_E (u, \mp P_m)^{1-1}. \]

So there exists a \( \mu_m < d_m/4 \) such that \( \text{dist}_E (\text{Proj}_m K_I(u), \mp P_m) \leq \mu_m \) for every \( u \in \mp D_0 (m, \mu_m) \). The conclusion follows. □

**Lemma 3.7.**
\[
\lim_{u \in E_{k+1}, \|u\|_E \to \infty} G(u) = -\infty, \quad \lim_{u \in E_{k+1}, \|u\|_E \to \infty} I(u) = -\infty.
\]

**Proof.** By the definition of \( E_{k+1} \), \((H_2)\) and \((H_4)\), it is readily shown that
\[
\lim_{u \in E_{k+1}, \|u\|_E \to \infty} \frac{\int_{\Omega} F(x, u) \, dx}{\|u\|_E^2} = \infty; \quad \limsup_{u \in E_{k+1}, \|u\|_E \to \infty} \frac{\int_{\Omega} P(x, u) \, dx}{\|u\|_E^2} \leq \infty.
\]

The conclusions of the lemma follow immediately. □

**Lemma 3.8.** For each fixed \( m > 0 \), there exists a \( \bar{c} > 0 \) such that \( \|u\|_1 + \|u\|_{E_{k+1}} \leq \bar{c} d \) for all \( u \in \pm U_\delta \cap \{ u \in E_m : I_m(u) \leq d \} \), where \( \bar{c} \) is independent of \( m, d > 0 \) and
\[
U_\delta := \left\{ u \in E_m : \|I_m'(u) - G_m'(u)\|_E > \frac{\|I_m'(u)\|_E}{\delta} \right\}.
\] (3.22)

**Proof.** Consider first the case, \( u \in U_\delta \cap \{ u : I_m(u) \leq d \} \). We have that
\[
\frac{1}{2} \|u\|_E^2 - \int_{\Omega} F(x, u) \, dx - \int_{\Omega} P(x, u) \, dx \leq d, \quad \|I_m'(u)\|_E < \delta \|I_m'(u) - G_m'(u)\|_E \quad (3.24)
\]

and
\[
|\langle I_m'(u), u \rangle_E| = \left| \|u\|_E^2 - \int_{\Omega} f(x, u)u \, dx - \int_{\Omega} p(x, u)u \, dx \right| \leq \|I_m'(u)\|_E \|u\|_E \leq \delta (\|I_m'(u) - G_m'(u)\|_E) \|u\|_E. \quad (3.25)
\]

Since \( \|I_m'(u) - G_m'(u)\|_E \leq c (\|u\|_2 + \|u\|_{E_0}^\gamma) \) (here and later the constant \( c \) is independent of \( m, d \)), we have by (3.25) that
\[
-\|u\|_E^2 \leq -\int_{\Omega} f(x, u)u \, dx - \int_{\Omega} p(x, u)u \, dx + c \|u\|_2 \|u\|_E + c \|u\|_{E_0}^\gamma \|u\|_E. \quad (3.26)
\]
Choose \( \omega_0 \in (2, \omega) \). By (3.23), (3.24) and \((H_2)\), we see that

\[
\left( \frac{\omega_0}{2} - 1 \right) \|u\|^2_E \leq \int_\Omega (\omega_0 F(x, u) - f(x, u)u) \, dx + \int_\Omega (\omega_0 P(x, u) - p(x, u)u) \, dx
\]
\[
+ c \|u\|_2 \|u\|_E + \omega_0 d + c \|u\|_{\omega_0}^\sigma \|u\|_E.
\]

This implies that

\[
\left( \frac{\omega_0}{2} - 1 \right) \|u\|^2_E + c \|u\|_{\omega_0}^\sigma 
\leq \left( \frac{\omega_0}{2} - 1 \right) \|u\|^2_E + \int_\Omega (f(x, u) - \omega_0 F(x, u)) \, dx + c \quad \text{(by \((H_2)\))}
\leq \int_\Omega (\omega_0 P(x, u) - p(x, u)u) \, dx
\]
\[
+ c \|u\|_2 \|u\|_E + \omega_0 d + c \|u\|_{\omega_0}^\sigma \|u\|_E + c
\leq c \|u\|^2_2 + c \|u\|_{\omega_0}^\sigma + c \|u\|_2 \|u\|_E + c \|u\|_{\omega_0}^\sigma \|u\|_E + \omega_0 d + c.
\]

Since \( \|u\|^2_2 \) and \( \|u\|_{\omega_0}^{\sigma+1} \) can be absorbed by \( \|u\|_{\omega_0}^\sigma \), it follows that

\[
c \|u\|^2_E + c \|u\|_{\omega_0}^\sigma \leq c \|u\|^2_2 \|u\|_E + c \|u\|_{\omega_0}^\sigma \|u\|_E + \omega_0 d + c.
\]

Here the constant \( c \) is independent of \( m, d \). By Cauchy’s inequality and the fact that \( 2\sigma < \omega \), we get \( \|u\|_{1+\sigma} \leq \tilde{c}d^{1/\omega} \).

Next, we turn to the second case, \( u \in -U_\delta \cap \{ u : I_m(u) \leq d \} \), that is, \( \|I'_m(-u)\|_E < \delta \|I'_m(-u) - G'_m(-u)\|_E \) and \( I_m(u) \leq d \). Then

\[
I_m(-u) = I_m(u) + \int_\Omega (P(x, u) - P(x, -u)) \, dx \leq d + c \|u\|^2_2 + c \|u\|_{\sigma+1}^{\sigma+1},
\]

\[
\frac{1}{2} \|u\|^2_E - \int_\Omega F(x, -u) \, dx - \int_\Omega P(x, -u) \, dx \leq d + c \|u\|^2_2 + c \|u\|_{\sigma+1}^{\sigma+1}, \quad (3.27)
\]

\[
\|I'_m(-u)\|_E \leq \delta \|I'_m(-u) - G'_m(-u)\|_E \quad (3.28)
\]

and

\[
\|I'_m(-u), -u\|_E = \left| \|u\|^2_E + \int_\Omega f(x, -u) u \, dx + \int_\Omega p(x, -u) u \, dx \right| \leq \|I'_m(-u)\|_E \|u\|_E \leq \delta(\|I'_m(-u) - G'_m(-u)\|_E) \|u\|_E. \quad (3.29)
\]
Note that \( \|I'_m(-u) - G'_m(-u)\|_E \leq c\|u\|_2 + c\|u\|_{G_0}^o \). Combining this with (3.29), we have that
\[
-\|u\|_E^2 \leq \int_{\Omega} f(x, -u) u \, dx + \int_{\Omega} p(x, -u) u \, dx + c\|u\|_2^2\|u\|_E + c\|u\|_{G_0}^o\|u\|_E.
\] (3.30)

Therefore, by (3.27)–(3.30), (H2) and (H4),
\[
\left(\frac{\omega_0}{2} - 1\right)\|u\|_E^2 + c\|u\|_G^o
\leq\left(\frac{\omega_0}{2} - 1\right)\|u\|_E^2 + \int_{\Omega} (-\omega_0 F(x, -u) - f(x, -u)u) \, dx + c
\leq\int_{\Omega} (\omega_0 P(x, -u) + p(x, -u)u) \, dx + c\|u\|_2\|u\|_E + \omega_0 d + c\|u\|_2^2
+c\|u\|_{G_0}^{\sigma+1} + c\|u\|_G^o\|u\|_E
\leq c\|u\|_2^2 + c\|u\|_2\|u\|_E + \omega_0 d + c\|u\|_2^2 + c\|u\|_{G_0}^{\sigma+1} + c\|u\|_G^o\|u\|_E,
\]
where the constant \( c \) is independent of \( m, d \). This gives the desired conclusion. \( \square \)

**Lemma 3.9.** Assume that \( u_m \in E_m \) is sign-changing and satisfies
\[
I'_m(u_m) = 0, \quad \sup_{m \geq 1} |I_m(u_m)| < \infty.
\]

Then \( \{u_m\} \) has a convergent subsequence whose limit is a sign-changing critical point of \( I \).

**Proof.** The proof of the existence of the convergent subsequence of \( \{u_m\} \) is the same as the proof of the (PS) condition in Lemma 3.4. We just prove that the limit of the subsequence is also sign-changing. Let \( u_m^\pm := \max\{\pm u_m, 0\} \). Then
\[
\|u_m^\pm\|_E^2 = \int_{\Omega} (f(x, u_m^\pm) u_m^\pm + p(x, u_m^\pm) u_m^\pm) \, dx.
\]

By \( (H_1)\)–\( (H_2) \), for any \( \varepsilon > 0 \), there exists a \( C_\varepsilon > 0 \) such that
\[
f(x, u)u + p(x, u)u \leq \varepsilon |u|^2 + C_\varepsilon |u|^s \quad \text{for all } u \in \mathbb{R} \text{ and a.e. } x \in \Omega.
\]

It follows that
\[
\|u_m^\pm\|_E^2 \leq \varepsilon\|u_m^\pm\|_E^2 + C\|u_m^\pm\|_s^2.
\]
Hence, $\|u_m^+\|_E \geq s_0 > 0$, where $s_0$ is a constant independent of $m$. This implies that the limit of the subsequence is also sign-changing. □

**Proof of Theorem 3.1.** Assume that there exists a $C_0 > 0$ such that $I$ has no sign-changing critical point with critical value greater than $C_0$. Choose $k_0 > 0$ such that $\delta_k > C_0$ for all $k > k_0$, where $\delta_k$ comes from Lemma 3.5. Let $m > k + 2 > k_0 + 2$. Then $E_k \subset E_m$. Consider $I_m := I|_{E_m}$. Let

$$N := E_k, M(m) = E_k^\perp \cap E_m, Q(\rho_k, m) := \{u \in M(m) : \|u\|_E = \rho_k\}.$$

Then by (3.17) and (3.19),

$$Q(\rho_k, m) \subset S_m.$$

Define

$$N^* = N \oplus \text{span}\{u^*\}, \quad u^* \in E_{k+1}, u^* \notin E_k;$$

$$N^*_+ := \{u + tu^* : u \in N, t \geq 0\}.$$

Then $N^* \cap E_{k+1} \neq \{0\}$, and both $N^*$ and $N^*_+$ are independent of $m$. We want to apply Theorem 2.1 to $I_m$. Evidently, by Lemma 3.7,

(i) $I_m(0) = 0$;
(ii) there exists a $R_1 > \rho_k$ independent of $m$ such that $I_m(u) \leq 0$ for all $u \in N$ with $\|u\| \geq R_1$;
(iii) there exists a $R_2 \geq R_1 > 0$ independent of $m$ such that $I_m(u) \leq 0$ for all $u \in N^*$ with $\|u\| \geq R_2$.

Let

$$\Gamma_m := \{\phi \in C(E_m, E_m) : \phi \text{ is odd }, \phi(D_m) \subset D_m, \phi(u) = u \text{ if } \max\{I_m(u), I_m(-u)\} \leq 0\}.$$

Define

$$\gamma^*_k(m) := \inf_{\phi \in \Gamma_m} \sup_{\phi(N^*_+ \cap S_m)} I_m, \quad \gamma^{**}_k(m) := \inf_{\phi \in \Gamma_m} \sup_{\phi(N \cap S_m)} I_m. \quad (3.31)$$

For any $\phi \in \Gamma_m$, by the Intersection Theorem (cf. Proposition 9.23 of [R2]), $\phi(N \cap B_{R_1}) \cap Q(\rho_k, m) \neq \emptyset$. Note that $Q(\rho_k, m) \subset S_m$, and recall Lemma 3.5. Thus, we have

$$\sup_{\phi(N \cap B_{R_1}) \cap S_m} I_m \geq \inf_{Q(\rho_k, m)} I_m \geq C_1 \delta_k s^{(1-2)/(s-2)} = \delta_k.$$

For any $\phi \in \Gamma_m$, by the Intersection Theorem (cf. Proposition 9.23 of [R2]), $\phi(N \cap B_{R_1}) \cap Q(\rho_k, m) \neq \emptyset$. Note that $Q(\rho_k, m) \subset S_m$, and recall Lemma 3.5. Thus, we have

$$\sup_{\phi(N \cap B_{R_1}) \cap S_m} I_m \geq \inf_{Q(\rho_k, m)} I_m \geq C_1 \delta_k s^{(1-2)/(s-2)} = \delta_k.$$
where \( x := n(1/2 - 1/s) \) and \( C_1 \) are independent of \( k, m \). Hence,

\[
\gamma^{**}_k(m) = \inf_{\phi \in \Gamma_m} \sup_{\phi(N) \cap S_m} I_m \geq C_1 \lambda_k s^{(1-2)/(s-2)} = \delta_k \to \infty \quad \text{as} \quad k \to \infty. \tag{3.32}
\]

We consider two cases.

**Case 1:** For \( k \geq k_0 \), if there exists a sequence \( m_i \to \infty \) as \( i \to \infty \) such that

\[
\gamma_k^*(m_i) > \gamma^{**}_k(m_i), \quad \text{for all} \quad i > 1,
\]

then by Theorem 2.1, there exists a sign-changing critical point \( u_{m_i} \) such that

\[
I'_{m_i}(u_{m_i}) = 0,
\]

\[
C_0 < \delta_k \leq \gamma^{**}_k(m_i) \leq I(u_{m_i}) \leq \sup_{N^*} I + 1.
\]

Here \( \sup_{N^*} I \) is a constant depending on \( k \) and independent of \( m_i \). By Lemma 3.9, \( \{u_{m_i}\} \) has a convergent subsequence whose limit \( u \) is a sign-changing critical point of \( I \), and \( I(u) \geq \delta_k > C_0 \). This contradicts the assumption.

**Case 2:** For \( k \geq k_0 \), there exists an \( m_k \) such that

\[
\gamma_k^*(m) = \gamma^{**}_k(m), \quad \text{for all} \quad m > m_k. \tag{3.33}
\]

Let \( K_{\text{com}}(m) \) denote the set of common critical points of \( G_m \) and \( I_m \). By \((H_4)\), \( K_{\text{com}}(m) = \{0\} \). Define

\[
V_{\delta} := \{ u \in E_m : \| u \|_E \leq \delta \}
\]

and let \( U_{\delta} \) be as in (3.22), containing all non-common critical points of \( G_m \) and \( I_m \). By Lemma 3.7, there exists a \( R_1 > \rho_k \) such that \( I_m(u) \leq 0 \) for all \( u \in N \) with \( \| u \| \geq R_1 \). Here \( R_1 \) is independent of \( m \). Combining the definition of \( \gamma_k^*(m) \) and (3.32), we find a \( \phi_0 \in \Gamma_m \) such that

\[
\sup_{\phi_0(N^*_m) \cap S_m} I = \sup_{\phi_0(N^*_m \cap B_R) \cap S_m} I \leq \gamma_k^*(m) + \frac{1}{\delta}. \tag{3.34}
\]

Let \( U_{\delta}^*(m) := V_{\delta} \cup U_{\delta} \cup (-U_{\delta}) \). Then \( U_{\delta}^* \) is a symmetric set and contains all critical points of \( G_m \) and \( I_m \). Define two non-negative continuous functions:

\[
\zeta_1(u) = \begin{cases} 0, & u \in U_{10}^*(m), \\ 1, & u \notin U_{12}^*(m), \quad (is \ even), \end{cases} \quad \zeta_2(u) = \begin{cases} 0, & u \leq 0, \\ 1, & u \geq 1, \end{cases}
\]
and a vector field

\[ V_m^*(u) = -\zeta_2 \left( \max\{I_m(u), I_m(-u)\} \right) \zeta_1(u) V_m(u), \]

where the pseudo gradient vector field \( V_m \) comes from Lemma 2.1 obtained for \( G_m \) (recall Lemma 3.6). Then \( V_m \) is odd since \( G_m \) is even. Hence, \( V_m^* \) is odd.

Let \( \sigma(t, u) \) denote the unique (odd in \( u \)) solution of the Cauchy initial value problem:

\[ \frac{d\sigma(t, u)}{dt} = V_m^*(\sigma(t, u)), \quad \sigma(0, u) = u \in E_m. \]

Then

\[ \frac{dI_m(\sigma(t, u))}{dt} \leq 0. \quad (3.35) \]

For any \( u \notin U_{\delta}^*(m) \), we have \( u \notin \pm U_{\delta} \), and by (3.22), we have

\[ \|G'_m(u)\| \leq \frac{\delta + 1}{\delta} \|I'_m(u)\|, \quad \|I'_m(u)\| \leq \frac{\delta}{\delta - 1} \|G'_m(u)\|. \]

Further, for all \( u \notin U_{\delta}^*(m) \),

\[ \langle I'_m(u), V_m(u) \rangle_E = \langle G'_m(u), V_m(u) \rangle_E - \langle G'_m(u) - I'_m(u), V_m(u) \rangle_E \]
\[ \geq \frac{1}{2} \|G'_m(u)\|_E^2 - 2\|G'_m(u)\|_E \|G'_m(u) - V_m(u)\|_E \]
\[ \geq \frac{(\delta - 1)^2 - 4(\delta + 1)}{2\delta^2} \|I'_m(u)\|^2 \]

and

\[ \|V_m(u)\| \leq 2\|G'_m(u)\| \leq \frac{2(\delta + 1)}{\delta} \|I'_m(u)\|. \]

Moreover, since \( u \notin U_{12}^*(m) \) implies \( \zeta_1(u) = 1 \), we see that \( I_m(u) > 1 \) implies

\[ \zeta_2 \left( \max\{I_m(u), I_m(-u)\} \right) = 1. \]
We have
\[
\frac{dI_m(\sigma(t, u))}{dt} \bigg|_{t=0} = \left( I'_m(\sigma(t, u)), \frac{d\sigma}{dt} \right)_E \bigg|_{t=0} \\
= \langle I'_m(\sigma(t, u)), V^*_m(\sigma(t, u)) \rangle_E \bigg|_{t=0} \\
= \left( I'_m(u), -\zeta_2 \left( \max\{I_m(\sigma(t, u)), I_m(-\sigma(t, u))\} \right) \zeta_1(\sigma(t, u)) V_m(\sigma(t, u)) \right)_E \\
= -\langle I'_m(u), V_m(u) \rangle_E \\
\leq - \frac{69}{288} \| I'_m(u) \|^2 
\]
for all \( u \notin U_{12}(m) \) satisfying \( I_m(u) > 1 \).

We claim that \( \sigma(t, \phi_0(\cdot)) \in \Gamma_m \) for any \( t \geq 0 \). In fact, \( \sigma(t, \phi_0(u)) \) is odd in \( u \) since \( \phi_0 \) and \( V^*_m \) are odd. Recall that \( \phi_0 \in \Gamma_m \). Thus, \( \phi_0(u) = u \) for \( u \) with \( \max\{I(u), I(-u)\} \leq 0 \). Hence, \( \sigma(t, \phi_0(u)) = \sigma(t, u), V^*_m(u) = 0 \). It follows that \( \sigma(t, u) = u \). As in the proof of (2.7) of Theorem 2.1, we conclude that
\[
\sigma(t, \phi_0(D_m)) \subset \sigma(t, D_m) \subset D_m, \quad \forall t \geq 0.
\]
Therefore, \( \sigma(t, \phi_0(u)) \in \Gamma_m \) for all \( t \geq 0 \). For any \( t \geq 0 \), we have the following estimates which lead to a contradiction.

\[
\gamma^*_k(m) + \frac{1}{2} \\
= \gamma^{**}_k(m) + \frac{1}{2} \quad \text{(by (3.33))} \\
\geq \sup_{\phi_0(N^+_m \cap B_{R_1}) \cap S_m} I \quad \text{(by (3.34))} \\
= \sup_{\phi_0(N^+_m \cap S_m)} I \quad \text{(by (3.34))} \\
\geq \sup_{\sigma(t, \phi_0(N^+_m)) \cap S_m} I \\
\geq \gamma^{**}_{k+1}(m) - \sup_{u \in \sigma(t, \phi_0(N^+_m)) \cap S_m} (I(-u) - I(u)) \quad \text{(since } \sigma(t, \phi_0(u)) \in \Gamma_m) 
\]
\[ \gamma_{k+1}^{**}(m) \geq \sup_{u \in \sigma(t, \phi_0(N^+_k)) \cap S_m \cap \{ u \in E_m : I_m(u) \leq \gamma_k^{**}(m) + 1/2 \}} (I(-u) - I(u)) \]
\[ \times \gamma_{k+1}^{**}(m) \geq \sup_{u \in \sigma(t, \phi_0(N^+_k)) \cap \{ u \in E_m : I_m(u) \leq \gamma_k^{**}(m) + 1/2 \}} (I(-u) - I(u)) \]
\[ \geq \gamma_{k+1}^{**}(m) - \sup_{u \in \sigma(t, \phi_0(N^+_k)) \cap \{ u \in E_m : I_m(u) \leq \gamma_k^{**}(m) + 1/2 \}} |I(-u) - I(u)| \] (by \(3.37\))
\[ \geq \gamma_{k+1}^{**}(m) - \sup_{u \in \sigma(t, \phi_0(N^+_k)) \cap \{ u \in E_m : I_m(u) \leq \gamma_k^{**}(m) + 1/2 \}} |I(-u) - I(u)| - c \]
\[ \times (\text{since } V_{12} \text{ is bounded}) \]
\[ \geq \gamma_{k+1}^{**}(m) - \sup_{u \in \sigma(t, \phi_0(N^+_k)) \cap \{ u \in E_m : I_m(u) \leq \gamma_k^{**}(m) + 1/2 \}} c \|u\|_{1+\sigma}^{1+\sigma} - c \]
\[ \geq \gamma_{k+1}^{**}(m) \geq \gamma_k^{**}(m) - c \gamma_k^{**}(m)^{(1+\sigma)/\omega} - c, \quad \text{(by Lemma 3.8)} \] (3.38)

where \(c\) is a constant independent of \(k, m\). Therefore, \(\gamma_{k+1}^{**}(m) \leq \gamma_k^{**}(m) + c(\gamma_k^{**}(m))^{(1+\sigma)/\omega}\). Hence, by iterating, we have
\[ \gamma_k^{**}(m) \leq c k^{\omega/(\omega-(1+\sigma))}. \] (3.39)

However, recall that by Lemma 3.1 and (3.32),
\[ \gamma_k^{**}(m) \geq C_1 \gamma_k^{(1-2)/(s-2)} \geq c k^{(2s-ns+2n)/(n(s-2))}, \]
which contradicts (3.39) in view of (3.2). \[\square\]

**Proof of Theorem 3.2.** It suffices to rewrite \(f(x, u)\) as follows
\[ f(x, u) = \frac{f(x, u) - f(x, -u)}{2} + \frac{f(x, u) + f(x, -u)}{2}. \] \[\square\]

**Proofs of Corollary 3.1.** This is trivial by letting \(\sigma = 0\) in Theorem 3.2. \[\square\]

**4. Application (II)**

Consider the following equation
\[ -\Delta u = \frac{|u|^{q-2}}{|x|^s} u + p(x, u) \quad \text{in } \Omega; \quad u = 0 \quad \text{on } \partial\Omega, \] (4.1)
where \( \Omega \subset \mathbb{R}^n (n \geq 3) \) is an open bounded domain with smooth boundary containing the origin \( 0, \ 0 \leq s < 2, \ 2 < q < 2^*(s) \), and \( 2^*(s) := \frac{n-s}{n-2} \) is the Hardy critical exponent. The main result of this section is the following theorem.

**Theorem 4.1.** Assume \((H_4)\) (see Section 3) with

\[
0 \leq \sigma < \min \left\{ q - 1 - \frac{qn(q-2)}{2q-2s-nq+2n}, \ q/2 \right\}.
\]

(4.2)

Then Eq. (4.1) has an unbounded sequence of sign-changing solutions.

Obviously, (4.2) becomes \( 0 \leq \sigma < q/2 \) if \( n \) is large enough.

Let \( E := H_0^1(\Omega) \) be the usual Hilbert space with inner product \( \langle u, v \rangle = \int_{\Omega} \nabla u \cdot \nabla v \, dx \) and norm \( \| u \| = \langle u, u \rangle^{1/2} \). Define

\[
G(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \frac{1}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} \, dx
\]

and

\[
I(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \frac{1}{q} \int_{\Omega} \frac{|u|^q}{|x|^s} \, dx - \int_{\Omega} P(x, u) \, dx, \quad u \in E.
\]

Then \( G \) and \( I \) are in \( C^1(E, \mathbb{R}) \). Let \( 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \) be the eigenvalues of \( -\Delta \) with zero boundary condition, and let \( N_k \) be the eigenspace of \( \lambda_k \). Then \( \dim N_k < \infty \). Consider an approximation of \( E: E_1 \subset E_2 \subset \cdots \subset E_k \subset \cdots \subset E_m \subset \cdots \), where \( \dim E_k < \infty \) for each \( k \). For each \( k > 2 \), define

\[
G_k := G|_{E_k}, \quad I_k := I|_{E_k}.
\]

Then \( G_k, I_k \in C^1(E_k, \mathbb{R}) \).

**Lemma 4.1.** Under the assumptions of Theorem 4.1, \( I_k \) (and hence \( G_k \)) satisfies the \((PS)\) condition in \( E_k \).

**Proof.** Assume that \( \{u_m\} \subset E_k \) is a \((PS)\) sequence: \( \sup_{m \geq 1} |I_k(u_m)| \) is bounded and \( I'_k(u_m) \to 0 \) as \( m \to \infty \). Then, for a renamed subsequence, 

\[
I_k(u_m) = \frac{1}{2} \int_{\Omega} |\nabla u_m|^2 \, dx - \frac{1}{q} \int_{\Omega} \frac{|u_m|^q}{|x|^s} \, dx - \int_{\Omega} P(x, u_m) \, dx = c + o(1)
\]

(4.5)
and
\[
\langle I_k'(u_m), u_m \rangle = \int_\Omega |\nabla u_m|^2 \, dx - \int_\Omega \frac{|u_m|^q}{|x|^s} \, dx - \int_\Omega p(x, u_m) u_m \, dx = o(1)\|u_m\|.
\]
(4.6)

Since \(1 + \sigma < q\), we take \(q_0 \in (1 + \sigma, q)\). Also, since \(\Omega\) is bounded,
\[
\int_\Omega |u|^{q_0} \, dx = \int_\Omega \frac{|u|^{q_0}}{|x|^{q_0s/q}} |x|^{q_0s/q} \, dx \\
\leq c \left( \int_\Omega \frac{|u|^q}{|x|^s} \, dx \right)^{q_0/q}.
\]
Therefore,
\[
\|u\|_{q_0}^q \leq c \int_\Omega \frac{|u|^q}{|x|^s} \, dx \quad \text{for all } u \in E,
\]
(4.7)

where \(c\) is a constant. By (4.5)–(4.7) and \((H_4)\),
\[
c\|u_m\|_{q_0}^q \leq \left( \frac{1}{2} - \frac{1}{q} \right) \int_\Omega \frac{|u_m|^q}{|x|^s} \, dx \\
= c + o(1)\|u_m\| + \int_\Omega \left( P(x, u_m) - \frac{1}{2} p(x, u_m) u_m \right) \, dx \\
\leq c + o(1)\|u_m\| + c\|u_m\|^{1+\sigma}.
\]
Hence, \(\|u_m\|_{q_0}^q \leq c + o(1)\|u_m\|\). By (4.5) and (4.6) again, we have
\[
\left( \frac{1}{2} - \frac{1}{q} \right) \|u_m\|^2 \\
= c + \int_\Omega \left( P(x, u_m) - \frac{1}{q} p(x, u_m) u_m \right) \, dx \\
+ o(1)\|u_m\| + o(1) \\
\leq c + o(1)\|u_m\| + c\|u_m\|^{1+\sigma} \\
\leq c + o(1)\|u_m\|.
\]

It follows that \(\{u_m\}\) is bounded. By the compactness of the Hardy–Sobolev embedding, \(\{u_m\}\) has a convergent subsequence. □
Lemma 4.2. There are constants $C_1, C_2 > 0$ (independent of $k$) such that
\[
I(u) \geq C_1 \lambda_k^{\frac{(q-s)(1-\frac{q}{2})}{q-2}}
\]
for $u \in E_{k-1}^\perp$ with
\[
\|u\| = \rho_k := \left( C_2 \left( \frac{1}{\lambda_k} \left( \frac{1}{2} \right) + \frac{1-\beta q}{2} \right) \right)^{-1/(q-2)},
\]
where $\lambda = \frac{n(q-2)}{2(q-s)} \in (0, 1)$. In particular, $\rho_k \to \infty$ as $k \to \infty$.

Proof. By the Hölder, Hardy (cf., e.g., [GY]) and Gagliardo–Nirenberg (cf. [N]) inequalities, we have that
\[
\int \frac{|u|^q}{|x|^s} \, dx = \int \frac{|u|^s}{|x|^s} |u|^{q-s} \, dx 
\leq \left( \int \frac{|u|^2}{|x|^2} \, dx \right)^{s/2} \left( \int |u|^{2(q-s)/(2-s)} \, dx \right)^{(2-s)/2}
\leq C_n \|u\|^s \left( \int |u|^{2(q-s)/(2-s)} \, dx \right)^{(2-s)/2}
\leq C_n \|u\|^s \|u\|^{q-s \alpha} \|u\|^{(q-s)(1-\alpha)}
\]
where $C_n$ is a constant depending on $n$ only, and $\alpha = \frac{n(q-2)}{2(q-s)} \in (0, 1)$ is a constant from the Gagliardo–Nirenberg inequality. Meanwhile, for any $\varepsilon > 0$, by $(H_4)$, there is a constant $C_\varepsilon > 0$ such that
\[
\int \Omega P(x, u) \, dx \leq \varepsilon \|u\|^2 + C_\varepsilon \|u\|_q^q \leq \varepsilon \|u\|^2 + C_\varepsilon \|u\|^{\beta q} \|u\|_2^{(1-\beta)q},
\]
where $\beta = n(1/2 - 1/q) \in (0, 1)$ is a constant from the Gagliardo–Nirenberg inequality. Since $\lambda_k \|u\|^2 \leq \|u\|^2$ for all $u \in E_{k-1}^\perp$, we have the following estimates:
\[
I(u) \geq \frac{1}{2} \|u\|^2 - \frac{1}{q} C_n \|u\|^q \lambda_k^{-(q-s)(1-\frac{q}{2})/2} - \varepsilon \|u\|^2 - C_\varepsilon \|u\|^{\beta q} \|u\|_2^{(1-\beta)q/2}
\geq \frac{1}{8} \rho_k^2
\geq C_1 \lambda_k^{\frac{(q-s)(1-\frac{q}{2})}{q-2}}
\]
for \( u \in E_{k-1}^\perp \) with

\[
\|u\| = \rho_k := \left( C_2 \left( \gamma_k^{-(q-2)(1-s)} + \gamma_k^{-(1-\beta q)/(2-q)} \right) \right)^{-1/(q-2)},
\]

where \( C_1 > 0, C_2 > 0, \) and \( \alpha \in (0, 1) \) are independent of \( k. \) Evidently, \( \rho_k \to \infty \) as \( k \to \infty. \)

Let \( m > k + 2 \) and \( E_m, Q(\rho_k, m), P_m, \mu_m, D_0(m, \mu_0) \) be defined as in Section 3, where \( \rho_k \) comes from Lemma 4.2. Consider \( I_m := I|_{E_m}, \; G_m := G|_{E_m}. \) Then

\[
I'_m(u) = u - \text{Proj}_m K_I u, \quad G'_m(u) = u - \text{Proj}_m K_G u, \; u \in E_m,
\]

where

\[
K_I(u) = (-\Delta)^{-1}(\|u\|^{q-2}u/|x|^s + p(x, u)); \quad K_G(u) = (-\Delta)^{-1}(\|u\|^{q-2}u/|x|^s).
\]

We have

**Lemma 4.3.** There exists a \( \mu_m \in (0, d_m/4) \) such that

\[
\text{Proj}_m K_I(\pm D_0(m, \mu_m)) \subset \pm D_0(m, \mu_m)
\]

and

\[
\text{Proj}_m K_G(\pm D_0(m, \mu_m)) \subset \pm D_0(m, \mu_m).
\]

**Proof.** The proof is similar to that of Lemma 3.4. However, since it involves Hardy’s potential, there are still things to be done. First, we have, for any \( u \in E_m, \)

\[
\|u^\pm\|_r = \min_{w \in (\mp P_m)} \|u - w\|_r \leq C_I \min_{w \in (\mp P_m)} \|u - w\| = C_I \text{dist}(u, \mp P_m) \quad (4.8)
\]

for each \( t \in [2, 2^*], \) where \( C_I > 0 \) is a constant depending on \( t. \) By Hardy’s inequality (cf., e.g., [GY]), we have that

\[
\left\| \frac{u^\pm}{|x|} \right\|_2 = \min_{w \in (\mp P_m)} \left\| \frac{u}{|x|} - w \right\|_2 \leq \min_{w \in (\mp P_m)} \left\| \frac{u}{|x|} - w \right\|_2 \leq c \text{dist}(u, \mp P_m). \quad (4.9)
\]
Let \( v = \text{Proj}_m K_G(u) \). Then by (4.8)–(4.9),
\[
\text{dist}(v, \mp P_m) \|v^\pm\| \\
\leq \|v^\pm\|^2 \\
= (v, v^\pm) \\
= \int_\Omega \left( \frac{|u|^{q-2}}{|x|^s} u + p(x, u) \right) v^\pm \, dx \\
\leq \int_\Omega \left( \frac{|u^\pm|^{q-1}}{|x|^s} + |p(x, u^\pm)| \right) |v^\pm| \, dx \\
\leq \left( \int_\Omega \frac{|u|^q}{|x|^s} \, dx \right)^{(q-1)/q} \left( \int_\Omega \frac{|v^\pm|^q}{|x|^s} \, dx \right)^{1/q} + \varepsilon \int_\Omega |u^\pm| |v^\pm| \, dx \\
+ C\varepsilon \int_\Omega |u^\pm|^{q-1} |v^\pm| \, dx \\
\leq \left( \left( \int_\Omega \frac{|u^\pm|^2}{|x|^s} \, dx \right)^{s/2} \left( \int_\Omega \frac{|v^\pm|^2}{|x|^s} \, dx \right)^{(2-s)/2} \right)^{(q-1)/q} \left( \int_\Omega \frac{|v|^q}{|x|^s} \, dx \right)^{1/q} \\
+ \varepsilon c \text{dist}(u, \mp P_m) \|v^\pm\| + (c \text{dist}(u, \mp P_m))^{q-1} \|v^\pm\| \\
\leq \left( \varepsilon c \text{dist}(u, \mp P_m) + c \text{dist}(u, \mp P_m)^{(q-1)} \right) \|v^\pm\|.
\]
Since \( q - 1 > 1 \), we may chose \( \mu_m < d_m/4 \) and \( \varepsilon \) small enough so that
\[
\text{dist}\left(\text{Proj}_m K_I(u), \mp P_m\right) \leq \mu_m
\]
for every \( u \in \mp D_0(m, \mu_m) \). The conclusion follows. \( \Box \)

**Lemma 4.4.** \( \lim_{u \in E_{k+1}, \|u\|_E \to \infty} G(u) = -\infty \), \( \lim_{u \in E_{k+1}, \|u\|_E \to \infty} I(u) = -\infty \).

**Proof.** By (4.7),
\[
I(u) \leq G(u) \leq \frac{1}{2} \|u\|^2 - c \|u\|^{q_0}. 
\]
Then the conclusions of the lemma are obvious since \( \dim E_{k+1} < \infty \). \( \Box \)

Choose \( q_0 \) such that \( 1 + \sigma < q_0 < q \), \( 2\sigma < q_0 < q \), and \( 2 < q_0 < q \). Then we have the following lemma.
Lemma 4.5. For each fixed $m > 0$, there exists a $\tilde{c} > 0$ such that $\|u\|_{q_0} \leq \tilde{c}d^{1/q}$ for all $u \in \pm U_\delta \cap \{u \in E_m : I_m(u) \leq d\}$, where $\tilde{c}$ is independent of $m, d > 0$ and

$$U_\delta := \left\{ u \in E_m : \|I'_m(u) - G'_m(u)\| > \frac{\|I'_m(u)\|}{\delta} \right\}. \quad (4.10)$$

Proof. Consider the first case, $u \in U_\delta \cap \{u : I_m(u) \leq d\}$. Then

$$\frac{1}{2} \|u\|^2 - \frac{1}{q} \int_\Omega \frac{|u|^q}{|x|^s} \, dx - \int_\Omega P(x, u) \, dx \leq d, \quad (4.11)$$

$$\|I'_m(u)\| < \delta \|I'_m(u) - G'_m(u)\|, \quad (4.12)$$

$$|\langle I'_m(u), u \rangle| = \|u\|^2 - \frac{1}{q} \int_\Omega \frac{|u|^q}{|x|^s} \, dx - \int_\Omega p(x, u) u \, dx \leq \|I'_m(u)\| \|u\| \leq \delta(\|I'_m(u) - G'_m(u)\| \|u\|). \quad (4.13)$$

By a simple calculation, $\|I'_m(u) - G'_m(u)\| \leq c(\|u\|_2 + \|u\|^{2\sigma})$. By (4.10) we have that

$$-\|u\|^2 \leq - \int_\Omega \frac{|u|^q}{|x|^s} \, dx - \int_\Omega p(x, u) u \, dx + c\|u\|_2 \|u\| + c\|u\|^{2\sigma} \|u\|. \quad (4.14)$$

By (4.11)–(4.13),

$$\left( \frac{q_0}{2} - 1 \right) \|u\|^2 \leq \int_\Omega \left( \frac{q_0}{q} - 1 \right) \frac{|u|^q}{|x|^s} \, dx + \int_\Omega (q_0 P(x, u) - p(x, u) u) \, dx$$

$$+ c\|u\|_2 \|u\| + q_0 d + c\|u\|^\sigma \|u\|. \quad (4.15)$$

It follows by (4.7) and $(H_4)$ that

$$c\|u\|^2 + c\|u\|_{q_0}^q \leq c\|u\|_2^2 + c\|u\|^{\sigma+1} + c\|u\|_2 \|u\| + q_0 d + c\|u\|^\sigma \|u\|.$$

By Cauchy’s inequality, this yields

$$c\|u\|^2 + c\|u\|_{q_0}^q \leq dq_0.$$
Therefore, \( \|u\|_{q_0} \leq \tilde{c}d^{1/q} \) where \( \tilde{c} \) is independent of \( m, d \). Similarly, we may prove this is true for the second case: \( u \in -U_\delta \cap \{u : I_m(u) \leq d\} \). \( \square \)

**Lemma 4.6.** Assume that \( u_m \in E_m \) is sign-changing and satisfies

\[
I'_m(u_m) = 0, \quad \sup_{m \geq 1} |I_m(u_m)| < \infty.
\]

Then \( \{u_m\} \) has a convergent subsequence whose limit is a sign-changing critical point of \( I \).

**Proof.** The proof of the existence of a convergent subsequence of \( \{u_m\} \) is the same as the proof of the (PS) condition of Lemma 4.1. We just prove that the limit of the subsequence is sign-changing. Let \( u_m^\pm := \max\{\pm u_m, 0\} \). Then

\[
\|u_m^\pm\|^2 = \int \Omega \left( \frac{|u_m^\pm|^q}{|x|^s} + p(x, u_m^\pm)u_m^\pm \right) dx.
\]

By \((H_4)\), for any \( \varepsilon \), there exists a \( C_\varepsilon > 0 \) such that

\[
p(x, u) \leq \varepsilon |u|^2 + C_\varepsilon |u|^q \quad \text{for all } u \in \mathbb{R} \text{ and a.e. } x \in \Omega.
\]

It follows by Hardy’s inequality that (cf., e.g., [GY])

\[
\|u_m^\pm\|^2 \leq \varepsilon \|u_m^\pm\|^2 + c \|u_m^\pm\|^q.
\]

Hence, \( \|u_m^\pm\| \geq \delta_0 > 0 \), where \( \delta_0 \) is a constant independent of \( m \). This implies that the limit of the subsequence is also sign-changing. \( \square \)

**Proof of Theorem 4.1.** The proof is almost the same as that of Theorem 3.1. We just mention the following difference. Similar to \((3.32)\), by Lemma 4.2

\[
\gamma_k^{**}(m) = \inf_{\phi \in \Gamma_m} \sup_{\phi \in (N) \cap S_m} I \geq C \lambda_k^{(q-s)(1-s)} \frac{(q-s)(1-s)}{d-s},
\]

where \( \varepsilon = \frac{n(q-2)}{2(q-s)} \in (0, 1) \). Since \( \lambda_k \geq c_k^{2/n} \) (cf. [LY]), we see that

\[
\gamma_k^{**}(m) \geq c_k^{2/(n(q-2))}. \tag{4.15}
\]
On the other hand,
\[
\gamma_k^*(m) + \frac{1}{2} \geq k(m) + 1 \geq \gamma_k^{**}(m) - \sup_{u \in (-U_{12} \cup U_{12}) \cap \{u \in E^*_m : I_m(u) \leq \gamma_k^{**}(m) + 1/2\}} \left| I(-u) - I(u) \right| - c
\]
\[
\geq \gamma_k^{**}(m) - \sup_{u \in (-U_{12} \cup U_{12}) \cap \{u \in E^*_m : I_m(u) \leq \gamma_k^{**}(m) + 1/2\}} c\|u\|^{1+\sigma} - c
\]
\[
\geq \gamma_k^{**}(m) - c\left(\gamma_k^{**}(m)\right)^{(1+\sigma)/q} - c. \quad \text{(by Lemma 4.5)}
\]

Therefore, \(\gamma_k^{**}(m) \leq \gamma_k^{**}(m) + c\left(\gamma_k^{**}(m)\right)^{(1+\sigma)/q}\). Hence, by iterating, we have
\[
\gamma_k^{**}(m) \leq c k^{q/(q-(1+\sigma))},
\]
which contradicts (4.15) in view of (4.2). □

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References


Further Reading


