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# On a Problem of Chowla

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The equation $\Sigma f(n)/n = 0$ is studied for periodic algebraically-valued functions $f$ and, in particular, a well known problem of Chowla in this context is resolved. The work depends on an application of a theorem of the first author concerning linear forms in the logarithms of algebraic numbers.

## 1. INTRODUCTION.

In a lecture at the Stony Brook conference on number theory in the summer of 1969, Chowla raised the question whether there exists a rational-valued function $f(n)$, periodic with prime period $p$, such that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0. \tag{1}$$

He proved some twenty years ago (cf. [3]) that this certainly could not hold for odd functions $f$ if $\frac{1}{2}(p-1)$ is a prime, a condition subsequently removed by Siegel, and recently he showed that the same is true for even

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functions $f$ provided only that $f(0) = 0$. In the present paper we solve this problem in general by proving that there is in fact no function $f$ with the above properties.[1] We shall indeed treat a somewhat wider problem in which $f$ is allowed to assume arbitrary algebraic values and the period is no longer restricted to primes.

By $\mathbf{Q}$ we shall mean, as usual, the rational number field. We denote by $q$ an arbitrary natural number and by $\Phi_q$ the $q$th cyclotomic polynomial. Our main theorem is as follows.

THEOREM 1. *If $f$ is a nonvanishing function defined on the integers with algebraic values and period $q$ such that* (i) $f(r) = 0$ *if* $1 < (r, q) < q$, (ii) $\Phi_q$ *is irreducible over* $\mathbf{Q}(f(1),\ldots, f(q))$, *then*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$

It will be seen that if $q$ is a prime then (i) is vacuous and if $f$ is rational-valued then (ii) holds trivially; thus, we have the solution to Chowla's problem mentioned at the beginning. Further, we observe that the theorem would become false if the condition (i) were omitted; for (1) certainly holds if, for instance, $q = p^2$, where $p$ is a prime, and $f$ is defined by

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = (1 - p^{1-s})^2 \zeta(s),$$

where $\zeta(s)$ denotes the Riemann zeta-function. When $q$ is composite it would perhaps look more natural to have

$$f(r) = 0 \qquad \text{if} \quad (r, q) > 1 \tag{2}$$

in place of (i), but our condition derives naturally from the case when $q$ is a prime and implies a slightly greater degree of generality. Also one cannot, in general, waive condition (ii); if, for instance, $\chi$, $\chi'$ are the quadratic characters mod 12 with conductors 3 and 4, respectively, then (1) holds with $f = 2\chi - \sqrt{3}\chi'$, since

$$L(1, \chi) = \frac{\pi}{2\sqrt{3}}, \qquad L(1, \chi') = \frac{\pi}{3}.$$

Other examples will be given by Theorem 2.

[1] While working on the manuscript we were informed by Professor Chowla that he had also solved the problem to the extent stated above.

There is an obvious analogy between our theorem and Dirichlet's famous result $L(1, \chi) \neq 0$ for a nonprincipal character $\chi$ mod $q$. In fact we use this result in the proof of our theorem and, indeed, the latter is a much more general statement for certain $q$, in particular when $q$ is a prime. Because of condition (ii), however, the theorem does not contain Dirichlet's result for all $q$. For let us denote by $\xi_l$ the $l$th root of unity $e^{2\pi i/l}$ and by $\mathbf{Q}_l = \mathbf{Q}(\xi_l)$ the $l$th cyclotomic field; then if $f$ is any primitive character mod $q$ condition (ii) demands that $\Phi_q$ be irreducible over $\mathbf{Q}_{\varphi(q)}$, that is

$$[\mathbf{Q}_{\varphi(q)}(\xi_q) : \mathbf{Q}_{\varphi(q)}] = \varphi(q).$$

The first of these fields is

$$\mathbf{Q}(\xi_{\varphi(q)}, \xi_q) = \mathbf{Q}(\xi_{[\varphi(q),q]}),$$

where the square brackets signify the least common multiple and which has degree

$$\varphi([\varphi(q), q]) = \frac{\varphi(\varphi(q))\,\varphi(q)}{\varphi((\varphi(q), q))};$$

the other has degree $\varphi(\varphi(q))$. Condition (ii), therefore, amounts to $\varphi((\varphi(q), q)) = 1$ or simply $(\varphi(q), q) \leqslant 2$. The only genuine case with $(\varphi(q), q) = 2$, incidentally, is $q = 4$, since for moduli $q = 2q'$ with odd $q'$ there are no primitive characters.

The failure of our theorem to cover Dirichlet's result in full is not due to unnecessary stringency in connection with condition (ii). In fact no theorem of this general nature can cover Dirichlet's result fully as becomes clear from the following assertion: *There are integers $q$ and functions $f$ periodic* mod $q$ *that take their values in the field* $\mathbf{Q}_{\varphi(q)}$ *but satisfy* (1) *and* (2). Obviously, therefore, for such $q$, the nonvanishing of $L(1, \chi)$ for a primitive character $\chi$ mod $q$ is due to properties not reflected in general statements about $\mathbf{Q}_{\varphi(q)}$, the field generated by the character values. The assertion will be verified after the formulation of Theorem 2.

The function $f = 2\chi - \sqrt{3}\chi'$ mentioned previously shows that relations of the kind

$$\sum_{\chi} a_\chi L(1, \chi) = 0,$$

where $\chi$ runs through all characters with a given modulus, can hold with algebraic $a_\chi$ not all 0. The question arises as to whether the same is true with rational $a_\chi$. We leave the question open, but note that Theorem 1 contains the following partial answers.

COROLLARY 1. *Let $(q, \varphi(q)) = 1$ and let $\chi$ run through the nonprincipal characters* mod $q$. *Then the numbers $L(1, \chi)$ are linearly independent over* $\mathbf{Q}$.

COROLLARY 2. *If $\chi_1,\ldots,\chi_k$ are any quadratic characters for which the associated primitive characters $\chi_1^*,\ldots,\chi_k^*$ are distinct then $L(1,\chi_1),\ldots,L(1,\chi_k)$ are linearly independent over* $\mathbf{Q}$.

Corollary 1 follows immediately from the theorem on noting that any

$$f=\sum_{\chi\neq\chi_0} a_\chi\chi$$

with rational $a_\chi$ fulfils the conditions of Theorem 1 since, as remarked before, $\Phi_q$ is irreducible over $\mathbf{Q}_{\varphi(q)}$. To verify Corollary 2 we observe that since $L(1,\chi)$ and $L(1,\chi^*)$ differ only by a rational factor we may assume, without loss of generality, that $\chi_1,\ldots,\chi_k$ are defined to a common modulus, namely the least common multiple of the conductors, and any function

$$f=\sum_{l=1}^{k} a_l\chi_l$$

with rational $a_l$ is rational-valued and so satisfies the hypotheses of Theorem 1. Actually Corollary 2 can be obtained more directly from Dirichlet's formulae for $L(1,\chi)$, where $\chi$ is the quadratic character; in fact $L(1,\chi)$ is an algebraic multiple of a logarithm of a unit in a quadratic field and the result, therefore, follows by an argument parallel to the proof of Lemma 2 on noting that the square roots of the discriminants of the fields that arise are linearly independent over the rationals. Moreover, if we restrict $\chi$ to even quadratic characters then the $L(1,\chi)$ are linearly independent over the field of all algebraic numbers; for in this case the units that arise are multiplicatively independent.

We have already remarked that (i) and (ii) are trivially valid if $q$ is a prime and $f$ is rational-valued. We show now that if we allow $f$ to take algebraic values then there will indeed be a wide class of solutions of (1) provided that $q>4$. It is in fact relatively easy to write down a basis for all odd functions with this property.

THEOREM 2. *Let $q\geqslant 3$ be a natural number. Then all odd algebraically-valued functions $f$, periodic* mod $q$, *for which* (1) *holds, are given by the totality of linear combinations with algebraic coefficients of the following* $[\frac{1}{2}(q-3)]$ *functions*:

$$f_l(n)=(-1)^{n-1}\left(\frac{\sin n\pi/q}{\sin \pi/q}\right)^l, \qquad l=3,5,\ldots,q-2, \quad \text{if } q \text{ is odd},$$

$$f_l(n)=(-1)^{n-1}\left(\frac{\cos n\pi/q}{\cos \pi/q}\right)\left(\frac{\sin n\pi/q}{\sin \pi/q}\right)^l, \quad l=3,5,\ldots,q-3, \quad \text{if } q \text{ is even}.$$

*The functions are linearly independent and take real values in* $\mathbf{Q}_q$.

It can further be verified that the functions $f$ are characterized alternatively by the condition

$$\sum_{r=1}^{q-1} f(r) \cot r\pi/q = 0,$$

the left side being in fact $(-2q/\pi) \sum f(n)/n$ if $f$ is odd.

If $q = 3$ or 4 the theorem implies that there are no nontrivial functions with the above properties. To confirm the assertion made before about the existence of functions $f$ in $\mathbf{Q}_{\varphi(q)}$ satisfying (1) and (2), we have only to take $f = \chi_0 f_1$, where $f_1$ is one of the functions given by Theorem 2 with prime period $p_1 \geqslant 5$ and $\chi_0$ is the principal character modulo any prime $p_2 \equiv 1 \pmod{p_1}$; then $f$ has period $q = p_1 p_2$, takes values in $\mathbf{Q}_{p_1} \subset \mathbf{Q}_{p_2-1} \subset \mathbf{Q}_{\varphi(q)}$, and clearly satisfies (2); further, since $f_1(np_2) = f_1(n)$ we have

$$\sum \frac{f(n)}{n} = \sum \frac{f_1(n)}{n} - \sum \frac{f_1(np_2)}{np_2} = 0.$$

Finally we prove that, when $f$ is not odd, (ii) is redundant and that, therefore, Theorem 2 is exhaustive.

THEOREM 3. *All algebraically-valued functions $f$, periodic* mod $q$, *for which* (i) *and* (1) *hold are odd.*

Thus, Theorem 2 lists all functions under consideration when $q$ is a prime. In particular for $q = 2$ and 3 there are no such functions and for $q = 5$ the only solutions are given by

$$f(0) = 0, \qquad f(1) = -f(4) = 1, \qquad f(2) = -f(3) = 2 + \sqrt{5}$$

and multiples thereof. For arbitrary $q$ condition (i) obviously defines a subspace of the set of functions specified in Theorem 2, and, as an incidental consequence of Theorem 1, we find that no nonvanishing element of this subspace can comply with condition (ii). Furthermore, as an immediate deduction from Theorem 3 we obtain the following.

COROLLARY 3. *If $\chi_1, \ldots, \chi_k$ are any even characters for which the associated primitive characters are distinct then $L(1, \chi_1), \ldots, L(1, \chi_k)$ are linearly independent over the field of all algebraic numbers.*

This generalizes our earlier remark concerning even quadratic characters.

The proofs of the theorems rest on an application of Theorem 1 of [1] relating to linear forms in the logarithms of algebraic numbers; otherwise the arguments run on classical lines. As an intermediate step in the

derivation of both Theorems 1 and 3, we show that if (1) holds then also

$$\sum_{n=1}^{\infty} \frac{f(hn)}{n} = 0 \qquad \text{for} \quad (h, q) = 1,$$

provided that $f$ satisfies suitable conditions. By appealing to more refined results on linear forms in logarithms (see Feldman [4]) we can easily strengthen the conclusion of these theorems to

$$\left| \sum_{n=1}^{\infty} \frac{f(n)}{n} \right| > cF^{-\kappa},$$

where $F$ denotes the maximum of the heights of $f(1),\ldots, f(q)$ and $c$, $\kappa$ denote positive numbers depending only on $q$ and the degrees of $f(1),\ldots, f(q)$.

## 2. Lemmas

By $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{A}$ we shall mean the sets of all rational integers, rational numbers, and algebraic numbers, respectively. We shall denote by $q$ any natural number $\geqslant 2$, and, as in Section 1, we put $\xi = \xi_q = e^{2\pi i/q}$, and we write $\mathbf{Q}_q = \mathbf{Q}(\xi_q)$.

We denote by $F_q$ the set of all functions $f: \mathbf{Z} \to \mathbf{A}$ with period $q$ such that (1) holds. By $G_q$ we signify the set of all $g$ defined on $\mathbf{Z}$ of the form

$$g(s) = q^{-1} \sum_{r=1}^{q} f(r)\, \xi_q^{-rs} \tag{3}$$

with $f$ in $F_q$. Clearly (3) is inverted by

$$f(r) = \sum_{s=1}^{q} g(s)\, \xi_q^{rs}, \tag{4}$$

and $F_q$ and $G_q$ are vector spaces of the same finite dimension over the algebraic numbers. Further we have

$$\sum_{r=1}^{q} f(r) = 0 \qquad \text{if} \quad f \in F_q, \tag{5}$$

for otherwise $\sum f(n)/n$ would diverge, and (5) implies that

$$g(0) = 0 \qquad \text{if} \quad g \in G_q.$$

Logarithms will have their principal values.

LEMMA 1. *$g \in G_q$ if and only if $g$ is algebraically-valued and*

$$\sum_{s=1}^{q-1} g(s) \log(1 - \xi^s) = 0. \tag{6}$$

*Proof.* The assertion is clear on noting that for every $f$, periodic mod $q$, for which $g(0) = 0$, where $g$ is given by (3), we have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = -\sum_{s=1}^{q-1} g(s) \log(1 - \xi^s).$$

This follows easily using the logarithmic expansion of the right side and substituting for $f$ on the left from (4).

LEMMA 2. *If $g \in G_q$ and $\sigma$ is any automorphism of* **A** *then $\sigma g \in G_q$.*

*Proof.* Let $\log \alpha_1, \ldots, \log \alpha_t$ be a maximal subset of the $\log(1 - \xi^s)$ with $s = 1, 2, \ldots, q - 1$, linearly independent with respect to **Q**. We have then

$$\log(1 - \xi^s) = \sum_{r=1}^{t} a_{rs} \log \alpha_r ,$$

where $a_{rs} \in \mathbf{Q}$. From (6) we obtain

$$\beta_1 \log \alpha_1 + \cdots + \beta_t \log \alpha_t = 0,$$

where

$$\beta_r = \sum_{s=1}^{q-1} g(s)\, a_{rs} .$$

Here the $\alpha_r$ and $\beta_r$ are algebraic and the $\log \alpha_r$ are linearly independent over **Q**. Hence, Theorem 1 of [1][2] implies that $\beta_r = 0$ for all $r$. But then also

$$\sum_{s=1}^{q-1} \sigma g(s)\, a_{rs} = 0$$

for any automorphism $\sigma$ and so

$$\sum_{s=1}^{q-1} \sigma g(s) \log(1 - \xi^s) = \sum_{r=1}^{t} \sum_{s=1}^{q-1} \sigma g(s)\, a_{rs} \log \alpha_r = 0.$$

In view of Lemma 1 this means that $\sigma g \in G_q$, as required.

[2] Alternatively the dependence of $g(1), \ldots, g(q - 1)$ can be deduced directly from Theorem 2 of [2].

LEMMA 3. *$G_q$ has a basis of functions $\gamma$: $\mathbf{Z} \to \mathbf{Z}$.*

*Proof.* Suppose that $g \in G_q$ and let $\beta_1, \ldots, \beta_k$ be a basis for $\mathbf{Q}(g(1), \ldots, g(q-1))$. We have

$$g(n) = \sum_{l=1}^{k} \beta_l g_l(n)$$

for some rational-valued $g_l(n)$, and clearly each $g_l(n)$ can be written as a linear combination of conjugates $\sigma g$. Hence, by Lemma 2, all $g_l \in G_q$. It follows now that any basis for the rational-valued $g \in G_q$ will be a basis for all of $G_q$, and the required integer-valued basis is given by multiplying through with a common denominator.

LEMMA 4. *If $f \in F_q$, $\sigma$ is any automorphism of $\mathbf{A}$ and $h$ is the integer defined* mod $q$ *by $\sigma^{-1}\xi = \xi^h$ then $f'(n) = \sigma f(hn)$ is also in $F_q$.*

*Proof.* By definition $\sigma\xi^h = \xi$, and (4) holds for some $g \in G_q$. Hence,

$$\sigma f(hr) = \sigma \sum_{s=1}^{q-1} g(s)\, \xi^{hrs} = \sum_{s=1}^{q-1} \sigma g(s)\, \xi^{rs}.$$

The lemma follows since $\sigma g \in G_q$ by Lemma 2.

LEMMA 5. *$f \in F_q$ if and only if the odd and even parts of $f$ are in $F_q$; and the same holds for $g \in G_q$.*

*Proof.* We need only consider real-valued $f$, for clearly $f \in F_q$ if and only if the real and imaginary parts of $f$ are in $F_q$. On applying Lemma 4 with $\sigma$ denoting complex conjugation, we see that $h = -1$ and $f'(s) = f(-s)$ is in $F_q$. Hence, the odd and even parts of $f$, namely

$$\tfrac{1}{2}(f(s) - f(-s)), \qquad \tfrac{1}{2}(f(s) + f(-s))$$

are also in $F_q$. This carries over to $G_q$ since

$$q^{-1} \sum_{r=1}^{q} f'(r)\, \xi^{-rs} = q^{-1} \sum_{r=1}^{q} f(r)\, \xi^{rs} = g(-s).$$

## 3. PROOF OF THEOREM 1

We show that if $f \in F_q$ and (i) and (ii) hold then $f$ vanishes identically.

In view of (ii), there exists, for any $h$ with $(h, q) = 1$, an automorphism $\sigma$ of $\mathbf{A}$ such that $\sigma^{-1}\xi = \xi^h$ and $\sigma f = f$. Thus, by Lemma 4,

$$f(hn) \in F_q \tag{7}$$

for all such $h$. Summing over $h$ we get

$$\sum_h \sum_n \frac{f(hn)}{n} = 0,$$

that is $\sum a(n)/n = 0$, where

$$a(n) = \sum_{\substack{h=1 \\ (h,q)=1}}^{q} f(hn).$$

Now $a(n) = \varphi(q) f(0)$ if $(n, q) = q$ and, by (i), $a(n) = 0$ if $1 < (n, q) < q$. If $(n, q) = 1$ we have from (i) and (5),

$$a(n) = a(1) = \sum_{h=1}^{q} f(h) - f(0) = -f(0).$$

But this gives

$$\sum_{n=1}^{\infty} \frac{a(n)}{n} = \sum_{m=0}^{\infty} \sum_{s=1}^{q} \frac{a(s)}{s + mq}$$

$$= f(0) \sum_{m=0}^{\infty} \left\{ \frac{\varphi(q)}{q + mq} - \sum_{\substack{s=1 \\ (s,q)=1}}^{q} \frac{1}{s + mq} \right\},$$

and the sum over $m$ here is plainly negative. Hence, we see that $f(0) = 0$ and so, by (5),

$$\sum_{h=1}^{q} \chi_0(h) f(h) = 0,$$

where $\chi_0$ denotes the principal character mod $q$.

Let now $\chi$ be any nonprincipal character mod $q$ and write

$$b(n) = \sum_{h=1}^{q} \chi(h) f(hn).$$

Then, by (7), $b \in F_q$. Further we see that $b(n) = b(1)\, \bar{\chi}(n)$ (the equation being trivial if $(n, q) > 1$) and thus $b(1)\, L(1, \bar{\chi}) = 0$. Since, by Dirichlet's theorem, $L(1, \bar{\chi}) \neq 0$, it follows that $b(1) = 0$. We have, therefore, shown that

$$\sum_{h=1}^{q} \chi(h) f(h) = 0$$

for all characters $\chi$. By the orthogonality properties of the characters, this implies that $f(h) = 0$ for all $h$ with $(h, q) = 1$, and since we know already that $f(h) = 0$ for all other $h$, the theorem is proved.

## 4. Proof of Theorem 2

In view of (3) and (4) it will suffice to determine all odd $g \in G_q$.

We note first that

$$1 - \xi^s = -(\xi^{s/2} - \xi^{-s/2})\,\xi^{s/2} = -2i(\sin s\pi/q)\,e^{s\pi i/q},$$

and so the principal value of the logarithm is

$$\log(1 - \xi^s) = \log\left(2 \sin \frac{s\pi}{q}\right) + \left(\frac{s}{q} - \frac{1}{2}\right)\pi i$$

if $1 \leqslant s < q$. Now $\sin s\pi/q$ is even, and, thus, we deduce from Lemma 1 that an odd function $g \in G_q$ if and only if

$$\sum_{s=1}^{q-1} sg(s) = 0.$$

There are $[\frac{1}{2}(q-1)]$ linearly independent odd functions periodic mod $q$, and subjecting them to a single linear relation reduces this number by 1. Thus, there are $[\frac{1}{2}(q-3)]$ linearly independent odd functions $g \in G_q$ and as many linearly independent odd functions $f \in F_q$. In particular, if $q = 3$ or 4, then there are no nontrivial odd $g \in G_q$ or $f \in F_q$.

We now observe that if $q \geqslant 5$ is odd then the functions

$$g_k(s) = (-1)^{s-k}\binom{q-2k}{s-k} \qquad (1 \leqslant k \leqslant \tfrac{1}{2}(q-3),\ 1 \leqslant s < q),$$

where the binomial coefficient is read as 0 if $s < k$ or $s > q - k$, are odd, linearly independent and satisfy

$$\sum_{s=1}^{q-1} sg_k(s) = 0. \tag{8}$$

For clearly

$$g_k(q-s) = (-1)^{q-s-k}\binom{q-2k}{q-k-s} = (-1)^{q+s-k}\binom{q-2k}{s-k} = -g_k(s);$$

the linear independence is easily verified on noting that $g_k(k) = 1$ and $g_k(s) = 0$ for $s = 1, 2, \ldots, k-1$; and the polynomial

$$\sum_{s=1}^{q-1} s g_k(s)\, x^{s-1}$$

is the derivative of

$$\sum_{s=1}^{q-1} g_k(s)\, x^s = x^k(1-x)^{q-2k},$$

whence (8) follows if $q - 2k \geqslant 3$. If $q$ is even we work similarly with the functions

$$g_k(s) = (-1)^{s-k}\left\{\binom{q-2k-1}{s-k} - \binom{q-2k-1}{s-k-1}\right\}$$
$$(1 \leqslant k \leqslant \tfrac{1}{2}(q-4),\ 1 \leqslant s < q),$$

which satisfy

$$\sum_{s=1}^{q-1} g_k(s)\, x^s = x^k(1+x)(1-x)^{q-2k-1}.$$

The corresponding functions $f$ are obtained from (4); we have

$$f_k(n) = \sum_{s=1}^{q-1} g_k(s)\, \xi_q^{ns}$$

and the expression on the right is given by

$$\xi_q^{kn}(1-\xi_q^{\,n})^{q-2k} \qquad \text{or} \qquad \xi_q^{kn}(1+\xi_q^{\,n})(1-\xi_q^{\,n})^{q-2k-1}$$

according as $q$ is odd or even. By normalizing so that $f(1) = 1$ and setting $l = q - 2k$ or $q - 2k - 1$ we obtain the functions of Theorem 2.

## 5. Proof of Theorem 3

Let $f \in F_q$ be even and suppose that $(h, q) = 1$. We shall prove that $f'(r) = f(hr)$ is also in $F_q$; this will suffice to establish the theorem for, as we have shown in the proof of Theorem 1, the latter fact together with (i) implies that $f$ vanishes identically and so, by Lemma 5, any $f \in F_q$ satisfying (i) must be odd.

From Lemmata 3 and 5 we see that $G_q$ has a basis of functions $\gamma\colon \mathbf{Z} \to \mathbf{Z}$ which are each either odd or even and clearly only the even $\gamma$ are needed

to represent the even $g \in G_q$. Since the function $g$ defined in terms of $f$ by (3) is even it suffices now to prove that for any $\gamma$ as before and for each $k$ with $(k, q) = 1$ we have $\gamma'(s) = \gamma(ks) \in G_q$; for then $g'(s) = g(ks) \in G_q$, and, choosing $k$ such that $hk \equiv 1 \pmod{q}$, we obtain

$$f(hr) = \sum_{s=1}^{q} g(s)\, \xi^{hrs} = \sum_{s=1}^{q} g(ks)\, \xi^{rs},$$

whence, by Lemma 1, $f' \in F_q$ as required.

Now by Lemma 1 again we see that

$$\sum_{s=1}^{q-1} \gamma(s) \log(1 - \xi^s) = 0,$$

and, thus,

$$\prod_{s=1}^{q-1} (1 - \xi^s)^{\gamma(s)} = 1.$$

On applying the automorphism $\sigma$ of $\mathbf{Q}_q$ given by $\sigma\xi = \xi^h$, it follows that

$$\prod_{s=1}^{q-1} (1 - \xi^{sh})^{\gamma(s)} = 1,$$

whence

$$\sum_{s=1}^{q-1} \gamma(s) \log(1 - \xi^{sh})$$

is a multiple of $2\pi i$. But since the imaginary part of $\log(1 - \xi^{sh})$ is an odd function of $s$, and, by definition, $\gamma$ is even, the foregoing sum is real and so must in fact vanish. Substituting $sk$ for $s$, where $hk \equiv 1 \pmod{q}$, we get

$$\sum_{s=1}^{q-1} \gamma(ks) \log(1 - \xi^s) = 0,$$

and, appealing once more to Lemma 1, we see that $\gamma' \in G_q$, as required.

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