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Closed-form dynamic response of damped mass–spring cascades

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Abstract

Closed-form responses due to step forces of finite and semi-infinite damped mass–spring cascades with variable boundary mass are derived. By convolution arbitrary responses may be obtained. Eigenfrequencies, eigenmodes and critical damping of nonuniform and uniform finite cascades are found, showing a decreasing critical damping with increasing eigenfrequency. Further the frequency-response functions of the cascades are given for three cases.

Keywords: Cascade; Mass–spring–damper system; Vibration; Wave propagation; Dynamics

1. Introduction

In many fields of engineering the dynamic behaviour of one-dimensional cascades of discrete elements is of importance for analysing vibrations and wave-propagation problems. In mechanical engineering cascades of mass–spring–damper systems are used to model the dynamic behaviour of, e.g. trains, high-rise buildings, piles, etc. In electrical engineering cascades of capacities, inductances and resistances are used in the modelling of electrical lines, chips, etc. In chemical engineering molecular lattices are modelled by mechanical cascades. Hydraulical cascades of capacities, inertias and resistances model the behaviour of all kinds of fluid systems.

In the past the effort of many famous researchers as Newton, John and Daniel Bernoulli, Taylor, Euler, Lagrange, Cauchy, Baden-Powell, Lord-Kelvin, etc., have given substantial insight in the behaviour of cascades as compiled in [5]. However until recently a closed-form solution for an arbitrary dynamic loading has not been found.

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In recent papers the responses due to a pulse loading of several types of finite and semi-infinite cascades of mass–spring systems with a uniform distibution of mass and stiffness for the internal bodies and springs have been derived analytically. Systems with specific values for the mass and the stiffness of the bodies and the springs at the boundaries (boundary) have been considered in [4]. Structural differences between the analytical solutions of discrete and continuous systems have been derived in [8]. The responses due to a pulse loading of discrete systems with variable boundary mass have been treated in [3]. With these results the dynamic behaviour of e.g. undamped trains can be determined analytically. In [7] simplified models and eigenfrequencies of semi-infinite cascades with variable boundary mass have been dealt with. In [6,9,10] mass–spring systems are linked with orthogonal polynomials and solutions for infinite cascades are presented.

In this paper the closed-form responses of finite and semi-infinite cascades of damped mass–spring systems due to a step-loading at a boundary mass will be studied. In the model the stiffness of the springs, the viscous friction of the dampers and the masses of the internal bodies are distributed uniformly. The velocities of the masses due to a pulse loading of a system are equal to the accelerations of these masses due to a step-loading of the same system. This can be shown by differentiating the velocity response due to a pulse and using the commutativity of the convolution with respect to differentiation. So the response due to a pulse loading and hence each transient and stationary dynamic behaviour can be derived by use of the convolution principle.

The eigenfrequencies and eigenmodes of the vibrations in finite systems will be derived. It will be shown that each mode has its own critical damping, which decreases with increasing eigenfrequency.

Some responses of the velocities of the masses due to step forces, the total forces in the interactions and the frequency response functions for a train model will be discussed. The case of a uniform mass distribution for all the bodies will be treated separately.

Finally the wave propagation in a semi-infinite-damped discrete system will be derived and discussed.

2. Model

For damped mass–spring cascades with free boundaries the following system of differential equations can be derived using the second law of Newton, where u_j denotes the displacement of the jth body, m_j is the mass of the jth body, k_j is the stiffness of the jth spring and c_j is the coefficient of viscous friction of the jth damper.

$$\begin{aligned}
m_1\ddot{u}_1 &= k_1(u_2-u_1)+c_1(\dot{u}_2-\dot{u}_1)+F_0U(t),\\
m_j\ddot{u}_j &= -k_{j-1}(u_j-u_{j-1})+k_j(u_{j+1}-u_j)-c_{j-1}(\dot{u}_j-\dot{u}_{j-1})\\
&\quad +c_j(\dot{u}_{j+1}-\dot{u}_j),\quad j=2,\dots,n-1,\\
m_n\ddot{u}_n &= -k_{n-1}(u_n-u_{n-1})-c_{n-1}(\dot{u}_n-\dot{u}_{n-1}).
\end{aligned} \tag{1}$$

The boundary mass m_1 is loaded by a constant stepforce F_0 at time $t=0$. The function $U(t)$ represents the heaviside unit-step function. The linearity of the equations means that for $t>0$ the velocities v_j obey the same equations, with the $F_0U(t)$ term omitted and with the initial conditions

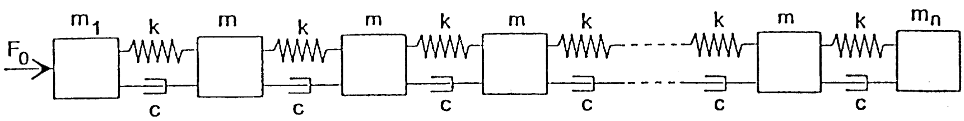

Fig. 1. Damped cascade.

$$v_j(0)=0, \quad j=1,\dots,n,$$
$$\dot v_1(0)=\frac{F_0}{m_1}, \quad \dot v_j(0)=0, \quad j=2,\dots,n. \tag{2}$$

We consider a cascade of damped mass–spring systems as depicted in Fig. 1. The mass of the first body m_1 and last body m_n may be different from the other masses. Inside the cascade the masses and stiffnesses are all equal to m and k, respectively. The coefficient of viscous friction for all interactions is c. Hence we take

$$k_j=k, \quad c_j=c, \quad j=1,\dots,n-1,$$

$$m_j=m, \quad j=2,\dots,n-1.$$

When we introduce the new time scale

$$\tau=\sqrt{\frac{k}{m}}t \tag{3}$$

and define ε,γ and δ by the equations

$$\varepsilon=\frac{c}{\sqrt{km}}, \quad 1-\gamma=\frac{m}{m_1}, \quad 1-\delta=\frac{m}{m_n}, \tag{4}$$

the velocity system takes the form

$$v_1''(\tau)=(1-\gamma)[(v_2-v_1)+\varepsilon(v_2'-v_1')], \tag{5}$$

$$v_j''(\tau)=(v_{j-1}-2v_j+v_{j+1})+\varepsilon(v_{j-1}'-2v_j'+v_{j+1}'), \tag{6}$$

$$v_n''(\tau)=(1-\delta)[(v_{n-1}-v_n)+\varepsilon(v_{n-1}'-v_n')]. \tag{7}$$

Without loss of the generality, the step F_0 can be chosen such that

$$v_j(0)=0, \quad j=1,\dots,n, \tag{8}$$

$$v_1'(0)=1, \quad v_j'(0)=0, \quad j=2,\dots,n. \tag{9}$$

3. Finite cascades

In Section 3.1 we derive the solution for the velocities of the masses due to a step-loading at mass m_1 for the case in which the boundary masses are different from the internal masses. The location of the eigenfrequencies and the critical damping per mode is investigated. By use of the computer program MAPLE the velocities, force distributions and frequency-response functions for some values of c are given.

In Section 3.2 we go into more detail for a special case in which the mass distribution is uniform. Then the eigenfrequencies, eigenmodes and critical damping per mode are calculated and the frequency-response spectra of the system for some values of the damping are discussed.

The results derived in this paper are applicable for an arbitrary number of masses in the cascade. For explanatory reasons only the cases $n = 10, 11$ are elaborated.

3.1. Nonuniform mass-distribution

3.1.1. Derivation of the solution

By Laplace transformation system (5)–(9) transforms to ($n \geqslant 3$)

$$s^2\bar{v}_1(s) - 1 = (1-\gamma)(1+\varepsilon s)(\bar{v}_2(s) - \bar{v}_1(s)), \tag{10}$$

$$s^2\bar{v}_j(s) = (1+\varepsilon s)(\bar{v}_{j-1}(s) - 2\bar{v}_j(s) + \bar{v}_{j+1}(s)), \quad j = 2, \dots, n-1, \tag{11}$$

$$s^2\bar{v}_n(s) = (1-\delta)(1+\varepsilon s)(\bar{v}_{n-1}(s) - \bar{v}_n(s)). \tag{12}$$

If we set

$$\frac{s^2}{1+\varepsilon s} + 2 = \beta^{-2} + \beta^2, \tag{13}$$

so that

$$s = \frac{(\beta^{-1}-\beta)\sqrt{\varepsilon^2(\beta^{-1}-\beta)^2+4} + \varepsilon(\beta^{-1}-\beta)^2}{2},$$

then the wave fields in the cascade are separated into left and right travelling parts. Eq. (11) then becomes

$$(\beta^{-2} + \beta^2)\bar{v}_j(s) = \bar{v}_{j-1}(s) + \bar{v}_{j+1}(s), \quad j = 2, \dots, n-1. \tag{14}$$

This equation has the solution of the form

$$\bar{v}_{j+1} = C(\beta)\beta^{2j} + D(\beta)\beta^{-2j}. \tag{15}$$

Substitution of (15) into (10) and (12) yields

$$C(\beta)(\gamma\beta^2 + \beta^{-2} - \gamma - 1) + D(\beta)(\gamma\beta^{-2} + \beta^2 - \gamma - 1) = \left(\frac{2}{\sqrt{\varepsilon^2(\beta^{-1}-\beta)^2+4} + \varepsilon(\beta^{-1}-\beta)}\right)^2,$$

$$C(\beta)(\delta\beta^{-2} + \beta^2 - \delta - 1)\beta^{2n-2} + D(\beta)(\delta\beta^2 + \beta^{-2} - \delta - 1)\beta^{-2n+2} = 0$$

or

$$C(\beta)(\beta^{-1} - \gamma\beta) + D(\beta)(\gamma\beta^{-1} - \beta) = \frac{1}{\beta^{-1}-\beta}\left(\frac{2}{\sqrt{\varepsilon^2(\beta^{-1}-\beta)^2+4} + \varepsilon(\beta^{-1}-\beta)}\right)^2,$$

$$C(\beta)(\delta\beta^{-1} - \beta)\beta^{2n-2} + D(\beta)(\beta^{-1} - \delta\beta)\beta^{-2n+2} = 0.$$

We easily find the following solution:

$$\bar{v}_{j+1} = \frac{1}{\beta^{-1}-\beta}\left(\frac{2}{\sqrt{\varepsilon^2(\beta^{-1}-\beta)^2+4}+\varepsilon(\beta^{-1}-\beta)}\right)^2 \times \frac{(\beta^{-2n+2j+1}+\beta^{2n-2j-1})-\delta(\beta^{-2n+2j+3}+\beta^{2n-2j-3})}{(\beta^{-2n}-\beta^{2n})-(\gamma+\delta)(\beta^{-2n+2}-\beta^{2n-2})+\gamma\delta(\beta^{-2n+4}-\beta^{2n-4})}.$$

If we set

$$\beta = \mathrm{e}^{\mathrm{i}(\zeta+\pi/2)}$$

then we find

$$\cos\zeta = \frac{\mathrm{i}s}{2\sqrt{1+\varepsilon s}}$$

and for $j \in \{0,1,\ldots,n-2\}$ obtain the result

$$\bar{v}_{j+1} = \frac{\mathrm{i}(-1)^j}{s\sqrt{1+\varepsilon s}}\frac{U_{2n-2j-2}(\mathrm{i}s/(2\sqrt{1+\varepsilon s}))+\delta U_{2n-2j-4}(\mathrm{i}s/(2\sqrt{1+\varepsilon s}))}{U_{2n-1}(\mathrm{i}s/(2\sqrt{1+\varepsilon s}))+(\gamma+\delta)U_{2n-3}(\mathrm{i}s/(2\sqrt{1+\varepsilon s}))+\gamma\delta U_{2n-5}(\mathrm{i}s/(2\sqrt{1+\varepsilon s}))}, \tag{16}$$

where $U_m(z)$ denotes the Chebyshev polynomial of the second kind given by

$$U_m(\cos\vartheta) = \frac{\sin((m+1)\vartheta)}{\sin\vartheta}. \tag{17}$$

For the last mass $j=n-1$ we find

$$\bar{v}_n = \frac{\mathrm{i}(-1)^{n-1}}{s\sqrt{1+\varepsilon s}}\frac{1-\delta}{U_{2n-1}(\mathrm{i}s/(2\sqrt{1+\varepsilon s}))+(\gamma+\delta)U_{2n-3}(\mathrm{i}s/(2\sqrt{1+\varepsilon s}))+\gamma\delta U_{2n-5}(\mathrm{i}s/(2\sqrt{1+\varepsilon s}))}. \tag{18}$$

With the results (16) and (18) the velocities of each mass due to a step-loading at the left mass are determined. For $c=0$, i.e., $\varepsilon=0$, the undamped results as given in [3] are confirmed. The responses in the time domain, the location of eigenfrequencies, the effects of damping, etc., are easily calculated from (16) and (18) by using the program MAPLE.

3.1.2. *Results*

The results shown in the Figs. 2–6 have been derived for the parameter values $m = 1$ kg, $k = 1$ N/m, and the stepforce is chosen such that the initial acceleration of the boundary mass m_1 is $a=1$ m/s^2. For the velocities the time scale as defined by (3) must be noted and the amplitudes of the fluctuations have to be multiplied by $\sqrt{m/k}$.

For several cases of a nonuniform mass distribution the velocities of the first, fifth, and last mass in a cascade of $n=10$ have been calculated as an example. The first case may model a train with a locomotive with a mass which is twice the other masses, i.e., $\gamma=0.5$ and $\delta=0$. The value of $\varepsilon=0.1$ yields a considerable (yet still undercritical) damping.

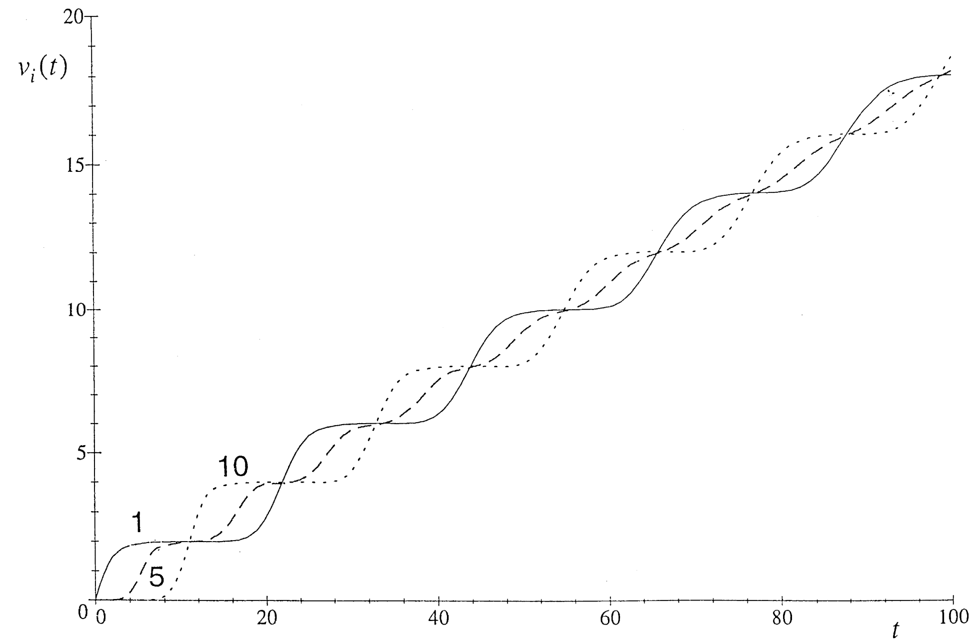

Fig. 2. Velocities of the masses 1, 5, 10; $m_1 = 2m, \varepsilon = 0.1$.

The velocities have been depicted in Fig. 2, showing that the high-frequency fluctuations immediately after each wave reflection, as found in [3], are now damped out. After several travelling times of the resulting wave in the cascade the system behaves more and more as a rigid body. For increasing damping this process is speeded up as expected. For larger values of the first mass the negative accelerations, as found in the undamped systems in [3], are still present.

The following results are concerned with the total forces (elastic and frictional) between the first and second mass, the fifth and sixth mass and the ninth and last mass, as depicted in Fig. 3. The parameter values are as before $\gamma = 0.5$, $\delta = 0$, $\varepsilon = 0.1$. The results show that the fluctuations are severe in the middle of the cascade during the first few travelling times of the compressional waves, despite the relatively high value of the damping. For large values of t the rigid-body movement of the train is confirmed. In Fig. 4 the frequency-response functions are given for the first, the fifth and the last mass, showing a phenomenon already found in [8] that higher frequencies are filtered out towards the end of a discrete cascade, owing to the dispersive nature of the (discrete) cascade.

We postpone a quantitative discussion of the location of the eigenfrequencies of an undercritically damped finite cascade to the next section. Here we will discuss the qualitative aspects of the location of the damped eigenfrequencies, the eigenmodes and the critical damping.

The eigenfrequencies, the damping of each mode as well as the critical damping can be found by calculating the zeros of the polynomial

$$U_{2n-1}(z) + (\gamma + \delta)U_{2n-3}(z) + \gamma\delta U_{2n-5}(z), \tag{19}$$

occurring in the denominator of (16). In [2] it has been shown that the zeros of (19) are all real and different and include 0, whereas the other zeros are located symmetrically around 0 in the interval

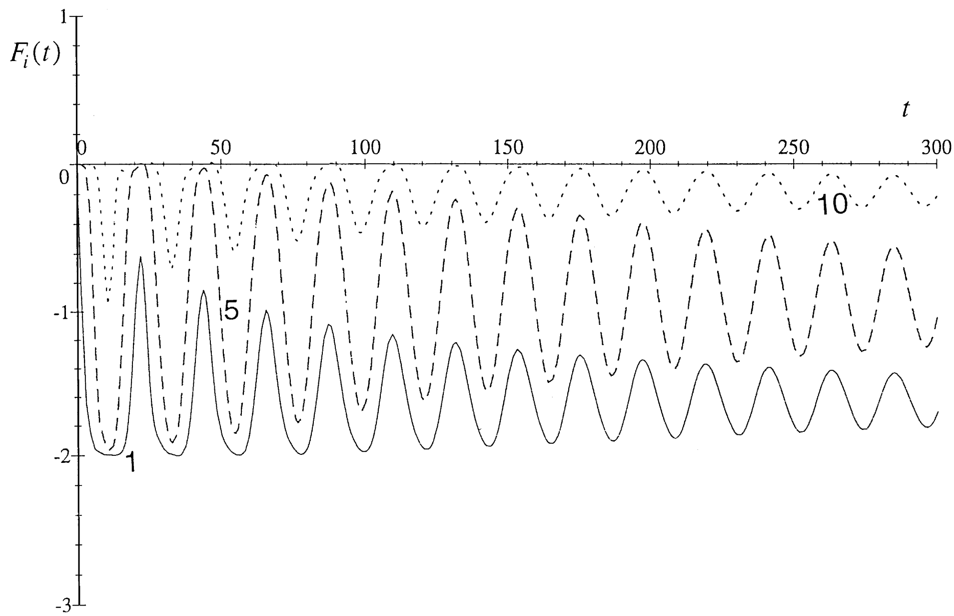

Fig. 3. Total forces between the masses 1,2; 5,6; 9,10; $m_1 = 2m, \varepsilon = 0.1$.

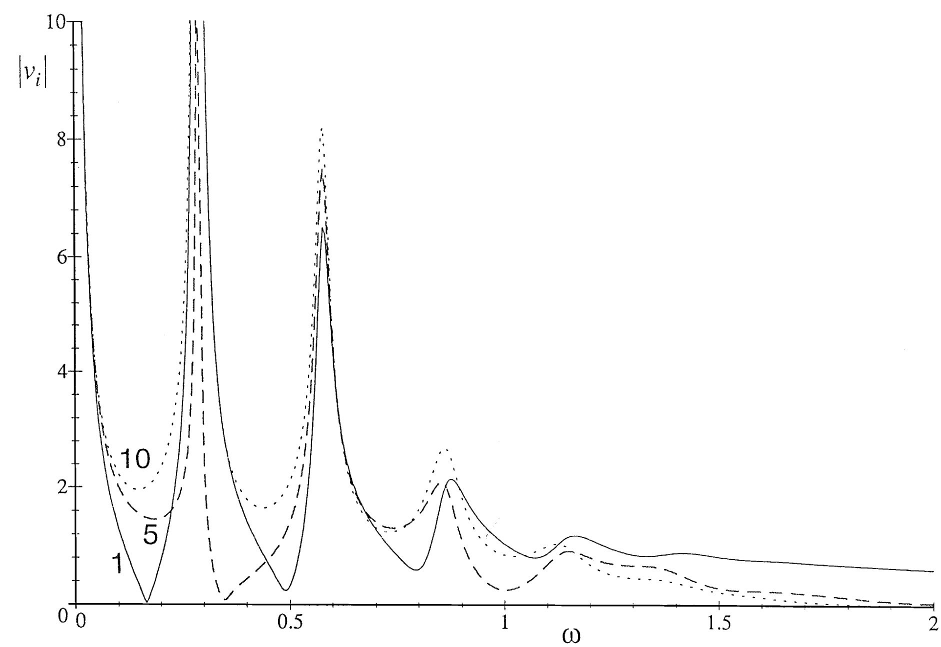

Fig. 4. Frequency-response functions of masses 1, 5, 10; $m_1 = 2m, \varepsilon = 0.1$.

$[-1,1]$ provided that $-1 \leqslant \gamma < 1, -1 \leqslant \delta < 1$. Let z_j $(j = 1, 2, \ldots, n-1)$ denote the positive zeros of (19) in increasing order. Since

$$z = \frac{\mathrm{i}s}{2(1+\varepsilon s)^{1/2}}, \tag{20}$$

we can calculate the corresponding complex conjugate roots s_j as

$$s_{j1,2} = -2\varepsilon z_j^2 \pm 2\mathrm{i}z_j(1-\varepsilon^2 z_j^2)^{1/2}.$$

For an undercritically damped case of the mode j, i.e., $\varepsilon < 1/z_j$, the imaginary part of s_j is the damped eigenfrequency, whereas the real part of s_j represents the damping coefficient of the eigenmodes of the form

$$A_j \mathrm{e}^{-\beta_j t} \cos(\omega_j t + \phi_j)$$

with $\omega_j = 2z_j(1-\varepsilon^2 z_j^2)^{1/2}$ and $\beta_j = 2\varepsilon z_j^2$. Note that $|s_j| = 2z_j$, i.e., all eigenfrequencies are located as complex conjugates on semi-circles in the left half of the complex plane.

For the critical damping of mode j, i.e., $\varepsilon = 1/z_j$, both roots coincide. It follows that the critical damping decreases with increasing mode, as known from practice in finite element calculations.

For $\varepsilon > 1/z_j$ the mode j will be overcritically damped, resulting in real roots $s_{1,2}$, the product of which is $4z_j^2$,

3.2. Uniform mass-distribution

The case elaborated here is concerned with a uniform distribution of mass (m), stiffness (k) and viscous friction (c) for a finite cascade (size n). Thus we have $\gamma = 0$, $\delta = 0$. The eigenfrequencies are derived from the zeros of (19) with these values of the parameters resulting in $U_{2n-1}(z) = 0$. Substitution of (20) in the definition of the Chebyshev polynomials (17) gives the eigenfrequency of the damped mode j as

$$\omega_j = 2(k/m)^{1/2} \cos\alpha_j (1-\varepsilon^2 \cos^2\alpha_j)^{1/2} \quad \text{with} \quad \alpha_j = \frac{(n-j)\pi}{2n}, \quad j = 1, \ldots, n-1$$

and the damping coefficient of mode j as

$$\beta_j = 2\frac{c}{m}\cos^2\alpha_j, \quad j = 1, \ldots, n-1.$$

Critical damping of mode j will occur when the value of ε is the inverse of the value of $\cos\alpha_j$. Again the critical damping is decreasing with increasing eigenfrequency, so with increasing values of the damping the higher modes will be critically damped before the lower modes.

For the undamped case the wave length λ_j of each of the eigenmodes can be determined from the phase-velocity c_f and the wave number κ. For the undamped case, see [8], this results in

$$\lambda_j = \frac{2\pi}{\kappa_j} = \pi d \arcsin \frac{\omega_j}{\omega_m},$$

where $\omega_m = 2(k/m)^{1/2}$ and d is the distance between the masses. For undercritically damped cases the wave number will be complex and the length of the eigenmodes will increase compared to the undamped case. The critically damped and overcritically damped eigenmodes can easily be derived.

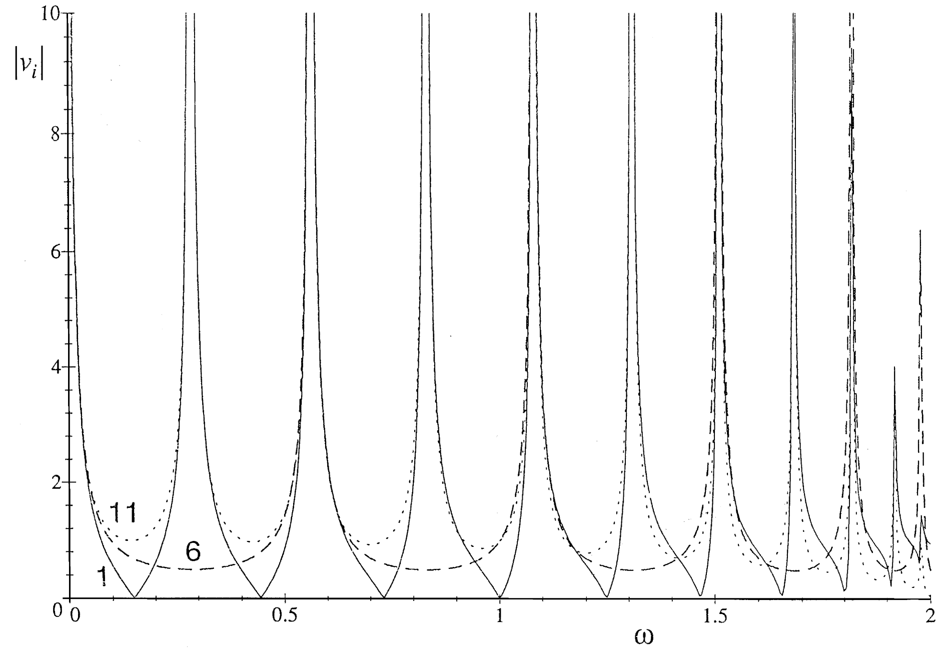

Fig. 5. Frequency-response functions of masses 1, 6, 11; $m_1 = m$, $\varepsilon = 0.001$.

Since the loading of the system is a step-force the frequency-response function of each mass can be computed by multiplying formula (16) by s and substituting $s = \mathrm{i}\omega$. In this way we have the transform of the response due to a pulse loading, which is the frequency-response function. Elaborating this for the case investigated shows a frequency-response function ($m=1, k=1$) of the form

$$H_{j+1}(\omega) = \frac{\mathrm{i}(-1)^j}{\sqrt{1+\mathrm{i}\varepsilon\omega}} \frac{U_{2n-2j-2}(\omega/(2\sqrt{1+\mathrm{i}\varepsilon\omega}))}{U_{2n-1}(\mathrm{i}s/(2\sqrt{1+\mathrm{i}\varepsilon\omega}))}, \quad j = 0, 1, \ldots, n-1. \tag{21}$$

In the case $n = 11$ the frequency responses of the first, sixth and last mass have been depicted for two different values of ε (0.001 and 0.1) in Fig. 5 and Fig. 6, respectively. The figures again show the phenomenon that higher frequencies are filtered out towards the end of a discrete cascade. From (16) it follows that in the case of an odd number of masses n we have for the middle ($j = (n-1)/2$)

$$\frac{U_{n-1}(\cos\vartheta)}{U_{2n-1}(\cos\vartheta)} = \frac{\sin n\vartheta}{\sin 2n\vartheta} = \frac{1}{2\cos n\vartheta},$$

which by (21) explains that in Figs. 5 and 6 the sixth mass only shows half of the eigenfrequencies of the other masses. By symmetry of the system this can easily be explained physically.

The eigenfrequencies, eigenmodes and frequency-response functions have been verified by using Modal-analysis calculations for the case of a fixed boundary at the right side ($\delta = 1$), showing the same results, within numerical errors. For this case further various numerical calculations have been performed as well by using a discrete element program with a Newmark-β solver also showing similar results for the cases inverstigated, see [11]. Numerical dispersion again turned out to be negligible, as already found in [8].

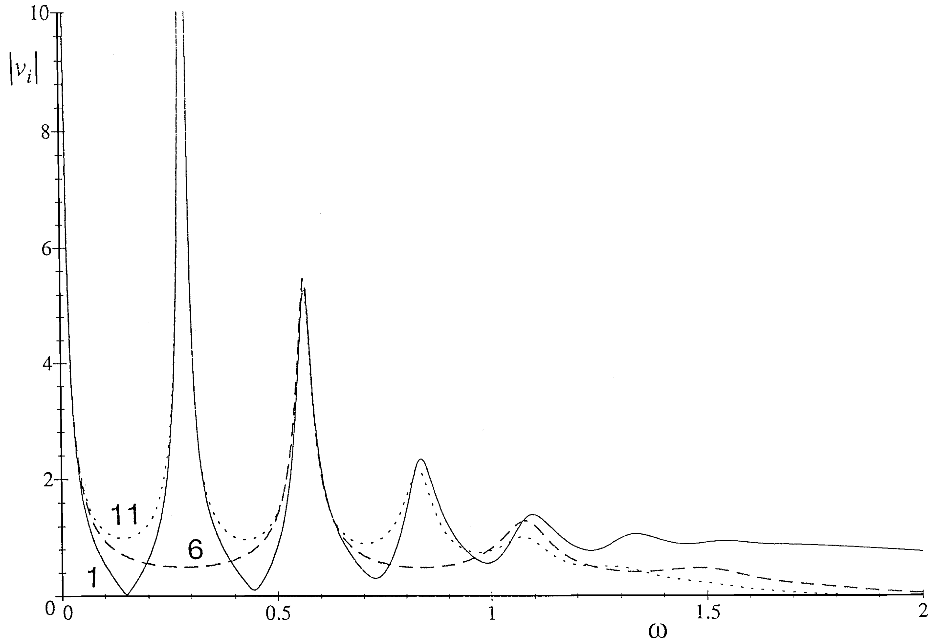

Fig. 6. Frequency-response functions of masses 1, 6, 11; $m_1 = m$, $\varepsilon = 0.1$.

4. The semi-infinite case with $\gamma = 0$

If we consider a semi-infinite system of a damped mass–spring system, then the following equations are applicable for the first and other masses, respectively,

$$s^2\bar{v}_1(s) - 1 = (1 + \varepsilon s)(\bar{v}_2(s) - \bar{v}_1(s)), \tag{22}$$

$$s^2\bar{v}_j(s) = (1 + \varepsilon s)(\bar{v}_{j-1}(s) - 2\bar{v}_j(s) + \bar{v}_{j+1}(s)), \quad j = 2, 3, \ldots \tag{23}$$

or after substitution of (13)

$$(\beta^{-2} + \beta^2 - 1)\bar{v}_1(s) - \left(\frac{2}{\sqrt{\varepsilon^2(\beta^{-1} - \beta)^2 + 4} + \varepsilon(\beta^{-1} - \beta)}\right)^2 = \bar{v}_2(s),$$

$$(\beta^{-2} + \beta^2)\bar{v}_j(s) = \bar{v}_{j-1}(s) + \bar{v}_{j+1}(s), \quad j = 2, \ldots .$$

Looking for a solution in the form (15), then $D(\beta) = 0$ and we obtain

$$C(\beta) = \frac{1}{\beta^{-2} - 1}\left(\frac{2}{\sqrt{\varepsilon^2(\beta^{-1} - \beta)^2 + 4} + \varepsilon(\beta^{-1} - \beta)}\right)^2 .$$

The solution becomes

$$\bar{v}_j = \frac{1}{\beta^{-1} - \beta}\left(\frac{2}{\sqrt{\varepsilon^2(\beta^{-1} - \beta)^2 + 4} + \varepsilon(\beta^{-1} - \beta)}\right)^2 \beta^{2j-1}.$$

Noticing that

$$\beta = \frac{\sqrt{s^2+4\varepsilon s+4}-s}{2\sqrt{1+\varepsilon s}},$$

we can express the velocity of the jth mass $\bar{v}_j$ as a function of s

$$\bar{v}_j(s) = \frac{1}{s(1+\varepsilon s)^j}\left(\frac{\sqrt{s^2+4\varepsilon s+4}-s}{2}\right)^{2j-1}$$

and for the Laplace transform of the acceleration a_j of the jth mass we have

$$\bar{a}_j(s) = \frac{1}{(1+\varepsilon s)^j}\left(\frac{\sqrt{s^2+4\varepsilon s+4}-s}{2}\right)^{2j-1}. \tag{24}$$

For $\varepsilon = 0$, we retrieve our earlier result (see [4]) by means of [1, formula 29.3.58]

$$a_j(\tau) = \frac{2j-1}{\tau}J_{2j-1}(2\tau)$$

and for $\varepsilon = 1$ the system is critically damped and we obtain as response due to a step force

$$a_j(\tau) = \mathrm{e}^{-\tau}\frac{\tau^{j-1}}{(j-1)!}.$$

For the general case ($\varepsilon \neq 0$) we rewrite (24)

$$\bar{a}_j(s) = \frac{1}{(1+\varepsilon s)^j}\sum_{r=0}^{2j-1}\binom{2j-1}{r}\left(\frac{\sqrt{(s+2\varepsilon)^2+4(1-\varepsilon^2)}-(s+2\varepsilon)}{2}\right)^{2j-1-r}\varepsilon^r,$$

which leads in the variable τ to the convolution

$$a_j(\tau) = \frac{(\varepsilon\tau)^{j-1}}{(j-1)!}\mathrm{e}^{-\tau/\varepsilon} + \sum_{r=0}^{2j-2}\binom{2j-1}{r}\varepsilon^r(1-\varepsilon^2)^{(2j-1-r)/2}$$
$$\times\left[\frac{\tau^{j-1}}{\varepsilon^j(j-1)!}\mathrm{e}^{-\tau/\varepsilon} * \mathrm{e}^{-2\varepsilon\tau}\frac{2j-1-r}{\tau}J_{2j-1-r}(2(1-\varepsilon^2)^{1/2}\tau)\right].$$

The solution for $\varepsilon = 1$ is easily verified as well as for the case $\varepsilon = 0$, which has been treated in [3].

In Fig. 7 the accelerations of the first, fifth and eleventh mass due to a step-loading of a finite cascade ($n = 11$) for $\varepsilon = 1$ have been shown. In Fig. 8 the same accelerations are depicted for a semi-infinite cascade with critical damping ($\varepsilon = 1$).

A comparison of the two solutions shows the typical differences between them. In the finite cascade the acceleration of each of the bodies converges to a finite value of $a = 0.09$ as to be expected from a rigid-body calculation. In the semi-infinite cascade the accelerations converge to zero after passing of the wavefront due to the step-load, as may be understood from a continuous one-dimensional system, where a constant velocity would result. Note that the amplitude of the acceleration of the eleventh (last) mass in the finite cascade is two times higher than in the semi-infinite cascade, due to the reflection at the boundary.

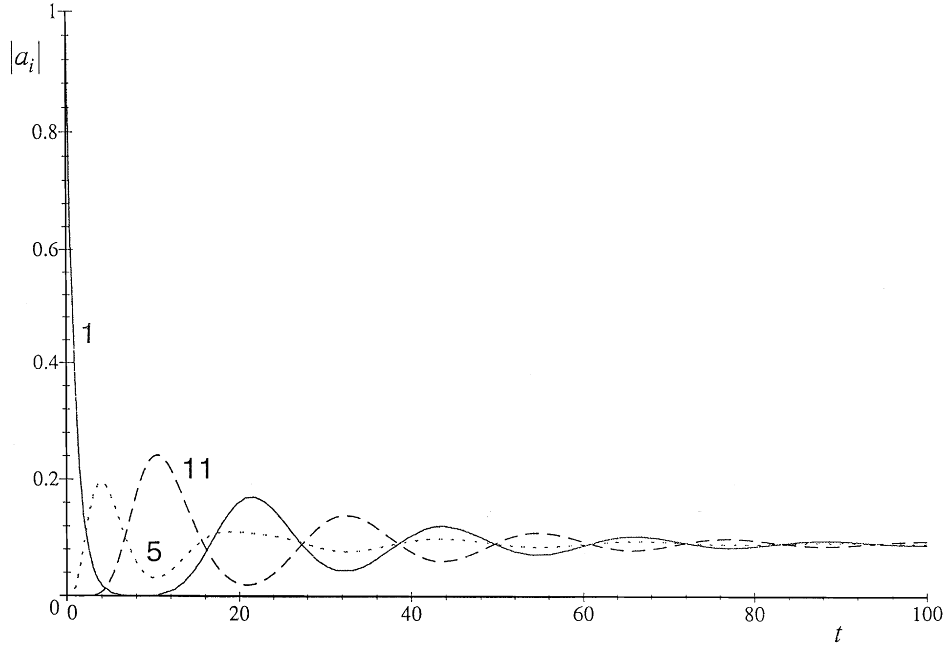

Fig. 7. Uniform finite cascade, accelerations of masses 1, 5, 11; $\varepsilon = 1$.

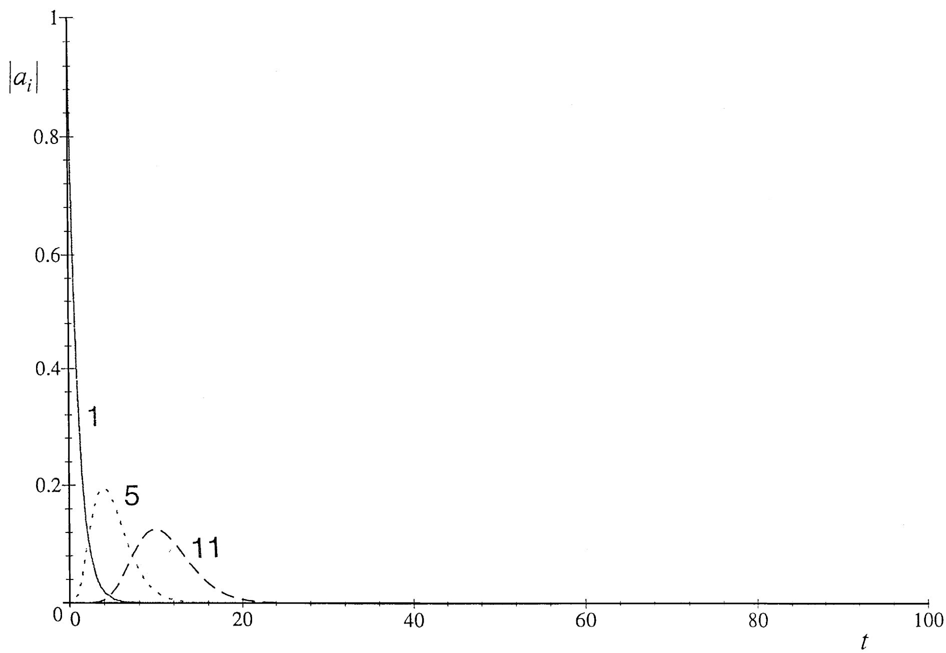

Fig. 8. Semi-infinite cascade, accelerations of masses 1, 5, 11; $m_1 = m$, $\varepsilon = 1$.

5. Conclusions

The responses due to step-forces of finite and semi-infinite cascades of damped mass–spring systems with a nonuniform mass-distribution at the boundaries have been derived analytically. The resulting expressions can be used to calculate the response due to a pulse loading and hence the dynamic behaviour of various cascades by using the convolution principle. Velocities, internal forces and frequency-response functions have been calculated by means of the computer program MAPLE. Especially the total forces in the connections in the middle of a cascade show severe fluctuations after the step-loading has been applied even in a relatively strong (undercritically) damped case.

The location of eigenfrequencies, the matching eigenmodes and critical damping of each mode have been derived qualitatively for cascades with a non-uniform mass-distribution at the boundaries. The analysis shows that each mode has its own critical damping, which is equal to the inverse of the corresponding zero of the linear combination of Chebyshev polynomials occurring in the denominator of the step-response functions. The critical damping per mode is decreasing with increasing eigenfrequency.

For the semi-infinite case the response due to a step-loading has been found in the form of a convolution integral. The solution for the case of critical damping of the whole cascade gives a very simple expression. For an undamped cascade the solution is equal to the results found in [3].

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