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Decomposition and convergence for tree martingales

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Abstract

In this paper, the authors firstly construct a graph-theoretic decomposition of an index set for tree martingales, and based on this decomposition, they give a locally finite tree martingale's notion and a tree martingale decomposition theorem. Secondly, they establish some relations between the locally finite tree martingales and the multiparameter martingales, and furthermore the convergence of tree martingales is shown by using the multiparameter martingale theory of Cairoli–Walsh. Finally, with some mild conditions, two inequalities for tree martingales are obtained by using their decomposition theorem and multiparameter martingale theory.

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Keywords: Tree martingales; Martingale convergence; Martingale decomposition; Directed locally finite tree; Partial ordering

1. Introduction

Definition 1.1 (*[1,2]*). Let $\mathbf{T}$ be a countable, upward-directed index set with respect to the partial ordering $\preceq$ satisfying the following two conditions:

(1) for every $t \in \mathbf{T}$, the set $\mathbf{T}^t := \{u \in \mathbf{T} : u \preceq t\}$ is finite;
(2) for every $t \in \mathbf{T}$, the set $\mathbf{T}_t := \{u \in \mathbf{T} : t \preceq u\}$ is linearly ordered.

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Thus $\mathbf{T}$ is a tree set and every nonempty subset of $\mathbf{T}$ has at least one minimum. A tree $\mathbf{T}$ is also a special partially ordered set with respect to the partial ordering $\preceq$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\mathcal{F} = (\mathcal{F}_t, t \in \mathbf{T})$ be a family of nondecreasing sub-σ-algebras of $\mathcal{F}$ with respect to the partial ordering $\preceq$, $(\mathcal{F}_s, s \in \mathbf{T}_t)$ can be linearly ordered and $\mathcal{F} = \sigma(\bigcup_{t \in \mathbf{T}} \mathcal{F}_t)$. Throughout this paper, without specification, we let E_t be the conditional expectation operator with respect to $\mathcal{F}_t$, and L_1 be the space of integrable measurable functions, the indicator function of a set $\mathbf{A}$ is denoted by $\chi_{\mathbf{A}}$. In the tree case, since conditional expectation operators are also projections, more generally, we work with projections instead of conditional expectation operators. Let $(\phi_t, t \in \mathbf{T})$ be a family of complex-value measurable functions with $|\phi_t| = 1$. For every $t \in \mathbf{T}$, set

$$P_t f = \phi_t E_t(f \bar{\phi}_t), \quad f \in L_1. \tag{1.1}$$

Then $(P_t, t \in \mathbf{T})$ is a family of projections since $\|P_t\| \le 1$ and $P_t \circ P_t = P_t$. It is clear that the conditional expectation operators are projections of the form (1.1).

Definition 1.2 ([1,2]). Let $(P_t, t \in \mathbf{T})$ be a family of projections as defined in (1.1). Then the family of $(\mathcal{F}_t, P_t : t \in \mathbf{T})$ is called a tree basis if

(1) for every $f \in L_1$ and $s \preceq t$ $(s, t \in \mathbf{T})$, $P_t f = \phi_s E_t(f \bar{\phi}_s)$;
(2) for every pair of incomparable s, t in $\mathbf{T}$, $P_t P_s = 0$.

It follows from the previous (1) that for any comparable $s, t \in \mathbf{T}$, if $s \preceq t$ then $P_t P_s = P_s P_t = P_s$. Now, tree martingales are to be defined according to the tree basis.

Definition 1.3 ([1,2]). We say that a family of $f = (f_t, t \in \mathbf{T})$ of integrable functions is a tree martingale if $s \preceq t$ implies $P_s f_t = f_s$.

Tree martingales have been studied by Schipp, Fridli, Weisz, Young, and others but there are many open problems. In 1980s, Schipp and Fridli [1,3], Weisz [2] have identified that a maximal inequality with respect to tree martingales as well as Burkholder–Gundy's inequality holds if $2 < p < \infty$, furthermore, this result has been extended to all $1 < p < \infty$ for a regular tree stochastic basis. Moreover, using the results on tree martingales, Schipp [4] and Gosselin [5] proved that for an arbitrary Vilenkin system and for $f \in L_p$, the Vilenkin–Fourier series of f converges in L_p-norm to itself. When we tackle tree martingales, however, there are two sources of difficulty that need to be overcome and they are both related to the fact that the tree $\mathbf{T}$ — cannot be well ordered in a useful way. The first problem is that there is no sensible way to uniquely define tree martingale's stopping times. Because it is not clear-and in general not true – that there is a uniquely minimum element $t \in \mathbf{T}$. The second source of difficulty with an index set for tree martingales is that there is a partial ordering-nonlinear. This is analogous to multiparameter martingales [6–11], and it is necessary to point out that tree martingale transforms cannot be defined as one-parameter martingales either. Schipp [3] obtained some results for tree martingales by using the convex method. He and Hou [12] have verified that some scalar-valued tree martingale inequalities hold, and they investigated vector-valued tree martingales and some inequalities for them by using the properties of Banach space in [13]. Using the UMD property, He and Shen [14] tried to investigate the tree martingale transform operator and shown that the maximal operators of $\mathbf{X}$-valued tree martingale transforms are norm-bounded in $L_p(\mathbf{X})$ provided $\mathbf{X}$ is a UMD space.

In spite of some problems on tree martingales having been proved, we still did not find that the convergence theorem for tree martingales have been given explicitly. Since the index set of tree martingales is nonlinear, there are many difficulties in defining tree martingale convergence. We think that tree martingale convergence is connected to not only L^1- boundedness for tree martingales but also the structures on the index set of tree martingales. The question is: Are there efficient ways to overcome these difficulties? Here we study this problem. In this paper, we try to answer the following problems:

- What is the structure of a tree set $\mathbf{T}$?
- What is the relation between tree martingales and multiparameter martingales?
- How to define the convergence for tree martingales and under what conditions is a tree martingale convergent?

The paper is concerned with constructing a graph-theoretic decomposition of an index set for tree martingales and some relations between the tree martingales and the multiparameter martingale theory of Cairoli–Walsh. The rest of the paper is concerned with applying the decomposition of tree martingales to obtain some tree martingale inequalities in harmonic analysis.

Notations: The collection of positive (nonnegative) integers is denoted by $\mathbb{N}(\mathbb{N}_0)$ and $\overline{\mathbb{N}} = \mathbb{N}_0 \bigcup \{\infty\}$, $\mathbb{N}^{N+1}$ denotes an $N+1$ dimensional Euclidean lattice, N denotes a fixed positive integer, C denotes a complex space, $A - B$ denotes the difference of two sets A and B or the deletion of B from the graph A, $\oplus$ denotes directed sum, namely, $A \oplus B = A \cup B - A \cap B$, and $|A|$ denotes the cardinality of a set A.

2. Decomposition of index set on tree martingales

In this chapter we shall construct a graph-theoretic decomposition of the index set of a tree martingale. Also, we shall show that the tree set $\mathbf{T}$ is isomorphic to an inside-directed locally finite forest in Theorem 2.1. Some graphic knowledge and elements of set theory can be found in [15,16], respectively.

Theorem 2.1. *A tree set $\mathbf{T}$ is isomorphic to an inside-directed locally finite forest.*

Before proving this, we need to obtain a way of characterizing the linear orderings on the family of sets $\{\mathbf{T}_t - \{t\}\}_{t\in\mathbf{T}}$, where $T_t - \{t\} = \{u \in \mathbf{T} : t \prec u\}$.

Lemma 2.2. *For every element $t \in \mathbf{T}$, a linear ordering on $\mathbf{T}_t - \{t\}$ is a well-ordering on $\mathbf{T}_t - \{t\}$.*

Proof. For any fixed element $t_0 \in \mathbf{T}$, there exists a linear ordering $\preceq$ at least on the set $\mathbf{T}_{t_0} - \{t_0\}$, since for every element $t \in \mathbf{T}$, $\mathbf{T}_t - \{t\}$ is linearly ordered. It is to be shown that the linear ordering $\preceq$ is a well-ordering on $\mathbf{T}_{t_0} - \{t_0\}$. Under the linear ordering $\preceq$, define the initial segment as the following:

$$\operatorname{seg}(t) = \{s | s \preceq t\} \cap (\mathbf{T}_{t_0} - \{t_0\}), \quad t \in \mathbf{T}_{t_0} - \{t_0\}. \tag{2.1}$$

And let B be any subset of $\mathbf{T}_{t_0} - \{t_0\}$, it is to be shown that either B has a unique minimum element or B is empty. Define the set A of strict lower bounds of B as the following:

$$A = \{t \in \mathbf{T}_{t_0} - \{t_0\} | t \prec s \text{ for every } s \in B\}. \tag{2.2}$$

Note that

$$A \cap B = \emptyset, \tag{2.3}$$

Fig. 1. $\mathbf{T}_t - \{t\}$ is nonempty.

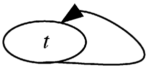

Fig. 2. $\mathbf{T}_t - \{t\}$ is empty.

otherwise, we have $t \prec t$, a contradiction. We consider the proposition (P): Assuming that for any element $t \in \mathbf{T}_{t_0} - \{t_0\}$, and any set $C \subseteq \mathbf{T}_{t_0} - \{t_0\}$ satisfying the condition

$$\mathrm{seg}(t) \subseteq C \Longrightarrow t \in C, \tag{2.4}$$

and want to know if the condition (P) holds for A.

Case I. Proposition (P) fails. Then there exists some $t \in \mathbf{T}_{t_0} - \{t_0\}$ with $\mathrm{seg}(t) \subseteq A$ but $t \notin A$. We claim that t is a unique minimum element of B, since $t \notin A$, there are some $s \in B$ with $s \preceq t$. However, s cannot belong to $\mathrm{seg}(t)$, which is disjoint from B. Thus $s = t$ and $t \in B$. And t is a unique minimum element in B, since anything smaller than t is in $\mathrm{seg}(t)$ and hence not in B.

Case II. Proposition (P) holds. Then for any element $t \in \mathbf{T}_{t_0} - \{t_0\}$, and the set $A \subseteq \mathbf{T}_{t_0} - \{t_0\}$, one sees easily that

$$\mathrm{seg}(t) \subseteq A \Longrightarrow t \in A, \tag{2.5}$$

namely,

$$A = \mathbf{T}_{t_0} - \{t_0\}. \tag{2.6}$$

Since $B \subseteq \mathbf{T}_{t_0} - \{t_0\}$, $B \subseteq A$. (2.3) implies $B = \emptyset$. This shows that either B has a unique minimum element or B is empty. Therefore, for every element $t \in \mathbf{T}$, a linear ordering $\preceq$ on $\mathbf{T}_t - \{t\}$ is a well-ordering on $\mathbf{T}_t - \{t\}$. This completes the proof of Lemma 2.2. □

Remark 1. Using Lemma 2.2 and the well-ordering's theorem (Hausdorff maximality theorem), we obtain easily that for every element $t \in \mathbf{T}$, if $\mathbf{T}_t - \{t\}$ is nonempty, then there is a unique minimum element in the set $\mathbf{T}_t - \{t\}$, and this minimum element is denoted by t^+, which is the succeeding element of t. □

2.1. The Proof of Theorem 2.1

Proof. Using the partial ordering $\preceq$ on the tree set $\mathbf{T}$, we can construct a mapping ϕ between the tree set $\mathbf{T}$ and a graph $G(\mathbf{T})$ as the following:

Case I. For any element $t \in \mathbf{T}$, if $\mathbf{T}_t - \{t\}$ is nonempty then draw a directed arc a_t between elements t and t^+ as in Fig. 1, where a_t is incident out of t and incident into t^+. Then t is an in-neighbor of t^+ and it follows from Lemma 2.2 that t^+ is an unique out-neighbor of t, that is, in this case, for every element $t \in \mathbf{T}$ there is a unique corresponding arc a_t such that vertices $v(t)$ and $v(t^+)$ are head and end respectively of the arc a_t. The direction that the arrow points toward shows that $t \preceq t^+$ in the tree $\mathbf{T}$.

Case II. For any element $t \in \mathbf{T}$, if $\mathbf{T}_t - \{t\}$ is empty then draw a directed arc a_t as in Fig. 2, where the arc a_t is incident out of t and incident into t, then it is a loop.

According to the previous statement, we can define exactly a mapping ϕ that for every element $t \in \mathbf{T}$,

$$\phi(t) = \begin{cases} a_t (\text{a directed arc}) \text{ and } I_{G(\mathbf{T})}(a_t) = \langle t, t^+ \rangle, & \text{if } \mathbf{T}_t - \{t\} \text{ is nonempty;} \\ a_t (\text{a loop}) \text{ and } I_{G(\mathbf{T})}(a_t) = \langle t, t \rangle, & \text{if } \mathbf{T}_t - \{t\} \text{ is empty.} \end{cases}$$

Such a mapping ϕ, which maps a tree set $\mathbf{T}$ onto a graph $G(\mathbf{T})$. Let $V(G(\mathbf{T}))$ be the set of vertices of $G(\mathbf{T})$, and $A(G(\mathbf{T}))$ be the set disjoint from $V(G(\mathbf{T}))$, i.e.,

$$A(G(\mathbf{T})) = \{a_t, t \in \mathbf{T}\}, \tag{2.7}$$

$I_{G(\mathbf{T})}$ be an incidence map that associates with each arc of $G(\mathbf{T})$ an ordered pair of vertices of $G(\mathbf{T})$, i.e.,

$$I_{G(\mathbf{T})}(a_t) = \langle t, t^+ \rangle. \tag{2.8}$$

Step I. It is to be shown that $(V(G(\mathbf{T})), A(G(\mathbf{T})), I_{G(\mathbf{T})})$ is an ordered triple. According to the definition of the mapping $\phi(t)$, one sees easily that for every element $t \in \mathbf{T}$ there exists a unique arc a_t such that $\phi(t) = a_t$, conversely, for every arc a_t there also exists a unique element $t \in \mathbf{T}$ such that $\phi^{-1}(a_t) = t$. Therefore, the mapping ϕ is a one-to-one mapping between $\mathbf{T}$ and $A(G(\mathbf{T}))$, i.e.,

$$\phi : \mathbf{T} \longrightarrow A(G(\mathbf{T})). \tag{2.9}$$

Moreover, this mapping ϕ is viewed as a one-to-one mapping between $\mathbf{T}$ and $V(G(\mathbf{T}))$, i.e.,

$$\phi : \mathbf{T} \longrightarrow V(G(\mathbf{T})).$$

Since for every element $t \in \mathbf{T}$ there exists a unique vertex $v(t)$, which is the head of arc a_t such that $\phi(t) = v(t)$, conversely, for every vertex $v(t)$ there also exists a unique element $t \in \mathbf{T}$ such that $\phi^{-1}(v(t)) = t$. We consider the reachable relation between any vertices $v(t)$ and $v(t')$ in $V(G(\mathbf{T}))$, and such this reachable relation is denoted by $v(t)\ \mathcal{R}\ v(t')$ if from the vertex $v(t)$ to the vertex $v(t')$ is reachable on the graph $G(\mathbf{T})$. For example, if $v(t_1)\ \mathcal{R}\ v(t_2)$, and $v(t_2)\ \mathcal{R}\ v(t_3)$ then there exists a unique directed path on the graph $G(\mathbf{T})$: $v(t_1)a_{t_1}v(t_2)a_{t_2}v(t_3)$, that is, $v(t_1)\ \mathcal{R}\ v(t_3)$. On the other hand, for every vertex $v(t) \in V(G(\mathbf{T}))$ we take as $v(t)\ \mathcal{R}\ v(t)$. In other words, $\mathcal{R}$ is a transitive relation and reflexive, thus $\mathcal{R}$ is a partial ordering on $V(G(\mathbf{T}))$. From this, we take the reachable relation between any vertices $v(t)$ and $v(t')$ in $V(G(\mathbf{T}))$ as a partial ordering $\mathcal{R}$ on $V(G(\mathbf{T}))$, then $(V(G(\mathbf{T})), \mathcal{R})$ is a partial ordering set. Next, we will prove that the mapping ϕ is an isomorphism from $(\mathbf{T}, \preceq)$ on to $(V(G(\mathbf{T})), \mathcal{R})$. Because ϕ is a one-to-one mapping between $\mathbf{T}$ and $V(G(\mathbf{T}))$, we only need to prove that ϕ preserves partial ordering. For any $t, t' \in \mathbf{T}$, without loss of generality, suppose that $t \preceq t'$, set

$$\Delta\mathbf{T}_t^{t'} = \mathbf{T}_t - (\mathbf{T}_{t'} - \{t'\}) = \{s | t \preceq s \preceq t'\}. \tag{2.10}$$

Note that if $\mathbf{T}_t - \{t\}$ is empty, then $t^+ = \inf\{\mathbf{T}_t - \{t\}\} = t$. Since $t \preceq t'$, then $t^+ \preceq t' \preceq (t')^+$, So

$$\begin{aligned} \inf\{\Delta\mathbf{T}_t^{t'} - \{t\}\} &= \min\{t^+, (t')^+\} \\ &= \inf\{(\Delta\mathbf{T}_t^{t'} - \{t\}) \oplus (\mathbf{T}_{t'} - \{t'\})\} \\ &= \inf\{\mathbf{T}_t - \{t\}\}. \end{aligned} \tag{2.11}$$

Therefore, for any element $t \in \mathbf{T}$,

$$I_{G(\mathbf{T})}(a_t) = \langle t, \inf\{\Delta\mathbf{T}_t^{t'} - \{t\}\}\rangle = \langle t, \inf\{\mathbf{T}_t - \{t\}\}\rangle, \tag{2.12}$$

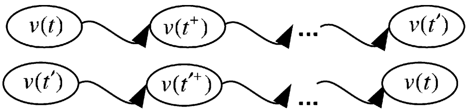

Fig. 3. $t \prec t'$ or $t \succ t'$.

and since $\Delta \mathbf{T}_t^{t'} - \{t\}$ is countable and finite, and the partial ordering $\preceq$ on $\Delta \mathbf{T}_t^{t'} - \{t\}$ is a well-ordering, so $\{I_{a_s}\}_{s \in \Delta \mathbf{T}_t^{t'} - \{t\}}$ is a directed path from the vertex $v(t)$ to the vertex $v(t')$, and is a unique directed path on the graph $G(\mathbf{T})$. In other words, from the vertex $v(t)$ to the vertex $v(t')$ is a reachable relation, i.e.,

$$v(t) \,\mathcal{R}\, v(t'), \quad \text{namely, } \phi(t) \,\mathcal{R}\, \phi(t').$$

Fig. 3 shows this reachable relation between two vertices $v(t)$ and $v(t')$.

Conversely, for any $v(t), v(t') \in G(\mathbf{T})$, without loss of generality, if $v(t) \,\mathcal{R}\, v(t')$, then there exists a unique directed path $\{I_{a_s}\}_{s \in \Delta \mathbf{T}_t^{t'} - \{t\}}$ that is from the vertex $v(t)$ to the vertex $v(t')$ on the graph $G(\mathbf{T})$. According to the definition of the mapping ϕ and the transitivity of elements in $\mathbf{T}$ and (2.12), we can obtain that $t \preceq t'$. In a word, for any elements $t, t' \in \mathbf{T}$, the one-to-one mapping ϕ satisfies that

$$t \preceq t' \quad \text{iff} \quad \phi(t) \,\mathcal{R}\, \phi(t'). \tag{2.13}$$

This shows that the one-to-one mapping ϕ is an isomorphism between $(\mathbf{T}, \preceq)$ and $(V(G(\mathbf{T})), \mathcal{R})$, i.e.,

$$(\mathbf{T}, \preceq) \cong (V(G(\mathbf{T})), \mathcal{R}). \tag{2.14}$$

Finally, since $\mathbf{T}$ is a countable, upward-directed index set with respect to the partial ordering $\preceq$, so $V(G(\mathbf{T}))$ is also a countable, upward-directed vertex set with respect to the reachable relation $\mathcal{R}$ on the graph $G(t)$. This shows that $(V(G(\mathbf{T})), A(G(\mathbf{T})), I_{G(\mathbf{T})})$ is an ordered triple. Therefore, this graph $G(\mathbf{T})$ is a directed graph.

Step II. Here, we will point out that this directed graph $G(\mathbf{T})$ contains the following two propositions:

(1) outdegree of every vertex $v(t)$ on this graph $G(\mathbf{T})$

$$d^+(v(t)) = 1, \quad t \in \mathbf{T}; \tag{2.15}$$

(2) indegree of every vertex $v(t)$ on this graph $G(\mathbf{T})$ $d^-(v(t))$ is finite.

Therefore, the degree of every vertex $v(t)$ on this graph $G(\mathbf{T})$, $d(v(t))$ is finite. For example, choose two incomparable elements $t_1, t_2 \in \mathbf{T}$ such that

$$\mathbf{T}_{t_1} - \{t_1\} = \mathbf{T}_{t_2} - \{t_2\}, \tag{2.16}$$

thus,

$$t_1^+ = \inf[\mathbf{T}_{t_1} - \{t_1\}] = \inf[\mathbf{T}_{t_2} - \{t_2\}] = t_2^+. \tag{2.17}$$

In other words, both $\mathbf{T}_{t_1} - \{t_1\}$ and $\mathbf{T}_{t_2} - \{t_2\}$ are totally ordered sets and their elements are completely identical, but they respectively contain a distinct well-ordering. Therefore, there exists a directed connection ϕ_{t_1} and ϕ_{t_2}, respectively, such that the vertices $v(t_1)$ and $v(t_2)$ are unreachable relations (refer to Fig. 4).

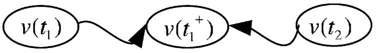

Fig. 4. t_1 and t_2 is incomparable and $\mathbf{T}_{t_1} - \{t_1\} = \mathbf{T}_{t_2} - \{t_2\}$.

In this case, it is clear that

$$2 \le d^+(v(t_1^+)) + d^-(v(t_1^+)) = d(v(t_1^+)) < \infty. \tag{2.18}$$

Now, the vertices of the directed graph $G(\mathbf{T})$ can be divided into three sorts of vertices.

- The vertices with $d^+(v(t)) = 1$ and $d^-(v(t)) \neq 0$ with a loop.
- The vertices with $d^+(v(t)) = 1$ and $d^-(v(t)) \neq 0$ without a loop.
- The vertices with $d^+(v(t)) = 1$ and $d^-(v(t)) = 0$.

Step III. In this step, the directed graph $G(\mathbf{T})$ will be represented as a direct sum of a sequence of inside-directed trees. Firstly, we shall construct a sequence child graphs $\{G_i(\mathbf{T})\}_{i=0}^{\infty}$ of the directed graph $G(\mathbf{T})$ by induction method.

1. Choose a vertex $v(t_0) \in G(\mathbf{T})$ arbitrarily, define a child graph

$$G_0(\mathbf{T}) = \{\text{the weakly connected graph including the vertex } v(t_0)\} \cap G(\mathbf{T}).$$

2. Choose a vertex $v(t_1) \in G(\mathbf{T}) - G(\mathbf{T}_0)$ arbitrarily, then $v(t_1) \neq v(t_0)$, we define a child graph

$$G_1(\mathbf{T}) = \{\text{the weakly connected graph including the vertex } v(t_1)\} \cap G(\mathbf{T}).$$

$\cdots$

n. Choose a vertex $v(t_{n-1}) \in G(\mathbf{T}) - \bigcup_{i=0}^{n-2} G_i(\mathbf{T})$ arbitrarily, then

$$v(t_{n-1}) \neq v(t_i), \quad i = 0, 1, 2, \ldots, n-2, \tag{2.19}$$

we define a child graph

$$G_{n-1}(\mathbf{T}) = \{\text{the weakly connected graph including the vertex } v(t_{n-1})\} \cap G(\mathbf{T}).$$

$\ldots,$

and so on. The directed graph $G(\mathbf{T})$ is divided into a sequence of child graphs $\{G_i(\mathbf{T})\}_{i=1}^{\infty}$ with

$$G_i(\mathbf{T}) \cap G_j(\mathbf{T}) = \emptyset, \qquad V[G_i(\mathbf{T})] \cap V[G_j(\mathbf{T})] = \emptyset, \quad i \neq j,\ i, j \in \overline{\mathbb{N}}. \tag{2.20}$$

Clearly,

$$G(\mathbf{T}) = G_0(\mathbf{T}) \oplus G_1(\mathbf{T}) \oplus \cdots \oplus G_i(\mathbf{T}) \oplus \cdots = \sum_{i=0}^{\infty} G_i(\mathbf{T}) \tag{2.21}$$

and

$$V[G(\mathbf{T})] = V[G_0(\mathbf{T})] \oplus V[G_1(\mathbf{T})] \oplus \cdots \oplus V[G_i(\mathbf{T})] \oplus \cdots = \sum_{i=0}^{\infty} V[G_i(\mathbf{T})], \tag{2.22}$$

where $\oplus$ denotes the direct sum of two disjoint child graphs or two disjoint subsets, i.e.,

$$G_i(\mathbf{T}) \oplus G_j(\mathbf{T}) = G_i(\mathbf{T}) \cup G_j(\mathbf{T}), \quad i \neq j,\ i, j \in \overline{\mathbb{N}}, \tag{2.23}$$

$$V[G_i(\mathbf{T})] \oplus V[G_j(\mathbf{T})] = V[G_i(\mathbf{T})] \cup V[G_j(\mathbf{T})], \quad i \neq j,\ i, j \in \overline{\mathbb{N}}. \tag{2.24}$$

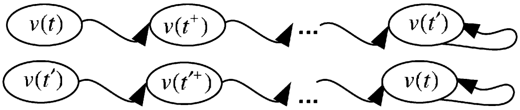

Fig. 5. Both $v(t)$ and $v(t')$ with a loop.

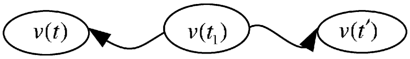

Fig. 6. $t_1 \prec t$ and $t_1 \prec t'$.

Moreover, $\{G_i(\mathbf{T})\}_{i=0}^{\infty}$ is a sequence of inside-directed locally finite trees. In fact, because of

$$G_i(\mathbf{T}) \subset G(\mathbf{T}), \quad i \in \overline{\mathbb{N}}$$

and $G(\mathbf{T})$ is a directed graph, then $G_i(\mathbf{T})$ is also a directed graph. For every $i \in \mathbb{N}_0$, choose a vertex $v(t) \in G_i(\mathbf{T})$, arbitrarily, if the vertex $v(t)$ with a loop, then there does not exists another vertex $v(t') \in G_i(\mathbf{T})$ with a loop, which is distinct with the vertex $v(t)$. Otherwise, assume that there exists a vertex $v(t') \in G_i(\mathbf{T})$ with loop and $v(t') \neq v(t)$. Then by applying the isomorphism ϕ to $v(t') \neq v(t)$, one obtains that $t' \neq t$.

Case I. As t' and t is comparable, we see easily that either $t' \succ t$ or $t' \prec t$.

(i) if $t' \succ t$, then, from the definition of the mapping ϕ, it follows that the vertex $v(t)$ without a loop on the graph $G(\mathbf{T})$ (refer to Fig. 5), a contradiction.

(ii) if $t' \prec t$, then, from the definition of the mapping ϕ, it follows that the vertex $v(t')$ without a loop on the graph $G(\mathbf{T})$ (refer to Fig. 5), a contradiction.

Case II. As t and t' is incomparable, it is clear that there exists no directed path between vertices $v(t)$ and $v(t')$ on the graph $G_i(\mathbf{T})$ by using the mapping ϕ, otherwise, t and t' are comparable (refer to Fig. 5), and since that both $v(t)$ and $v(t')$ have a distinct loop, one sees that corresponding t and t' are maximal elements in the tree set $\mathbf{T}$, and since $G_i(\mathbf{T})$ is a weakly connected graph and $v(t), v(t') \in G_i(\mathbf{T})$, one sees easily that $v(t)$ and $v(t')$ are also weak connected on the graph $G_i(\mathbf{T})$. Therefore, there exists at least an element t_1 with $t_1 \prec t$ and $t_1 \prec t'$ such that $v(t_1)\ \mathcal{R}\ v(t)$ and $v(t_1)\ \mathcal{R}\ v(t')$ (refer to Fig. 6), then $d^+(v(t_1)) \geq 2$, a contradiction.

In a word, for any $i \in \overline{\mathbb{N}}$, there exists a unique vertex $v(t)$ with loop on the directed graph $G_i(\mathbf{T})$. Moreover, it follows from the propositions of the directed graph $G(\mathbf{T})$ that $G_i(\mathbf{T})$ is a directed inside tree for any $i \in \overline{\mathbb{N}}$, and it follows from those results of *Step* II that $G_i(\mathbf{T})$ is an inside-directed locally finite tree. That is, the directed graph sequence of $\{G_i(\mathbf{T})\}_{i=0}^{\infty}$ is an inside-directed locally finite forest. Therefore, it follows from (2.14) that the tree set $\mathbf{T}$ is isomorphic to an inside-directed locally finite forest. The proof is complete. $\square$

Remark 2. It follows from the proof of Theorem 2.1 that for any $i \in \overline{\mathbb{N}}$, $G_i(\mathbf{T})$ is an inside-directed locally finite tree. On every inside-directed locally finite tree $G_i(\mathbf{T})$, the vertex with a loop is its tree root, those vertices $(v(t), t \in \mathbf{T})$ with $d^-(v(t)) = 0$ are tree leaves and the number of the leaves on $G_i(\mathbf{T})$ is finite, those vertices $(v(t), t \in \mathbf{T})$ with $d^-(v(t)) \neq 0$ and without a loop are branching vertices. $\square$

We follow the strategy in order to show that a decomposition of the tree set $\mathbf{T}$ by using Theorem 2.1. $V(G_i(\mathbf{T}))$ is the vertex set of the inside-directed locally finite tree $G_i(\mathbf{T})$. ϕ^{-1} is the inversion of the mapping ϕ. In the following, ϕ denotes the mapping in the proof of Theorem 2.1.

Theorem 2.3. *If* $\mathbf{T}$ *is a tree set, then there exists a sequence of sets* $\{\mathbf{T}_i\}_{i=0}^{\infty}$ *with*

$$\mathbf{T}_i \cap \mathbf{T}_j = \emptyset, \quad i \neq j,\ i, j \in \overline{\mathbb{N}} \tag{2.25}$$

such that

$$\mathbf{T} = \mathbf{T}_0 \oplus \mathbf{T}_1 \oplus \cdots \oplus \mathbf{T}_i \oplus \cdots = \sum_{i=0}^{\infty} \mathbf{T}_i, \tag{2.26}$$

where $\mathbf{T}_i = \phi^{-1}[V(G_i(\mathbf{T}))]$ $(i \in \overline{\mathbb{N}})$.

Proof. Since the mapping ϕ is an isomorphism between $\mathbf{T}$ and $V(G(\mathbf{T}))$, then the inversion of ϕ exists and is denoted by ϕ^{-1}, which is also an isomorphism between $V(G(\mathbf{T}))$ and $\mathbf{T}$. In terms of the existence of $G_i(\mathbf{T}) \subset G(\mathbf{T})$ and $\mathbf{T}_i = \phi^{-1}[V(G_i(\mathbf{T}))]$ $(i \in \overline{\mathbb{N}})$, one can derive from (2.20) that

$$\begin{aligned} \mathbf{T}_i \cap \mathbf{T}_j &= \phi^{-1}[V(G_i(\mathbf{T}))] \cap \phi^{-1}[V(G_j(\mathbf{T}))] \\ &= \phi^{-1}[V(G_i(\mathbf{T})) \cap V(G_j(\mathbf{T}))] = \emptyset, \end{aligned} \tag{2.27}$$

and $\mathbf{T}_i, \mathbf{T}_j \subset \mathbf{T}$. Moreover, one can derive from (2.22) that

$$\mathbf{T} = \phi^{-1}[V(G(\mathbf{T}))] = \sum_{i=0}^{\infty} \phi^{-1}[V(G_i(\mathbf{T}))] = \sum_{i=0}^{\infty} \mathbf{T}_i. \tag{2.28}$$

The proof is complete. □

Remark 3. In fact, the idea of the proof of Theorem 2.3 can be represented as

$$\mathbf{T} \xrightarrow{\ \phi\ } G(\mathbf{T}) = \sum_{i=0}^{\infty} G_i(\mathbf{T}) \xrightarrow{\ \phi^{-1}\ } \sum_{i=0}^{\infty} \mathbf{T}_i = \mathbf{T},$$

where $\mathbf{T}$ is a tree set, for any $i \in \mathbb{N}$, $G_i(\mathbf{T})$ is an inside-directed locally finite tree and $\mathbf{T}_i$ is a single tree set. □

Considering the mapping ϕ is an isomorphism between $\mathbf{T}$ and $G(\mathbf{T})$, one obtains Corollary 2.4 immediately.

Corollary 2.4. *If* $\{\mathbf{T}_i\}_{i=0}^{\infty}$ *is a sequence of sets as in Theorem 2.3, then each* $\mathbf{T}_i$ $(i \in \overline{\mathbb{N}})$ *is respectively isomorphic to an inside-directed locally finite tree.*

Definition 2.5. Let $\mathbf{T}^l$ be a countable, upward-directed index set with respect to the partial ordering $\preceq$ and $\mathbf{T}^l$ be isomorphic to an inside-directed locally finite tree, then we called $\mathbf{T}^l$ a locally finite tree set.

Here, we need to point out that $\mathbf{T}^l$ is the same as $\mathbf{T}_i$ in Theorem 2.3. To facilitate the investigation for tree martingales, we shall give an equivalent definition on the locally finite tree set.

Definition 2.6. Let $\mathbf{T}^l$ be a countable, upward-directed index set with respect to the partial ordering $\preceq$ satisfying the following three conditions:

(1) for every $t \in \mathbf{T}^l$, the set $\mathbf{T}^{lt} := \{s \in \mathbf{T}^l : s \preceq t\}$ is finite;
(2) for every $t \in \mathbf{T}^l$, the set $\mathbf{T}^l_t := \{s \in \mathbf{T}^l : t \preceq s\}$ is linearly ordered;
(3) there is an unique maximum element in $\mathbf{T}^l$.

Then we called the set a locally finite tree set.

Proposition 2.7. *Assume that $\mathbf{T}^l$ is a locally finite tree set. Then its supremum exists and for any $t, t' \in \mathbf{T}^l$, it holds that*

$$\sup[\mathbf{T}^l_t] = \sup[\mathbf{T}^l_{t'}] = \sup \mathbf{T}^l. \tag{2.29}$$

Obviously, a locally finite tree set contains Proposition 2.7, which may not be contained by a tree set, since a tree set may have many maximal elements.

Theorem 2.8. *Definition 2.5 is equivalent to Definition 2.6.*

Proof. Suppose that $\mathbf{T}^l$ is a set in Definition 2.5. Then it follows from Theorem 2.1 that there exists an isomorphism ϕ such that

$$\phi : \mathbf{T}^l \to V[G(\mathbf{T}^l)], \tag{2.30}$$

where $G(\mathbf{T}^l)$ is an inside-directed locally finite tree, $(V[G(\mathbf{T}^l)], A[G(\mathbf{T}^l)], I_{G(\mathbf{T}^l)})$ is an ordered triple. Therefore, $\mathbf{T}^l = \phi^{-1}(V[G(\mathbf{T}^l)]) \subset \mathbf{T}$, $\mathbf{T}$ is a tree set. Then

(i) for every $t \in \mathbf{T}^l$, the set

$$\mathbf{T}^{lt} = \{s \in \mathbf{T}^l : s \preceq t\} = \mathbf{T}^l \cap \mathbf{T}^t \subset \mathbf{T}^t, \tag{2.31}$$

$\mathbf{T}^{lt}$ is finite since $\mathbf{T}^t$ is finite for every $t \in \mathbf{T}$.

(ii) for every $t \in \mathbf{T}^l$, the set

$$\mathbf{T}^l_t = \{s \in \mathbf{T}^l : t \preceq s\} = \mathbf{T}^l \cap \mathbf{T}_t \subset \mathbf{T}_t, \tag{2.32}$$

$\mathbf{T}^l_t$ is linearly-ordered, and the linear-ordering on $\mathbf{T}^l_t$ is a well-ordering since $\mathbf{T}_t$ is linearly-ordered, and the linear-ordering on $\mathbf{T}_t$ is a well-ordering.

(iii) assume that there exist two distinct maximal elements $t_{\max}, t'_{\max} \in \mathbf{T}^l$, then $\mathbf{T}_{t_{\max}} - \{t_{\max}\}$, $\mathbf{T}_{t'_{\max}} - \{t'_{\max}\}$ are empty. According to the definition of the isomorphism ϕ, we see that both vertices $v(t_{\max})$ and $v(t'_{\max})$ on $G(\mathbf{T}^l)$ have a loop, but $G(\mathbf{T}^l)$ is an inside-directed tree, which is a contradiction. Therefore, there is a unique maximum element in $\mathbf{T}^l$. Conversely, assume that $\mathbf{T}^l$ satisfies the three conditions in Definition 2.6. Then $\mathbf{T}^l \subset \mathbf{T}$, which is a tree set. By applying Theorem 2.1 to $\mathbf{T}^l$, one sees that there is an isomorphism ϕ between $\mathbf{T}^l$ and $V[G(\mathbf{T}^l)]$, which is the vertex set of a graph $G(\mathbf{T}^l)$, such that this graph $G(\mathbf{T}^l)$ is an inside-directed locally finite forest. Now, if $t_{\max}$ is a unique maximum in $\mathbf{T}^l$, then for any $t \in \mathbf{T}^l$ has $t \preceq t_{\max}$, and by using the isomorphism ϕ, it is obtained that

$$\phi(t)\mathcal{R}\phi(t_{\max}), \quad \text{i.e.,} \ v(t)\ \mathcal{R}\ v(t_{\max}), \tag{2.33}$$

and $v(t_{\max})$ is unique. That is, for any vertex $v(t) \in V[G(\mathbf{T}^l)]$ has $v(t)\mathcal{R}\, v(t_{\max})$ on this inside-directed locally finite tree forest $G(\mathbf{T}^l)$, therefore, there is a unique tree root $v(t_{\max})$ on this inside-directed locally finite tree forest $G(\mathbf{T}^S)$, then $G(\mathbf{T}^l)$ is an inside-directed locally finite tree. □

From proof of Theorem 2.8 and Corollary 2.4, it is easy to see that the supremum of the locally finite tree set $\mathbf{T}^l$ is identified with the root of the corresponding inside-directed locally finite tree $G(\mathbf{T}^l)$, and these minimal elements of the locally finite tree set $\mathbf{T}^l$ are respectively identified with those leaves of the corresponding inside-directed locally finite tree $G(\mathbf{T}^l)$. Two important notions will be introduced in Definition 2.9., they will be used in the chapter 4.

Definition 2.9. On an inside-directed locally finite tree $G(\mathbf{T}^l)$, a *ray* is an infinite path from each leaf tending towards the root that does not backtrack. We call the set of rays of an inside-directed locally finite tree $G(\mathbf{T}^l)$ the *boundary* of $G(\mathbf{T}^l)$, denoted $\partial G(\mathbf{T}^l)$.

Here, the *ray* and *boundary* are different from Lyons [17] (refer to p: 4, 13). Now, we turn to the important business of constructing an index set decomposition for tree martingales by using the previous theorems and definitions.

Theorem 2.10 (*Index Set Decomposition Theorem*). *Assume that* $\mathbf{T}$ *is a tree set. Then* $\mathbf{T}$ *can be represented as*

$$\mathbf{T} = \mathbf{T}_0^l \oplus \mathbf{T}_1^l \oplus \cdots \oplus \mathbf{T}_i^l \oplus \cdots = \sum_{i=0}^{\infty} \mathbf{T}_i^l, \tag{2.34}$$

where $\mathbf{T}_i^l$ $(i \in \overline{\mathbb{N}})$ *is either a locally finite tree set or an empty set.*

Proof. Theorem 2.10 comes from Theorems 2.3 and 2.8, Corollary 2.4 and Definitions 2.5 and 2.6 immediately. □

3. Decomposition for tree martingales

Before using Theorem 2.10 we can obtain a decomposition for tree martingales, we need introduce two definitions.

Definition 3.1. Assume that $\mathbf{T}^l$ is a locally finite tree set and $(P_t, t \in \mathbf{T}^l)$ be a family of projections as defined in (1.1). Then the family of $(\mathcal{F}_t, P_t : t \in \mathbf{T}^l)$ is called a locally finite tree basis if

(1) for every $f \in L_1$ and $s \preceq t$ $(s, t \in \mathbf{T}^l)$, $P_t f = \phi_s E_t(f\bar{\phi}_s)$;
(2) For every pair of incomparable s, t in $\mathbf{T}^l$, $P_t P_s = 0$.

Definition 3.2. We say that a family of projections $P = (P_t, t \in \mathbf{T}^l)$ is a locally finite tree martingale if for every $f \in L_1$ then

(1) for every pair of comparable $s, t \in \mathbf{T}^l$, if $s \preceq t$ implies
$$P_t P_s(f) = P_s P_t(f) = P_s(f); \tag{3.1}$$
(2) for every pair of incomparable $s, t \in \mathbf{T}^l$, $P_t P_s(f) = 0$.

Now, a decomposition for tree martingales shall be constructed by using this locally finite tree martingale.

Theorem 3.3 (*Decomposition of Tree Martingales*). *Assume that for any* $f \in L_1$, $P(f) = (P_t(f), t \in \mathbf{T})$ *is a tree martingale as in Definition 1.3. Then*

$$P(f) = P_0(f) \oplus P_1(f) \oplus \cdots \oplus P_i(f) \oplus \cdots = \sum_{i=0}^{\infty} P_i(f), \tag{3.2}$$

where $P_i(f) = (P_t(f), t \in \mathbf{T}_i^l)$ *is a locally finite tree martingale,* $\{\mathbf{T}_i^l\}_{i=0}^{\infty}$ *is a sequence of locally finite tree sets such that*

$$\mathbf{T} = \mathbf{T}_0^l \oplus \mathbf{T}_1^l \oplus \cdots \oplus \mathbf{T}_i^l \oplus \cdots = \sum_{i=0}^{\infty} \mathbf{T}_i^l.$$

Proof. For any $f \in L_1$, $P(f)$ is viewed as a mapping from a tree set $\mathbf{T}$ on to the set $\{P_t(f), t \in \mathbf{T}\}$, which is a family of projections as (1.1), i.e.,

$$P(f) : t \to P(f)(t) = P_t(f), \quad t \in \mathbf{T}, \tag{3.3}$$

then $(P_t(f), t \in \mathbf{T})$ is a family of tree martingales. Clearly, $P(f)$ is an one-to-one mapping from a tree set $\mathbf{T}$ on to $\{P_t(f), t \in \mathbf{T}\}$. It follows from Theorem 2.10 that there exists a sequence of locally finite tree sets $\{\mathbf{T}_i^S\}_{i=0}^{\infty}$ such that

$$\mathbf{T} = \mathbf{T}_0^l \oplus \mathbf{T}_1^l \oplus \cdots \oplus \mathbf{T}_i^l \oplus \cdots = \sum_{i=0}^{\infty} \mathbf{T}_i^l, \tag{3.4}$$

by applying this mapping $P(f)$ to (3.4), one obtains that

$$\begin{aligned} P(f)(\mathbf{T}) &= P(f)[\mathbf{T}_0^l] \oplus P(f)[\mathbf{T}_1^l] \oplus \cdots P(f)[\mathbf{T}_i^l] \oplus \cdots \\ &= \sum_{i=0}^{\infty} P(f)(\mathbf{T}_i^l), \end{aligned} \tag{3.5}$$

let $P_i(f) = (P_t(f), t \in \mathbf{T}_i^l) = P(f)(\mathbf{T}_i^l)$, and $P(f) = P(f)(\mathbf{T}) = (P_t(f), t \in \mathbf{T})$, then it follows from (3.5) that

$$P(f) = \sum_{i=0}^{\infty} P_i(f). \tag{3.6}$$

And it follows from Definition 3.2 that $P_i(f) = (P_t(f), t \in \mathbf{T}_i^l)$ is a locally finite tree martingale. The proof is complete. $\square$

Theorem 3.3 shows that some problems for the tree martingales can be translated into some corresponding problems for the locally finite tree martingales. And from Theorem 3.3, it is clear that if a locally finite tree martingale $P_i(f) = (P_t(f), t \in \mathbf{T}_i^l)$ is convergent, then it does not mean that the tree martingales $P(f) = (P_t(f), t \in \mathbf{T})$ are also convergent. In the following, our main interest is studying how to define the convergence for tree martingales and under what conditions is a tree martingale convergent.

4. Convergence for tree martingales

The index set of tree martingales $\mathbf{T}$ is a tree set, which has a few of maximal elements, and maybe its supremum does not exist, since that the tree set $\mathbf{T}$ is a partially ordered set. We might do some thinking from another angle, this tree set $\mathbf{T}$ is isomorphic to an inside-directed locally finite forest $G(\mathbf{T})$, this forest maybe has many topological ends. However, if the index set of tree martingales $\mathbf{T}$ is not only a tree set but also a locally finite tree set, then it follows from Proposition 2.7 that $\sup\{\mathbf{T}^l\}$ exists. Then how to define a tree martingale limit? If $\mathbf{T}^l$ is a locally finite tree set, then for any $t \in \mathbf{T}^l$, t tends to $\sup\{\mathbf{T}^l\}$ in the partial ordering $\preceq$ on $\mathbf{T}^S$, that is,

$$t \underset{\preceq}{\longrightarrow} \sup\{\mathbf{T}^l\}, \quad t \in \mathbf{T}^l.$$

If we can prove that for any $f \in L_1$ it holds

$$\lim_{t \underset{\preceq}{\longrightarrow} \sup\{\mathbf{T}^S\}} P_t f = P_{\sup\{\mathbf{T}^l\}} f, \quad t \in \mathbf{T}^l, \tag{4.1}$$

then the convergence theorem for the locally finite tree martingales shall be established.

Definition 4.1. Let $\mathbf{T}^l$ is a locally finite tree set, in which there are N minimal elements, we say that $\mathbf{T}^l$ is a so-called locally finite tree set with N-minimal element, and is denoted by $\mathbf{T}_N^S$.

Remark 4. By applying the isomorphism ϕ to $\mathbf{T}_N^S$, one obtains that

$$\phi : \mathbf{T}_N^l \to \phi(\mathbf{T}_N^l) = G(\mathbf{T}_N^l),$$

and it follows from Corollary 2.4 that $G(\mathbf{T}_N^l)$ is an inside-directed locally finite tree with N leaves. Note that Definition 2.9, it is easy to see that on an inside-directed locally finite tree with N leaves $G(\mathbf{T}_N^l)$, each leaf is identified with a ray. That is, there exists a bijective homomorphism such that the set of leaves is isomorphic to $\partial G(\mathbf{T}_N^l)$, then the cardinality of $\partial G(\mathbf{T}_N^l)$, $|\partial G(\mathbf{T}_N^l)| = N$. Conversely, if an inside directed locally finite tree with $|\partial G(\mathbf{T}^l)| < \infty$, then $\mathbf{T}^l$ has a finite number of minima.

Theorem 4.2. *Assume that $G(\mathbf{T}_N^l)$ is an inside-directed locally finite tree with N leaves. Then $V(G(\mathbf{T}_N^l))$ is isomorphic to a subset $\mathbb{N}(\mathbf{T}_N^l)$ of the $N+1$ dimensional Euclidean lattice $\mathbb{N}^{N+1}$.*

Proof. Firstly, we shall show that $V(G(\mathbf{T}_N^l))$ can be embedded into an $N+1$ dimensions discrete Euclidean space $\mathbb{N}^{N+1}$.

Case I. If the height of $G(\mathbf{T}_N^l)h$ is finite and $h \in \mathbb{N}$. We will construct a mapping ψ such that $V(G(\mathbf{T}_N^l))$ can be mapped into Euclidean space $\mathbb{N}^{N+1}$ by using coordinates of some points in $N+1$ dimensions Euclidean lattice $\mathbb{N}^{N+1}$ to represent the vertices on an inside-directed locally finite tree with N leaves.

Step 1. In Euclidean lattice $\mathbb{N}^{N+1}$, the root of $G(\mathbf{T}_N^l)$ is represented by the coordinates $(h, \underbrace{N, \ldots, N}_{N})$, where h is viewed as the hth level of $G(\mathbf{T}_N^l)$, and since the in-degree of every vertex on $G(\mathbf{T}_N^l)$ is finite, let us suppose the in-degree of the root $d^-(root) = r$, $r \in \mathbb{N}_0$ and $r \leq N$. Then there are r vertices in the $(h-1)$th level, from the left side to the right side, the coordinates of the r vertices are respectively denoted by

$$(h-1, N, \underbrace{N, \ldots, N}_{N-1}),\ (h-1, N-1,\ \underbrace{N, \ldots, N}_{N-1}),$$
$$\ldots, (h-1, N-(r-1), \underbrace{N, \ldots, N}_{N-1}).$$

Step 2. Each vertex on the $(h-1)$th level is viewed as a root, respectively, by repeating *Step* 1, in Euclidean lattices $\mathbb{N}^{N+1}$, the coordinates of all vertices on the $h-2$th level can be obtained. For example, the coordinates of a vertex with $d^- = 3$ is $(h-1, N-1, \underbrace{N, \ldots, N}_{N-1})$, then its child generation coordinates are the following:

$$(h-2, N-1, N, \underbrace{N, \ldots, N}_{N-2}),\ (h-2, N-1, N-1, \underbrace{N, \ldots, N}_{N-2}),$$
$$\ldots, (h-2, N-1, N-2, \underbrace{N, \ldots, N}_{N-2}).$$

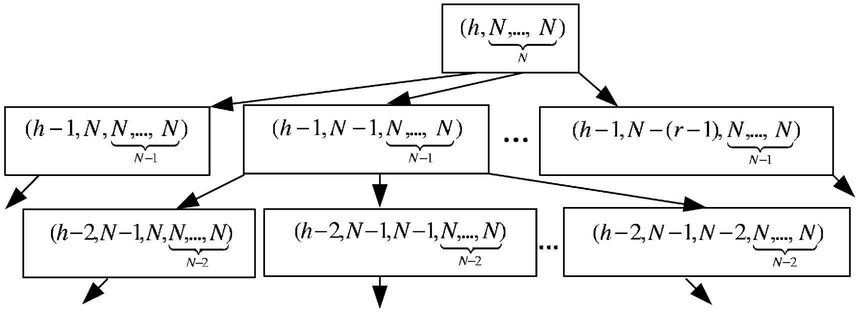

Fig. 7. As the height of $G(\mathbf{T}_N^l)h$ is finite, $V(G(\mathbf{T}_N^l))$ can be embedded into an $N+1$ dimensional Euclidean lattice $\mathbb{N}^{N+1}$.

Such this, by $(h-1)$ steps, for every vertex $v(t)$ on $G(\mathbf{T}_N^l)$, correspondingly, there is a unique point $n(t)$ in Euclidean lattices $\mathbb{N}^{N+1}$, the coordinates of this point $n(t)$ are denoted by

$$n(t) = (h-m, l_1, l_2, \ldots, l_k, \underbrace{N, \ldots, N}_{N-k}), \tag{4.2}$$

where $0 \le m \le h$, $0 \le k \le h$, $1 \le l_i \le N$, $m, k, l_i \in \mathbb{N}_0$ $(i = 1, 2, \ldots, k)$ and

$$N-k = \begin{cases} N-k, & 0 \le k \le N \\ 0, & k > N, \end{cases}$$

$$l_k = \begin{cases} l_k, & 0 \le k \le N \\ \emptyset, & k > N. \end{cases}$$

The collection of such coordinates is denoted by $\mathbb{N}(\mathbf{T}_N^l)$ and $\mathbb{N}(\mathbf{T}_N^l) \subset \mathbb{N}^{N+1}$. Then such a mapping ψ is a one-to-one mapping between $\mathbf{T}_N^l$ and $\mathbb{N}(\mathbf{T}_N^l) \subset \mathbb{N}^{N+1}$, i.e.,

$$\psi : V(G(\mathbf{T}_n^l)) \to \psi[V(G(\mathbf{T}_N^l))] = \mathbb{N}(\mathbf{T}_N^l), \tag{4.3}$$

$V(G(\mathbf{T}_N^l))$ is embedded into an $N+1$ dimensional Euclidean lattice $\mathbb{N}^{N+1}$ (refer to Fig. 7).

Case II. If the height of $G(\mathbf{T}_N^l)h$ is infinite. Let $h \to \infty$, by using the methods of *Case* I and by induction, we also obtain such a mapping ψ as (4.3), which is an one-to-one mapping between $\mathbf{T}_N^l$ and $\mathbb{N}(\mathbf{T}_N^l)$.

Second, the relation $\preccurlyeq$ on the Euclidean lattice $\mathbb{N}(\mathbf{T}_N^l)$ is to be defined as the following: for any

$$n(t_k) = (h-m, l_1, l_2, \ldots, l_k, \underbrace{N, \ldots, N}_{N-k}), \quad m \in \mathbb{N}_0, \tag{4.4}$$

and

$$n(t_{k+j}) = (h-m-j, l_1', l_2', \ldots, l_k', \ldots, l_{k+j}' \underbrace{N, \ldots, N}_{N-k-j}), \quad j \in \mathbb{N}_0, \tag{4.5}$$

the relation between $n(t_k)$ and $n(t_{k+j})$ is to be defined as the following:

$$n(t_{k+j}) \preccurlyeq n(t_k) \quad \text{iff} \quad h-m-j \le h-m,\ l_1 = l_1', \ldots, l_k = l_k', \tag{4.6}$$

otherwise, $n(t_k)$ and $n(t_{k+j})$ is incomparable. Clearly, the relation $\preccurlyeq$ is a transitive relation and reflexive. Therefore, this relation $\preccurlyeq$ is a partial ordering on $\mathbb{N}(\mathbf{T}_N^l)$. Note that $\preccurlyeq$ is a partial ordering which is the same as the special partial ordering on $\mathbb{N}^{N+1}$ (refer to [8]), even $\preccurlyeq$ is more special.

Third, it is to be shown that $V(G(\mathbf{T}_N^l))$ is isomorphic to $\mathbb{N}(\mathbf{T}_N^l)$ preserving the partial ordering $\preccurlyeq$, i.e.,

$$(V(G(\mathbf{T}_N^l)), \mathcal{R}) \cong (\mathbb{N}(\mathbf{T}_N^l), \preccurlyeq).$$

For any two vertices $v(t)$ and $v(t')$, if $v(t)\ \mathcal{R}\ v(t')$ then there is one, and only one directed path

$$v(t)v(t_1)\dots v(t_\tau)v(t'), \quad 0 \le \tau \le h-2,\ \tau \in \mathbb{N}_0,$$

on $G(\mathbf{T}_N^l)$, when $\tau = 0$ set $v(t_0) = v(t')$. Without loss of generality, set

$$\psi(v(t)) = n(t) = (h-m, l_1, l_2, \dots, l_k, \underbrace{N, \dots, N}_{N-k}), \tag{4.7}$$

then there exists a unique vertex $v(t_1)$ on the $h-m+1$th level of $G(\mathbf{T}_N^l)$ such that $v(t)\ \mathcal{R}\ v(t_1)$, according to the definition of the mapping ψ, one obtains that

$$\psi(v(t_1)) = n(t_1) = (h-m+1, l_1, l_2, \dots, l_{k-1}, \underbrace{N, \dots, N}_{N-k+1}), \tag{4.8}$$

it follows from (4.6)–(4.8) and the definition of the partial ordering $\preccurlyeq$ that $n(t) \preccurlyeq n(t_1)$. By using transfinite induction, it can be shown that

$$n(t_1) \preccurlyeq n(t_2), \dots, n(t_{\tau-1}) \preccurlyeq n(t_\tau),\ n(t_\tau) \preccurlyeq n(t'), \tag{4.9}$$

by furthermore using the transition of the partial ordering $\preccurlyeq$ and (4.9), we have $n(t) \preccurlyeq n(t')$. This shows that the one-to-one mapping ψ is an isomorphism between $V(G(\mathbf{T}_N^l))$ and $\mathbb{N}(\mathbf{T}_N^l)$, and this isomorphism ψ preserves the partial ordering $\preccurlyeq$. □

Corollary 4.3. *If $\mathbf{T}_N^l$ is a locally finite tree set with N-minimal element, then $\mathbf{T}_N^l$ is isomorphic to $\mathbb{N}(\mathbf{T}_N^l)$.*

Proof. It follows from Theorems 2.1 and 4.2 that ϕ is an isomorphism between $\mathbf{T}_N^l$ and $V(G(\mathbf{T}_N^S))$, and ψ is an isomorphism between $V(G(\mathbf{T}_N^S))$ and $\mathbb{N}(\mathbf{T}_N^l)$, set $\theta = \psi \circ \phi$, then θ is an isomorphism between $\mathbf{T}_N^S$ and $\mathbb{N}(\mathbf{T}_N^l)$, i.e.,

$$(\mathbf{T}_N^l, \preceq) \cong (\mathbb{N}(\mathbf{T}_N^l), \preccurlyeq). \quad \square \tag{4.10}$$

Let $(\Omega, \mathcal{F}, \mu)$ be a complete probability space, where Ω is a so-called sample space, $\mathcal{F}$ is a σ-field of subsets of Ω, and μ is a probability measure on $\mathcal{F}$. Now, two families of σ-fields on $(\Omega, \mathcal{F}, \mu)$ are to be defined. One is that for any index set $\mathbf{T}_N^l$, $(\mathcal{F}_t, t \in \mathbf{T}_N^l)$ is a family of nondecreasing σ-fields with respect to the partial ordering $\preceq$, and write $\bigcup_{t\in\mathbf{T}_N^l} \mathcal{F}_t \subset \mathcal{F}$; the other is that for any index set $\mathbb{N}(\mathbf{T}_N^l)$, $(\mathcal{F}_{n(t)}, n(t) \in \mathbb{N}(\mathbf{T}_N^l))$ is a family of nondecreasing σ-fields with respect to the partial ordering $\preccurlyeq$, we write $\bigcup_{n(t)\in\mathbb{N}(\mathbf{T}_N^l)} \mathcal{F}_{n(t)} \subset \mathcal{F}$ for the smallest σ-field that contains all of the σ-fields $(\mathcal{F}_{n(t)}, n(t) \in \mathbb{N}(\mathbf{T}_N^l))$. By using the two families of σ-fields and for any $f \in L_1$, we can define

$$P_t f = \phi_t E_t(f\bar{\phi}_t), \quad t \in \mathbf{T}_N^l, \tag{4.11}$$

$$P_{n(t)} f = \phi_{n(t)} E_{n(t)}(f\bar{\phi}_{n(t)}), \quad n(t) \in \mathbb{N}(\mathbf{T}_N^l), \tag{4.12}$$

where $(\phi_t, t \in \mathbf{T}_N^l)$, $(\phi_{n(t)}, n(t) \in \mathbb{N}(\mathbf{T}_N^l))$ is a family of complex-value measurable functions with $|\phi_t| = 1$ and $|\phi_{n(t)}| = 1$, respectively. Then it follows from Definitions 3.1 and 3.2 that $(P_t f, t \in \mathbf{T}_N^l)$ is a locally finite tree martingale, and it is verified that $(P_{n(t)} f, n(t) \in \mathbb{N}(\mathbf{T}_N^l))$ is a multiparameter martingale with respect to the partial ordering $\preccurlyeq$. And since $\mathbb{N}(\mathbf{T}_N^l) \subset \mathbb{N}^{N+1}$, it follows from the definition of orthomartingale [8] (p:16) that $(P_{n(t)} f, n(t) \in \mathbb{N}(\mathbf{T}_N^l))$ is an orthomartingale and is a special orthomartingale.

Theorem 4.4. *$(P_t f, t \in \mathbf{T}_N^l)$ is isomorphic to $(P_{n(t)} f, n(t) \in \mathbb{N}(\mathbf{T}_N^l))$.*

Proof. Let $M = (Pf)^{-1} \circ \phi \circ \psi \circ (Pf)$, then

$$\begin{tikzcd}
(P_t f, t \in \mathbf{T}_N^l) \arrow[r, "\text{M is onto}"] \arrow[d, "(Pf)^{-1}"'] & (P_{n(t)} f, n(t) \in \mathbb{N}(\mathbf{T}_N^l)) \\
\mathbf{T}_N^l \arrow[r, "\theta"'] & \mathbb{N}(\mathbf{T}_N^l), \arrow[u, "Pf"']
\end{tikzcd}$$

$$\begin{tikzcd}
(P_t f, t \in \mathbf{T}_N^l) & (P_{n(t)} f, n(t) \in \mathbb{N}(\mathbf{T}_N^l)) \arrow[l, "\text{Inversion of M}"'] \arrow[d, "(Pf)^{-1}"] \\
\mathbf{T}_N^l \arrow[u, "Pf"] & \mathbb{N}(\mathbf{T}_N^l). \arrow[l, "\theta^{-1}"]
\end{tikzcd}$$

Diagrams shows that M is an one-to-one mapping between $(P_t f, t \in \mathbf{T}_N^l)$ and $(P_{n(t)} f, n(t) \in \mathbb{N}(\mathbf{T}_N^l))$. Furthermore, M preserves the partial ordering, and for any $f_1, f_2 \in L_1$ and $\alpha, \beta \in C$, we have

$$M(\alpha f_1 + \beta f_2) = \alpha M(f_1) + \beta M(f_2). \tag{4.13}$$

Therefore, M is an isomorphism between $(P_t f, t \in \mathbf{T}_N^l)$ and $(P_{n(t)} f, n(t) \in \mathbb{N}(\mathbf{T}_N^l))$. $\square$

By applying Cairoli's Strong (p, p) inequality [8] to a family of special orthosmartingales $(P_{n(t)} f, n(t) \in \mathbb{N}(\mathbf{T}_N^l))$, the following Lemma can be obtained.

Lemma 4.5 (*Cairoli's Strong (p, p) Inequality*). *Suppose that $Pf = (P_{n(s)} f, n(s) \in \mathbb{N}(\mathbf{T}_N^l))$ is a nonnegative orthosmartingale with respect to one-parameter filtration $\mathcal{F}^1, \ldots, \mathcal{F}^{N+1}$. Then, for all $n(t) \in \mathbb{N}(\mathbf{T}_N^l) \subset \mathbb{N}^{N+1}$ and $p > 1$,*

$$E[\max_{n(s) \preccurlyeq n(t)} (P_{n(s)} f)^p] \leq \left(\frac{p}{p-1}\right)^{(N+1)p} E[P_{n(t)} f]^p. \tag{4.14}$$

Note that Theorem 2.5.1 [8], Cairoli's Second Convergence Theorem [8] can also be translated into the family of special orthosmartingales $(P_{n(t)} f, n(t) \in \mathbb{N}(\mathbf{T}_N^l))$. Next, by applying Lemma 4.5 to the locally finite tree martingales $(P_t f, t \in \mathbf{T}_N^l)$, and combining with Theorem 4.4, we obtain that Theorem 4.6 for a locally finite tree martingale $(P_t f, t \in \mathbf{T}_N^l)$.

Theorem 4.6. *Suppose that $Pf = (P_t f, t \in \mathbf{T}_N^l)$ is a nonnegative locally finite tree martingale. Then, for all $t \in \mathbf{T}_N^l$ and $p > 1$,*

$$E[\max_{s \preceq t} (P_s f)^p] \leq \left(\frac{p}{p-1}\right)^{(N+1)p} E[P_t f]^p. \tag{4.15}$$

We are very interested in the problem: What tree is a tree set $\mathbf{T}$ isomorphic to, and are the corresponding tree martingales $Pf = (P_t f, t \in \mathbf{T})$ convergent? In the following, an attempt shall be made to answer this problem.

Theorem 4.7. *Assume that for some $p > 1$, the tree martingales $Pf = (P_t f, t \in \mathbf{T})$ satisfy $\sup_{t\in\mathbf{T}} E[|P_t f|]^p < \infty$. If $\mathbf{T}$ is a locally finite tree set with a finite number of minima, then a limit of the tree martingales Pf exists.*

Proof. By applying Theorems 4.4 and 4.6 to the tree martingales $Pf = (P_t f, t \in \mathbf{T})$, it is clear that a limit of the tree martingales Pf exists. □

Note that Remark 4, the following corollary will be obtained immediately.

Corollary 4.8. *Assume that for some $p > 1$, the tree martingales $Pf = (P_t f, t \in \mathbf{T})$ satisfy $\sup_{t\in\mathbf{T}} E[|P_t f|]^p < \infty$. If $\mathbf{T}$ is isomorphic to an inside directed locally finite tree $G(\mathbf{T})$ with $|\partial G(\mathbf{T})| < \infty$, then the limit of the tree martingales Pf exists.*

Theorem 4.9. *Assume that for some $p > 1$, the tree martingales $Pf = (P_t f, t \in \mathbf{T})$ satisfy $\sup_{t\in\mathbf{T}} E[|P_t f|]^p < \infty$. If $\mathbf{T}$ is a tree set that has a decomposition each part of which has a finite number of minima, then a limit of the tree martingales Pf exists.*

Proof. Based on the assumptions of Theorems 4.9 and 3.3, the tree set $\mathbf{T}$ has a decomposition

$$\mathbf{T} = \mathbf{T}_0^l \oplus \mathbf{T}_1^l \oplus \cdots \oplus \mathbf{T}_i^l \oplus \cdots = \sum_{i=0}^{\infty} \mathbf{T}_i^l,$$

where $\mathbf{T}_i^l$ is a locally finite tree set with a finite number of minima, such that

$$P(f) = P_0(f) \oplus P_1(f) \oplus \cdots \oplus P_r(f) = \sum_{i=0}^{\infty} P_i(f), \tag{4.16}$$

where $P_i(f) = (P_t(f), t \in \mathbf{T}_i^l)$ is a locally finite tree martingale. Obviously,

$$\sup_{t\in\mathbf{T}_\mathbf{i}^\mathbf{l}} E[|P_t f|]^p \leq \sup_{t\in\mathbf{T}} E[|P_t f|]^p < \infty.$$

It follows from Theorem 4.7 that for each $i \in \mathbb{N}$, the limit of the locally finite tree martingales $P_i(f) = (P_t(f), t \in \mathbf{T}_i^l)$ exists. That is, the tree martingales $Pf = (P_t f, t \in \mathbf{T})$ have a decomposition each part of which is convergent. Therefore, this implies the existence of a limit of the tree martingales Pf. □

Corollary 4.10. *Assume that for some $p > 1$, the tree martingales $Pf = (P_t f, t \in \mathbf{T})$ satisfy $\sup_{t\in\mathbf{T}} E[|P_t f|]^p < \infty$. If $\mathbf{T}$ is a tree set that has a decomposition each part of which is isomorphic to an inside-directed locally finite tree with the finite boundary, then implies the existence of a limit of the tree martingales Pf.*

An interesting problem is that what relations between tree martingales and uniform spanning forests have (Benjamini, Lyons, Peres and Schramm [18], Lyons, Morris, and Schramm [19], Alexander [20], Pemantle [21], HÄggström [22])? Can a more general convergence theorem for tree martingales be obtained by using the uniform spanning forests?

5. Applications

We shall make analogies to the earlier tree martingale work of Schipp, Weisz in harmonic analysis. It is known that there is a close connection among the maximal function, the one-parameter martingale transform and the quadratic variation, which is defined by

$$S(f) = \left(\sum_{n=0}^{\infty} |f_n - f_{n-1}|^2 \right)^{\frac{1}{2}}, \quad \text{where } f_{-1} = 0. \tag{5.1}$$

In the tree martingale case, Fridli, Schipp, and Weisz [2] defined the tree martingale maximal function, the tree martingale transform and the tree martingale quadratic variation, which is defined by

$$S_t(f) = \left(\sum_{s \in \mathbf{T}_t} |f_{t^+} - f_t|^2 \right)^{\frac{1}{2}}, \qquad S(f) = \sup_{t \in \mathbf{T}} S_t(f), \tag{5.2}$$

and they proved that there is a close connection among the tree martingale maximal function, the tree martingale transform and the tree martingale quadratic variation in L^p-norm provided $2 < p < \infty$. Applying the modified Riesz–Thorin-type interpolation theorem, Fridli, Schipp [1] showed that the operator $S(f)$ is of type (p, p) provided $2 < p < \infty$. Here, by using the decomposition theorem of tree martingales and multiparameter martingale theory, with some mild conditions, we can proved that the maximal function, and the operator $S(f)$ are of type (p, p) provided $1 < p < \infty$.

Theorem 5.1. *Assume that $Pf = (P_t f, t \in \mathbf{T})$ is a nonnegative tree martingale. If $\mathbf{T}$ is a tree set that has a finite decomposition each part of which has a finite number of minima, then for $p > 1$,*

$$E[\sup_{t \in \mathbf{T}} (P_t f)^p] \le C_{pr} E[|P_t f|]^p. \tag{5.3}$$

In some sense, Theorem 5.1 is Doob's inequality for tree martingales.

Proof. Since the tree set $\mathbf{T}$ has a finite decomposition, without loss of generality, assume that there exists a finite number r such that

$$\mathbf{T} = \mathbf{T}_0^l \oplus \mathbf{T}_1^l \oplus \cdots \oplus \mathbf{T}_r^l = \sum_{i=0}^{r} \mathbf{T}_i^l, \tag{5.4}$$

where $\mathbf{T}_i^l$ is a locally finite tree set with a finite number of minima, by applying Theorem 3.3 to tree martingales Pf, we obtain that

$$P(f) = P_0(f) \oplus P_1(f) \oplus \cdots \oplus P_r(f) = \sum_{i=0}^{r} P_i(f), \tag{5.5}$$

where $P_i(f) = (P_t(f), t \in \mathbf{T}_i^l)$ is a locally finite tree martingale. Next, applying Theorem 4.6 to $P_i(f) = (P_t(f), t \in \mathbf{T}_i^l)$ one obtains that

$$E[\sup_{t \in \mathbf{T}_i^l} (P_t f)^p] \le \left(\frac{p}{p-1} \right)^{(N_i+1)p} E[P_t f]^p, \quad i = 1, 2, \ldots, r, \tag{5.6}$$

where N_i $(i = 1, 2, \ldots, r)$ is the number of the minima of each locally finite tree set $\mathbf{T}_i^l$, respectively. It follows from (5.5) and (5.6) that

$$E[\sup_{t \in \mathbf{T}} (P_t f)^p] \le C_{pr} E[|P_t f|]^p, \tag{5.7}$$

where C_{pr} depending on p and r. The proof is complete. □

Theorem 5.2. *Assume that $Pf = (P_t f, t \in \mathbf{T})$ is a tree martingale. If $\mathbf{T}$ is a tree set that has a finite decomposition each part of which has a finite number of minima, then the operator $S(f)$ is of type (p, p) provided $1 < p < \infty$.*

We only give a profile of the proof of Theorem 5.2. The detail proof is omitted.

Proof. *Step* I. We can construct a finite decomposition of tree martingales as the proof of Theorem 5.1.

Step II. Extend Burkholder–Davis–Gundy's inequality of two-parameter martingales to multiparameter martingales (refer to [8] [p: 256–257], [23]).

Step III. Applying Theorem 4.4, and Burkholder–Davis–Gundy's inequality of multiparameter martingales to the tree martingales $(P_t f, t \in \mathbf{T})$, we can obtain that Theorem 5.2. □

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