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Blaschke Products and Singular Functions with Prescribed Boundary Values

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This paper shows that there exists a Blaschke product having a prescribed radial limit at each point of a prescribed finite subset of the unit circle. In addition, an analogue for singular inner functions is proved; and an extension dealing with tangential limits is established.

1. INTRODUCTION

Let C denote the complex plane; let U denote the open unit disk, $\{z \in C: |z| < 1\}$; let $\bar{U}$ denote the closed unit disk, $\{z \in C: |z| \leqslant 1\}$; and let T denote the unit circle, $\{z \in C: |z| = 1\}$.

A Blaschke product is a (holomorphic) function, B, defined for each z in the complement of the closure of $\{1/\bar{z}_1, 1/\bar{z}_2, 1/\bar{z}_3, \ldots\}$ by

$$B(z) = z^p \prod_k \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \bar{z}_k z}$$

where p is a nonnegative integer and $Z = (z_1, z_2, z_3, \ldots)$ is a sequence—empty, finite, or infinite—of complex numbers in $U - \{0\}$ satisfying the conditon $\sum_k (1 - |z_k|) < \infty$. (If Z is empty, then $B(z)$ reduces to z^p; if, in addition, $p = 0$, then $B(z) = 1$ for each $z \in C$.) If Z is empty or finite, then B is called a finite Blaschke product.

Throughout this paper a function of the form cB where $c \in T$ and B is a Blaschke product will be called an extended Blaschke product. Clearly, if b is an extended finite Blaschke product and if $b(0) > 0$, then b is a finite Blaschke product.

An inner function, I, is a function that is bounded and holomorphic in U and has a radial limit, $I^*(\alpha) = \lim_{r \to 1-} I(r\alpha)$, of modulus one at almost every point $\alpha \in T$. Every extended Blaschke product is an inner function; and every inner

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function can be factored into the product of an extended Blaschke product and a singular inner function, which is an inner function without zeros in U that is positive at the origin. (Cf. [4] or [8].) If I is an inner function, then, according to a theorem of 0. Frostman [3, p. 113; 2, p. 37], there exists a subset, L, of U such that the logarithmic capacity of L is zero and, for each $a \in U - L$, $(a - I)/(1 - \bar{a}I)$ is an extended Blaschke product.

Cantor and Phelps [1] proved

THEOREM CP. *Let n be a positive integer, let $(\alpha_1, \alpha_2, \dots, \alpha_n)$ be an n-tuple of distinct complex numbers of modulus one, and let $(\beta_1, \beta_2, \dots, \beta_n)$ be an n-tuple of complex numbers of modulus one. Then there exists a finite Blaschke product, B, such that $B^*(\alpha_k) = \beta_k$ for each $k \in \{1, 2, \dots, n\}$.*

Actually, Cantor and Phelps proved the weaker version of this theorem that results when "finite Blaschke product" is replaced by "extended finite Blaschke product". To prove Theorem CP, we proceed as follows. For each $k \in \{1, 2, \dots, n\}$, define γ_k to be $\{(\frac{1}{2}) - \beta_k\}/\{\alpha_k(1 - (\frac{1}{2})\beta_k)\}$ and note that $\gamma_k \in T$. According to the Cantor-Phelps theorem, there exists an extended finite Blaschke product, b_1, such that $b_1(\alpha_k) = \gamma_k$ for each $k \in \{1, 2, \dots, n\}$. Let $b_2(z) = zb_1(z)$ for each z in the domain of b_1. Then b_2 is an extended finite Blaschke product, $b_2(0) = 0$, and $b_2(\alpha_k) = \{(\frac{1}{2}) - \beta_k\}/\{1 - (\frac{1}{2})\beta_k\}$ for each $k \in \{1, 2, \dots, n\}$. Let $b_3(z) = \{(\frac{1}{2}) - z\}/\{1 - (\frac{1}{2})z\}$ whenever $z \neq 2$, and denote the composite function $b_3 \circ b_2$ by B. Since B is the composition of two extended finite Blaschke products, it is an extended finite Blaschke product. Moreover, since $B(0) = 1/2 > 0$, B is a finite Blaschke product. Finally, a routine calculation, which we omit, shows that $B(\alpha_k) = \beta_k$ for each $k \in \{1, 2, \dots, n\}$.

If in Theorem CP the restriction that each β_k have modulus one is replaced by the weaker restriction that each β_k have modulus at most one, then the resulting assertion is false, since $B(T) \subset T$ for each extended finite Blaschke product B. The primary purpose of this paper is to prove that a correct assertion (Theorem 1) does result if, in addition, B is permitted to be a Blaschke product.

Two distinct proofs of this modification of Theorem CP are given. The second proof yields a stronger result (Theorem 2), in which each radial limit is replaced by a tangential limit of (arbitrarily high) prescribed order of contact. From a logical point of view, the proof of Theorem 1 is superfluous, since it is superceded by that of Theorem 2. However, the existence of a Blaschke product with a prescribed boundary value at each point of a prescribed countably infinite set has not been established; therefore, the inclusion of two proofs in the finite case seems to be justified, since it furnishes a broader base for future research. An additional justification for including the proof of the weaker result is that the proof can be modified so as to yield an analogous result for singular inner functions (Theorem 3); whereas the proof of the stronger result cannot be used in a similar manner.

2. Blaschke Products with Prescribed Radial Limits

THEOREM 1. *Let n be a positive integer, let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be an n-tuple of distinct complex numbers of modulus one, and let $(\beta_1, \beta_2, \ldots, \beta_n)$ be an n-tuple of complex numbers of modulus at most one. Then there exists a Blaschke product, B, such that $B^*(\alpha_k) = \beta_k$ for each $k \in \{1, 2, \ldots, n\}$.*

Proof. The proof consists of the following steps. First, we prove that there exist a Blaschke product, B_1, and an n-uple, $(\gamma_1, \gamma_2, \ldots, \gamma_n)$, of complex numbers of modulus one such that $B_1^*(\gamma_k) = \beta_k$ for each $k \in \{1, 2, \ldots, n\}$. Second, we note that there exists a finite Blaschke product, B_2, such that $B_2(0) = 0$ and $B_2(\alpha_k) = \gamma_k$ for each $k \in \{1, 2, \ldots, n\}$. Third, we prove that the composite function $B_1 \circ B_2$ is a Blaschke product. Finally, we prove that $(B_1 \circ B_2)^* (\alpha_k) = \beta_k$ for each $k \in \{1, 2, \ldots, n\}$.

The existence of B_1 is a consequence of the fact that *there exists a Blaschke product, B_1, such that $\{\gamma \in T: B_1^*(\gamma) = \beta\}$ is uncountable for each $\beta \in \bar{U}$.* To prove this, first recall that, using the theory of Riemann surfaces, M. Ohtsuka [7] has constructed an inner function, I, such that $\{\gamma \in T: I^*(\gamma) = \alpha\}$ is uncountable for each $\alpha \in \bar{U}$. According to the previously mentioned theorem of Frostman, there exist a constant $a \in U$, a constant $c \in T$, and a Blaschke product, B_1, such that $\{a - I(z)\}/\{1 - \bar{a}I(z)\} = cB_1(z)$ for each $z \in U$. Let $\beta \in \bar{U}$, and consider the corresponding point $\alpha = \{a - c\beta\}/\{1 - \bar{a}c\beta\}$. Clearly, $\alpha \in \bar{U}$, since $a \in U$ and $c\beta \in \bar{U}$. A routine argument, which we omit, shows that, if $\gamma \in T$, then $I^*(\gamma) = \alpha$ if and only if $B_1^*(\gamma) = \beta$. Since $\{\gamma \in T: I^*(\gamma) = \alpha\}$ is uncountable, so is $\{\gamma \in T: B_1^*(\gamma) = \beta\}$; and this completes the proof.

According to Theorem CP, there exists a finite Blaschke product, b, such that $b(\alpha_k) = \gamma_k/\alpha_k$ for each $k \in \{1, 2, \ldots, n\}$. Let $B_2(z) = zb(z)$ for each z in the domain of b. Then B_2 is a finite Blaschke product such that $B_2(0) = 0$ and $B_2(\alpha_k) = \gamma_k$ for each $k \in \{1, 2, \ldots, n\}$.

Next, let us prove that $B_1 \circ B_2$ is a Blaschke product. We know that B_1 has a representation of the form

$$B_1(z) = z^p \prod_k \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \bar{z}_k z} \qquad (z \in U)$$

where p is a nonnegative integer and $Z = (z_1, z_2, z_3, \ldots)$ is a sequence–empty, finite, or infinite—of complex numbers in $U - \{0\}$ satisfying the condition $\sum_k (1 - |z_k|) < \infty$. Since

$$B_1 \circ B_2(z) = \{B_2(z)\}^p \prod_k \frac{|z_k|}{z_k} \frac{z_k - B_2(z)}{1 - \bar{z}_k B_2(z)}$$

for each $z \in U$ and since $\{B_2(z)\}^p$ defines a finite Blaschke product, it will suffice to prove that the function, b, defined by the formula

$$b(z) = \prod_k \frac{|z_k|}{z_k} \frac{z_k - B_2(z)}{1 - \bar{z}_k B_2(z)} \qquad (z \in U) \tag{1}$$

is a Blaschke product.

If Z is empty, then $b(z) = 1$ for each $z \in U$; and we are done. Assume that Z is not empty.

Suppose that the finite Blaschke product B_2 is of order m, that is, assume that B_2 has m zeros counted according to multiplicity in U. Then, for each k, the function defined by the formula

$$\frac{|z_k|}{z_k} \frac{z_k - B_2(z)}{1 - \bar{z}_k B_2(z)} \tag{2}$$

is a finite Blaschke product of order m. In fact, (2) represents an extended finite Blaschke product of order m, since it is the composition of a finite Blaschke product of order m and a finite Blaschke product of order 1; and this extended finite Blaschke product is a finite Blaschke product, since (2) reduces to the positive number $|z_k|$ when $z = 0$. Hence, for each k, there exists an m-tuple, $(z_{k1}, z_{k2}, \ldots, z_{km})$, of complex numbers in $U - \{0\}$ such that

$$\frac{|z_k|}{z_k} \frac{z_k - B_2(z)}{1 - \bar{z}_k B_2(z)} = \prod_{j=1}^{m} \frac{|z_{kj}|}{z_{kj}} \frac{z_{kj} - z}{1 - \bar{z}_{kj} z} \tag{3}$$

for each $z \in U$.

If Z is finite, then it follows at once from (1) and (3) that b is a finite Blaschke product, as desired.

Finally, assume that Z is an infinite sequence. Then, from (1) and (3), it follows that, for each $z \in U$,

$$\begin{aligned} b(z) = &\left\{\left(\frac{|z_{11}|}{z_{11}} \frac{z_{11} - z}{1 - \bar{z}_{11} z}\right)\left(\frac{|z_{12}|}{z_{12}} \frac{z_{12} - z}{1 - \bar{z}_{12} z}\right) \cdots \left(\frac{|z_{1m}|}{z_{1m}} \frac{z_{1m} - z}{1 - \bar{z}_{1m} z}\right)\right\} \\ &\times \left\{\left(\frac{|z_{21}|}{z_{21}} \frac{z_{21} - z}{1 - \bar{z}_{21} z}\right)\left(\frac{|z_{22}|}{z_{22}} \frac{z_{22} - z}{1 - \bar{z}_{22} z}\right) \cdots \left(\frac{|z_{2m}|}{z_{2m}} \frac{z_{2m} - z}{1 - \bar{z}_{2m} z}\right)\right\} \\ &\times \left\{\left(\frac{|z_{31}|}{z_{31}} \frac{z_{31} - z}{1 - \bar{z}_{31} z}\right)\left(\frac{|z_{32}|}{z_{32}} \frac{z_{32} - z}{1 - \bar{z}_{32} z}\right) \cdots \left(\frac{|z_{3m}|}{z_{3m}} \frac{z_{3m} - z}{1 - \bar{z}_{3m} z}\right)\right\} \cdots . \end{aligned} \tag{4}$$

Next, let us prove that the removal of the curly brackets from the right-hand side of (4) yields an infinite product that converges and has the same value as the original product. (As the convergent infinite product

$$\{(-1)(-1)\}\{(-1)(-1)\}\{(-1)(-1)\}\cdots$$

clearly shows, the removal of brackets must be treated with care.)

By hypothesis, $\sum_{k=1}^{\infty}(1-|z_k|)<\infty$ and $0<|z_k|<1$ for each positive integer k; hence $\prod_{k=1}^{\infty}|z_k|>0$. (Cf. [8, p. 322].)

Setting $z=0$ in (3), we conclude that $|z_k|=|z_{k1}||z_{k2}|\cdots|z_{km}|$ for each positive integer k. Hence, the sequence of partial products of $|z_1||z_2||z_3|\cdots$ (which converges to a positive real number) is a subsequence of the decreasing (and, hence, convergent) sequence of partial products of

$$|z_{11}||z_{12}|\cdots|z_{1m}||z_{21}||z_{22}|\cdots|z_{2m}|\cdots. \tag{5}$$

Consequently, (5) is a convergent infinite product (in the strict sense); and, hence, if the curly brackets are removed from the right-hand side of (4), the resulting infinite product converges for each $z\in U$ and represents a Blaschke product. (Cf. [4, p. 64].)

A simple argument shows that, if brackets are inserted in a convergent infinite product, the resulting infinite product is convergent and has the same value as the original product. Hence, b is a Blaschke product.

Finally, let us prove that $(B_1\circ B_2)^*(\alpha_k)=\beta_k$ for each $k\in\{1,2,\ldots,n\}$. We prove the following more general result, since it is needed in the proof of Theorem 3: *If B_1 is a bounded holomorphic function defined on U, B_2 is a nonconstant extended finite Blaschke product, $\alpha\in T$, $\gamma=B_2(\alpha)$, and B_1 has a radial limit, $B_1^*(\gamma)$, at γ, then $B_1\circ B_2$ has a radial limit at α and $(B_1\circ B_2)^*(\alpha)=B_1^*(\gamma)$.*

Since B_1 is bounded and holomorphic in U and has a radial limit at γ, a classical theorem of Montel implies that B_1 has an angular limit at γ, that is, if K is the convex hull of a set $\{\gamma,p,q\}$ where $p,q\in U$ (such a K is called a Stolz triangle at γ), then

$$\lim_{\substack{z\to\gamma\\ z\in K}} B_1(z)$$

exists and is equal to $B_1^*(\gamma)$. (The same conclusion also follows from Lindelöf's theorem, [2, p.19].)

Next, consider the nonconstant extended finite Blaschke product B_2. Clearly, B_2 can be expressed in the form

$$B_2(z)=c\prod_{k=1}^{m}\frac{z_k-z}{1-\bar{z}_k z}\qquad (z\in U) \tag{6}$$

where $c \in T$, m is a positive integer, and $(z_1, z_2, \ldots, z_m)$ is an m-tuple of complex numbers in U. A rather long but straightforward calculation (which we omit) using the logarithmic derivative shows that

$$B_2'(z) = \frac{B_2(z)}{z} \sum_{k=1}^{m} \frac{1 - |z_k|^2}{|z - z_k|^2} \tag{7}$$

for each $z \in T$. This implies that the radius of $\bar{U}$ terminating at α is mapped onto a curve that (lies in $\bar{U}$ and) is tangent at the point γ to the radius of $\bar{U}$ terminating at γ. Indeed, the radial image under B_2 has the parametric representation $z(t) = B_2(t\alpha)$, $0 \leqslant t \leqslant 1$. The tangent line to the radial image at $z(1) = B_2(\alpha) = \gamma$ is parallel to the vector representation of

$$z'(1) = B_2'(\alpha)\,\alpha = B_2(\alpha) \sum_{k=1}^{m} \{1 - |z_k|^2\}/\{|\alpha - z_k|^2\}.$$

Since $z'(1)$ is a positive multiple of $\gamma = B_2(\alpha)$, the desired conclusion follows immediately. (The same conclusion is a consequence of the conformality of B_2 at α, which holds since $B_2'(\alpha) \neq 0$ in virtue of (7), and the fact that $B_2(T) \subset T$.)

Next, let K be a Stolz triangle at γ whose angle at γ is bisected by the radius of $\bar{U}$ terminating at γ. In virtue of the foregoing observations, it is geometrically clear and not very difficult to prove analytically (although we omit the proof because of its length) that, as z approaches α radially, $B_2(z)$ eventually enters and remains in K and approaches γ, and, hence, that $B_1(B_2(z))$ approaches $B_1^*(\gamma)$. Analytically, this means that $(B_1 \circ B_2)^*(\alpha)$ exists and is equal to $B_1^*(\gamma)$.

3. Blaschke Products with Prescribed Tangential Limits

In this section we prove that there exists a Blaschke product having a prescribed tangential limit (of arbitrarily high order of contact) at each point of a prescribed finite subset of T. First, we prove that the boundary behavior can be prescribed at a single point while control is maintained off that point.

Lemma 1. *Suppose that $\alpha \in T$ and $\beta \in \bar{U}$; and let J be a Jordan curve such that $J \subset \bar{U}$ and $J \cap T = \{\alpha\}$. Then there exists an extended Blaschke product, B, such that B is holomorphic in $\bar{U} - \{\alpha\}$ and*

$$\lim_{\substack{z \to \alpha \\ z \in J \cup I(J)}} B(z) = \beta \tag{8}$$

where $I(J)$ denotes the interior of J.

Proof. If $|\beta| = 1$, then it suffices to take $B(z) = \beta$ for each complex number z.

Next, suppose that $\beta = 0$. According to a theorem of Linden and Somadasa [5], which was proved independently by Beurling and published by Weiss [9], there exists a Blaschke product, B_0, such that the sequence of zeros of B_0 converges to α and

$$\lim_{\substack{z \to \alpha \\ z \in J \cup I(J)}} B_0(z) = 0. \tag{9}$$

Since the sequence of zeros of B_0 converges to α, B_0 is holomorphic in $\bar{U} - \{\alpha\}$ and has modulus one at each point of $T - \{\alpha\}$ (cf. [4, p. 68]).

Finally, assume that $0 < |\beta| < 1$. Let B_0 be the Blaschke product just described; and define a function, B, by means of the formula

$$B(z) = \frac{\beta - B_0(z)}{1 - \bar{\beta} B_0(z)}$$

for each z in the domain of B_0 such that $B_0(z) \neq 1/\bar{\beta}$. Clearly, B is an inner function, B is holomorphic in $\bar{U} - \{\alpha\}$, and B has modulus one at each point of $T - \{\alpha\}$. *A fortiori,* $B^*(\zeta)$ exists and has modulus one for each $\zeta \in T - \{\alpha\}$. In virture of Lindelöf's theorem (cf. [6, p. 5]), $B_0^*(\alpha) = 0$. Hence, $B^*(\alpha) = \beta$. Consequently, zero is not a radial limit of the nonconstant inner function B. Hence, B is an extended Blaschke product (cf. [6, p. 33]).

Since (9) holds, it follows that (8) holds, as desired.

THEOREM 2. *Let n be a positive integer; let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be an n-tuple of distinct complex numbers of modulus one; let $(\beta_1, \beta_2, \ldots, \beta_n)$ be an n-tuple of complex numbers of modulus at most one; and, for each $k \in \{1, 2, \ldots, n\}$, let J_k be a Jordan curve such that $J_k \subset \bar{U}$ and $J_k \cap T = \{\alpha_k\}$. Then there exists a Blaschke product, B, such that B is holomorphic in $\bar{U} - \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and for each $k \in \{1, 2, \ldots, n\}$*

$$\lim_{\substack{z \to \alpha_k \\ z \in J_k \cup I(J_k)}} B(z) = \beta_k$$

where $I(J_k)$ denotes the interior of J_k.

Proof. Suppose that $j \in \{1, 2, \ldots, n\}$. According to Lemma 1, there exists an extended Blaschke product, b_j, such that b_j is holomorphic in $\bar{U} - \{\alpha_j\}$ and $b_j(z) \to \beta_j$ as $z \to \alpha_j$ through $J_j \cup I(J_j)$. Then there exist a constant $c_j \in T$ and a Blaschke product, B_j, such that $b_j = c_j B_j$.

Since B_j is holomorphic in $\bar{U} - \{\alpha_j\}$, B_j is holomorphic and has modulus one at α_k if $k \in \{1, 2, \ldots, n\}$ and $k \neq j$. *A fortiori*, for each such k,

$$\lim_{\substack{z \to \alpha_k \\ z \in J_k \cup \bar{I}(J_k)}} B_j(z)$$

exists and is equal to $B_j(\alpha_k)$ where $B_j(\alpha_k) \in T$.

According to Theorem CP, there exists a finite Blaschke product, B_{n+1}, such that

$$B_{n+1}(\alpha_k) = c_k \Big/ \prod_{\substack{j=1 \\ j \neq k}}^{n} B_j(\alpha_k)$$

for each $k \in \{1, 2, \ldots, n\}$.

Let B denote the product $B_1 B_2 \cdots B_n B_{n+1}$. Then B is a Blaschke product, since the product of a finite number of Blaschke products is a Blaschke product.

Clearly, B is holomorphic in $\bar{U} - \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

Finally, for each $k \in \{1, 2, \ldots, n\}$,

$$\begin{aligned}
\lim_{\substack{z \to \alpha_k \\ z \in J_k \cup \bar{I}(J_k)}} B(z) &= \left\{ \prod_{\substack{j=1 \\ j \neq k}}^{n} \lim_{\substack{z \to \alpha_k \\ z \in J_k \cup \bar{I}(J_k)}} B_j(z) \right\} \left\{ \lim_{\substack{z \to \alpha_k \\ z \in J_k \cup \bar{I}(J_k)}} B_k(z) \right\} \left\{ \lim_{\substack{z \to \alpha_k \\ z \in J_k \cup \bar{I}(J_k)}} B_{n+1}(z) \right\} \\
&= \left\{ \prod_{\substack{j=1 \\ j \neq k}}^{n} B_j(\alpha_k) \right\} \left\{ \frac{\beta_k}{c_k} \right\} \left\{ \frac{c_k}{\prod_{j=1, j \neq k}^{n} B_j(\alpha_k)} \right\} \\
&= \beta_k,
\end{aligned}$$

as desired.

4. Singular Functions with Prescribed Radial Limits

In this section we prove an analogue of Theorem 1 for singular inner functions.

THEOREM 3. *Let n be a positive integer, let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be an n-tuple of distinct complex numbers of modulus one, and let $(\beta_1, \beta_2, \ldots, \beta_n)$ be an n-tuple of complex numbers of modulus at most one. Then there exists a singular inner function, S, such that $S^*(\alpha_k) = \beta_k$ for each $k \in \{1, 2, \ldots, n\}$.*

Proof. The following proof is patterned after the proof of Theorem 1. Without loss of generality, we may (and do) assume that $0 \in \{\beta_1, \beta_2, \ldots, \beta_n\}$.

First, let us prove the existence of an inner function having each β_k ($k = 1, 2, \ldots, n$) as a radial limit. Let W denote the universal covering surface of $U - \{\beta_1, \beta_2, \ldots, \beta_n\}$, let F denote a univalent conformal mapping of W onto U, and let S_1 denote the composition of F^{-1} with the conformal projection of W onto $U - \{\beta_1, \beta_2, \ldots, \beta_n\}$. A well-known argument [3, p. 113; 2, p. 37] shows that $S_1(U) = U - \{\beta_1, \beta_2, \ldots, \beta_n\}$ and that S_1 is an inner function. According to a theorem of Seidel [2, p. 107], if $k \in \{1, 2, \ldots, n\}$ and $\beta_k \in U$, then there exists a point $\gamma_k \in T$ such that $S_1^*(\gamma_k) = \beta_k$. According to a theorem of Calderón, González–Domínguez, and Zygmund [6, p. 37], if $k \in \{1, 2, \ldots, n\}$ and $\beta_k \in T$, then there exists a point $\gamma_k \in T$ such that $S_1^*(\gamma_k) = \beta_k$.

Since $S_1(U) = U - \{\beta_1, \beta_2, \ldots, \beta_n\}$, there exists a point $a \in U$ such that $S_1(a) > 0$. Let f denote the function such that $f(z) = (a - z)/(1 - \bar{a}z)$ for each $z \in C - \{1/\bar{a}\}$, let $\delta_k = (a - \gamma_k)/(1 - \bar{a}\gamma_k)$ for each $k \in \{1, 2, \ldots, n\}$, and note that $\delta_k \in T$ for each $k \in \{1, 2, \ldots, n\}$.

According to the second step in the proof of Theorem 1, there exists a finite Blaschke product, B_2, such that $B_2(0) = 0$ and $B_2(\alpha_k) = \delta_k$ for each $k \in \{1, 2, \ldots, n\}$.

Let S denote the composite function $S_1 \circ (f \circ B_2)$, and note that $f \circ B_2$ is a nonconstant extended finite Blaschke product. According to the last step in the proof of Theorem 1, $S^*(\alpha_k)$ exists and is equal to β_k for each $k \in \{1, 2, \ldots, n\}$. Since $0 \notin S(U)$ and $S(0) > 0$, we can conclude that S is a singular inner function once we know that S is an inner function.

That S is an inner function follows from the known fact that the composition of two nonconstant inner functions is again an inner function. However, since in the special case that we are now considering one of the inner functions is a nonconstant extended finite Blaschke product, we can give an elementary proof that S is an inner function. One way to proceed is as follows. Let λ denote one-dimensional Lebesgue measure on T, and let E denote the set $T - \{\zeta \in T$: $S_1^*(\zeta)$ exists and has modulus one$\}$. Since S_1 is an inner function, $\lambda(E) = 0$. Let B denote the nonconstant extended finite Blaschke product $f \circ B_2$. If $\lambda(B^{-1}(E)) = 0$, then the last step in the proof of Theorem 1 implies that $S_1 \circ B = S$ is an inner function, as desired.

There are a number of ways to prove that $\lambda(B^{-1}(E)) = 0$, some of which are quite sophisticated. One elementary proof consists of assuming that B is represented by the right-hand side of (6), letting ϵ be a positive real number, letting $A_1, A_2, A_3, \ldots$ be a sequence of open arcs of T such that $E \subset \bigcup_{n=1}^{\infty} A_n$ and

$$\sum_{n=1}^{\infty} \lambda(A_n) < \epsilon \left\{ \frac{1}{m} \sum_{k=1}^{m} \frac{1 - |z_k|}{1 + |z_k|} \right\},$$

and then using (7) to infer that the sequence $B^{-1}(A_1)$, $B^{-1}(A_2)$, $B^{-1}(A_3), \ldots$ determines in a natural way a sequence of open arcs of T whose union contains

$B^{-1}(E)$ and the sum of whose lengths is less than ϵ. Because of space considerations, we omit the somewhat lengthy details.

Alternatively, once we know that $B^{-1}(E)$ is λ-measurable, we can use an extension of Löwner's lemma (cf. [6, p. 34]) to conclude that $\lambda(B^{-1}(E)) = 0$. To prove that $B^{-1}(E)$ is λ-measurable, first note that S_1^* is a Borel measurable function on the $F_{\sigma\delta}$ subset $\{\zeta \in T: S_1^*(\zeta) \text{ exists}\}$ of T (cf. [2, p. 23]) and, hence, that $T - E = (S_1^*)^{-1}(T)$ is a Borel set relative to the topological space consisting of the set $\{\zeta \in T: S_1^*(\zeta) \text{ exists}\}$ with its inherited topology. Since the class of Borel sets is a minimal σ-algebra, a routine argument (which we omit) shows that $T - E$ is a Borel set relative to T. Hence, E is a Borel set relative to T. Since B determines a continuous mapping from T onto T, $B^{-1}(E)$ is a Borel set relative to T; and, thus, $B^{-1}(E)$ is λ-measurable, as desired. (Incidentally, if E were countable, the reasoning above would be somewhat heavy handed; however, if $U \cap \{\beta_1, \beta_2, \ldots, \beta_n\}$ contains more than one point, a theorem of Lohwater (cf. [2, p. 109]) shows that the logarithmic capacity of E is not zero.)

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