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Well-behaved Beurling primes and integers

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Abstract

In this paper, we study generalised prime systems for which both the prime and integer counting functions are asymptotically well-behaved, in the sense that they are approximately $\mathrm{li}(x)$ and ρx, respectively (where ρ is a positive constant), with error terms of order $O(x^{\theta_1})$ and $O(x^{\theta_2})$ for some $\theta_1, \theta_2 < 1$. We show that it is impossible to have both θ_1 and θ_2 less than $\frac{1}{2}$.

Keywords: Beurling's generalized primes

1. Introduction

A *generalised prime system* (or *g-prime system*) $\mathcal{P}$ is a sequence of positive reals $p_1, p_2, p_3, \ldots$ satisfying

$$1 < p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n \leqslant \cdots$$

and for which $p_n \to \infty$ as $n \to \infty$. From these can be formed the system $\mathcal{N}$ of *generalised integers* or *Beurling integers*; that is, the numbers of the form

$$p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k},$$

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where $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in \mathbb{N}_0$.[1] This system generalises the notion of prime numbers and the natural numbers obtained from them. Such systems were first introduced by Beurling [2] and have been studied by many authors since then (see in particular [1]).

Much of the theory concerns connecting the asymptotic behaviour of the g-prime and g-integer counting functions, $\pi_{\mathcal{P}}(x)$ and $N_{\mathcal{P}}(x)$, defined, respectively, by[2]

$$\pi_{\mathcal{P}}(x) = \sum_{p\in\mathcal{P},\, p\leqslant x} 1 \quad \text{and} \quad N_{\mathcal{P}}(x) = \sum_{n\in\mathcal{N},\, n\leqslant x} 1.$$

The methods invariably involve the associated (Beurling) zeta function, defined by

$$\zeta_{\mathcal{P}}(s) = \prod_{p\in\mathcal{P}} \frac{1}{1-p^{-s}} = \sum_{n\in\mathcal{N}} \frac{1}{n^s}.$$

It is often more useful to connect the functions $\psi_{\mathcal{P}}(x)$ and $N_{\mathcal{P}}(x)$, where $\psi_{\mathcal{P}}(x)$ is the function

$$\psi_{\mathcal{P}}(x) = \sum_{p^k\leqslant x,\, p\in\mathcal{P}, k\in\mathbb{N}} \log\, p = \sum_{n\leqslant x, n\in\mathcal{N}} \Lambda_{\mathcal{P}}(n),$$

where $\Lambda_{\mathcal{P}}$ denotes the (generalised) von Mangoldt function, defined by $\Lambda_{\mathcal{P}}(n) = \log\, p$ if $n = p^m$ for some $p \in \mathcal{P}$ and $m \in \mathbb{N}$, and $\Lambda_{\mathcal{P}}(n) = 0$ otherwise. This is because these functions are directly related to $\zeta_{\mathcal{P}}(s)$ via

$$\zeta_{\mathcal{P}}(s) = s\int_1^\infty \frac{N_{\mathcal{P}}(x)}{x^{s+1}}\, dx \quad \text{and} \quad -\frac{\zeta'_{\mathcal{P}}(s)}{\zeta_{\mathcal{P}}(s)} = s\int_1^\infty \frac{\psi_{\mathcal{P}}(x)}{x^{s+1}}\, dx.$$

We shall denote $-\frac{\zeta'_{\mathcal{P}}(s)}{\zeta_{\mathcal{P}}(s)}$ by $\phi_{\mathcal{P}}(s)$. In this paper, we shall be concerned with systems for which both $\psi_{\mathcal{P}}(x)$ and $N_{\mathcal{P}}(x)$ are 'well-behaved', in the sense that

$$\psi_{\mathcal{P}}(x) = x + O(x^{\theta_1}) \quad \text{and} \quad N_{\mathcal{P}}(x) = \rho x + O(x^{\theta_2}), \tag{1.1}$$

hold simultaneously, for some $\theta_1, \theta_2 < 1$ and $\rho > 0$. Note that the former is equivalent to

$$\pi_{\mathcal{P}}(x) = \mathrm{li}(x) + O(x^{\theta'_1}) \quad \text{for some } \theta'_1 < 1.$$

[1] Here and henceforth, $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

[2] We write $\sum_{p\in\mathcal{P}}$ to mean a sum over all the g-primes, counting multiplicities. Similarly for $\sum_{n\in\mathcal{N}}$.

For example, for the rational primes when $\mathcal{N} = \mathbb{N}$, assuming the Riemann hypothesis, these asymptotic relations are true with $\theta_2 = 0$ and any $\theta_1 > \frac{1}{2}$.

The relations (1.1) are equivalent to knowing that $\zeta_{\mathcal{P}}(s)$ has an analytic continuation to some vertical strip to the left of $\Re s = 1$ except for a simple pole at $s = 1$ (with residue ρ), is *zero-free* in this strip, and has *finite order* [3] here (see [5]). Our main result in this paper is to show that this strip cannot have width greater than $\frac{1}{2}$. In particular, this means that it is impossible for both θ_1 and θ_2 to be less than $\frac{1}{2}$.

2. Main results

2.1. $[\alpha, \beta]$-*systems*

For $0 \leqslant \alpha, \beta < 1$, we define an $[\alpha, \beta]$-*system* to be a generalised prime-system for which

$$\psi_{\mathcal{P}}(x) = x + O(x^{\alpha+\varepsilon}), \tag{2.1}$$

$$N_{\mathcal{P}}(x) = \rho x + O(x^{\beta+\varepsilon}) \quad (\text{for some } \rho > 0) \tag{2.2}$$

hold for all $\varepsilon > 0$, but for no $\varepsilon < 0$.

It is clear that $\alpha, \beta \geqslant 0$ is necessary, since $N(x) - \rho x = \Omega(1)$ (for every ρ) and $\psi(x) - x = \Omega(\log x)$ in any case. Note that (2.2) implies that

$$\psi_{\mathcal{P}}(x) = x + O(xe^{-c\sqrt{\log x}}),$$

for some $c > 0$ (see [7]), and this is best possible in the sense that there exist systems for which (2.2) holds, but $\psi_{\mathcal{P}}(x) - x = \Omega(xe^{-c\sqrt{\log x}})$ (see [4]). In the other direction, (2.1) implies

$$N_{\mathcal{P}}(x) = \rho x + O(xe^{-c\sqrt{\log x \, \log\log x}})$$

for some $c > 0$ (see [5]). It is not clear if this is best possible.

If $\mathcal{P}$ is the set of rational primes, so that $\mathcal{N} = \mathbb{N}$, then (2.2) holds with $\beta = 0$ (and $\rho = 1$) and if the Riemann hypothesis is true, (2.1) holds for $\alpha = \frac{1}{2}$. This would then demonstrate the existence of a $[\frac{1}{2}, 0]$-system. Further examples arise if we are prepared to assume other conjectures, such as the generalised Riemann hypothesis. For example,

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[3] $f(s)$ has finite order in the strip where $\Re s \in [a, b]$ if $f(\sigma + it) = O(|t|^A)$ as $|t| \to \infty$ for some constant A, uniformly for $\sigma \in [a, b]$. If there is no such A, we say that f is of infinite order in this strip.

We define, as usual, the *order* $\mu_f(\sigma)$ to be the infimum of all real numbers λ such that $f(\sigma + it) = O(|t|^\lambda)$. It is well-known that, as a function of σ, $\mu_f(\sigma)$ is non-negative, decreasing, and convex.

for the Gaussian integers of the field $\mathbb{Q}(i)$, the Dedekind zeta-function is given by

$$\zeta_{\mathcal{P}}(s) = \frac{1}{1-2^{-s}} \prod_p \Big(\frac{1}{1-p^{-s}}\Big)^2 \prod_q \Big(\frac{1}{1-q^{-2s}}\Big) = \frac{1}{4}\sum_{n=1}^{\infty} \frac{r(n)}{n^s},$$

where p and q run over the rational primes $1 \pmod 4$ and $3 \pmod 4$, respectively, and $r(n)$ is the number of ways of writing n as a^2+b^2 with $a, b \in \mathbb{Z}$. The corresponding g-prime system $\mathcal{P}$ therefore consists of 2, the rational primes $p \equiv 1 \pmod 4$ occuring with multiplicity two, and the squares of the primes of the form $3 \pmod 4$. Thus

$$\pi_{\mathcal{P}}(x) = 1 + 2\pi_{1,4}(x) + \pi_{3,4}(\sqrt{x}),$$

where $\pi_{k,m}(x)$ is the number of primes less than or equal to x of the form $k \pmod m$. On the generalised Riemann hypothesis, one has

$$\pi_{\mathcal{P}}(x) = \mathrm{li}(x) + O(x^{\frac{1}{2}+\varepsilon}) \quad \text{for all } \varepsilon > 0.$$

On the other hand, it is known that (see [6])

$$N_{\mathcal{P}}(x) = \frac{1}{4}\sum_{n \leqslant x} r(n) = \frac{\pi}{4}x + O(x^{\frac{23}{73}})$$

and it is conjectured that the exponent in the error term is actually $\frac{1}{4}+\varepsilon$ for all $\varepsilon > 0$. Hence, assuming these conjectures, $\mathcal{P}$ is an example of a $[\frac{1}{2}, \frac{1}{4}]$-system. Further examples of such $[\frac{1}{2}, \beta]$-systems (with $\beta < \frac{1}{2}$) based on Dedekind zeta functions, can be conjectured to exist.

However, at present it seems that no actual examples of $[\alpha, \beta]$-systems are known. [4]

The best possible system would be one where $\alpha, \beta = 0$. However, we show that such systems are impossible. Indeed, we find that α and β cannot *both* be less than $\frac{1}{2}$.

Theorem 1. *Let $\mathcal{P}$ be an $[\alpha, \beta]$-system. Then $\Theta = \max\{\alpha, \beta\} \geqslant \frac{1}{2}$.*

Corollary 2. (a) *If $\psi_{\mathcal{P}}(x) = x + O(x^{\alpha})$ for some constant $\alpha < \frac{1}{2}$ (which implies that $N_{\mathcal{P}}(x) \sim \rho x$ for some $\rho > 0$), then for every $\eta \in (\alpha, \frac{1}{2})$, $N_{\mathcal{P}}(x) - \rho x = \Omega(x^{\eta})$ and $\zeta_{\mathcal{P}}(s)$ does not have finite order throughout the strip $\{s \in \mathbb{C} : \eta < \Re s < 1\}$.*

[4] In a recent personal communication, H. Montgomery told me that he believes that the methods employed in [4] can be used to find examples of $[\alpha, \beta]$-systems for any $\alpha, \beta \geqslant \frac{1}{2}$.

Of course, if we allow *continuous* systems, where $N_{\mathcal{P}}(x)$ and $\psi_{\mathcal{P}}(x)$ may vary continuously, then the existence of such systems is trivial; e.g. take $N_{\mathcal{P}}(x) = \psi_{\mathcal{P}}(x) = x - 1$ for $x \geqslant 1$, and 0 otherwise; then $\zeta_{\mathcal{P}}(s) = \int_0^{\infty} x^{-s}\, dN_{\mathcal{P}}(x) = \frac{1}{s-1}$.

(b) *If* $N_{\mathcal{P}}(x) = \rho x + O(x^{\beta})$ *for some constants* $\rho > 0$ *and* $\beta < \frac{1}{2}$, *then for every* $\eta' \in (\beta, \frac{1}{2})$, $\psi_{\mathcal{P}}(x) - x = \Omega(x^{\eta'})$ *and* $\zeta_{\mathcal{P}}(s)$ *has infinitely many zeros in the strip* $\{s \in \mathbb{C} : \eta' < \Re s < 1\}$.

3. Proofs

For the proofs we recall, from [5], Theorem 2.3 (which is a generalisation of the implication '*Riemann hypothesis implies Lindelöf hypothesis*') and the remark following it.

Theorem A (*Hilberdink and Lapidus [5, Theorem 2.3]*). *Let* $\mathcal{P}$ *be a* $[\alpha, \beta]$-*system. Then for* $\sigma > \Theta = \max\{\alpha, \beta\}$, *and uniformly for* $\sigma \geqslant \Theta + \delta$ (*any* $\delta > 0$),

$$\phi_{\mathcal{P}}(\sigma + it) = O((\log|t|)^{\frac{1-\sigma}{1-\Theta}+\varepsilon}) \quad \text{and} \quad \zeta_{\mathcal{P}}(\sigma + it) = O(\exp\{(\log|t|)^{\frac{1-\sigma}{1-\Theta}+\varepsilon}\}),$$

for all $\varepsilon > 0$. *In particular, both* $\phi_{\mathcal{P}}(s)$ *and* $\zeta_{\mathcal{P}}(s)$ *have zero order for* $\Re s > \Theta$.

Remark B. (i) If $\alpha < \beta$ and we already know that $\zeta_{\mathcal{P}}(s)$ is of finite order for $\sigma > \eta$ for some $\eta \in (\alpha, \beta)$, then $\zeta_{\mathcal{P}}(s)$ and $\phi_{\mathcal{P}}(s)$ have zero order in this range.

(ii) If $\beta < \alpha$ and we already know that $\phi_{\mathcal{P}}(s)$ has only finitely many poles for $\sigma > \eta'$ (equivalently, $\zeta_{\mathcal{P}}(s)$ has finitely many zeros here), then $\zeta_{\mathcal{P}}(s)$ and $\phi_{\mathcal{P}}(s)$ have zero order in this range.

Proof of Theorem 1. The result would follow immediately from a theorem of Carlson [3, p. 7] concerning general Dirichlet series if we assume that

$$n' > n + \frac{1}{n^A} \quad \text{for some } A \geqslant 0, \tag{3.1}$$

where n and n' are consecutive g-integers. For Theorem A tells us that the function $f(s)$ defined by

$$f(s) = \zeta_{\mathcal{P}}(s) - \rho\phi_{\mathcal{P}}(s) = \sum_{n \in \mathcal{N}} \frac{1 - \rho\Lambda_{\mathcal{P}}(n)}{n^s},$$

which is analytic for $\Re s > \Theta$, would have order 0 in this half-plane. By Carlson's result, if (3.1) holds, a mean-value would exist here and

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^2 \, dt = \sum_{n \in \mathcal{N}} \frac{(1 - \rho\Lambda_{\mathcal{P}}(n))^2}{n^{2\sigma}}.$$

This is plainly absurd if $\Theta < \frac{1}{2}$, as the final sum diverges for $\sigma \leqslant \frac{1}{2}$. Hence $\Theta \geqslant \frac{1}{2}$.

However, we do not want to restrict the size of $n'-n$ by assuming something like (3.1).

Suppose, for a contradiction, that we have $\Theta < \frac{1}{2}$. Let $\zeta_N(s) = \sum_{n \leqslant N} n^{-s}$, where the sum ranges over $n \in \mathcal{N}$ (for clarity, we shall drop the subscript $\mathcal{P}$ throughout this proof). Consider

$$\int_0^T |\zeta_N(\sigma+it)|^2\,dt = \frac{1}{2}\int_{-T}^T |\zeta_N(\sigma+it)|^2\,dt$$

for fixed $\sigma \in (\Theta, \frac{1}{2})$. We have

$$\frac{1}{2}\int_{-T}^T |\zeta_N(\sigma+it)|^2\,dt = \frac{1}{2}\int_{-T}^T \sum_{n\leqslant N}\frac{1}{n^{\sigma+it}}\sum_{m\leqslant N}\frac{1}{m^{\sigma-it}}\,dt = \sum_{m,n\leqslant N}\frac{S_{m,n}(T)}{(mn)^\sigma},$$

where $S_{n,n}(T) = T$ and for $m \neq n$,

$$S_{m,n}(T) = \frac{\sin(T\log(n/m))}{\log(n/m)}.$$

Note that $S_{m,n}(T) = S_{n,m}(T)$. Hence

$$\int_0^T |\zeta_N(\sigma+it)|^2\,dt = T\sum_{n\leqslant N}{}^{*}\,\frac{1}{n^{2\sigma}} + 2\sum_{n\leqslant N}\frac{1}{n^\sigma}\sum_{m<n}\frac{S_{m,n}(T)}{m^\sigma},$$

the $*$ indicating that the multiplicities must be squared. In any case, we have $\sum_{n\leqslant N}^{*} n^{-2\sigma} \geqslant \sum_{n\leqslant N} n^{-2\sigma} \geqslant k_1 N^{1-2\sigma}$ for some $k_1 > 0$.[5] For $m \leqslant \frac{n}{2}$, we have $|S_{m,n}(T)| \leqslant 1/\log 2$, so that

$$\left|2\sum_{n\leqslant N}\frac{1}{n^\sigma}\sum_{m\leqslant n/2}\frac{S_{m,n}(T)}{m^\sigma}\right| \leqslant \frac{2}{\log 2}\sum_{n\leqslant N}\frac{1}{n^\sigma}\sum_{m\leqslant n/2}\frac{1}{m^\sigma} = O\Big(\sum_{n\leqslant N} n^{1-2\sigma}\Big) = O(N^{2-2\sigma}).$$

Thus, for some positive constants k_1, k_2, independent of T and N,

$$\int_0^T |\zeta_N(\sigma+it)|^2\,dt \geqslant k_1 T N^{1-2\sigma} + 2\sum_{n\leqslant N}\frac{1}{n^\sigma}\sum_{\frac{n}{2}<m<n}\frac{S_{m,n}(T)}{m^\sigma} - k_2 N^{2-2\sigma}. \tag{3.2}$$

[5] It follows readily from $N_{\mathcal{P}}(x) \sim \rho x$ that $\sum_{n\leqslant x} n^\lambda \sim \frac{\rho}{1+\lambda}x^{1+\lambda}$ for fixed $\lambda > -1$.

Now put $T = 2r - 1$ for $r = 1, 2, \ldots, R$, and sum both sides. Observe that

$$\sum_{r=1}^{R} \sin\Big((2r-1)\log\frac{n}{m}\Big) = \frac{\sin^2(R\,\log\,n/m)}{\sin(\log\,n/m)} \geqslant 0,$$

since $0 < \log\,n/m < \log\,2$. Thus (3.2) yields

$$\sum_{r=1}^{R}\int_0^{2r-1} |\zeta_N(\sigma+it)|^2\,dt \geqslant k_1 R^2 N^{1-2\sigma} - k_2 R N^{2-2\sigma} = R N^{1-2\sigma}(k_1 R - k_2 N).$$

In particular, for $N \leqslant \frac{k_1}{2k_2} R$,

$$\sum_{r=1}^{R}\int_0^{2r-1} |\zeta_N(\sigma+it)|^2\,dt \geqslant \frac{k_1}{2} R^2 N^{1-2\sigma}. \tag{3.3}$$

Now let $c > 1 - \sigma$ and $N \notin \mathcal{N}$. Then

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\Big(\frac{N}{n}\Big)^w \frac{dw}{w} = O\Big(\frac{(N/n)^c}{T|\log\,N/n|}\Big) + \begin{cases} 1 & \text{if } n < N \\ 0 & \text{if } n > N \end{cases},$$

where the implied constant is independent of n and N. Multiply through by $n^{-s} = n^{-\sigma-it}$, where $|t| < T$, and sum over all $n \in \mathcal{N}$. Thus for $N \notin \mathcal{N}$, we have

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{\zeta(s+w)N^w}{w}\,dw = \zeta_N(s) + O\Bigg(\frac{N^c}{T}\sum_{n\in\mathcal{N}}\frac{1}{n^{c+\sigma}|\log\,N/n|}\Bigg).$$

For $n \leqslant \frac{N}{2}$ and $n \geqslant 2N$, $|\log N/n| \geqslant \log\,2$, so

$$\begin{aligned}
\zeta_N(s) &= \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{\zeta(s+w)N^w}{w}\,dw + O\Bigg(\frac{N^c}{T}\sum_{n\in\mathcal{N}}\frac{1}{n^{c+\sigma}}\Bigg) \\
&\quad + O\Bigg(\frac{N^c}{T}\sum_{\frac{N}{2}<n<2N}\frac{1}{n^{c+\sigma}|\log\,N/n|}\Bigg) \\
&= \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{\zeta(s+w)N^w}{w}\,dw + O\Big(\frac{N^c}{T(c+\sigma-1)}\Big) \\
&\quad + O\Bigg(\frac{N^{1-\sigma}}{T}\sum_{\frac{N}{2}<n<2N}\frac{1}{|n-N|}\Bigg),
\end{aligned}$$

since $\zeta(x) = O(\frac{1}{x-1})$ and $|\log N/n| = |\log(1 + \frac{n-N}{N})| \asymp \frac{|n-N|}{N}$ for $\frac{N}{2} < n < 2N$. For the integral on the right, we push the contour as far as $\Re w = -\eta$, for some $\eta \in (0, \sigma - \Theta)$, picking up residues at $w = 0$ and $w = 1 - s$ (since $|t| < T$). The contribution along the horizontal line $[-\eta + iT, c + iT]$ is, in modulus, less than

$$\frac{1}{2\pi} \int_{-\eta}^{c} \frac{N^y |\zeta(\sigma + y + i(t + T))|}{\sqrt{y^2 + T^2}} \, dy = O(N^c T^{\varepsilon - 1}) \quad \text{for all } \varepsilon > 0,$$

since $\zeta(s)$ has zero order in this range after Theorem A. Similarly on $[-\eta - iT, c - iT]$. For the integral along $\Re w = -\eta$, we have

$$\left| \frac{1}{2\pi i} \int_{-\eta - iT}^{-\eta + iT} \frac{\zeta(s + w) N^w}{w} \, dw \right| \leqslant \frac{N^{-\eta}}{2\pi} \int_{-T}^{T} \frac{|\zeta(\sigma - \eta + i(t + y))|}{\sqrt{\eta^2 + y^2}} \, dy = O(N^{-\eta} T^{\varepsilon})$$

for all $\varepsilon > 0$. The residues at $w = 0$ and $w = 1 - s$ are, respectively, $\zeta(s)$ and $\rho N^{1-s}/(1 - s) = O(\frac{N^{1-\sigma}}{|t|})$. Putting these observations together and letting $c = 1 - \sigma + \frac{1}{\log N}$ (so that $N^c = e N^{1-\sigma}$), we have

$$\begin{aligned} \zeta_N(s) &= \zeta(s) + O\left(\frac{N^{1-\sigma}}{|t|}\right) + O(N^{-\eta} T^{\varepsilon}) + O(N^{1-\sigma} T^{\varepsilon - 1}) + O\left(\frac{N^{1-\sigma} \log N}{T}\right) \\ &\quad + O\left(\frac{N^{1-\sigma}}{T} \sum_{\frac{N}{2} < n < 2N} \frac{1}{|N - n|}\right). \end{aligned}$$

Suppose now that $N \to \infty$ in such a way that $(N - \frac{1}{N}, N + \frac{1}{N}) \cap \mathcal{N} = \emptyset$. (This is possible since otherwise $n_{k+1} < n_k + O(\frac{1}{n_k})$ (where n_k is the kth g-integer), which leads to $n_k = O(\sqrt{k})$—a contradiction.) Then $|N - n| \geqslant \frac{1}{N}$ for every $n \in \mathcal{N}$, and the last sum is at most $\sum_{n<2N} N = O(N^2)$. Taking T to be a sufficiently large power of N, say $T = N^5$, we have

$$\zeta_N(\sigma + it) = \zeta(\sigma + it) + O\left(\frac{N^{1-\sigma}}{|t|}\right) + o(1)$$

for $|t| < N^5$. But, by Theorem A, $\zeta(\sigma + it) = O(|t|^{\varepsilon})$ for all $\varepsilon > 0$, so that for $N^{1-\sigma} \leqslant |t| < N^5$,

$$|\zeta_N(\sigma + it)| = O(|t|^{\varepsilon}), \tag{3.4}$$

as $N \to \infty$ such that $(N - \frac{1}{N}, N + \frac{1}{N}) \cap \mathcal{N} = \emptyset$. We show that this is incompatible with (3.3).

For, in any case, $|\zeta_N(\sigma + it)| \leqslant \zeta_N(\sigma) = O(N^{1-\sigma})$. Hence

$$\sum_{r \leqslant \sqrt{R}} \int_0^{2r-1} |\zeta_N(\sigma + it)|^2 \, dt = O(RN^{2-2\sigma})$$

and

$$\sum_{r \leqslant R} \int_0^{\sqrt{r}} |\zeta_N(\sigma + it)|^2 \, dt = O(R^{3/2} N^{2-2\sigma}).$$

Thus, if R is chosen of larger order than N^2, say $R = N^4$, then (3.3) implies

$$\sum_{\sqrt{R} < r \leqslant R} \int_{\sqrt{r}}^{2r-1} |\zeta_N(\sigma + it)|^2 \, dt \geqslant c R^2 N^{1-2\sigma} \tag{3.5}$$

for some positive constant c. But on the left, t ranges between $R^{1/4}$ and $2R - 1$; i.e. between N and $2N^4 - 1$. This lies in the range $[N^{1-\sigma}, N^5)$, so that from (3.4), the LHS of (3.5) is

$$O\Bigg(\sum_{\sqrt{R} < r \leqslant R} \int_{\sqrt{r}}^{2r-1} t^{\varepsilon} \, dt \Bigg) = O(R^{2+\varepsilon}) \qquad \text{for all } \varepsilon > 0,$$

which contradicts (3.5). $\square$

Remark C. Note that in the proof of Theorem 1, no explicit use was made of the fact that $\zeta_{\mathcal{P}}(s)$ has no zeros for $\sigma > \alpha$ (which follows from (2.1)). This was only implicitly used (in Theorem A) to show that $\zeta_{\mathcal{P}}(s)$ has zero order for $\sigma > \Theta$. In particular, after Remark B(ii), this means that the same proof holds for the following: *for an* $[\alpha, \beta]$*-system with* $\beta < \alpha$*, if* $\zeta_{\mathcal{P}}(s)$ *has finitely many zeros for* $\sigma > \eta'$ *with* $\eta' \in (\beta, \alpha)$*, then* $\eta' \geqslant \frac{1}{2}$.

To prove Corollary 2, we shall first require the following Tauberian result connecting the Dirichlet series $f(s) = \sum_{n \in \mathcal{N}} \frac{a_n}{n^s}$ and the asymptotic behaviour of the sum $A(x) = \sum_{n \leqslant x} a_n$, for a given non-negative sequence a_n defined on the g-integers.

Proposition 3. *Let* $\{a_n\}_{n \in \mathcal{N}}$ *be a non-negative sequence such that* $a_n = O(n^{\varepsilon})$ *for every* $\varepsilon > 0$. *Let* $f(s)$ *and* $A(x)$ *be defined as above for* $\Re s > 1$ *and* $x \geqslant 0$, *respectively. Suppose that for some* $\theta \in [0, 1)$, $f(s)$ *has an analytic continuation to the half-plane* $\{s \in \mathbb{C} : \Re s > \theta\}$, *except for a simple (non-removable) pole at* $s = 1$ *with residue* a.

Further assume that $|f(\sigma+it)| = O(|t|^{\varepsilon})$ *for all* $\varepsilon > 0$, *uniformly for* $\sigma \geqslant \theta+\delta$ *for any* $\delta > 0$. *Then*

$$A(x) = ax + O(x^{\theta+\varepsilon}) \quad \textit{for all } \varepsilon > 0.$$

Proof. Let $c > 1$, $T, x > 0$ such that $x \notin \mathcal{N}$. Then, for $n \in \mathcal{N}$,

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(\frac{x}{n}\right)^{s}\frac{ds}{s} = O\left(\frac{(x/n)^{c}}{T|\log x/n|}\right) + \begin{cases} 1 & \text{if } n < x, \\ 0 & \text{if } n > x, \end{cases}$$

where the implied constant is independent of n and x. Multiply through by a_n and sum over all $n \in \mathcal{N}$. Thus for $x \notin \mathcal{N}$, we have

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{f(s)x^{s}}{s}\,ds = A(x) + O\left(\frac{x^{c}}{T}\sum_{n\in\mathcal{N}}\frac{a_n}{n^{c}|\log x/n|}\right).$$

For $n \leqslant \frac{x}{2}$ and $n \geqslant 2x$, $|\log x/n| \geqslant \log 2$, so

$$\begin{aligned} A(x) &= \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{f(s)x^{s}}{s}\,ds + O\left(\frac{x^{c}}{T}\sum_{n\in\mathcal{N}}\frac{a_n}{n^{c}}\right) + O\left(\frac{x^{c}}{T}\sum_{\frac{x}{2}<n<2x}\frac{a_n}{n^{c}|\log x/n|}\right) \\ &= \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{f(s)x^{s}}{s}\,ds + O\left(\frac{x^{c}}{T(c-1)}\right) + O\left(\frac{x^{1+\varepsilon}}{T}\sum_{\frac{x}{2}<n<2x}\frac{1}{|n-x|}\right), \end{aligned} \tag{3.6}$$

since $f(c) = O(\frac{1}{c-1})$ and $|\log x/n| = |\log(1+\frac{n-x}{x})| \asymp \frac{|n-x|}{x}$ for $\frac{x}{2} < n < 2x$.

Now consider the integral on the right-hand side of (3.6). We can push the contour past the pole at $s=1$ to the line $\Re s = \sigma$ for any $\sigma > \theta$. The residue at 1 is ax. Hence

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{f(s)x^{s}}{s}\,ds = ax + \frac{1}{2\pi i}\left(\int_{c-iT}^{\sigma-iT} + \int_{\sigma-iT}^{\sigma+iT} + \int_{\sigma+iT}^{c+iT}\right)\frac{f(s)x^{s}}{s}\,ds.$$

We estimate these integrals in turn, using $f(s) = O(|t|^{\varepsilon})$. We have

$$\left|\frac{1}{2\pi i}\int_{\sigma+iT}^{c+iT}\frac{f(s)x^{s}}{s}\,ds\right| \leqslant \frac{x^{c}}{2\pi T}\int_{\sigma}^{c}|f(y+iT)|\,dy = O(x^{c}T^{-1+\varepsilon})$$

and similarly for $\int_{c-iT}^{\sigma-iT}$, while

$$\left|\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{f(s)x^{s}}{s}\,ds\right| \leqslant \frac{x^{\sigma}}{2\pi}\int_{-T}^{T}\frac{|f(\sigma+it)|}{\sqrt{\sigma^2+t^2}}\,dt = O(x^{\sigma}T^{\varepsilon}).$$

Now choose $c = 1 + \frac{1}{\log x}$. Then (3.6) gives

$$A(x) = ax + O(xT^{-1+\varepsilon}) + O(x^\sigma T^\varepsilon) + O\left(\frac{x^{1+\varepsilon}}{T}\right) + O\left(\frac{x^{1+\varepsilon}}{T} \sum_{\frac{x}{2}<n<2x} \frac{1}{|n-x|}\right) \tag{3.7}$$

for $x \notin \mathcal{N}$ and every $\varepsilon > 0$. We need to bound the sum on the right-hand side but this is difficult in general as x can be arbitrarily close to a g-integer. So let us suppose that x is such that there are no g-integers n with $|n-x| < \frac{1}{x^2}$; i.e. $(x - \frac{1}{x^2}, x + \frac{1}{x^2}) \cap \mathcal{N} = \emptyset$. Then

$$\sum_{\frac{x}{2}<n<2x} \frac{1}{|n-x|} \leqslant x^2 \sum_{\frac{x}{2}<n<2x} 1 \leqslant x^2 N(2x) = O(x^3).$$

Taking $T = x^4$, (3.7) gives $A(x) = ax + O(x^{\sigma+\varepsilon})$ for all $\varepsilon > 0$. This holds for all $\sigma > \theta$ so

$$A(x) = ax + O(x^{\theta+\varepsilon}),$$

whenever $x \to \infty$ in such a way that $(x - \frac{1}{x^2}, x + \frac{1}{x^2}) \cap \mathcal{N} = \emptyset$.

Now we show that this is sufficient to prove the theorem. More precisely, we show the following: *for all x sufficiently large for which $(x - \frac{1}{x^2}, x + \frac{1}{x^2}) \cap \mathcal{N} \neq \emptyset$, $\exists$ $x_1 \in (x-3, x)$ and $x_2 \in (x, x+3)$ such that*

$$\left(x_1 - \frac{1}{x_1^2}, x_1 + \frac{1}{x_1^2}\right) \cap \mathcal{N} = \emptyset \quad \text{and} \quad \left(x_2 - \frac{1}{x_2^2}, x_2 + \frac{1}{x_2^2}\right) \cap \mathcal{N} = \emptyset. \tag{3.8}$$

Then the result will follow since $x = x_r + O(1)$ and $A(x_r) = ax_r + O(x_r^{\theta+\varepsilon})$ (for $r = 1, 2$), so that

$$A(x) \leqslant A(x_2) = ax_2 + O(x_2^{\theta+\varepsilon}) = ax + O(x^{\theta+\varepsilon})$$

and

$$A(x) \geqslant A(x_1) = ax_1 + O(x_1^{\theta+\varepsilon}) = ax + O(x^{\theta+\varepsilon}).$$

It remains to prove (3.8).

Suppose, for a contradiction, that there is no such x_2. Let $y_n = \sqrt[3]{x^3 + 9n}$, for $n \in \mathbb{N}$. Thus each interval $(y_n - \frac{1}{y_n^2}, y_n + \frac{1}{y_n^2})$ contains an element of $\mathcal{N}$ whenever $y_n < x + 3$; i.e. for $n < x^2 + 3x + 3$. It is elementary to show that

$$y_n + \frac{1}{y_n^2} < y_{n+1} - \frac{1}{y_{n+1}^2},$$

so that these intervals are non-overlapping. This means that $N(x+3) - N(x) \geqslant x^2$. But this is false for all x sufficiently large, as $N(x) = O(x)$.

The existence of x_1 is shown in a similar way using the sequence $z_n = \sqrt[3]{x^3 - 9n}$, leading to $N(x) - N(x-3) \geqslant x^2$. □

Proof of Corollary 2. (a) The assumptions imply that $N(x) \sim \rho x$ for some $\rho > 0$. From Theorem 1, it is immediate that $N_{\mathcal{P}}(x) - \rho x = \Omega(x^{\eta})$ for every $\eta < \frac{1}{2}$.

Now suppose $\zeta_{\mathcal{P}}(s)$ has finite order in some strip $\{s \in \mathbb{C} : \eta < \Re s < 1\}$ with $\eta \in (\alpha, \frac{1}{2})$. Then $\mathcal{N}$ is an $[\alpha', \beta']$-system for some α', β' with $\alpha' \leqslant \alpha$. By Remark B(i), $\zeta_{\mathcal{P}}(s)$ has zero order in the strip where $\sigma > \eta$. Now apply Proposition 3 with $a_n = 1$, so that $A(x) = N_{\mathcal{P}}(x)$ and $f(s) = \zeta_{\mathcal{P}}(s)$. Then $f(s)$ has a simple pole at $s = 1$ with residue ρ, and satisfies the conditions of Proposition 3 with $\theta = \eta$. Hence, $N_{\mathcal{P}}(x) = \rho x + O(x^{\eta+\varepsilon})$ for all $\varepsilon > 0$. This contradicts Theorem 1 since $\eta < \frac{1}{2}$.

(b) It follows immediately from Theorem 1 that $\psi_{\mathcal{P}}(x) - x = \Omega(x^{\eta'})$ for every $\eta' < \frac{1}{2}$.

Now suppose $\zeta_{\mathcal{P}}(s)$ has only finitely many zeros in some strip $\{s \in \mathbb{C} : \eta' < \Re s < 1\}$ with $\eta' \in (\beta, \frac{1}{2})$. Then $\mathcal{N}$ is an $[\alpha'', \beta'']$-system for some α'', β'' with $\beta'' \leqslant \beta$. By Remark B(ii), $\zeta_{\mathcal{P}}(s)$ has zero order in the strip where $\sigma > \eta'$. But then by Remark C, $\eta' \geqslant \frac{1}{2}$—a contradiction. □

4. Final discussion

After Theorem 1, we see that the best possible systems are $[\frac{1}{2}, 0]$ and $[0, \frac{1}{2}]$. The existence of the former is conjectured for $\mathbb{N}$, but there is no apparent candidate for the latter system. We can certainly find a system for which $\alpha = 0$, that is,

$$\psi_{\mathcal{P}}(x) = x + O(x^{\varepsilon}) \quad \text{for all } \varepsilon > 0$$

by choosing p_n appropriately. For example, let $p_n = R^{-1}(n)$, where R is the strictly increasing function on $[1, \infty)$ defined by

$$R(x) = \sum_{k=1}^{\infty} \frac{(\log x)^k}{k!k\zeta(k+1)},$$

where $\zeta(\cdot)$ is the classical Riemann zeta-function. For then, $\pi_{\mathcal{P}}(x) \leqslant R(x) < \pi_{\mathcal{P}}(x) + 1$ and hence

$$\begin{aligned}
\Pi_{\mathcal{P}}(x) &\stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{\pi_{\mathcal{P}}(x^{1/n})}{n} = \sum_{1 \leqslant n \leqslant A \log x} \frac{R(x^{1/n})}{n} + O(\log\log x) \quad (\text{for some } A > 0) \\
&= \sum_{1 \leqslant n \leqslant A \log x} \frac{1}{n} \sum_{k=1}^{\infty} \frac{(\log x^{1/n})^k}{k!k\zeta(k+1)} + O(\log\log x) \\
&= \sum_{k=1}^{\infty} \frac{(\log x)^k}{k!k\zeta(k+1)} \sum_{1 \leqslant n \leqslant A \log x} \frac{1}{n^{k+1}} + O(\log\log x) \\
&= \sum_{k=1}^{\infty} \frac{(\log x)^k}{k!k\zeta(k+1)} \left(\zeta(k+1) + O\left(\frac{1}{k(\mathrm{A} \log x)^k} \right) \right) + O(\log\log x) \\
&= \operatorname{li}(x) + O(\log\log x).
\end{aligned}$$

By integration, one obtains

$$\psi_{\mathcal{P}}(x) = \int_0^x \log t \, d\Pi_{\mathcal{P}}(t) = x + O(\log x \log\log x).$$

The question is now whether the corresponding counting function $N_{\mathcal{P}}(x)$ behaves according to (2.2). We know that the error $|N_{\mathcal{P}}(x) - \rho x|$ is $O(xe^{-c\sqrt{\log x \log\log x}})$ for some $c > 0$, and after Theorem 1, it is $\Omega(x^{\frac{1}{2}-\delta})$ for every $\delta > 0$.

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