Spectral Flow, Eta Invariants, and von Neumann Algebras

VARGHESE MATHAI

Department of Pure Mathematics, The University of Adelaide, Adelaide, South Australia

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In this paper, we prove that the spectral flow of a path of tangential signature operators on a closed odd dimensional manifold parametrized by unitary representations of the fundamental group is a homotopy invariant of the manifold and the path in the space of unitary representations, for a large class of fundamental groups with the help of results due to Weinberger (Proc. Nat. Acad. Sci. U.S.A. 85 (1988), 5362-5363). We also prove that an eta invariant for normal covering spaces introduced by Cheeger and Gromov (J. Differential Geom. 21 (1985), 1-34) and defined using the von Neumann trace, is a homotopy invariant for Bieberbach groups. Finally, we compute this invariant for closed locally symmetric spaces, and obtain the surprising result that the eta invariant for such spaces is a differential invariant. © 1992 Academic Press, Inc.

INTRODUCTION

In this paper, we study the homotopy invariance of several analytically defined invariants on closed manifolds. These have been shown to be differential invariants by their inventors, but we determine under some assumptions on the fundamental group, when these differential invariants are in fact homotopy invariants.

In the first section, we study the spectral flow of a path of tangential signature operators on an odd dimensional, closed manifold, parametrized by unitary representations of the fundamental group of the manifold. We show that the spectral flow is a homotopy invariant of the manifold and the path in the space of unitary representations of the fundamental group, for many classes of fundamental groups, with the help of results due to Weinberger [13]. Next we describe the regularization procedure for the signature of Hermitian forms invariant under the action of a finite von Neumann algebra, which we use in the rest of the paper. We then prove that an eta invariant for normal covering spaces which was introduced by Cheeger and Gromov [4] and defined using the von Neumann trace is a differential invariant.
homotopy invariant for Bieberbach groups. Finally we compute this invariant for closed locally symmetric spaces with the help of results due to Moscovici and Stanton [10], and we get the surprising result that the eta invariant of a locally symmetric space is a differential invariant. (In general, only a difference of eta invariants is a differential invariant). In this case, we conjecture that the eta invariant is actually a homotopy invariant.

1. SPECTRAL FLOW AND REDUCED ETA INVARIANTS

Let $\pi$ be a discrete group and let $X$ be a smooth, compact, connected, oriented manifold of dimension $2n - 1$. Fix a homomorphism $h: \pi_1(X) \to \pi$ where $\pi_1(X)$ is the fundamental group of $X$. Let $\beta: \pi \to U(p)$ be a unitary representation of $\pi$ and let $\alpha: \pi_1(X) \to U(p)$ be the unitary representation of $\pi_1(X)$ which factors through $\pi$. Given a Riemannian metric on $X$, one can define the tangential signature operator, also known as the Atiyah, Patodi, and Singer (APS) operator $B_{\alpha}$ as

$$(B_{\alpha})\phi = i^n(-1)^{j+1}(\ast d - d \ast)\phi, \quad \phi \in \Omega^{2j}(X, E,),$$

where $E, \phi$ is the flat unitary vector bundle over $X$ determined by $\alpha$ and $\Omega^{\ast}(X, E,)$ is the space of differential forms on $X$ with coefficients in $E, \phi$.

It is easily seen that $B_{\alpha}$ is an elliptic, self-adjoint differential operator acting on $\Omega^{even}(X, E, \phi)$. APS [2, Part II] define an invariant

$$\tilde{\eta}(X, \alpha, \alpha') = (\eta_{\alpha}(0) - \eta_{\alpha'}(0)),$$

where $\alpha': \pi_1(X) \to \pi \to U(p)$ is the unitary representation of $\pi_1(X)$ which factors through $\pi$. Here $\eta_{\alpha}(0)$ and $\eta_{\alpha'}(0)$ are the eta invariants of the elliptic, self-adjoint differential operators $B_{\alpha}$ and $B_{\alpha'}$, respectively.

Note that $\tilde{\eta}(X, \alpha, \alpha')$ takes values in $\mathbb{R}$. Here

$$\eta_{\alpha}(0) = \frac{1}{2} \left\{ \sum_{\lambda \neq 0} \sign \lambda |\lambda|^{-s} |_{s = 0} + \dim \ker B_{\alpha} \right\}, \quad (*)$$

where $\lambda \in \text{spectrum } (B_{\alpha})$ and the right hand side of $(*)$ is defined via meromorphic continuation. APS [2, Part II] define an invariant which we misname the reduced eta invariant by

$$\tilde{\xi}(X, \alpha, \alpha') = (\eta_{\alpha}(0) - \eta_{\alpha'}(0)) \quad (\text{mod } 1).$$

Note that $\tilde{\xi}(X, \alpha, \alpha')$ takes the values in $\mathbb{R}/\mathbb{Z}$. They show that the right hand side of $(*)$ has no pole at $s = 0$ and they show that $\tilde{\eta}(X, \alpha, \alpha')$ is independent of the metric on $X$, i.e., $\tilde{\eta}(X, \alpha, \alpha')$ is a differential invariant of
(X, \alpha, \alpha'). They also show that the reduced eta invariant \( \xi(X, \alpha, \alpha') \) is independent of the metric on \( X \) and is a cobordism invariant of \((X, \alpha, \alpha')\), i.e., \( \xi(X, \alpha, \alpha') = 0 \) if \( X \) is the boundary of \( Y \), where \( Y \) is a smooth, compact, connected, oriented manifold with boundary and \( \alpha, \alpha' \) extending to unitary representations of \( \pi_1(Y) \).

We can formulate the problem of the homotopy invariance of \( \eta(X, \alpha, \alpha') \) and \( \xi(X, \alpha, \alpha') \) as follows: Let \( g: Y \to X \) be a homotopy equivalence of smooth, compact, connected, oriented, \( 2n-1 \) dimensional manifolds. If we identify the fundamental group of \( Y \) with the fundamental group of \( X \) via the isomorphism induced by \( g \), then is it true that \( \eta(X, \alpha, \alpha') = \eta(Y, \alpha, \alpha') \) and \( \xi(X, \alpha, \alpha') = \xi(Y, \alpha, \alpha') \) for all unitary representations \( \alpha \) and \( \alpha' \) of \( \pi_1(X) \) which factor through \( \pi \) via \( h \)? By the formula given in [2, Part II] for the eta invariant of lens spaces, it can be seen that both \( \eta(X, \alpha, \alpha') \) and \( \xi(X, \alpha, \alpha') \) are different on lens spaces which are homotopy equivalent but which are not diffeomorphic. So neither of \( \xi(X, \alpha, \alpha') \), \( \eta(X, \alpha, \alpha') \) is a homotopy invariant in this case.

We establish the homotopy invariance of the reduced eta invariant \( \xi(X, \alpha, \alpha') \) under the assumption that the unitary representation \( \beta: \pi \to U(p) \) can be connected to the unitary representation \( \beta': \pi \to U(p) \) via a smooth path \( \gamma \) in the space \( R = \text{Hom}(\pi, U(p)) \). Under the same assumptions as above, we prove that \( \eta(X, \alpha, \alpha') \) is a homotopy invariant of \((X, \alpha, \alpha')\) if and only if the spectral flow of the family of APS operators parametrized by \( \gamma \) is a homotopy invariant of \((X, \gamma)\). Hence with the help of results due to Weinberger [13], we are able to prove that this spectral flow is a homotopy invariant of \((X, \gamma)\) for a large class of groups \( \pi \). Our main construction is that of a generalized Poincaré Hermitian vector bundle over \( B\pi \times R \).

We thank S. Weinberger and D. G. Quillen for some helpful comments on this section.

We first construct our generalized Poincaré Hermitian vector bundle. Let \( E \pi \to B \pi \) be a principal \( \pi \) bundle over the paracompact space \( B \pi \) with contractible total space \( E \pi \). Then \( B \pi \) is called the classifying space of the group \( \pi \). Let \( f: X \to B \pi \) be a continuous map classifying the \( \pi \) covering \( \tilde{X} \times_{\pi} \pi \), where \( \tilde{X} \) is the universal cover of \( X \). We construct a tautological rank \( p \) Hermitian vector bundle \( F \to B \pi \times R \) over \( B \pi \times R \) as follows.

Consider the action of \( \pi \) on \( E \pi \times R \times \mathbb{C}^p \) given by

\[
(E \pi \times R \times \mathbb{C}^p) \times \pi \to E \pi \times R \times \mathbb{C}^p \\
((q, \alpha, v), \tau) \to (q\tau, \alpha, \alpha(\tau)v).
\]

Define the Poincaré Hermitian vector bundle \( F \) over \( B \pi \times R \) to be \( (E \pi \times R \times \mathbb{C}^p) / \pi \). \( F \) has the property that \( F|_{B \pi \times x} \) is the flat Hermitian vector bundle over \( B \pi \) defined by \( x \). Let \( I \) denote the closed unit interval \([0, 1]\)
and $\gamma: I \to R$ be a smooth path in $R$ joining the unitary representation $\beta$ and the representation $\beta'$. Define $E = (f \times \gamma)^* F \to X \times I$ to be the Hermitian vector bundle over $X \times I$. By the Kunneth theorem in cohomology, we have $\text{ch}(F) = \sum_i x_i \xi_i$, where $\text{ch}(F)$ is the Chern character of $F$, $x_i \in H^*(B\pi, \mathbb{R})$, and $\xi_i \in H^*(\mathbb{R}, \mathbb{R})$. It follows that if $y_i = f^*(x_i)$ and $\mu_i = \gamma^*(\xi_i)$, then $\text{ch}(E) = \sum_i y_i \mu_i$. Note that the pullback connection makes $E$ into a Hermitian vector bundle over $X \times I$. Let $D$ be the signature operator on the Riemannian manifold $X \times I$ with the product metric, and $D \otimes E$ be the signature operator on $X \times I$ with coefficients in the Hermitian vector bundle $E$. Let $\text{ind}(D \otimes E)$ denote the index of the operator $D \otimes E$ given the global boundary conditions of [2, Part I]. Then we have

**Lemma 1.1.** (1) $\tilde{\eta}(X, \alpha, \alpha') = \sum_i \int_X L(X) y_i \int_I \mu_i - \text{sf}([B \otimes \alpha_i]_{i \in I})$, where $\int_I$ picks up the degree one component in $H^*(\mathbb{R}, \mathbb{R})$ and $\text{sf}([B \otimes \alpha_i]_{i \in I})$ denotes the spectral flow of the family of operators $[B \otimes \alpha_i]_{i \in I}$. (Here $\alpha_i = \gamma(t) \circ h$ for $t \in I$).

(2) $\overline{\eta}(X, \alpha, \alpha') = \sum_i \int_X L(X) y_i \int_I \mu_i (\text{mod } 1)$.

**Proof.** By the APS signature theorem for manifolds with boundary [2, Part I], we see that

$$\text{ind}(D \otimes E) = \int_{X \times I} L(X) \text{ch}(E) (\eta_\alpha(0) - \eta_{\alpha'}(0)), \quad (+)$$

since $l_*(X \times I) = L(X)$. Here $L(X)$ is the $L$-characteristic class of $X$. By the discussion above we see that

$$\int_{X \times I} L(X) \text{ch}(E) = \sum_i \int_X L(X) y_i \int_I \mu_i.$$

From [2, Part III], it follows that $\text{ind}(D \otimes E)$ is the spectral flow of the family $[B \otimes \alpha_i]_{i \in I}$.

By reducing equation $(+)$ modulo one, we obtain (2).

**Definition.** We let $\text{sf}(X, \gamma)$ be the spectral flow of the family of self adjoint operators $([B \otimes \alpha_i]_{i \in I})$ (which can be thought of as the net number of eigenvalues of that family which cross zero).

By the results of [2, Part II] and Lemma 1.1 Part (1), $\text{sf}(X, \gamma)$ is independent of the choice of metric on $X$, hence it is well defined. We also observe that $\text{sf}(X, \gamma)$ depends only on the homotopy class of the path $\gamma$ (with fixed endpoints) and that the spectral flow along the reverse path of $\gamma$ is the negative of the spectral flow along $\gamma$. 


THEOREM 1.2. Let $X$ be smooth, compact, connected, oriented manifold of odd dimension. Let $\pi$ be a discrete group and $\beta: \pi \to U(p)$ be any unitary representation which can be connected by a smooth path $\gamma$ to the unitary representation $\beta': \pi \to U(p)$ in the space $\mathbb{R}$. Then for unitary representations $\alpha: \pi_1(X) \to U(p)$ and $\alpha': \pi_1(X) \to U(p)$ (where $\beta$ and $\beta'$ are as described above) of $\pi_1(X)$ which factor through $\pi$,

(1) $\zeta(X, \alpha, \alpha')$ is a homotopy invariant of $(X, \alpha, \alpha')$;

(2) $\bar{\eta}(X, \alpha, \alpha')$ is a homotopy invariant of $(X, \alpha, \alpha')$ if and only if $\text{sf}(X, \gamma)$ is a homotopy invariant of $(X, \gamma)$;

(3) if $\gamma$ and $\gamma'$ are two paths in $\mathbb{R}$ joining $\beta$ and $\beta'$ as above, then $\text{sf}(X, \gamma) - \text{sf}(X, \gamma')$ is a homotopy invariant of $X$.

Proof. (1) We observe that the unitary connection induced on $E$ has curvature which is a multiple of $dt$, so that $\text{ch}(E) = p + c_1(E)$ where $c_1(E)$ is the first Chern class of $E$, $t$ is the variable on the interval $I$ and $p$ is the rank of $E$. It follows that $\text{ch}(E) = p + y\mu$, where $y \in H^1(X, \mathbb{R})$ and $\mu \in H^1(I, \mathbb{R})$, since $c_1(E)$ can be represented by the trace of the curvature of a unitary connection on $E$. Since $c_1(E) = (f \times \gamma)^* c_1(F)$ we see that $y = f^*(x)$ and $\mu = \gamma^*(\xi)$ for some $x \in H^1(B\pi, \mathbb{R})$ and $\xi \in H^1(R, \mathbb{R})$. Now Hsiang [7], Kasparov [8], and Lusztig [9] have proved that the Novikov conjecture holds for all one dimensional cohomology classes, i.e., $\int_X L(X) f^*(x)$ are homotopy invariants of $X$ for all $x \in H^1(B\pi, \mathbb{R})$. So the right hand side of the equality in Lemma 1.1 Part (2) depends only on the homotopy type of $X$ and the path $\gamma$ in $\mathbb{R}$.

But the left hand side of the equality in Part (2) of Lemma 1.1 does not depend on the path $\gamma$ in $\mathbb{R}$ joining the unitary representations $\beta$ and $\beta'$. This implies that the right hand side of the equality in Lemma 1.1 Part (1) depends only on the homotopy type of $X$ and the unitary representations $\alpha$ and $\alpha'$.

(2) We can prove this by a similar argument to the above, but using Part (1) of Lemma 1.1.

(3) $\text{sf}(X, \gamma) - \text{sf}(X, \gamma')$ is equal to $\text{sf}(X, \gamma'')$ where $\gamma''$ is the loop starting from $\alpha$ formed by going along the path $\gamma$ and then along the reverse path of $\gamma'$. By the equality in Part (1) of Lemma 1.1, we see that

$$\text{sf}(X, \gamma'') = \sum_i \int_X L(X) f^*(x_i) \int_t \gamma''^*(\xi_i).$$

The right hand side is a homotopy invariant of $X$ as observed in Part (1) above.

We now state the following important result due to Shmuel Weinberger.
THEOREM 1.3 (Weinberger [13]). Let \((X, \alpha, \alpha')\) be as in Theorem 1.2 above and \(\pi\) be a Bieberbach group, or the fundamental group of a complete hyperbolic manifold, or is torsion-free poly-finite-or-cyclic, or lies in the Cappell–Waldhausen class of groups. Then \(\tilde{h}(X, \alpha, \alpha')\) is a homotopy invariant of \((X, \alpha, \alpha')\).

We can now state a straightforward corollary of Theorems 1.2 and 1.3 above.

COROLLARY 1.4. Let \((X, \gamma)\) be as in Theorem 1.2 above. Then \(sf(X, \gamma)\) is a homotopy invariant of \((X, \gamma)\) if \(\pi\) is a Bieberbach group, or the fundamental group of a complete hyperbolic manifold, or is torsion-free poly-finite-or-cyclic, or lies in the Cappell–Waldhausen class of groups.

2. VON NEUMANN SIGNATURES

Let \(\mathcal{U}\) be a finite von Neumann algebra, i.e., \(\mathcal{U}\) is a von Neumann algebra with a faithful finite trace \(\tau\). Let \(l^2(\mathcal{U})\) denote the Hilbert space completion of \(\mathcal{U}\) with respect to the inner product \((a, b) \equiv \tau(a^*b)\). Let \(H\) be a separable Hilbert space and \(H_{\mathcal{U}} \equiv l^2(\mathcal{U}) \otimes H\) be the Hilbert tensor product of \(l^2(\mathcal{U})\) and \(H\). \(\mathcal{U}\) acts on \(l^2(\mathcal{U})\) as multiplication operators, called the left regular representation, and hence \(\mathcal{U}\) acts on \(H_{\mathcal{U}}\) via the left regular representation on \(l^2(\mathcal{U})\) and trivially on \(H\).

DEFINITION 2.1. A bounded operator \(A\) on \(H_{\mathcal{U}}\) is called a \(\mathcal{U}\)-operator if it commutes with the action \(\mathcal{U}\) on \(H_{\mathcal{U}}\).

A \(\mathcal{U}\)-operator \(A\) is in the commutant \(\mathcal{U}'\) of \(\mathcal{U}\) acting on \(H_{\mathcal{U}}\), and the commutant theorem [14, p. 81] says that \(\mathcal{U}'\) can be naturally identified with \(\mathcal{U} \otimes B(H)\), where \(B(H)\) denotes all bounded linear operators on \(H\). Observe that there is a natural trace on \(\mathcal{U} \otimes B(H)\) given by \(\tau \otimes \text{Tr}\), where \(\text{Tr}\) denotes the usual trace on \(B(H)\). We denote \(\tau \otimes \text{Tr}\) by \(\tau\).

(I) Assume that \(A\) is a \(\mathcal{U}\)-operator which is self-adjoint and possibly unbounded, but such that \(\tau(e^{-tA})\) and \(\psi_A(t) \equiv \tau(Ae^{-tA})\) are well defined for all \(t > 0\).

(II) Next we assume that \(\psi_A(t)\) has an asymptotic expansion near \(t = 0\) given by

\[
\psi_A(t) \sim \sum_{k=0}^{\infty} C_{\alpha_k} t^{\alpha_k},
\]

where \(\alpha_k \in \mathbb{R} - (-\frac{1}{2}) \forall k \geq 0, C_{\alpha_k} \neq 0, -\infty < \alpha_0 < \alpha_1 < \cdots < \alpha_\delta < \cdots \alpha_k \to \infty\) as \(k \to \infty\).
Consider the $L^2$-eta function of $A$ which is given formally by

$$
\eta_A(s) = \frac{1}{\Gamma((1+s)/2)} \int_0^\infty t^{(s-1)/2} \psi_A(t) \, dt.
$$

**Theorem 2.2.** $\eta_A(s)$ is uniquely defined as an analytic function near $s = 0$.

**Proof.** Let $\eta_A(s) = \eta_1(s) + \eta_2(s)$ where

$$
\eta_1(s) = \frac{1}{\Gamma((1+s)/2)} \int_0^1 t^{(s-1)/2} \psi_A(t) \, dt
$$

and

$$
\eta_2(s) = \frac{1}{\Gamma((1+s)/2)} \int_1^\infty t^{(s-1)/2} \psi_A(t) \, dt.
$$

The theorem follows from the following two lemmas.

**Lemma 2.3.** $\eta_1(s)$ is a holomorphic function for $\text{Re}(s) > -2\alpha_0 - 1$ and has a meromorphic extension to $\mathbb{C}$ with no pole at $s = 0$.

**Proof.**

$$
\eta_1(s) = \frac{1}{\Gamma((1+s)/2)} \sum_{k=0}^N \int_0^1 t^{(s-1)/2} C_{2k} t^{2k} \, dt + E_N(s)
$$

$$
= \sum_{k=0}^N \frac{2C_{2k}}{\Gamma((1+s)/2)(s+2\alpha_k+1)} + E_N(s), \quad (**)
$$

where $E_N(s)$ is analytic for $\text{Re}(s) > -2\alpha_N - 1$, hence (** is the meromorphic extension of $\eta_1(s)$. Since the Gamma function has poles on the non-positive integers, it follows that $\eta_1(s)$ has simple poles only at $-2\alpha_k - 1$, if $\alpha_k$ is not a non-negative odd integer. In particular $\eta_1(s)$ has no pole at $s = 0$, by the assumption on the powers of $t$ in the asymptotic expansion for $\psi_A(t)$.

**Lemma 2.4.** $\eta_2(s)$ is a holomorphic function in the strip $1 > \text{Re}(s) > -1$.

**Proof.**

$$
|\eta_2(s)| = \frac{1}{|\Gamma((s+1)/2)|} \left| \int_1^\infty t^{(s-1)/2} \int_1^\infty \lambda e^{-t^2} \, d\phi_A \, dt \right|
$$

$$
\leq \frac{1}{|\Gamma((s+1)/2)|} \int d\phi_A \int_1^\infty t^{(\text{Re}(s)-1)/2} |\lambda| e^{-t^2} \, dt
$$

$$
= \frac{1}{|\Gamma((s+1)/2)|} \int d\phi_A e^{-\lambda^2} \int_1^\infty t^{(\text{Re}(s)-1)/2} |\lambda| e^{-(t-1)^2} \, dt.
$$
Let \( u = (t - 1) \lambda^2 \) be a change of variable

\[
\int_{-\infty}^{\infty} \frac{1}{\Gamma((s + 1)/2)} \int_0^\infty d\rho \rho e^{-\rho^2} \int_0^\infty (u + \lambda^2)^{\frac{\text{Re}(s)}{2}} e^{-u} du
\]

and clearly \( \frac{\partial}{\partial s} \tau(s) = 0 \).

If \( A \) is a self adjoint \( \mathbb{U} \)-operator on \( H_u \), then \( A \) defines a Hermitian form

\[
Q_A(u) = (u, Au)
\]

for \( u \in H_u \).

The positive and negative spectral subspaces of \( Q_A(u) \) are Hilbert spaces which are \( \mathbb{U} \)-invariant. The von Neumann signature of \( Q_A(u) \) is defined to be the difference of the von Neumann dimensions of the positive and negative spectral subspaces of \( Q_A(u) \). This would be a finite number if \( H \) were a finite dimensional Hilbert space, but in the general case, this is essentially the number which we regularize.

**Definition 2.5.** Let \( A \) be a '2 operator which is self adjoint and satisfies conditions (I) and (II). Then the regularized von Neumann signature of its associated Hermitian form is defined to be

\[
\text{Sign}_{\tau}(A) = \eta_{\tau}(0).
\]

**Lemma 2.6.** Suppose that \( A \) is a \( \mathbb{U} \) operator which is self adjoint on \( H_u \) and satisfies conditions (I) and (II). Then

1. \( \text{Sign}_{\tau}(\lambda A) = 0 \)
2. \( \text{Sign}_{\tau}(cA) = (c/|c|) \text{Sign}_{\tau}(A) \) for \( c \in \mathbb{R} - \{0\} \)
3. If \( U \) is a unitary in \( \mathbb{U}' \), then \( \text{Sign}_{\tau}(U^*AU) = \text{Sign}_{\tau}(A) \).

**Proof.** Result (1) follows from the definition of \( \text{Sign}_{\tau}(A) \), while result (2) follows from the fact that \( \eta_{\tau}(s) = (c/|c|)^{s+1} \eta_{\tau}(s) \). For result (3), note that if \( U \) is a unitary in \( \mathbb{U}' \), then \( \text{Sign}_{\tau}(U^*AU) = \text{Sign}_{\tau}(A) \) since the von Neumann trace \( \tau \) is invariant under conjugation by such a unitary.

3. THE VON NEUMANN ETA INVARIANT FOR COVERING SPACES

In Section 4 of their paper, Cheeger and Gromov [4] studied an eta invariant which was defined using the von Neumann trace. We will now describe this invariant in greater detail.
Let $X$ be a closed, connected, oriented manifold of dimension $2n - 1$ and $\pi \rightarrow \hat{X} \rightarrow X$ denote a normal covering space of $X$ having the discrete group $\pi$ as a structure group.

Let $L: \Omega^*_2(\hat{X}) \rightarrow \Omega^*_2(\hat{X})$ be a bounded linear operator acting on $L^2$ differential forms on $\hat{X}$ which commutes with the action of $\pi$ on $\Omega^*_2(\hat{X})$. Assume also that the Schwartz kernel $k_L$ of $L$ is continuous. Since $L$ commutes with the action of $\pi$, it follows that

$$k_L(\gamma x, \gamma y) = k_L(x, y) \quad \text{for all} \quad \gamma \in \pi,$$

i.e., $k_L(x, y)$ descends to $(\hat{X} \times \hat{X})/\pi$. Then the von Neumann trace of $L$ is given by the expression (see [11])

$$\tau(L) = \int_X \text{tr}(k_L(x, x)) \, dx.$$

Let $B$ denote the APS operator on $X$ and $\hat{B}$ denote its lift to $L^2$ differential forms of odd degree on $\hat{X}$. Define the von Neumann eta invariant

$$\eta_{(2)}(\hat{X}, \hat{g}) \equiv \frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2} \tau(\hat{B} e^{-t\hat{g}}) \, dt$$

and the associated reduced von Neumann eta invariant

$$\rho_{(2)}(\hat{X}, \pi) \equiv \eta_{(2)}(\hat{X}, \hat{g}) - \eta(X, g).$$

Then Cheeger and Gromov prove that $\rho_{(2)}(\hat{X}, \pi)$ is a differential invariant of $X$.

We will prove in this section that in fact

**Theorem 3.1.** $\rho_{(2)}(\hat{X}, \pi)$ is a homotopy invariant of $X$ if $\pi$ is a Bieberbach group.

We also conjecture that $\rho_{(2)}(\hat{X}, \pi)$ is a homotopy invariant of $X$ when $\pi$ is a torsion free discrete group. The techniques of this section to not generalize easily to other groups. It is possible that the techniques in [13] could be generalized to prove this conjecture for the same class of groups as in Theorem 1.3. However we use a different approach in a paper which is in preparation [16] in order to study this conjecture, for groups $\pi$ for which the regular representation can be connected to the trivial one by a path of weakly continuous representations.

We use the direct integral decomposition of the Bieberbach group $\pi$ to reduce the problem of the homotopy invariance of $\rho_{(2)}(\hat{X}, \pi)$ to that of $\bar{\eta}(X, \alpha, \text{id})$ for

$$\alpha_\lambda: \pi_1(X) \rightarrow \pi \rightarrow U(q)$$
a representation induced from the character $\lambda \in (\mathbb{Z}^k)^* = \mathbb{T}^k$ as below. Here $\pi_1(X) \to \pi$ denotes the projection homomorphism defining the covering $\hat{X}$ of $X$. Hence we show that Theorem 3.1 follows from Weinberger's theorem 1.3.

By the Bieberbach theorem [15], $\pi$ can be realized as a group extension

$$1 \to \mathbb{Z}^k \xrightarrow{i} \pi \xrightarrow{\rho} \Phi \to 1,$$

(*)&

where $\Phi$ is a finite group of some order $q$. Then any non-trivial 1-dimensional unitary representation $\rho$ of $\mathbb{Z}^k$ (which is parametrized by $(\mathbb{Z}^k)^* = \mathbb{T}^k$ a torus) induces a non-trivial $q$-dimensional unitary representation of $\alpha_\lambda$ of $\pi$.

$$d\lambda = \text{normalized Haar measure on } \mathbb{T}^n.$$ 

Using [14, Theorem 2, Chap. 6, Part II], and (*), we observe that

$$l^2(\pi) \simeq \int_{\lambda \in \mathbb{T}^k} \mathcal{H}_\lambda d\lambda,$$

where the above map can be written explicitly as follows:

$$f(g) \to \int_{\lambda \in \mathbb{T}^k} F(\lambda, g) d\lambda.$$

Here

$$F(\lambda, g) = \sum_{z \in \mathbb{Z}^k} \hat{\lambda}(z) f(gz) \quad \text{and} \quad f \in l^2(\pi),$$

where $(\mathcal{H}_\lambda, \alpha_\lambda)$ is the $q$-dimensional unitary representation space of $\pi$ which is induced by the character $\lambda \in (\mathbb{Z}^k)^* = \mathbb{T}^k$. Note that the regular representation and the von Neumann trace under this decomposition become

$$\rho = \int_{\lambda \in \mathbb{T}^k} \alpha_\lambda d\lambda,$$

$$\tau(\rho(\gamma)) = \int_{\lambda \in \mathbb{T}^k} \frac{1}{q} \text{tr}(\alpha_\lambda(\gamma)) d\lambda.$$

$\gamma \in \pi$, respectively.

Let $k_\gamma(x, y)$ denote the Schwartz kernel of $\hat{B}e^{-i\beta}$. Then the Schwartz kernel of the operator $(B \otimes \alpha_\lambda)e^{-i\beta(B \otimes \alpha_\lambda)^2}$ is given by

$$k_\gamma(x, y) = \sum_{\gamma \in \pi} k_\gamma(x, y \gamma) \alpha_\lambda(y).$$

(*)
The sum on the right hand side converges uniformly over compact subsets of \( \hat{X} \times \hat{X} \) using well known estimates on the Schwartz kernel of \( e^{-t\hat{B}^2} \) (see, e.g. [6]). Here \( \lambda(\gamma) = \exp(2\pi i \gamma \cdot \lambda) \), where \( \gamma \in \mathbb{Z}^k \subset \mathbb{R}^k \) and \( \lambda \in F_k \subset \mathbb{R}^k \). Here \( F \) denotes a fundamental domain for the action of \( \pi \) on \( \hat{X} \) and \( F_k \) denotes a fundamental domain for the standard action of \( \mathbb{Z}^k \) on \( \mathbb{R}^k \). Note that \( F_k \) and \( F \) are relatively compact subsets of \( \mathbb{R}^k \) and \( \hat{X} \), respectively.

**Theorem 3.2.**

\[
\tau(\hat{B} e^{-t\hat{B}^2}) = \int_{F_k} \frac{1}{q} \operatorname{Tr}((B \otimes \alpha_\lambda) e^{-t(B \otimes \alpha_\lambda)^2}) \, d\lambda. 
\]

**Proof.** The RHS of (1) is equal to

\[
\int_{F_k} \left\{ \int_{F} \frac{1}{q} \operatorname{tr}(k_{\gamma}(x, x)) \, dx \right\} \, d\lambda = \sum_{\gamma \in \pi} \int_{F} \operatorname{tr}(k_{\gamma}(x, \gamma x)) \, dx \int_{F_k} \frac{1}{q} \operatorname{tr}(\alpha_\lambda(\gamma)) \, d\lambda,
\]

where we use (*) and justify the interchange of the sum and the integral by the uniform convergence of the sum. But

\[
\int_{F_k} \frac{1}{q} \operatorname{tr}(\alpha_\lambda(\gamma)) \, d\lambda = \begin{cases} 0 & \text{if } \gamma \neq 0 \\ 1 & \text{if } \gamma = 0. \end{cases}
\]

Hence the RHS of (1) equals

\[
\int_{F} \operatorname{tr}(k_{\gamma}(x, x)) \, dx = \tau(\hat{B} e^{-t\hat{B}^2}). 
\]

**Theorem 3.3.** \( \rho_{(2)}(\hat{X}, \pi) = \int_{F_k} \eta(X, \alpha_\lambda, \text{id}) \, d\lambda. \)

**Proof.**

\[
\eta_{(2)}(\hat{X}, \hat{g}) = \frac{1}{\Gamma(1/2)} \int_{0}^{\infty} t^{-1/2} \tau(\hat{B} e^{-t\hat{B}^2}) \, dt
\]

\[
= \frac{1}{\Gamma(1/2)} \int_{0}^{\infty} t^{-1/2} \left( \int_{F_k} \frac{1}{q} \operatorname{Tr}((B \otimes \alpha_\lambda) e^{-t(B \otimes \alpha_\lambda)^2}) \, d\lambda \right) \, dt,
\]

which by Theorem 3.2

\[
= \int_{F_k} \frac{1}{q} \left\{ \int_{0}^{\infty} t^{-1/2} \operatorname{Tr}((B \otimes \alpha_\lambda) e^{-t(B \otimes \alpha_\lambda)^2}) \, dt \right\} \, d\lambda,
\]

where we justify the interchange of integrals by Fubini's theorem, since
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Tr((B \otimes \alpha_j) e^{-nt(B \otimes \alpha_j)^2}) is non-negative and \( \eta_{(12)}(\tilde{X}, \tilde{g}) < \infty \) by [4, Sect. 4] so the integrand is absolutely integrable, etc. Thus

\[
\eta_{(12)}(\tilde{X}, \tilde{g}) = \int_{F_k} \frac{1}{q} \eta_{\alpha}(0) \, d\lambda
\]

so that

\[
\rho_{(12)}(\tilde{X}, \pi) = \int_{F_k} \frac{1}{q} \eta(\lambda, \alpha, \text{id}) \, d\lambda.
\]

**Proof of Theorem 3.1.** By Weinberger's Theorem 1.3, \( \tilde{\eta}(X, \alpha, \text{id}) \) is a homotopy invariant of \((X, \alpha)\). Hence, by Theorem 3.3, \( \rho_{(12)}(\tilde{X}, \pi) \) is a homotopy invariant of \( X \).

4. **Calculations on Locally Symmetric Spaces**

In this section we compute the invariant \( \rho_{(12)}(\tilde{X}, \Gamma) \) for semi-simple locally symmetric spaces \( X = \Gamma \backslash G/K \) (\( K \) is a maximal compact subgroup of \( G \), and \( \Gamma \) is a torsion-free cocompact discrete subgroup of \( G \)) which have the property that the dimension of \( G/K \) is odd. With the help of results due Moscovici and Stanton [10], we are able to express the \( L^2 \)-invariant explicitly in terms of a Selberg type zeta function.

Let \( Y \) be a closed Riemannian manifold such that a normal covering \( \tilde{Y} \) has an orientation reversing diffeomorphism \( \theta \) such that the composition \( \theta \circ \cdots \circ \theta \) is the identity on \( \tilde{Y} \). Let \( g \) be a metric on \( Y \) such that the induced metric \( \tilde{g} \) on \( \tilde{Y} \) is \( \theta \) invariant. Let \( \mathcal{M}_Y \) denote the space of such metrics on \( Y \). Clearly \( \mathcal{M}_Y \) is not empty, since given any metric \( g \) on \( Y \), we can average the induced metric \( \tilde{g} \) on \( \tilde{Y} \) with respect to the finite group generated by \( \theta \) and push down the resulting metric to \( Y \) to obtain a metric \( \tilde{g} \in \mathcal{M}_Y \).

**Lemma 4.1.** If \( Y \) is as above and \( g \in \mathcal{M}_Y \). Then \( \eta_{(12)}(\tilde{Y}, \tilde{g}) = 0 \).

**Proof.** Since \( g \in \mathcal{M}_Y \), it follows that \( \theta \) is an orientation reversing isometry on \((\tilde{Y}, \tilde{g})\) which anti-commutes with the Tangential Signature operator \( \tilde{B} \) of Atiyah, Patodi, and Singer on \((\tilde{Y}, \tilde{g})\). Hence the result follows by observing that

\[
\tau(\tilde{B} \exp(-i\tilde{B}^2)) = \tau(\theta \tilde{B} \exp(-i\tilde{B}^2) \theta^{-1}) = -\tau(\tilde{B} \exp(-i\tilde{B}^2)).
\]

**Definition 4.2.** If \( Y \) is as described above, then it's eta invariant \( \eta(Y) \equiv \eta(Y, g) \) is independent of the choice of \( g \in \mathcal{M}_Y \) (since it then equals the differential invariant \( \rho_{(12)}(\tilde{Y}, \Gamma) \)) and hence is (an oriented) differential invariant of \( Y \).
Let $X = \Gamma \backslash G/K$ be a closed locally symmetric space. Then $\tilde{X} = G/K$ has an orientation reversing involution called the Cartan involution, and it is easy to see that there is a homogeneous metric in $\mathcal{M}_X$. Hence we obtain by Lemma 4.1, the surprising result that the eta invariant of $X$ is a differential invariant. In general, only a difference of eta invariants is a differential invariant.

**Lemma 4.3.** The $L^2$-$\rho$ invariant of the closed locally symmetric space $X$ is given by

$$\rho_{(2)}(\tilde{X}, \Gamma) = -\eta(X).$$

The following lemma follows directly from the above and [10, Lemma 4.4]:

**Lemma 4.4.** If $G$ has no simple factors locally isomorphic to $SO_o(p, q)$, $pq$ odd or $SL(3, \mathbb{R})$, then

$$\rho_{(2)}(\tilde{X}, \Gamma) = 0.$$

We now describe a Selberg type zeta function which is used by Moscovici and Stanton to compute the eta invariant on a semisimple locally symmetric space $X$. The fixed point set of the geodesic flow, acting on the unit sphere bundle $T^1X$, is a disjoint union of submanifolds $X_\gamma$, parametrized by the nontrivial conjugacy classes $[\gamma] \neq 1$ in $\Gamma = \pi_1(X)$. Each $X_\gamma$ is itself a (possibly flat) locally symmetric manifold of nonpositive sectional curvature. Let us denote by $\tilde{\mathcal{E}}(\Gamma)$ the set of those conjugacy classes $[\gamma]$ for which $X_\gamma$, has the property that the Euclidean de Rham factor of $\tilde{X}_\gamma$, is 1-dimensional. Thus, for $[\gamma] \in \tilde{\mathcal{E}}(\Gamma)$, $\tilde{X}_\gamma \cong \mathbb{R} \times \tilde{X}'_\gamma$ and the lines $\mathbb{R} \times \{x'\}$, $x' \in \tilde{X}'_\gamma$ are the axes of $\gamma$. Projected down to $X_\gamma$, they become closed geodesics, $c_\gamma$, which foliate $X_\gamma$. The space of leaves $\tilde{X}_\gamma$ turns out to be an orbifold. The eigenvalues of absolute value 1 of the linear Poincaré map $P(\gamma)$ determine a bundle $C\tilde{X}_\gamma$ over $\tilde{X}_\gamma$ (the "center" bundle), and the parallel translation along the leaves $c_\gamma$ gives rise to an orthogonal transformation $\tilde{\tau}_\gamma$ of $C\tilde{X}_\gamma$. $C\tilde{X}_\gamma$ contains the tangent bundle $T\tilde{X}_\gamma$ and we let $N\tilde{X}_\gamma$ denote the orthogonal complement of $T\tilde{X}_\gamma$ in $C\tilde{X}_\gamma$. Since $T\tilde{X}_\gamma$ corresponds to the eigenvalue 1 of $\tilde{\tau}_\gamma$, $N\tilde{X}_\gamma$ decomposes as

$$N\tilde{X}_\gamma = N\tilde{X}_\gamma(-1) \oplus \sum_{0 < \theta < \pi} N\tilde{X}_\gamma(\theta),$$

according to the other eigenvalues $-1, e^{1+i\theta}$ ($0 < \theta < \pi$).

The restriction to $X_\gamma$ of the vector bundle $A^{\text{even}}T^*X$, can be pushed down to a vector bundle $A^{\text{even}}T^*X_\gamma$ over $\tilde{X}_\gamma$, which splits into subbundles $A^{\text{even}}T^*X_\gamma^\pm$ corresponding to the eigenvalue $\pm i$ of the symbol of $B$. One
thus obtains a $\hat{\tau}_\gamma$-equivariant complex $\hat{\sigma}_\gamma^B : \Lambda^{even}T^*X^+_\gamma \to \Lambda^{even}T^*X^-_\gamma$ over $T\hat{X}_\gamma$ and, therefore, a class $[\hat{\sigma}_\gamma^B] \in K^0_\gamma(T\hat{X}_\gamma)$, the $\hat{\tau}_\gamma$-equivariant $K$-theory group (with compact supports) of $T\hat{X}_\gamma$. As in Atiyah and Singer [3], we can then form the cohomology class $\text{ch} \hat{\sigma}_\gamma^B(\hat{\tau}_\gamma) \in H^{even}(T\hat{X}_\gamma; \mathbb{C})$. By analogy with the Lefschetz formula of Atiyah and Singer [3], and using the stable characteristic classes $\chi, \eta$, and $\eta'$ defined therein, we set:

$$L(\gamma, B) = \left\{ \frac{\text{ch} \hat{\sigma}_\gamma^B(\hat{\tau}_\gamma) \mathbb{R}(N\hat{X}_\gamma(-1)) \prod_{0 < \alpha \leq \infty} \mathcal{S}^{\alpha}(N\hat{X}_\gamma(\theta)) \mathcal{S}(\hat{\tau}_\gamma)}{\det(I - \hat{\tau}_\gamma|N\hat{X}_\gamma)} \right\} [T\hat{X}_\gamma].$$

For $[\gamma] \neq 1$, the closed geodesics $c_\gamma$ in the free homotopy class associated to $[\gamma] \neq 1$ have the same length $l_\gamma$. If $[\gamma] \in \mathcal{E}_1(\Gamma)$, then $q = \frac{1}{2} \dim N\hat{X}_\gamma$ is integer and independent of $\gamma$. Also, for $[\gamma] \in \mathcal{E}_1(\Gamma)$, $\Gamma^*_\gamma = \Gamma \cap C_\gamma$, where $C_\gamma$ is the connected center of $G_\gamma$, is infinite cyclic; we let $m_\gamma = [\Gamma^*_\gamma : Z_\gamma]$, where $Z_\gamma$ is the group generated by $\gamma$ in $\Gamma$. Again for $[\gamma] \in \mathcal{E}_1(\Gamma)$, we denote by $P_h(\gamma)$ the hyperbolic part of the linear Poincaré map $P(\gamma)$. For more details, see [10].

The main result in [10] establishes that a Selberg type zeta function can be defined, initially for $\Re(s) \geq 0$, by the formula

$$\log Z(s, B) = (-1)^q \sum_{[\gamma] \in \mathcal{E}_1(\Gamma)} \frac{L(\gamma, B)}{|\det(I - P_h(\gamma))|^{1/2}} m_\gamma e^{-st},$$

and furthermore that $Z(s, B)$ has a meromorphic extension to the entire complex plane with no pole at $s = 0$. Finally we can state the result that we are interested in

**THEOREM 4.5 (Moscovici and Stanton [10]).** The eta invariant of a closed locally symmetric space $X$ as described above, can be calculated in terms of the Selberg type zeta function,

$$\eta(X, g) = \frac{1}{\pi i} \log Z(0, B).$$

Hence we can calculate the $L^2$-$\rho$ invariant in terms of the Selberg type zeta function by Theorem 4.3 and Lemma 4.1.

Let $\varphi: \pi_1(X) \to U(n)$ denote a unitary representation of the closed locally symmetric space $X$. We now show how to improve on the Corollaries 7.3 and 7.4 of [10].

**COROLLARY 4.6.** The following is an oriented differential invariant of $X$

$$\eta_\varphi(B_\varphi) = \frac{(-1)^q}{\pi i} \sum_{[\gamma] \in \mathcal{E}_1(\Gamma)} \text{Tr} \varphi(\gamma) \frac{L(\gamma, B)}{|\det(I - P_h(\gamma))|^{1/2}} m_\gamma e^{-st} \bigg|_{s = 0}. $$
Proof. By [2, Part II]
\[ \hat{\eta}_\phi(B) = \eta_\phi(X, g) - \eta_{ad}(X, g) \]
is a differential invariant. Hence by Lemma 4.3, \( \eta_\phi(B) \) is a differential invariant denoted by \( \eta_\phi(X) \). By the argument above and Theorem 7.2 in [10], we get the right hand side of the corollary. This corollary improves Corollary 7.3 of [10] where it shows only that the difference of two eta invariants is a differential invariant.

**Corollary 4.7.** Assume that \( \eta_\phi(X) \neq 0 \) for some unitary representation \( \phi \) of \( \pi_1(X) \). Then \( G \) contains factors locally isomorphic to \( SL(3, \mathbb{R}) \) or \( SO_o(p, q) \), \( pq \) odd.

**Proof.** This follows from Lemma 4.4 and the above.

**References**