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Multivariate Spline Functions. I. Construction, Properties and Computation

D. W. ARTHUR

Applied Mathematics, University of Edinburgh, Edinburgh, Scotland

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A construction is given which allows the Hilbert space treatment of spline functions to be applied to the case of more than one variable, when the basic operator is a linear partial differential one. The particular case of the tensor product polynomial spline in two variables is then studied using a reproducing kernel, and its main properties, including the minimization ones, are deduced. A stable computational method is then given for this spline function, with certain point evaluation functionals. Finally, extensions are discussed, for more general linear functionals, for more general differential operators, and for more than two variables.

1. INTRODUCTION

The theory of spline functions in one variable is a well-understood one. There are difficulties in extending the results to more variables, not the least of which is the question; What is the "natural" extension? The major mathematical difficulty is that the operators involved, being partial differential operators, do not have a finite dimensional null-space. This prohibits application of the general theory of Atteia [3], and Anselone and Laurent [1] to define and determine the properties of multivariate spline functions.

In Section 2 we remove this obstacle by attaching further operators to the operator, by a construction similar to that used by Atteia [4] in a different context. This enables the general theory mentioned above to be invoked to define multivariate splines.

The most fruitful results are obtainable for "tensor product" splines, defined in n-dimensional rectangles. We shall restrict ourselves to this case from Section 3 onwards; and also, for simplicity, we shall deal only with the two-variable polynomial case. Extensions to more variables and to splines defined by other operators will be discussed in Section 7.

The main tool used in proving the properties of these splines is a reproducing kernel, which we prove the existence of, and indeed construct, in Section 3. Our spline functions are then constructed in Section 4, and their principal properties shown in Section 5.

A computational method for these spline functions is given in Section 6. It is based on the one-variable method of Greville [6, 7].

Notation. We shall, throughout this paper, use the notation, for partial derivatives;

$$f_j^i(x, y) \equiv \frac{\partial^{i+j}}{\partial x^i \partial y^j} f(x, y). \tag{1.1}$$

2. Definition of Multivariate Spline Functions

Let L be a linear partial differential operator in m variables $x_1, \ldots, x_m$; of order α_i in x_i; defined in a closed region Γ. Let M_i, $i = 1, \ldots, \mu$ be linear differential operators of order at most α_j in x_j; defined on a subset Γ_i of Γ, of dimension less than m.

Let H be the Hilbert Space containing functions f, defined in Γ, which are continuously differentiable $\alpha_i - 1$ times in x_i for all i, have their derivative of order $\alpha_i - 1$ in x_i for all i absolutely continuous, and that of order α_i in x_i for all i in $\mathscr{L}_2(\Gamma)$, the usual Lebesgue space. Let the scalar product be

$$(f, g) = \sum_{p_1=0}^{\alpha_1-1} \cdots \sum_{p_m=0}^{\alpha_m-1} \frac{\partial^{p_1+\cdots+p_m}}{\partial x_1^{p_1} \cdots \partial x_m^{p_m}} f(z) \frac{\partial^{p_1+\cdots+p_m}}{\partial x_1^{p_1} \cdots \partial x_m^{p_m}} g(z)$$
$$+ \sum_{i=1}^{\mu} \int_{\Gamma_i} M_i f M_i g + \int_{\Gamma} Lf Lg, \tag{2.1}$$

where $z = (z_1, \ldots, z_m) \in \Gamma$.

We assume that $\{M_i\}$ is minimal; and, if N is the set of $f \in H$ such that

$$\begin{aligned} Lf &= 0, \\ M_i f &= 0, \qquad i = 1, \ldots, \mu, \end{aligned} \tag{2.2}$$

that N has finite dimension q. Note that this latter assumption will mean that the M_i must contain operators whose domains are of every dimension less than m (dimension 0, viz. evaluation at a point, is not necessary, but is not prohibited).

Suppose a set, Λ, of n continuous linear functionals, λ_i, $i = 1, \ldots, n$ are defined on H, with representers $k_i \in H$. Assume that they are linearly independent and span a subspace $K \subset H$.

Assume

$$\text{(i)} \qquad n \geqslant q, \tag{2.3}$$

$$\text{(ii)} \qquad K^{\perp} \cap N = \{0\}. \tag{2.4}$$

These assumptions allow us to define a unique spline function.

Define

$$Z = \mathscr{L}_2(\Gamma) \times \mathscr{L}_2(\Gamma_1) \times \cdots \times \mathscr{L}_2(\Gamma_\mu), \tag{2.5}$$

where, if Γ_i has dimension 0, we interpret it as $\mathbb{R}$.

If

$$z_1 = (f, f_1, ..., f_\mu), \qquad f \in \mathscr{L}_2(\Gamma), \qquad f_i \in \mathscr{L}_2(\Gamma_i),$$

and

$$z_2 = (g, g_1, ..., g_\mu), \qquad g \in \mathscr{L}_2(\Gamma), \qquad g_i \in \mathscr{L}_2(\Gamma_i),$$

then define

$$(z_1, z_2)_Z = \int_\Gamma fg + \sum_{i=1}^{\mu} \rho_i \int_{\Gamma_i} f_i g_i, \qquad \rho_i > 0. \tag{2.6}$$

This form is a scalar product for Z, making it a Hilbert Space.

Define a map T from H into Z by

$$Tf = (Lf, M_1 f, ..., M_\mu f) \in Z \qquad \text{for all } f \in H. \tag{2.7}$$

Then T is linear and continuous from H onto $U = TH$ and has null-space N.

Define $r = (r_1, r_2, ..., r_n) \in E^n$, and

$$U(r) = \{f \in H; (k_i, f)_H = r_i, i = 1, ..., n\}. \tag{2.8}$$

Then, following Atteia [3] and Anselone and Laurent [1], we define an *interpolating spline function*, $s(x_1, ..., x_m)$ for r w.r.t. L and the M_i, as any element of $U(r)$ such that

$$\| Ts \|_Z = \min_{f \in U(r)} \| Tf \|_Z. \tag{2.9}$$

The theory of the two papers quoted can now be invoked, not only to prove existence and uniqueness of s, but also its main properties and a method of construction.

In what follows, however, we shall examine a particular type of spline function, the polynomial spline function in two variables, defined in a rectangle in E^2. This spline has important applications, and our method of approach and construction will yield more practical results than that referred to above, although we shall use the definition embodied in (2.9).

3. A Reproducing Kernel for Polynomial Spline Functions

Let us fix our attention on the case

$$L = (\partial^{m+n}/\partial x^m \partial y^n), \tag{3.1}$$

$$M_i = \left.\frac{\partial^m}{\partial x^m}\right|_{y=y_i}, \qquad i = 1,..., n,$$
$$M_{n+i} = \left.\frac{\partial^n}{\partial y^n}\right|_{x=x_i}, \qquad i = 1,..., m, \tag{3.2}$$

where $\{x_i\}_{i=1}^{m}$ and $\{y_j\}_{j=1}^{n}$ are distinct points in $[a, b]$ and $[c, d]$, respectively.

We choose as scalar product for H the form (similar to (2.1)),

$$(f, g) = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, g(x_i, y_j) + \sum_{j=1}^{n} \int_a^b f_0^{m}(x, y_j)\, g_0^{m}(x, y_j)\, dx$$
$$+ \sum_{i=1}^{m} \int_c^d f_n^{0}(x_i, y)\, g_n^{0}(x_i, y)\, dy + \int_a^b \int_c^d f_n^{m}(x, y) g_n^{m}(x, y)\, dy\, dx. \tag{3.3}$$

The null-space we are concerned with is now

$$N = \left\{ f \in H: f(x, y) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \beta_{ij} x^i y^j,\ \beta_{ij} \text{ constant} \right\}, \tag{3.4}$$

which is of *finite* dimension mn.

Our construction requires a *reproducing kernel*. The idea of using such a kernel in spline function theory originates with de Boor and Lynch [5]. If H is a Hilbert space of functions, then K is a reproducing kernel if $K(\cdot, y) \in H$ for all fixed y, and

$$f = (K(x, \cdot), f(x)), \qquad \text{for all } f \in H. \tag{3.5}$$

K then has the property that if λ is a linear functional on H, then λ is bounded if and only if $\phi \in H$, where

$$\phi = \lambda K(\cdot, y). \tag{3.6}$$

ϕ is then the representer of λ.

Following de Boor and Lynch, define

$$c_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{m} \frac{x - x_j}{x_i - x_j}, \qquad i = 1,..., m, \tag{3.7}$$

$$d_j(y) = \prod_{\substack{i=1 \\ i \neq j}}^{n} \frac{y - y_i}{y_j - y_i}, \qquad j = 1,..., n, \tag{3.8}$$

$$g(x, s) = \frac{(x - s)_+^{m-1}}{(m - 1)!} - \sum_{i=1}^{m} \frac{(x_i - s)_+^{m-1}}{(m - 1)!} c_i(x), \tag{3.9}$$

$$h(y, t) = \frac{(y - t)_+^{n-1}}{(n - 1)!} - \sum_{j=1}^{n} \frac{(y_j - t)_+^{n-1}}{(n - 1)!} d_j(y), \tag{3.10}$$

where

$$z_+ = \begin{cases} z & \text{if } z \geqslant 0, \\ 0 & \text{if } z < 0. \end{cases} \tag{3.11}$$

Also, define

$$\begin{aligned} K_1(x, s) = \sum_{i=1}^{m} c_i(x)c_i(s) + \frac{(-1)^m}{(2m - 1)!} \Big[&(x - s)_+^{2m-1} \\ &+ \sum_{i=1}^{m} \sum_{j=1}^{m} (x_i - x_j)_+^{2m-1} c_i(x)c_j(s) \\ &- \sum_{i=1}^{m} \{(x - x_i)_+^{2m-1} c_i(s) + (x_i - s)_+^{2m-1} c_i(x)\}\Big], \end{aligned} \tag{3.12}$$

with a similar expression for $K_2(y, t)$, and

$$K(x, s; y, t) = K_1(x, s)\, K_2(y, t). \tag{3.13}$$

We now have

THEOREM 3.1. *H, with scalar product* (3.3), *is a Hilbert space, with reproducing kernel K given by* (3.13).

Proof. This is proved in a similar way to the result in de Boor and Lynch. It is first shown (details omitted) that the Lagrangian formula

$$\begin{aligned} f(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_i(x)d_j(y) f(x_i, y_j) &+ \sum_{i=1}^{m} c_i(x) \int_c^d h(y, t) f_n{}^0(x_i, t)\, dt \\ &+ \sum_{j=1}^{n} d_j(y) \int_a^b g(x, s) f_0{}^m(s, y_j)\, ds \\ &+ \int_a^b \int_c^d g(x, s)\, h(y, t) f_n{}^m(s, t)\, dt\, ds, \end{aligned} \tag{3.14}$$

holds true in H. This formula allows us to show that H is a Hilbert space with scalar product (3.3).

It is then shown that K has properties:

$$K(\cdot, s; \cdot, t) \in H, \tag{3.15}$$

$$f = (K(x, \cdot; y, \cdot), f(x, y)), \tag{3.16}$$

i.e., it is a reproducing kernel for H.

COROLLARY 3.2. *The linear functional* λ, *where*

$$\lambda f = f(\xi, \eta), \qquad \xi \in [a, b], \qquad \eta \in [c, d], \tag{3.17}$$

is bounded.

Proof. It is easily seen that

$$\phi = K(\cdot, \xi; \cdot, \eta) \in H,$$

which proves λ is bounded, with representer ϕ.

4. CONSTRUCTION OF POLYNOMIAL SPLINES

We choose the linear functionals in Section 2 to be those defining point evaluation at (x_i, y_j) for $i = 1,..., k$ and $j = 1,..., l$, where all the x_i lie in $[a, b]$ and all the y_j in $[c, d]$. We assume that the x_i and y_j used already in (3.2) coincide with the corresponding ones in $\{x_i\}_{i=1}^{k}$ and $\{y_j\}_{j=1}^{l}$. We thereby assume that $k \geqslant m$ and $l \geqslant n$, which implies that the number of linear functionals, kl, is greater than or equal to the dimension of N, mn, as required in (2.4).

Let

$$k_{ij}(x, y) = K(x, x_i; y, y_j), \qquad i = 1,..., k, \qquad j = 1,..., l, \tag{4.1}$$

and define the subspace, S, of H to be that spanned by the functions $\{k_{ij}\}_{i=1,j=1}^{k,l}$. S has dimension kl, and so we can fix functions in S by imposing the interpolation conditions

$$\lambda_{ij}s = s(x_i, y_j) = r_{ij}, \qquad i = 1,..., k, \qquad j = 1,..., l. \tag{4.2}$$

Let P_S be the orthogonal projection of H onto S, i.e., $P_S f$ is the best approximation to f by an element in S with respect to the norm in H.

THEOREM 4.1. *$P_S f$ is the spline function defined in Section* 2.

Proof.

$$(P_S f, h) = (P_S f, P_S h) = (f, P_S h), \qquad \text{for all } f, h \in H \tag{4.3}$$

$$\|f\|^2 = \|f - P_S f\|^2 + \|P_S f\|^2, \qquad \text{for all } f \in H. \tag{4.4}$$

Define

$$w_f = \{h: h \in H, \lambda_{ij}h = \lambda_{ij}f, i = 1,..., k; j = 1,..., l\}. \tag{4.5}$$

Substitute $h = k_{ij}$ in (4.3) to give

$$\lambda_{ij}(P_S f) = \lambda_{ij}f, \qquad \text{for all } f \in H, \tag{4.6}$$

and so $P_S f$ is the unique element in S interpolating to f with respect to $\{\lambda_{ij}\}_{i=1,j=1}^{k,l}$. Thus,

$$P_S h = P_S f, \qquad \text{for all } h \in w_f, \tag{4.7}$$

and so

$$\| P_S f \| \leqslant \| h \| \qquad \text{for all } h \in w_f, \tag{4.8}$$

with equality if and only if $h = P_S f$.

This shows that $P_S f$ minimizes

$$\sum_{j=1}^{n} \int_a^b (h_0{}^m(x, y_j))^2 \, dx + \sum_{i=1}^{m} \int_c^d (h_n{}^0(x_i, y))^2 \, dy + \int_a^b \int_c^d (h_n{}^m(x, y))^2 \, dy \, dx, \tag{4.9}$$

i.e.,

$$\sum_{j=1}^{n} \int_a^b (M_j h)^2 \, dx + \sum_{i=1}^{m} \int_c^d (M_{i+n} h)^2 \, dy + \int_a^b \int_c^d (Lh)^2 \, dy \, dx, \tag{4.10}$$

subject to constraints (4.2), where $\lambda_{ij}h = r_{ij} = \lambda_{ij}f$, $i = 1,..., k; j = 1,..., l$. This is precisely how the spline function was defined in Section 2 (we have chosen $\rho_i = 1$ for all i).

COROLLARY 4.2. *S is the set of all such spline functions.*

5. PROPERTIES OF POLYNOMIAL SPLINES

Consider k_{ij}, one of the basis for S. (4.1) and (3.13) tell us that

$$k_{ij}(x, y) = K_1(x, x_i) \, K_2(y, y_j), \tag{5.1}$$

and hence we can readily deduce properties of k_{ij}, and hence of all members of S, from the properties of $K_1(x, x_i)$. This latter function, in fact, has the form of a one-variable spline function. In particular, K_1 is a polynomial of degree at most $2m - 1$ in each interval $[x_p, x_{p+1}]$, $p = 1,..., k - 1$, it has continuous derivatives up to and including order $2m - 2$, and is a polynomial of degree at most $m - 1$ in $[a, x_1]$ and $[x_k, b]$.

These properties tell us that our two-variable spline $s(x, y)$ has the form of a double polynomial

$$\sum_{i=0}^{2m-1} \sum_{j=0}^{2n-1} \eta_{ij}\, x^i y^j, \tag{5.2}$$

in each rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $i = 1,..., k-1, j = 1,..., l-1$. It has continuous derivatives up to and including that of order $2m-2$ in x and $2n-2$ in y. For $x \notin [x_1, x_k]$, the form is $\sum_{i=0}^{m-1} \sum_{j=0}^{2n-1} \eta_{ij} x^i y^j$, and for $y \notin [y_1, y_l]$, it is $\sum_{i=0}^{2m-1} \sum_{j=0}^{n-1} \eta_{ij} x^i y^j$. These latter conditions are equivalent to

$$\frac{\partial^{m+i} s}{\partial x^{m+i}} = 0, \qquad \text{for} \quad x < x_1 \quad \text{or} \quad x > x_k, \quad y \in [c, d],$$
$$i = 0, 1,..., m-1, \tag{5.3a}$$

$$\frac{\partial^{n+j} s}{\partial y^{n+j}} = 0, \qquad \text{for} \quad y < y_1 \quad \text{or} \quad y > y_l, \quad x \in [a, b],$$
$$j = 0, 1,..., n-1. \tag{5.3b}$$

Further analysis of this case will be found in Section 6, when computation is considered.

There are two minimization properties of these spline functions which we prove using the methods of de Boor and Lynch [5].

THEOREM 5.1. *For given $f \in H$, of all $\tilde{s} \in S$, the interpolating function in the sense of* (4.2), *s, has the property that*

$$\sum_{j=1}^{n} \int_a^b (f_0^m(x, y_j) - \tilde{s}_0^m(x, y_j))^2\, dx + \sum_{i=1}^{m} \int_c^d (f_n^0(x_i, y) - \tilde{s}_n^0(x_i, y))^2\, dy + \int_a^b \int_c^d (f_n^m(x, y) - \tilde{s}_n^m(x, y))^2\, dy\, dx$$
$$\geqslant \sum_{j=1}^{n} \int_a^b (f_0^m(x, y_j) - s_0^m(x, y_j))^2\, dx + \sum_{i=1}^{m} \int_c^d (f_n^0(x_i, y) - s_n^0(x_i, y))^2\, dy + \int_a^b \int_c^d (f_n^m(x, y) - s_n^m(x, y))^2\, dy\, dx, \tag{5.4}$$

with equality if and only if

$$s = s + \psi, \qquad \textit{where} \quad \psi(x, y) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \eta_{ij} x^i y^j, \quad \textit{for some} \quad \eta_{ij}. \tag{5.5}$$

Proof. This result follows since $s = P_S f$ is the best approximation to f with respect to the norm

$$\| h \|^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} (h(x_i, y_j))^2 + \sum_{j=1}^{n} \int_a^b (h_0{}^m(x, y_j))^2 \, dx$$

$$+ \sum_{i=1}^{m} \int_c^d (h_n{}^0(x_i, y))^2 \, dy + \int_a^b \int_c^d (h_n{}^m(x, y))^2 \, dy \, dx. \quad (5.6)$$

The second result is a generalization of Schoenberg's Theorem (Schoenberg [9]). Suppose λ is a linear functional belonging to

$$\mathscr{L}^{(m,n)} = \left\{ \lambda : \lambda f = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \int_a^b \int_c^d f_j{}^i(x, y) d\,\mu_{ij}(x, y), \right.$$

$$\left. \mu_{ij} \text{ of bounded variation} \right\}. \quad (5.7)$$

Following Sard [8], we approximate λ by λ^* where

$$\lambda^* f = \sum_{i=1}^{k} \sum_{j=1}^{l} \gamma_{ij} f(x_i, y_j) = \sum_{i=1}^{k} \sum_{j=1}^{l} \gamma_{ij} \lambda_{ij} f, \quad (5.8)$$

the γ_{ij} being constants, and the λ_{ij} bounded linear functionals, as in (4.2). The γ_{ij} are chosen as follows.

First, λ^* is enforced to be exact for $f \in N$, i.e., for functions of the form $\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \eta_{ij} x^i y^j$, η_{ij} constant. Now apply $R = \lambda - \lambda^*$ to (3.14):

$$Rf = \sum_{i=1}^{m} \int_c^d V_i(t) f_n{}^0(x_i, t) \, dt + \sum_{j=1}^{n} \int_a^b U_j(s) f_0{}^m(s, y_j) \, ds$$

$$+ \int_a^b \int_c^d W(s, t) f_n{}^m(s, t) \, dt \, ds, \quad (5.9)$$

where

$$U_j(s) = R^{(x,y)}[d_j(y)\, g(x, s)] = R^{(x,y)} \left[d_j(y) \frac{(x - s)_+^{m-1}}{(m-1)!} \right], \quad (5.10)$$

$$V_i(t) = R^{(x,y)}[c_i(x)\, h(y, t)] = R^{(x,y)} \left[c_i(x) \frac{(y - t)_+^{n-1}}{(n-1)!} \right], \quad (5.11)$$

$$W(s, t) = R^{(x,y)}[g(x, s)\, h(y, t)]$$

$$= R^{(x,y)} \left[\frac{(x - s)_+^{m-1} (y - t)_+^{n-1}}{(m-1)!\,(n-1)!} \right] - \sum_{j=1}^{n} \frac{(y_j - t)_+^{n-1}}{(n-1)!} U_j(s)$$

$$- \sum_{i=1}^{m} \frac{(x_i - s)_+^{m-1}}{(m-1)!} V_i(t). \quad (5.12)$$

We can now estimate (5.9) by

$$\| Rf \| \leqslant \left[\sum_{i=1}^{m} \int_c^d (V_i(t))^2\,dt + \sum_{j=1}^{n} \int_a^b (U_j(s))^2\,ds + \int_a^b \int_c^d (W(s,t))^2\,dt\,ds\right]^{1/2}$$

$$\times \left[\sum_{i=1}^{m} \int_c^d (f_n{}^0(x_i, t))^2\,dt + \sum_{j=1}^{n} \int_a^b (f_0{}^m(s, y_j))^2\,ds \right.$$

$$\left. + \int_a^b \int_c^d (f_n{}^m(s,t))^2\,dt\,ds\right]^{1/2}, \tag{5.13}$$

splitting the contributions of f and the γ_{ij}. The remaining degrees of freedom in the γ_{ij} are now removed by minimizing

$$\sum_{i=1}^{m} \int_c^d (V_i(t))^2\,dt + \sum_{j=1}^{n} \int_a^b (U_j(s))^2\,ds + \int_a^b \int_c^d (W(s,t))^2\,dt\,ds. \tag{5.14}$$

Such approximations are exceedingly difficult to construct, in all but the simplest cases. The task, however, may be by-passed by our generalization of Schoenberg's Theorem.

THEOREM 5.2. *λ^* satisfies*

$$\lambda^* f = \lambda(P_S f). \tag{5.15}$$

Proof. Firstly, we construct another approximation to λ, of the same form as λ^*.

$$\bar{\lambda} f = \sum_{i=1}^{k} \sum_{j=1}^{l} \xi_{ij}\,\lambda_{ij} f. \tag{5.16}$$

Let h be the representer of λ in H, and $\bar{h}$ that of $\bar{\lambda}$. k_{ij}, the representers of λ_{ij}, span S. Let

$$\bar{R} = \lambda - \bar{\lambda}, \tag{5.17}$$

then

$$\| \bar{R} \| = \sup_{\| f \| \leqslant 1} | \bar{R} f |. \tag{5.18}$$

Choose ξ_{ij}, and hence fix $\bar{\lambda}$, to minimize $\| \bar{R} \|$. But

$$\| \bar{R} \| = \left\| h - \sum_{i=1}^{k} \sum_{j=1}^{l} \xi_{ij} k_{ij} \right\|, \tag{5.19}$$

so $\bar{h} = P_S h$, and $\bar{\lambda}$ satisfies property (5.15).

We finish the proof by showing $\lambda^* = \bar{\lambda}$. Note that $\bar{\lambda}$, from (5.15), is exact for $f \in S$, and, *a fortiori*, for $f \in N$. But $\bar{\lambda}$ also minimizes $\| \bar{R} \|$ over $\lambda' = \sum_{i=1}^{k} \sum_{j=1}^{l} \beta_{ij} \lambda_{ij}$, and hence is identical to λ^*.

6. Computation of Polynomial Splines

We now consider the computation of the polynomial spline function of degree $(2m-1, 2n-1)$ (i.e., $2m-1$ in x and $2n-1$ in y), with interpolation conditions

$$s(x_i, y_j) = r_{ij}, \qquad i = 1,..., k, \qquad j = 1,..., l. \tag{6.1}$$

Define $\{x_1,..., x_k\}$ and $\{y_1,..., y_l\}$ to be *knot generating sets*, assuming

$$\begin{aligned} a \leqslant x_1 < x_2 < \cdots < x_k \leqslant b, \\ c \leqslant y_1 < y_2 < \cdots < y_l \leqslant d. \end{aligned} \tag{6.2}$$

We use (m, n) to represent the degree of a polynomial or spline function if it is of degree m in x and n in y.

It can be shown easily that s takes the form

$$\begin{aligned} s(x, y) = \sum_{i=1}^{k} \sum_{j=1}^{l} \alpha_{ij}(x - x_i)_+^{2m-1}(y - y_j)_+^{2n-1} + \sum_{i=1}^{k} p_i(y)(x - x_i)_+^{2m-1} \\ + \sum_{j=1}^{l} q_j(x)(y - y_j)_+^{2n-1} + P(x, y), \end{aligned} \tag{6.3}$$

where P is a polynomial of degree $(m-1, n-1)$, p_i and q_j are polynomials of degree $n-1$ and $m-1$, respectively, and the α_{ij} are constants. We also have conditions

$$\sum_{i=1}^{k} \alpha_{ij} x_i^r = 0, \qquad r = 0, 1,..., m-1, \qquad j = 1,..., l, \tag{6.4a}$$

$$\sum_{j=1}^{l} \alpha_{ij} y_j^s = 0, \qquad s = 0, 1,..., n-1, \qquad i = 1,..., k, \tag{6.4b}$$

$$\sum_{i=1}^{k} x_i^r p_i(y) \equiv 0, \qquad r = 0, 1,..., m-1, \tag{6.4c}$$

$$\sum_{j=1}^{l} y_j^s q_j(x) \equiv 0, \qquad s = 0, 1,..., n-1, \tag{6.4d}$$

This approach to the problem of computation is ruled out in the one-variable case due to the ill-conditioning of the equations. The situation here is even worse, since (6.1) and (6.4) provide $kl + kn + lm + 2mn$ equations for the $kl + kn + lm + mn$ unknowns in (6.3). (Observe, for example, that both (6.4a) and (6.4b) imply $\sum_{i=1}^{k} \sum_{j=1}^{l} \alpha_{ij} = 0$.)

Greville ([6] and, in more detail, [7]) gives a more stable construction for the one-variable spline. We adapt it, in what follows, to the two-variable case.

Using the notation

$$f(x_i, x_{i+1}, \ldots, x_{i+m}), \tag{6.5}$$

for the divided difference of f of order m for the points listed, we define the "B-splines":

$$\begin{aligned} N_\nu^{(m)}(x) &= N^{(m)}(x; x_\nu, \ldots, x_{\nu+m}), \qquad \nu = 1, \ldots, k-m, \\ N_\tau^{(n)}(y) &= N^{(n)}(y; y_\tau, \ldots, y_{\tau+n}), \qquad \tau = 1, \ldots, l-n, \end{aligned} \tag{6.6}$$

where

$$\begin{aligned} N^{(m)}(x; s) &= (x-s)_+^{2m-1}, \\ N^{(n)}(y; t) &= (y-t)_+^{2n-1}. \end{aligned} \tag{6.7}$$

We have seen in Section 5 that our spline function $s(x, y)$ may be written in the form

$$s(x, y) = \sum_{i=1}^{kl} \zeta_i u_i(x)\, v_i(y), \tag{6.8}$$

where ζ_i are constant, and $u_i(x)$ and $v_i(y)$ are spline functions with knots given by the knot generating sets, and of degrees $2m-1$ and $2n-1$, respectively. Greville proves that

$$\begin{aligned} u_i(x) &= \sum_{\nu=1}^{k-m} \xi_{i\nu} N_\nu^{(m)}(x) + P_i(x), \\ v_i(y) &= \sum_{\tau=1}^{l-n} \eta_{i\tau} N_\tau^{(n)}(y) + Q_i(y), \end{aligned} \tag{6.9}$$

where $\xi_{i\nu}$ and $\eta_{i\tau}$ are constants, and P_i and Q_i are polynomials of degree $m-1$ and $n-1$, respectively.

If we substitute (6.9) into (6.8), we can write the result in the form

$$s(x, y) = \sum_{i=1}^{k} \sum_{j=1}^{l} \beta_{ij} M_i(x)\, N_j(y), \tag{6.10}$$

where

$$M_i(x) = \begin{matrix} N_i^{(m)}(x), & i = 1, \ldots, k-m, & \\ x^r, & r = i-(k-m+1), & i = k-m+1, \ldots, k, \end{matrix} \tag{6.11}$$

and

$$N_j(y) = \begin{matrix} N_j^{(n)}(y), & j = 1, \ldots, l-n, & \\ y^s, & s = j-(l-n+1), & j = l-n+1, \ldots, l, \end{matrix} \tag{6.12}$$

and the β_{ij} are constants. This form for $s(x, y)$ now has kl unknowns, determined from the kl conditions (6.1). These unknowns are calculated by obtaining various divided differences of s in both variables, both from (6.1) and (6.10), and equating the results.

We define the following matrices

$$M\colon \{M_i(x_\mu\,, x_{\mu+1}\,,\dots, x_{\mu+m})\}_{i=1,\mu=1}^{k-m,k-m}\,, \tag{6.13}$$

$$N\colon \{N_j(y_\nu\,, y_{\nu+1}\,,\dots, y_{\nu+n})\}_{j=1,\nu=1}^{l-n,l-n}\,, \tag{6.14}$$

$$\Gamma\colon \{s(x_\mu\,,\dots, x_{\mu+m}\,; y_\nu\,,\dots, y_{\nu+n})\}_{\mu=1,\nu=1}^{k-m,l-n}\,, \tag{6.15}$$

$$B\colon \{\beta_{\mu\nu}\}_{\mu=1,\nu=1}^{k-m,l-n}\,, \tag{6.16}$$

$$\Gamma_x : \{s(x_1\,, x_2\,,\dots, x_\mu\,; y_\nu\,,\dots, y_{\nu+n})\}_{\mu=1,\nu=1}^{m,l-n}\,, \tag{6.17}$$

$$\Gamma_y : \{s(x_\mu\,,\dots, x_{\mu+m}\,; y_1\,, y_2\,,\dots, y_\nu)\}_{\mu=1,\nu=1}^{k-m,n}\,, \tag{6.18}$$

$$P\colon \{M_i(x_1\,,\dots, x_\mu)\}_{i=1,\mu=1}^{k-m,m}\,, \tag{6.19}$$

$$Q\colon \{N_j(y_1\,,\dots, y_\nu)\}_{j=1,\nu=1}^{l-n,n}\,, \tag{6.20}$$

$$B_x : \{\beta_{i+(k-m),j}\}_{i=1,j=1}^{m,l-n}\,, \tag{6.21}$$

$$B_y : \{\beta_{i,j+(l-n)}\}_{i=1,j=1}^{k-m,n}\,, \tag{6.22}$$

$$D_x : \{f_i(x_1\,,\dots, x_\mu)\}_{i=1,\mu=1}^{m,m}\,, \qquad \text{where} \quad f_i(s) = s^{i-1}, \tag{6.23}$$

$$D_y : \{g_j(y_1\,,\dots, y_\nu)\}_{j=1,\nu=1}^{n,n}\,, \qquad \text{where} \quad g_j(t) = t^{j-1}, \tag{6.24}$$

$$\Gamma_{xy} : \{s(x_1\,,\dots, x_\mu\,; y_1\,,\dots, y_\nu)\}_{\mu=1,\nu=1}^{m,n}\,, \tag{6.25}$$

$$B_{xy} : \{\beta_{i+(k-m),j+(l-n)}\}_{i=1,j=1}^{m,n}\,. \tag{6.26}$$

It is straightforward to construct Γ, Γ_x, Γ_y, and Γ_{xy} directly and to equate the results with the formulae obtained from (6.10) to find,

$$B = M^{-1}\Gamma N^{-1}, \tag{6.27a}$$

$$B_x = (D_x^T)^{-1}[\Gamma_x N^{-1} - P^T B], \tag{6.27b}$$

$$B_y = [M^{-1}\Gamma_y - BQ](D_y)^{-1}, \tag{6.27c}$$

$$B_{xy} = (D_x^T)^{-1}[\Gamma_{xy} - P^T BQ - D_x^T B_x Q - P^T B_y D_y](D_y)^{-1}. \tag{6.27d}$$

The four parts of (6.27) give the values of all the β_{ij}, and hence determine $s(x, y)$.

D_x and D_y are readily inverted as they are triangular matrices. M^{-1} is easily found (using, say, Cholesky's Method) since M is symmetric, $(2m-1)$-banded, and positive or negative definite according as $(-1)^m$ is $+1$ or -1, respectively. N^{-1} is found similarly.

$s(x, y)$ can be readily evaluated from (6.10) since $M_i(x)$ and $N_j(y)$ are easily computed.

In the one-variable case, coincidence of p knots may be interpreted as interpolation to the first $p-1$ derivatives, with a drop in continuity across the knot, of order $p-1$. The situation in the multivariate case is similar, and coincidence in the knot generating sets can be handled by our algorithm, provided the necessary derivatives are supplied, and the usual interpretation is placed on the resultant divided differences, viz. they become derivatives.

7. Extensions

We have used a Λ containing only point evaluation functionals of a certain form. This restriction may be lifted, although the reproducing kernel analysis of Sections 3, 4 and 5 depends on Λ containing mn functionals which may be split in the form

$$\lambda_{ij} = \mu_i^{(x)} \nu_j^{(y)}, \qquad i = 1,..., m, \qquad j = 1,..., n. \tag{7.1}$$

Provided this condition is satisfied, Λ may also contain any functionals in $\mathscr{L}^{(m,n)}$ as long as the resultant set satisfies linear independence, and the conditions (2.3) and (2.4). Indeed in Part Two of this paper [2], we require to add linear functionals to Λ which alter it from the form used above.

The algorithm of Section 6, however, is not adaptable to general linear functionals, and Λ must be of a form containing kl functionals, each splittable as in (7.1), and involving derivative evaluation of *all* derivatives up to a certain order. Should Λ not be of this strict form, s can be constructed numerically using Section 4, although that method tends to be an ill-conditioned one.

Our analysis can be adapted to cover more general linear operators. Suppose L is a differential operator in m variables $x^{(1)}, x^{(2)},..., x^{(m)}$, and

$$L = L_1 L_2 ,..., L_m , \tag{7.2}$$

where L_i operates on $x^{(i)}$ only, and is of order α_i, $i = 1,..., m$.

Impose interpolation conditions

$$f(x_{\beta_1}^{(1)}, x_{\beta_2}^{(2)},..., x_{\beta_m}^{(m)}) = r_{\beta_1\beta_2\cdots\beta_m} , \tag{7.3}$$

where $\beta_i = 1,..., \tau_i$, $\tau_i \geqslant \alpha_i$, $i = 1,..., m$.

Now choose

$$M_i^{(p)} = \prod_{\substack{k=1\\k\neq i}}^{m} L_k \,|_{x^{(i)}=x_p^{(i)}}\,, \quad i = 1,\dots, m,\ p = 1,\dots, \alpha_i\,,$$

$$M_{ij}^{(p,q)} = \prod_{\substack{k=1\\k\neq i,j}}^{m} L_k \,|_{x^{(i)}=x_p^{(i)},\,x^{(j)}=x_q^{(j)}}$$

$$i, j = 1,\dots, m,\ i \geqslant j \qquad p = 1,\dots, \alpha_i\,,\ q = 1,\dots, \alpha_j\,, \tag{7.4}$$

$$\vdots$$

$$M_{1,2,\dots,k-1,k+1,\dots,m}^{(\beta_1,\dots,\beta_{k-1},\beta_{k+1},\dots,\beta_m)} = L_k \,|_{x^{(i)}=x_{\beta_i}^{(i)}}$$

$$k = 1,\dots, m,\ \beta_i = 1,\dots, \alpha_i\,,\ (i = 1,\dots, m,\ i \neq k)$$

and minimize

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_m}^{b_m}(L_1\cdots L_m f(\xi_1,\dots,\xi_m))^2\,d\xi_m\cdots d\xi_1$$

$$+\sum_{i=1}^{m}\sum_{p=1}^{\alpha_i}\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}(M_i^{(p)}f)^2\,d\xi_m\cdots d\xi_{i+1}\,d\xi_{i-1}\cdots d\xi_1+\cdots \tag{7.5}$$

$$+\sum_{k=1}^{m}\sum_{\beta_1=1}^{\alpha_1}\cdots\sum_{\beta_m=1}^{\alpha_m}\int_{a_k}^{b_k}(M_{1,2,\dots,k-1,k+1,\dots,m}^{(\beta_1,\dots,\beta_{k-1},\beta_{k+1},\dots,\beta_m)}f)^2\,d\xi_k\,,$$

subject to the constraints (7.3).

The result is a spline function in m variables, which is a piecewise function, each piece being of the form

$$\sum_{\beta_1=1}^{\alpha_1}\cdots\sum_{\beta_m=1}^{\alpha_m}\eta_{\beta_1,\dots,\beta_m}\,u_{\beta_1}^{(1)}\cdots u_{\beta_m}^{(m)}, \tag{7.6}$$

where $u_{\beta_k}^{(k)}$ is one of the basis elements of the null-space of L_k. Explicit construction of the spline function is possible using a reproducing kernel, which turns out to be the product of m one-variable reproducing kernels.

We can also adapt the construction of Section 6 to the case of m variables, for $m > 2$, but the algorithm is no longer expressible in neat matrix form.

8. Conclusion

The properties shown in Section 5 make spline functions valuable for practical use in interpolation, numerical differentiation and quadrature. In Part Two of this paper, we consider the computation of "best" error bounds for application of the two-variable splines to these problems.

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