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ON THE NORMAL SUBGROUPS OF SL(2, A)

Douglas L. COSTA and Gordon E. KELLER
Department of Mathematics, University of Virginia, Charlottesville, VA 22903, U.S.A.

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Let A be a commutative ring having 2 in the stable range. Let N be a subgroup of SL(2, A) having level ideal J. It is shown that if either A is von Neumann regular or 2 is invertible in A, then N is normal in SL(2, A) if and only if N contains the commutator group $H(J)=[E(2,A), L(2,A,J)]$. Structure theorems for normal subgroups of SL(2, A) are deduced from this result.

Introduction

Suppose that A is a commutative ring such that whenever $aA+bA=A$ there exists an x in A with $a+bx$ a unit, i.e., A satisfies the Bass SR_2 condition. In this paper we give a complete description of the normal subgroups of SL(2, A) whenever A is a commutative SR_2-ring with $\frac{1}{2}\in A$ (cf. Theorems 6.11, 6.12). We also classify the normal subgroups of SL(2, A) whenever A is a commutative von Neumann regular ring (cf. Theorem 6.10). These theorems provide a partial answer to a question of Bass [3].

To provide a framework in which to discuss our results we begin with a short survey of the previous work on normal subgroups of SL(n, A) for commutative rings A.[1]

In 1901, L.E. Dickson showed that if A is a field, then the only normal subgroups of SL(n, A) are subgroups of the scalar matrices except when $n=2$ and $A=\mathrm{GF}(2)$ or GF(3).

If A is not a field, and J is any ideal in A, then there is a natural homomorphism from SL(n, A) into SL(n, A/J) induced by the natural homomorphism from A to A/J. The kernel of this map, SL(n, A; J) is a normal subgroup of SL(n, A). Any subgroup of SL(n, A) containing SL(n, A; J) whose image is contained in the center of SL(n, A/J) is normal in SL(n, A).

If N is a subset of SL(n, A), we denote by $l(N)$ the smallest ideal in A such that under the natural homomorphism into SL(n, $A/l(N)$) the elements of N map to scalar matrices. This ideal is called the *level ideal* of N. We set

[1] We have not included noncommutative rings and have specialized many results to SL(n, A). See [16] for a more general survey.

$$L(n, A; J) = \{T \in \mathrm{SL}(n, A) \mid l(T) \subseteq J\}\,.$$

In 1961, Klingenberg [7] shows that if A is a local ring and $n \geq 3$, a subgroup N of $\mathrm{SL}(n, A)$ is normal if and only if there is an ideal J with $\mathrm{SL}(n, A; J) \subseteq N \subseteq L(n, A; J)$. (Obviously $J = l(N)$.) For $n = 2$ he obtained the same conclusion provided $\frac{1}{2} \in A$ and the residue class field is not GF(3). Lacroix [8] dealt with the case $\frac{1}{2} \notin A$ and $n = 2$, reaching the same conclusion when the residue class field was not GF(2).

Now any subgroup of $\mathrm{SL}(n, A)$ which contains $\mathrm{SL}(n, A; J)$ for some nonzero ideal J is called a congruence subgroup. In [4], Bass, Milnor, and Serre solved what was known as the congruence subgroup problem: If A is an arithmetic Dedekind domain, is every subgroup of finite index a congruence subgroup? Although the answer was no in general, [2] and [4] gave a picture of the normal subgroup structure of $\mathrm{SL}(n, A)$ when A satisfied stable range conditions SR_m. Dedekind rings (e.g., $\mathbb{Z}$) satisfy SR_3.

The important groups needed in this description are $E(n, A)$, the group generated by the elementary matrices and $E(n, A; J)$, the smallest normal subgroup of $E(n, A)$ containing all J-elementary matrices (i.e., elementary matrices in $\mathrm{SL}(n, A; J)$).

Suppose A satisfies $\mathrm{SR}_m(A)$ with $n \geq m$ and $n \geq 3$. Then a subgroup N of $\mathrm{SL}(n, A)$ is normalized by $E(n, A)$ if and only if there is an ideal J with $E(n, A; J) \subseteq N \subseteq L(n, A; J)$. Furthermore, $[\mathrm{SL}(n, A), L(n, A; J)] = [E(n, A), L(n, A; J)] \subseteq E(n, A; J)$, so that $L(n, A; J)/E(n, A; J)$ is a central section of $\mathrm{SL}(n, A)$. Thus a subgroup N of level J is normal if and only if $E(n, A; J) \subseteq N$ (cf. [3, p. 240]).

Wilson [17] showed that a normal subgroup of level J contains $E(n, A; J)$ for any commutative ring provided $n \geq 4$. Furthermore, the work of Golubchik [6], and Suslin [13], showed that a subgroup N of $\mathrm{GL}(n, A)$ is normalized by $E(n, A)$ if and only if $[E(n, A), E(n, J)] \subseteq N$, where $J = l(N)$, when $n \geq 3$.

Now when $n = 2$ much less is known. Serre [12] extended the solution to the congruence subgroup problem to $\mathrm{SL}(2, A)$, where A was an arithmetic Dedekind domain with infinitely many units. Vaserstein [14] made a substantial contribution to our understanding of the subgroup structure under the same hypotheses. We mention several related papers in the bibliography.

It is already evident in Serre [12], that to compensate for the restrictions of $n = 2$ the units of the ring had to be exploited. For instance, the group $\mathrm{SL}(2, \mathbb{Z})$ has a complex normal subgroup structure as evidenced by the fact that $\mathrm{PSL}(2, \mathbb{Z})$ is isomorphic to the free product of cyclic groups of orders 2 and 3.

In [11] McDonald showed that if $\frac{1}{2} \in A$ and A has the property that any polynomial f in $A[x]$ whose coefficients generate A has a unit in its range, then any normal subgroup N of $\mathrm{SL}(2, A)$ of level J satisfies $\mathrm{SL}(2, A; J) \subseteq N$.

The papers of Abe [1], McDonald, Lacroix, and Klingenberg seem to constitute the literature on $\mathrm{SL}(2, A)$ with A an SR_2-ring.

We now review the results of this paper. Let $H(J) = [E(2, A), L(2, A, J)]$ where J is an arbitrary ideal in a commutative ring A. Let A be an SR_2-ring with 2 invertible. We show in Theorem 6.11 that $L(2, A, J)/H(J)$ is a central section of SL(2, A) and that a subgroup N of SL(2, A) is normal if and only if $H(J) \subseteq N$, where J is the level ideal of N. Thus, $H(J)$ plays precisely the role played by $E(n, A; J)$ in the work of Bass cited earlier.

Under the same hypotheses, the largest principal congruence subgroup in $H(J)$ is SL(2, A; vn(J)), where vn(J) is generated by the image of J under the map $x \mapsto x^3 - x$. (See Lemma 6.1 and Theorem 6.5.) Hence we see that every normal subgroup N contains SL(2, A; J), where $J = l(N)$ if and only if vn(J) $= J$ for every ideal J. This is in fact true if and only if vn(A) $= A$. If A is local, vn(A) $= A$ if and only if the residue class ring is not GF(3), giving Klingenberg's result. And we certainly have vn(A) $= A$ if $x^3 - x$ has a unit value, which gives McDonald's result. If 6 is a unit, then vn(A) $= A$ because $6 = 2^3 - 2$, and thus a normal subgroup N of level J contains SL(2, A; J). (This is actually Theorem 2.6.)

Furthermore, if N is a normal subgroup of level J, then $N = H(J)U(N)$ where $U(N)$ is the group of upper triangular matrices in N. This gives a factorization in the same spirit as those in higher dimensions, where $N = E(n, A; J)Q$ and Q is a lower dimensional group. (See [3, p. 240, (4.1)a].)

The structure of $H(J)$ can also be given (see Theorem 6.5), and it sheds light on the role played by GF(3).

Our method for proving these theorems necessitated that we first analyze the structure of normal subgroups of SL(2, A) for A a commutative von Neumann regular ring. All of the results just mentioned for SR_2-rings with 2 invertible hold also for commutative von Neumann regular rings.

As is often the case, our method of discovery is not evident in our presentation. We include here a brief sketch of what actually led to our results in the belief that it may prove helpful to the reader.

From the outset, our objective was to determine the normal subgroups of SL(2, A) under the assumption that A is an SR_2-ring. Inspired by formulae in Serre [12], we saw that for N a normal subgroup of SL(2, A) there were certain ideals J' for which one could force $E(2, A; J') \subseteq N$. The SR_2 hypothesis, however, implies that SL(2, A; J') $= E(2, A; J')$, so that SL(2, A; J') $\subseteq N$ for the appropriate ideals J' (cf. Lemmas 1.1, 1.2).

It is not hard to see that there is a largest ideal J_0 such that SL(2, A; J_0) $\subseteq N$. From this observation we were naturally led to the following simple strategy: Consider the image N' of N in SL(2, A/J_0). By construction, N' is a normal subgroup of SL(2, A/J_0) which contains no nonzero principal congruence subgroups. Therefore the Serre formulae force $J' = 0$ for certain ideals J'in A/J_0, i.e., A/J_0 must satisfy some identities.

This strategem quickly yielded Lemma 2.3 and the concomitant realization that under the added hypothesis that 2 be invertible, the level ideal $J/J_0 = l(N)$ must be generated by idempotents e in A/J_0 with the property that $(A/J_0)e$ is a von

Neumann regular ring which is locally either GF(2) or GF(3). It thus became imperative to determine the normal subgroups of SL(2, A) for rings A which were either Boolean or locally GF(3). We were able to accomplish this by thinking 'locally' or 'coordinate–wise', since SL(2, A) is 'locally' SL(2, 2) or SL(2, 3) in these cases, and using well-known facts about SL(2, 2) and SL(2, 3). The commutator groups of these groups are the only noncentral normal subgroups, and so play a major role in understanding normal subgroups of SL(2, A). This is what ultimately led us to focus on the corresponding subgroup $H(J)$ as the critical subgroup in our analysis.

Our success in the Boolean and locally GF(3) cases allowed us to describe all the normal subgroups N of SL(2, A) for A von Neumann regular or an SR_2-ring with $\frac{1}{2} \in A$, but these were descriptions of N modulo J_0, a highly indeterminate ideal. Happily, it became clear that the ideal $\mathrm{vn}(J) = \mathrm{vn}(l(N))$, completely determined by N, was always contained in J_0 and that our descriptions of N still held modulo vn(J). Early versions of Theorems 6.10, 6.11, and 6.12 then followed.

Finally, hindsight made it clear that full-blown hypotheses on the whole ring A were not necessary to our arguments. We were thus able to give the present descriptions of normal subgroups having fixed level ideal J by making von Neumann regularity or stable range assumptions on the ideal J itself.

We conclude this section by giving a glossary of terms and notations used in this paper. Most of them are standard and are included here for convenience.

Let A be a commutative ring and J an ideal in A.

vn(J)	– The ideal $\sum_{x \in J} A(x^3 - x)$.
GL(n, A)	– The group of invertible $n \times n$ matrices with coefficients in A.
SL(n, A)	– The subgroup of GL(n, A) of matrices T with $\det(T) = 1$.
$E(n, A)$	– The subgroup of SL(n, A) generated by elementary matrices.
SL(n, A; J)	– The group of all matrices $T \in \mathrm{SL}(n, A)$ with $T \equiv I \bmod J$ (also known as a principal congruence subgroup).
J-elementary matrix	– Any elementary matrix in SL(n, A; J).
$E(n, A; J)$	– Smallest normal subgroup of $E(n, A)$ containing all J-elementary matrices.
$l(N)$	– For any subset N of GL(n, A) the smallest ideal modulo which every element of N is scalar. This ideal is also known as the level ideal of N. If $l(N) = eA$ with e an idempotent we may also write $l(N) = e$.
$L(n, A; J)$	– Group of all matrices $T \in \mathrm{SL}(n, A)$ with $l(T) \subseteq J$.
J is 2-divisible	– $2xA = xA$ for every x in J so that multiplication by 2 is a bijection on any ideal contained in J.
$[S, T]$	– $[S, T] = S^{-1}T^{-1}ST$ where S, T are elements of any group.

$[M, N]$	– The group generated by $[S, T]$ where $S \in M$, $T \in N$ where M, N are subsets of a group.
$H(J)$	– $[E(2, A), L(2, A; J)]$.
$U(N)$	– The group of all upper triangular matrices contained in the subgroup N of GL(n, A).
$D(N)$	– The group of all diagonal matrices contained in the subgroup N of GL(n, A).

1. Preliminary results

Most of our results are based on the following elementary lemmas:

Lemma 1.1. *Let J be an ideal of a commutative ring A, and let N be a subgroup of* GL(2, A) *normalized by $E(2, A)$. Suppose that N contains an element of the form $T = \left[\begin{smallmatrix} v & x \\ 0 & u \end{smallmatrix}\right]$. Then*

$$E(2, A; (uv^{-1} - 1)J) \subseteq [E(2, A; J), N] .$$

Proof. Let $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2, A)$, and let $h \in A$. Let $\delta = ad - bc$. Then we have the commutator formulae

$$\delta^{-1}\begin{bmatrix} 1 & -h \\ 0 & 1 \end{bmatrix}\begin{bmatrix} d & -b \\ c & a \end{bmatrix}\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$= \delta^{-1}\begin{bmatrix} \delta + dch + c^2h^2 & (d^2 - \delta)h + dch^2 \\ -c^2h & \delta - dch \end{bmatrix} \tag{1}$$

and

$$\delta^{-1}\begin{bmatrix} 1 & 0 \\ -h & 1 \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} 1 & 0 \\ h & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$= \delta^{-1}\begin{bmatrix} \delta - abh & -b^2h \\ (a^2 - \delta)h + abh^2 & \delta + abh + b^2h^2 \end{bmatrix} . \tag{2}$$

(We include both formulae as a matter of convenience.)

Setting $a = v$, $b = x$, $c = 0$, $d = u$ in (1) and letting h be an arbitrary element of J shows that

$$\left[\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}, T\right] = \begin{bmatrix} 1 & h(uv^{-1} - 1) \\ 0 & 1 \end{bmatrix} \in [E(2, A; J), N]$$

for every $h \in J$. Since $[E(2, A; J), N]$ is normalized by $E(2, A)$, it follows that $E(2, A; (uv^{-1} - 1)J) \subseteq [E(2, A; J), N]$. $\square$

Lemma 1.2. *Let J be an ideal in a commutative ring A and let N be a subgroup of* GL(2, A) *normalized by $E(2, A)$. Let $T = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in N$ and let $q \in A$ with $qc^2 = 0$. Then $E(2, A; 2dcqJ) \subseteq [E(2, A; J), N]$.*

Proof. Let $\delta = \det(T)$ and set $h = q\delta$ in (1). Then we have $[\begin{smallmatrix} 1+dcq & * \\ 0 & 1-dcq \end{smallmatrix}] \in N$ and we are in the situation of Lemma 1.1 with $u = (1 - dcq)$. Now $uv^{-1} - 1 = -2dcq$, so we are done. $\square$

The following lemma is essentially due to Serre [12, p. 492]:

Lemma 1.3. *Let J be an ideal in a commutative ring A, and let N be a subgroup of* $\mathrm{SL}(2, A)$ *normalized by $E(2, A)$. Let $[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]$ be an element of N. If u is any unit of A with $u^2 \equiv 1 \bmod cJ$, then $E(2, (u^4 - 1)J) \subseteq [E(2, A; J), N]$.*

Proof. Choose $x \in J$ so that $u^2 = 1 + cx$ and let $t = ax$. We have the conjugation formulae

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a + ct & b - (a-d)t - ct^2 \\ c & d - ct \end{bmatrix}, \tag{3}$$

$$\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} = \begin{bmatrix} a - bt & b \\ c + (a-d)t - bt^2 & d + bt \end{bmatrix}. \tag{4}$$

Let $S = [\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}][\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}][\begin{smallmatrix} 1 & -t \\ 0 & 1 \end{smallmatrix}]$, and observe that $a + ct = au^2$ so that $S = [\begin{smallmatrix} au^2 & * \\ c & d-tc \end{smallmatrix}]$. Next, let

$$T = \begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u^{-1} & 0 \\ 0 & u \end{bmatrix} = \begin{bmatrix} a & u^2 b \\ u^{-2}c & d \end{bmatrix}.$$

It is well known that $[\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}] \in E(2, A)$ and hence both S and T are in N. (See [3, p. 227]). Hence, $Y = S^{-1}T \in N$. But

$$Y = \begin{bmatrix} d - ct & * \\ -c & au^2 \end{bmatrix}\begin{bmatrix} a & u^2 b \\ u^{-2}c & d \end{bmatrix}$$

is a matrix of determinant 1 whose bottom row has entries $0, u^2$. Hence, $Y = [\begin{smallmatrix} u^{-2} & * \\ 0 & u^2 \end{smallmatrix}]$ and an application of Lemma 1.1 completes the proof. $\square$

Lemma 1.4. *Let A be a commutative ring and let N be a subgroup of* $\mathrm{GL}(2, A)$ *normalized by $E(2, A)$. Then $l(N)$ is generated by lower left corner entries of matrices in N, i.e., $l(N) = (\{c \mid [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in N\})$. (The specification of corner is for convenience. We could have said off-diagonal entries or upper right corner entries.)*

Proof. For any matrix $T = [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in N$, $l(T) = (b, c, a - d)$. Now $[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}] \in E(2, A)$ (see [3]), so that $[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}]T[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}] \in N$ and $-b$ is a lower left corner. By (4), $c + (a - d) - b$ is a lower left corner. Hence, the ideal generated by lower left corners contains $(c, -b, c + (a - d) - b)$, which contains $l(T)$. The lemma is now clear. $\square$

Recall that an ideal J in a ring A is said to satisfy $\mathrm{SR}_2(A, J)$ if whenever

(i) $a \equiv 1 \bmod J$, and

(ii) $b \in J$ with $aA + bA = A$,

there exists $x \in A$ such that $a + bx$ is a unit. The element-wise definition easily yields that if $J_0 \subseteq J$, then $\mathrm{SR}_2(A, J_0)$ and $\mathrm{SR}_2(A/J_0, J/J_0)$ follow from $\mathrm{SR}_2(A, J)$. Recall also that $\mathrm{SR}_2(A, J)$ implies that $E(2, A; J) = \mathrm{SL}(2, A; J)$ [3, p. 240].

Lemma 1.5. *Let A be a commutative ring and let N be a subgroup of* $\mathrm{SL}(2, A)$. *If J_1 is an ideal in A with* $\mathrm{SL}(2, A, J_1) \subseteq N$, *then there exists an ideal J_0 with $J_1 \subseteq J_0$ and maximal with the property* $\mathrm{SL}(2, A; J_0) \subseteq N$. *($J_1$ may be zero.)*

Furthermore, if $l(N) = J$ satisfies $\mathrm{SR}_2(A, J)$, *then J_0 is the largest ideal with* $\mathrm{SL}(2, A; J_0) \subseteq N$.

Proof. A simple application of Zorn's lemma gives the existence of J_0.

Suppose $\mathrm{SR}_2(A, J)$ is satisfied and J_2 is any ideal with $\mathrm{SL}(2, A; J_2) \subseteq N$. By [3, p. 240],

$$\begin{aligned}
\mathrm{SL}(2, A; J_0 + J_2) &= E(2, A; J_0 + J_2) \\
&\subseteq E(2, A; J_0)E(2, A; J_2) \\
&= \mathrm{SL}(2, A; J_0)\mathrm{SL}(2, A; J_2) \subseteq N,
\end{aligned}$$

and since J_0 was maximal we have $J_2 \subseteq J_0$. Since J_2 was an arbitrary ideal with $\mathrm{SL}(2, A; J_2) \subseteq N$, $\mathrm{SL}(2, A; J_0)$ is the largest principal congruence subgroup contained in N. $\square$

2. The reduction hypothesis and its consequences

Despite its simplicity, Lemma 1.5 provides us with a key technique.

Suppose that N is a subgroup of $\mathrm{SL}(2, A)$ normalized by $E(2, A)$, that $l(N) = J$ satisfies $\mathrm{SR}_2(A, J)$, and that J_0 is the ideal guaranteed by Lemma 1.5. Then the image N' of N in $\mathrm{SL}(2, A/J_0)$ is normalized by $E(2, A/J_0)$, and by Lemma 1.5, N' contains no nontrivial congruence subgroup. In this manner, we can reduce the analysis of N to that of N', and hence in this section we adopt the following working hypotheses:

Reduction hypothesis 2.1. (i) A is a commutative ring with N a subgroup of $\mathrm{SL}(2, A)$ normalized by $E(2, A)$;

(ii) $l(N) = J$, where J satisfies $\mathrm{SR}_2(A, J)$;

(iii) N contains no nontrivial congruence subgroup of $\mathrm{SL}(2, A)$.

We shall now see that Reduction hypothesis 2.1 is strong enough to force certain identities to hold in A.

Lemma 2.2. *Assume that Reduction hypothesis* 2.1 *holds, that* $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in N$, *and that* $u \in A$ *is a unit with* $u^2 \equiv 1 \bmod cA$. *Then* $u^4 = 1$.

Proof. Taking $J = A$ in Lemma 1.3, it follows that $E(2, A, (u^4 - 1)A) = \mathrm{SL}(2, A, (u^4 - 1)A) \subseteq N$. Since N contains no nontrivial congruence subgroup, $u^4 - 1 = 0$. □

Lemma 2.3. *Suppose* A *is a commutative ring,* $c \in A$ *with* cA *satisfying* $\mathrm{SR}_2(A, cA)$, *and whenever* $u \in A$ *is a unit with* $u \equiv 1 \bmod cA$, *it follows that* $u^4 = 1$. *Then for every* y *in* A *there exists an element* t *in* A *with* $y^2c^2t = 4yc$.

In particular, if Reduction hypothesis 2.1 *holds and* $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in N$, *then the conclusion holds for the element* c.

Proof. Since $(y^2c^2, 1 + yc)A = A$, there exists $r \in A$ such that $u = 1 + yc + ry^2c^2$ is a unit in A. Let $z = 1 + ryc$, so that $u = 1 + zyc$. By hypothesis $u^4 = 1$, and therefore

$$4zyc + 6z^2y^2c^2 + 4z^3y^3c^3 + z^4y^4c^4 = 0 .$$

Since $z \equiv 1 \bmod ycA$ we have $4yc \equiv 0 \bmod y^2c^2A$ as desired. The rest follows from Lemma 2.2. □

The identities found in Lemmas 2.2 and 2.3 are already enough to reduce the analysis of Reduction hypothesis 2.1 to the case of von Neumann regular rings which are locally GF(2), GF(3), or GF(5). In the main theorem of this section, Theorem 2.5, we will avoid the GF(2) and GF(3) possibilities by assuming in effect that 6 is invertible. The next lemma will be necessary in order to show that GF(5) does not actually occur.

Lemma 2.4. *Let* k *be a field having more than* 2 *elements. Let* $S = k^X$ *be the ring of all functions from some nonempty set* X *into* k, *and let* R *be the subring of* S *consisting of functions with finite range. Let* A *be any* k*-subalgebra of* R. *As a vector space over* k, A *is spanned by its units. If* k *has more than* 3 *elements,* A *is spanned by the squares of its units.*

Proof. To prove the lemma it clearly suffices to handle the case $A = k[f]$, where $f : X \to k$ is any function with $f(X) = \{a_1, \ldots, a_n\}$ finite. Assume that $a_1, \ldots, a_n$ are distinct and for each $i = 1, \ldots, n$, let $X_i = \{x \in X \mid f(x) = a_i\}$. Then every element of $k[f]$ is constant on each X_i, so we consider each X_i to be a point, and we may therefore consider the elements of $k[f]$ as vectors of length n over k. Since $a_1, \ldots, a_n$ are distinct, the Vandermonde determinant shows that $1, f, \ldots, f^{n-1}$ are linearly independent over k, and hence that $k[f]$ is the set of all functions from $\{X_1, \ldots, X_n\}$ into k. Thus $A \cong k^n$, the direct product of n

copies of k. Choose a unit $u \neq 1$ in k, and with u a square if k has more than 3 elements. The n elements $(1, 1, \dots, 1) = 1$, $(1, u, 1, \dots, 1)$, $(1, 1, u, \dots, 1)$, $(1, 1, \dots, u)$ of k^n have determinant $(u-1)^{n-1} \neq 0$, whence they are linearly independent and therefore span k^n. This completes the proof. □

We now come to the main theorem of this section.

Theorem 2.5. *Suppose Reduction hypothesis* 2.1 *is satisfied and J is* 6*-divisible. Then N consists of scalar matrices.*

Proof. We assume N contains a nonscalar matrix and proceed by contradiction. In the first part of this proof we find a nontrivial elementary matrix in N. Since N contains a nonscalar matrix, Lemma 1.4 will produce a matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in N$ with $c \neq 0$. By Lemma 2.3, $c^2 s = c$ for some value of s, so that $cs = e$ is an idempotent with $cA = eA$. Now $u = 2e + (1-e)$ is a unit since $2e$ is a unit in Ae. By Lemma 2.2, $0 = u^4 - 1 = 15e$. Since $3e$ is invertible in Ae, $5e = 0$. Using (1) with $h = -s^2 e$, N contains a matrix of the form $\left[\begin{smallmatrix} x & y \\ 1 & z \end{smallmatrix}\right] e + (1-e)I$. Applying (3) we get $\left[\begin{smallmatrix} * & * \\ 1 & 3 \end{smallmatrix}\right] e + (1-e)I$ in N. Finally, apply (1) with $h = -1$ and get $S = \left[\begin{smallmatrix} -1 & -5 \\ 1 & 4 \end{smallmatrix}\right] e + (1-e)I = \left[\begin{smallmatrix} 1-2e & 0 \\ e & 1-2e \end{smallmatrix}\right]$ in N so that N contains the elementary matrix

$$T = S^4 = \begin{bmatrix} 1 & 0 \\ e & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} e + I(1-e)\,.$$

If the smallest normal subgroup M of SL(2, Ae) containing $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right] e$ contains SL(2, Ae; J_0), then J_0 is an ideal in A and N contains SL(2, A; J_0). Therefore, Ae and M satisfy Reduction hypothesis 2.1 with $l(M) = Ae$.

Lemmas 2.2 and 2.3 now imply Ae is von Neumann regular of characteristic 5 with $u^4 = 1$ for every unit u in Ae. Since Ae is an SR_2-ring every unit in a homomorphic image is a homomorphic image of a unit. It follows that every residue class field of Ae is isomorphic to GF(5), and hence that Ae is isomorphic to a subdirect product of copies of GF(5). By Lemma 2.4, Ae is spanned over GF(5) by the squares of its units, so the squares of units generate A as an abelian group. Now

$$\begin{bmatrix} u^{-1} & 0 \\ 0 & u \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} e \begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ u^2 & 1 \end{bmatrix} e \in M$$

for any unit u in Ae, and therefore SL(2, A, Ae) $\subseteq M$. This contradiction completes the proof. □

Theorem 2.6. *If A is an* SR_2*-ring with* 6 *invertible and N is a normal subgroup of* SL(2, A) *with $l(N) = J$, then* SL(2, A; J) $\subseteq N$.

Proof. By Lemma 1.5 there exists a largest ideal J_0 with SL(2, A, J_0) $\subseteq N$. The ring A/J_0 together with the image of N under the natural map induced on $\mathrm{SL}(2, A)$ by the canonical epimorphism from A to A/J_0 satisfy Reduction hypothesis 2.1 with J/J_0 as the level ideal. By Theorem 2.5, $J/J_0 = 0$ so that $J = J_0$ and $\mathrm{SL}(2, A, J) \subseteq N$. □

Remark. In an earlier version of Theorem 2.5 we used the invertibility of 6 in the following way. A lemma of Bass [2] states that if A contains units u, v such that $u^2 + v = 1$, then $\mathrm{SL}(2, A, J) = [\mathrm{SL}(2, A, J), \mathrm{SL}(2, A)]$ if $\mathrm{SR}_2(A, J)$ holds. (This is immediate from Lemma 1.1.) With 6 invertible, the lemma holds because we may take $u = \frac{1}{2}$, $v = \frac{3}{4}$. In fact, it is not hard to see that 6 is invertible if and only if there exist units $u = a/b$, $v = c/d$ with a, b, c, d in the prime subring of A such that $u^2 + v = 1$.

In the presence of the reduction hypothesis we are now able to deduce some properties of the ideal J which will prove useful in the sequel.

Lemma 2.7. *Assume that Reduction hypothesis* 2.1 *holds. Let* $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in N$, *and let* λ *be an element of* A *such that* $(\lambda, 6)A = A$. *Then* $c \in \lambda A$.

Proof. By Lemma 2.3 there exists $t \in A$ such that $\lambda^2 c^2 t = 4\lambda c$. If $\lambda c^2 t = 4c$, then c is a multiple of λ because $(\lambda, 4)A = A$. Hence we may assume that $c(\lambda ct - 4) \neq 0$. Let $w = \lambda ct - 4$. Then $w \neq 0$, but $\lambda cw = 0$. Choose $x, s \in A$ so that $4s = 1 + x\lambda$. Applying Lemma 2.3 again, choose $r \in A$ so that $(cw)^2 r = 4cw$. Then $(cw)^2 rs = 4scw = (1 + x\lambda)cw = cw$. Hence, $e = cwrs$ is an idempotent in A and $Ae = Acw$. Since $\lambda e = 0$, 6 is a unit in Ae. Using (1) with $h = e$ we see that N contains a normal subgroup of $\mathrm{SL}(2, A, Ae)$ of level e. Therefore Theorem 2.6 implies $\mathrm{SL}(2, A, Ae) \subseteq N$, contradicting the reduction hypothesis. □

Theorem 2.8. *If Reduction hypothesis* 2.1 *holds, then* $24 \cdot J = 0$ *and* $(6 \cdot J)^3 = 0$.

Proof. First we show that $6 \cdot J \subseteq \mathrm{nil}(A)$, the nilradical of A. By Lemma 1.4 it suffices to show that if $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in N$, then $6c \in \mathrm{nil}(A)$. Let P be any prime ideal of A, and suppose that $6c \not\subseteq P$. Setting $y = 24$ in Lemma 2.3 there exists $t \in A$ with $(24)^2 c^2 t = 4(24)c$. Thus $96c(6ct - 1) = 0$. Since $96c \not\subseteq P$, $\lambda = 6ct - 1 \in P$, and $(\lambda, 6)A = A$. Then Lemma 2.7 implies that $c \in \lambda A \subseteq P$, a contradiction. As we have now shown that $6c \in P$ for every prime ideal P, it follows that $6c \in \mathrm{nil}(A)$.

Next, observe again that to show $24J = 0$, Lemma 1.4 implies that it suffices to show $24c = 0$ if $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in N$. By (1) there is a matrix in N of the form $\left[\begin{smallmatrix} * & * \\ c^2 r & u \end{smallmatrix}\right]$ with r still arbitrary and u a unit. We see from Lemma 1.2 and Reduction hypothesis 2.1 that if $r, q \in A$ and $q(rc^2)^2 = 0$, then $2qrc^2 = 0$.

As above, we have $96c(6ct - 1) = 0$ for some t. But $6c \in \mathrm{nil}(A)$ so that $6ct - 1$ is a unit, and $96c = 0$. Thus $6(4c^2)^2 = 0$ and therefore $2 \cdot 6 \cdot 4c^2 = 48c^2 = 0$. Then $3 \cdot (4c^2)^2 = 0$, and again we have $2 \cdot 3 \cdot 4c^2 = 24c^2 = 0$. Since $6c$ is nilpotent, $u' = 1 + 6c$ is a unit. By the reduction hypothesis and Lemma 1.3, we have

$(u')^4 = 1$ and hence $0 = (6c)^4 + 4(6c)^3 + 6(6c)^2 + 4(6c) = 24c$. This completes the proof that $24 \cdot J = 0$. That $(6 \cdot J)^3 = 0$ is immediate. □

3. Von Neumann regular rings and ideals

Motivated by the appearance of von Neumann regular rings when using the reduction hypothesis, we study commutative von Neumann regular rings and ideals in this section. Such rings are of course SR_2-rings. In fact, if A is von Neumann regular and $a, b \in A$, then $(a, b)A = (a + u(1-e)b)A$, where u is any unit and e is the idempotent generator of aA.

We begin with a result on conjugacy in $GL(2, A)$.

Theorem 3.1. *Let A be a commutative von Neumann regular ring, and let $S, T \in GL(2, A)$. Then S and T are conjugate in $GL(2, A)$ if and only if*

(i) $tr(S) = tr(T)$;

(ii) $\det(S) = \det(T)$;

(iii) $l(S) = l(T)$;

(iv) $S \equiv T \pmod{l(T)}$.

Proof. The necessity of (i) and (ii) is well known. The necessity of (iii) follows from the fact that $l(RTR^{-1}) \subseteq l(T)$ for any $R \in GL(2, A)$, as an easy calculation shows. Since T is a scalar matrix mod $l(T)$, (iv) is obvious. (Note that necessity of (i)–(iv) holds for all commutative rings.)

For the sufficiency, first observe that if $T = [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]$ is any matrix in $GL(2, A)$, then $l(T) = (b, c, a-d)A$ is a principal ideal generated by $b - (1-e)(c + (1-f)(a-d)) = p$, where e, f are the idempotent generators of bA, cA, respectively.

Let $x = e + (1-e)(1-f)$ and $y = 1 - e$. Then

$$X = \begin{bmatrix} x & y \\ -y & e \end{bmatrix} \in SL(2, A)$$

and

$$T' = XTX^{-1} = \begin{bmatrix} * & x^2b - y^2c + xy(a-d) \\ * & * \end{bmatrix} = \begin{bmatrix} * & p \\ * & * \end{bmatrix}.$$

Now $p = ug$ where u is a unit in A and g is an idempotent. If $U = [\begin{smallmatrix} u^{-1} & 0 \\ 0 & 1 \end{smallmatrix}]$, then $T'' = UT'U^{-1} = [\begin{smallmatrix} w & g \\ * & * \end{smallmatrix}]$. Now $w \equiv a \bmod gA$ so applying (4) we have a conjugate $T''' = [\begin{smallmatrix} a & g \\ r & s \end{smallmatrix}]$. Now $s = tr(T) - a$ and $as - rg = \det(T)$, implying $r = as - \det(T)$. If $S = [\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}]$, S must be conjugate to $[\begin{smallmatrix} a' & g \\ * & * \end{smallmatrix}]$ in the same way T was conjugate to T'''. Now $a \equiv a' \bmod gA$ so applying (4), S is conjugate to $S' = [\begin{smallmatrix} a & g \\ r' & s' \end{smallmatrix}]$, but since $tr(T) = tr(S)$ and $\det(T) = \det(S)$, we get $S' = T'''$ and the proof is complete. □

Definition 3.2. An ideal J in a commutative ring A is called *von Neumann regular* if $c^2A = cA$ for every $c \in J$.

It can be easily verified that every finitely generated ideal contained in a von Neumann regular ideal J is generated by a unique idempotent and that if e is any idempotent in J, then Ae is a von Neumann regular ring with identity e. Just as we observed that a von Neumann regular ring is an SR_2-ring, a von Neumann regular ideal J satisfies $\mathrm{SR}_2(A, J)$.

We have introduced von Neumann regular ideals because the reduction hypothesis leads to them.

Lemma 3.3. *Suppose Reduction hypothesis* 2.1 *holds and J is* 2*-divisible. Then J is von Neumann regular.*

Proof. Since J is 2-divisible, Lemma 2.3 implies cA is von Neumann regular for any off-diagonal entry of any matrix in J. By Lemma 1.4, J is generated by such off-diagonal elements.

Now suppose J_1 and J_2 are von Neumann regular ideals and let x be an element of $J_1 + J_2$, $x = x_1 + x_2$ with x_i in J_i for $i = 1, 2$. Let e_i be the idempotent generator of x_iA for $i = 1, 2$. Ideals contained in von Neumann regular ideals are obviously von Neumann regular so that e_1A and $(1 - e_1)e_2A$ are von Neumann regular. Since $x \in (e_1 + e_2 - e_1e_2)A$ we have

$$x^2A = x^2e_1A + x^2(1 - e_1)e_2A = xe_1A + x(1 - e_1)e_2A = xA .$$

Therefore, $J_1 + J_2$ is von Neumann regular and by induction the sum of any finite number of von Neumann regular ideals is von Neumann regular. As J is the directed union of such finite sums, J is von Neumann regular as claimed. $\square$

Many of our calculations require locating elements of $l(N)$ as off-diagonal entries for matrices in N. The following lemma guarantees that we will find them if J is von Neumann regular:

Lemma 3.4. *Let J be a von Neumann regular ideal in a commutative ring A, and let N be a subgroup of* $\mathrm{SL}(2, A)$ *normalized by* $E(2, A)$ *with $l(N) = J$. If $x \in J$, then there exist S in N and an elementary matrix R such that x is an off-diagonal entry of $X = [R, S]$ and $X \in \mathrm{SL}(2, A; Ax)$.*

Proof. Let $T = [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]$ be an arbitrary element of N. Choose f so that $c^2f = c$ and hence $e = cf$ is the idempotent generator of cA. Taking $h = -f^2ye$ and applying the reduction hypothesis we get an element of $\mathrm{SL}(2, A; Ae)$ with lower left corner ey, an arbitrary element of Ae.

Now let x be an arbitrary element of J. By Lemma 1.4, $x \in c_1A + \cdots c_tA = Ae_0$ for some finite set of off-diagonal entries in J, which by the first paragraph may be taken as idempotent entries in matrices $T_i = [\begin{smallmatrix} * & * \\ c_i & * \end{smallmatrix}] \in \mathrm{SL}(2, A; Ac_i) \cap N$. We show that e_0 is an off-diagonal entry of a matrix in $N \cap \mathrm{SL}(2, A, Ae_0)$. The proof is an obvious induction on t which depends on the case $t = 2$.

Suppose e, e' are off-diagonal entries of matrices in $N \cap \mathrm{SL}(2, A, Ae)$ and $N \cap \mathrm{SL}(2, A, Ae')$, respectively. Applying the first paragraph and then (3), we get $\left[\begin{smallmatrix} * & * \\ e & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & * \\ (1-e)e' & * \end{smallmatrix}\right]$ in N. Multiplying shows that the idempotent generator $e + e' - ee'$ of $Ae + Ae'$ is an off-diagonal entry of a matrix in N. By the first paragraph there is such a matrix in $\mathrm{SL}(2, A; A(e + e' - ee'))$. By induction e_0 is an off-diagonal entry and applying the reduction hypothesis with $h = xe_0 = x$ gives the desired result. $\square$

If J is an ideal in a commutative ring A, we let $H(J) = [E(2, A), L(2, A; J)]$. Clearly any subgroup of level J containing $H(J)$ is normalized by $E(2, A)$. In the next theorem we show that if J is von Neumann regular, then $H(J) = [\mathrm{SL}(2, A), L(2, A; J)]$ so that subgroups of level J in $\mathrm{SL}(2, A)$ containing $H(J)$ will be normal.

Theorem 3.5. *Let J be a von Neumann regular ideal in a commutative ring A. Then $H(J) = [\mathrm{SL}(2, A), L(2, A; J)]$.*

Proof. Certainly $[\mathrm{SL}(2, A), L(2, A; J)] \supseteq [E(2, A), L(2, A; J)] = H(J)$. Let $T \in [\mathrm{SL}(2, A), L(2, A; J)]$ and let $l(T) = e$. Then $Te \in [\mathrm{SL}(2, Ae), L(2, Ae; Ae)] = [E(2, Ae), L(2, Ae; Ae)]$. Therefore $T \in [E(2, A; Ae), L(2, A; Ae)] \subseteq [E(2, A), L(2, A; J)]$. $\square$

Lemma 3.6. *Suppose J is an ideal in a commutative ring A and φ is a ring epimorphism defined on A. Letting φ also denote the natural map induced on* $\mathrm{SL}(2, A)$, *suppose that*

$$\mathrm{SL}(2, \varphi(A); \varphi(J)) \subseteq \varphi(\mathrm{SL}(2, A)) .$$

Then $\varphi(H(J)) = H(\varphi(J))$.

In particular, if $\varphi(J)$ satisfies $\mathrm{SR}_2(\varphi(A), \varphi(J))$, *then $\varphi(H(J)) = H(\varphi(J))$.*

Proof. Let $T \in L(2, \varphi(A); \varphi(J))$. T is congruent to a scalar matrix mod $(\varphi(J))$. Hence there is a matrix S in $E(2, \varphi(A))$ with $S \equiv T \bmod \varphi(J)$. (See [3, p. 227]). Therefore $T \in \mathrm{SL}(2, \varphi(A); \varphi(J))S \subseteq \varphi(\mathrm{SL}(2, A))$. Since T was arbitrary, $L(2, \varphi(A); \varphi(J)) \subseteq \varphi(\mathrm{SL}(2, A))$.

Let $C = \{[x, y] \mid x \in E(2, A), y \in L(2, A, J)\}$. By the previous paragraph

$$\varphi(C) = \{[s, t] \mid s \in E(2, \varphi(A)), y \in L(2, \varphi(A), \varphi(J))\} .$$

Since the groups $H(J)$ and $H(\varphi(J))$ are generated by C and $\varphi(C)$ respectively, $\varphi(H(J)) = H(\varphi(J))$ as claimed.

If $\varphi(J)$ satisfies $\mathrm{SR}_2(\varphi(A), \varphi(J))$,

$$\mathrm{SL}(2, \varphi(A), \varphi(J)) = E(2, \varphi(A), \varphi(J)) \subseteq E(2, \varphi(A)) = \varphi(E(2, A)) .$$

By the first part of the lemma $\varphi(H(J)) = H(\varphi(J))$. $\square$

Definition 3.7. For J any ideal in a commutative ring A, let $\mathrm{vn}(J) = \sum_{x \in J} A(x^3 - x)$.

The reason for studying $\mathrm{vn}(J)$ is made apparent by the next theorem.

Theorem 3.8. *Suppose that Reduction hypothesis* 2.1 *holds and that J is von Neumann regular. Then* $\mathrm{vn}(J) = 0$.

Proof. Let e be an arbitrary idempotent in J, and let $N' = \{T \in N \mid l(T) \subseteq Ae\}$. By Lemma 3.4, $l(N') = Ae$. Applying Lemma 3.4 to N' we see that if $x \in A$, then there exists $S = \left[\begin{smallmatrix} * & * \\ xe & * \end{smallmatrix}\right]$ in $N'' = [E(2, A), N']$, and that $S = Se + I(1 - e)$ since $l(N') = Ae$. Therefore $N''e$ satisfies the reduction hypothesis since any congruence subgroup of $N''e$ gives a congruence subgroup of N.

In the ring Ae, $l(N''e) = Ae$. Let M be an arbitrary maximal ideal in Ae and let $z \in Ae - M$. Since z is a unit mod M and Ae is von Neumann regular, there exists a unit u in Ae with $u \equiv z \bmod M$. By Lemma 2.2, $u^4 = 1$ and hence $z^4 \equiv 1 \bmod M$. Now z was arbitrary so that Ae/M must have 2, 3 or 5 elements. By Theorem 2.8, $6z = 0$ so Ae/M is GF(2) or GF(3).

Thus for any $x \in Ae$, $x^3 - x \in M$. Since M was arbitrary $x^3 - x$ is in the Jacobson radical and hence $x^3 - x = 0$ since the Jacobson radical of a von Neumann regular ring is 0.

Finally, since e was arbitrary $x^3 - x = 0$ for every $x \in J$ and the proof is complete. $\square$

We now record some elementary properties of $\mathrm{vn}(J)$.

Lemma 3.9. *Let J be an ideal in a commutative ring A. Then $J' = J/\mathrm{vn}(J)$ is a von Neumann regular ideal in $A' = A/\mathrm{vn}(J)$ and $6J' = 0$. If e is any idempotent in J', then eA' is a von Neumann regular ring. In fact, $eA' = (3e)A' + (4e)A'$ where $(3e)A'$ is a Boolean ring and $(4e)A'$ is locally* GF(3).

Proof. Suppose $x \in J$. Then $x^2 \cdot x = x^3 \equiv x \bmod \mathrm{vn}(J)$ implying J' is von Neumann regular. Now $6x \equiv 6x^3 \equiv (2x)^3 - 2x^3 \equiv 2x - 2x \equiv 0 \bmod \mathrm{vn}(J)$ so that $6J' = 0$.

Clearly, if e is an idempotent in J', eA' is a von Neumann regular ring. Since $6e = 0$, $3e$ and $4e$ are orthogonal idempotents and $eA' = (3e)A' + (4e)A'$. Since $(3e)A'$ is of characteristic 2 and satisfies the identity $z^3 = z$, $(3e)A'$ is a Boolean ring. Similarly $(4e)A'$ is of characteristic 3 and satisfies the identity $z^3 = z$ from which we conclude $(4e)A'$ is locally GF(3). $\square$

Lemma 3.10. *Let A be a commutative ring, J an ideal in A, and φ a ring epimorphism defined on A. Then* $\varphi(\mathrm{vn}(J)) = \mathrm{vn}(\varphi(J))$.

Proof. If $x \in J$, then $x^3 - x \in \mathrm{vn}(J)$, and

$$\varphi(x^3 - x) = \varphi(x)^3 - \varphi(x) \in \mathrm{vn}(\varphi(J)).$$

Since the epimorphic image of an ideal is an ideal and $\mathrm{vn}(J)$ is generated by elements $x^3 - x$ with x in J we have $\varphi(\mathrm{vn}(J)) \subseteq \mathrm{vn}(\varphi(J))$.

Suppose $y \in \varphi(J)$, and thus $y = \varphi(x)$ for some x in J. We have $y^3 - y = \varphi(x)^3 - \varphi(x) = \varphi(x^3 - x)$ so that $\mathrm{vn}(\varphi(J))$ is generated by images of elements in $\mathrm{vn}(J)$. This establishes the equality. □

4. The Boolean case

We now examine in detail the case in which A is a Boolean ring. (A Boolean ring is a ring in which every element is idempotent.) If A is Boolean, then 1 is the only unit in A and hence $\mathrm{GL}(2, A) = \mathrm{SL}(2, A)$. In addition, every finitely generated ideal is principal and has a unique generator. In particular, if $T = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2, A)$, then we let $l(T) = e$, where e is the unique element generating $(b, c, a - d)A$.

Lemma 4.1. *Let A be a Boolean ring, and let $T = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2, A)$. Then $l(T) = b + c + bc$, $\mathrm{tr}(T) = \mathrm{tr}(T)l(T)$, and T is conjugate to $\left[\begin{smallmatrix} 1 & l(T) \\ \mathrm{tr}(T) & 1 + \mathrm{tr}(T) \end{smallmatrix}\right]$. Consequently, if $S \in \mathrm{GL}(2, A)$, S is conjugate to T if and only if $l(T) = l(S)$ and $\mathrm{tr}(T) = \mathrm{tr}(S)$.*

Proof. Since $\mathrm{char}\, A = 2$, $bc = 1 + ad$ and $d - a = \mathrm{tr}(T) = (a + d)(1 + ad) = (a + d)bc \in (b, c)A$. Now $(b, c)A = (b + c + bc)A$ so that $l(T) = b + c + bc$ and $\mathrm{tr}(T) = \mathrm{tr}(T)l(T)$. Now T is a scalar matrix modulo $l(T)$, and since $A/l(T)A$ is Boolean, $T \equiv \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] (\mathrm{mod}\, l(T)A)$.

By Theorem 3.1, T is conjugate to $\left[\begin{smallmatrix} 1 & l(T) \\ \mathrm{tr}(T) & 1 + \mathrm{tr}(T) \end{smallmatrix}\right]$ and a matrix S in $\mathrm{GL}(2, A)$ is conjugate to this matrix and hence to T if and only if $l(T) = l(S)$ and $\mathrm{tr}(T) = \mathrm{tr}(S)$. □

Lemma 4.2. *Let A be a Boolean ring and suppose $T \in \mathrm{GL}(2, A)$. Then*

(i) $\mathrm{o}(T) = 1, 2, 3$, *or* 6, *where* $\mathrm{o}(T)$ *is the order of T in* $\mathrm{GL}(2, A)$,

(ii) $T^2 = I$ *if and only if* $\mathrm{tr}(T) = 0$,

(iii) $T^3 = I$ *if and only if* $l(T) = \mathrm{tr}(T)$,

(iv) $l(T^2) = \mathrm{tr}(T) = \mathrm{tr}(T^2)$,

(v) $l(T^3) = l(T) + \mathrm{tr}(T)$,

(vi) $T^3 = I$ *and* $l(T) = e$ *if and only if* $T = \left[\begin{smallmatrix} 1 + ex & e \\ e & 1 + ex + e \end{smallmatrix}\right]$ *for some x in A. Furthermore if $S^3 = T^3 = I$, then $ST = TS$.*

Proof. By the previous lemma there is a conjugate S of T with $S = \left[\begin{smallmatrix} 1 & e \\ t & 1 + t \end{smallmatrix}\right]$, with

$et = t$. Then $S^2 = \left[\begin{smallmatrix} 1+t & t \\ t & 1 \end{smallmatrix}\right]$, $S^3 = \left[\begin{smallmatrix} 1 & t+e \\ 0 & 1 \end{smallmatrix}\right]$, and $S^6 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. This gives parts (i)–(v).

Suppose $T^3 = I$ and $l(T) = e$. Then $T = \left[\begin{smallmatrix} 1+ex & ey \\ ez & 1+ew \end{smallmatrix}\right]$, for some $x, y, z, w \in A$. By (iii), $e = \mathrm{tr}(T) = ex + ew$ so that $ew = e + ex$. Then $\det(T) = (1 + ex)(1 + e + ex) + eyz = 1 + e + eyz$, so that $eyz = e$. By the uniqueness of generators of principal ideals we have $ey = ez = e$, and the first part of (vi) is clear. Proving $ST = TS$ is now just a simple computation. □

Lemma 4.3. *Let A be a Boolean ring and let N be a normal subgroup of* $\mathrm{SL}(2, A)$ *with $l(N) = J$. Let $H_3(J) = \{T \in \mathrm{SL}(2, A, J) \mid T^3 = I\}$. Then $[\mathrm{SL}(2, A), N] = H_3(J) = H(J)$.*

Proof. Let M be the normal subgroup of $\mathrm{SL}(2, A)$ generated by all $[X, T]$ where X is elementary and T is in N. Clearly $\mathrm{SL}(2, A)$ acts trivially on N/M since all elementary matrices act trivially and they generate $\mathrm{SL}(2, A)$. It follows that $M = [\mathrm{SL}(2, A), N]$.

If $T = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and $X = \left[\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right]$, then (1) gives

$$S = [X, T] = \begin{bmatrix} 1 + dch + ch & (d+1)h + dch \\ ch & 1 + dch \end{bmatrix}.$$

Now $ad - bc = 1$ so that $ad - dbc = d$, $ad + 1 - dbc = d + 1$ and hence $bc - dbc = d + 1$. Therefore $d + 1$ is in cA and it is now clear that $\mathrm{tr}(S) = ch = l(S)$, so that by Lemma 4.2(iii), $S^3 = I$. Certainly the same conclusion may be reached when X is a lower elementary matrix and therefore Lemma 4.2(vi) implies M is an elementary abelian 3-group, so that $M \subseteq H_3(J)$.

Now let T vary and set $h = 1$, getting elements S of M with $S^3 = I$ and $l(S) = e$ for any idempotent e in J by Lemma 3.4. Since all elements of order 3 of the same level are conjugate by Lemma 4.1 and Lemma 4.2(iii), $H_3(J) \subseteq M$. Applying this result when $N = L(2, A; J)$ we get $H_3(J) = M = [\mathrm{SL}(2, A), L(2, A; J)] = H(J)$. □

Theorem 4.4. *Let A be a Boolean ring and let N be a subgroup of* $\mathrm{SL}(2, A)$ *with $l(N) = J$. Then N is normal in* $\mathrm{SL}(2, A)$ *if and only if $H(J) \subseteq N$.*

Proof. The theorem follows from Lemma 4.3 and the fact that $H(J) = [\mathrm{SL}(2, A), L(2, A; J)]$. □

5. Rings locally isomorphic to GF(3)

Boolean rings can be described as rings which are locally isomorphic to GF(2). These rings will arise naturally in attacking the general case, as will von Neumann regular rings of one other special type: the rings which are locally isomorphic to GF(3).

Lemma 5.1. *Let A be a commutative ring. Then the following are equivalent*:

(1) *For each prime ideal P of A, $A_P \cong \mathrm{GF}(3)$*;

(2) *The characteristic of A is 3, and for each x in A, $x^3 = x$*;

(3) *The characteristic of A is 3, and for each x in A, $x^5 = x$.*

Proof. That (1) implies (2) and (3) is clear since any equation in A holds globally if and only if it holds locally.

If (2) holds, then A is von Neumann regular. If P is any prime ideal of A, then A/P is a field in which $x^3 = x$ holds, so that $A/P \cong \mathrm{GF}(3)$. But we also have $A_P \cong A/P$ since A is von Neumann regular.

Finally, suppose (3) holds. Again, A is von Neumann regular and for any prime ideal P of A, A/P is a field of characteristic 3 in which $x^5 = x$ holds for all elements. This implies A/P has 5 or fewer elements so that $A/P \cong \mathrm{GF}(3)$. □

For the rest of this subsection we assume that A is locally isomorphic to GF(3). In this case there is a natural embedding $A \to \prod_{P \in \mathrm{Spec}\, A} A_P \cong \prod_{P \in \mathrm{Spec}\, A} \mathrm{GF}(3)$, via which we may regard A as a subring of the ring of functions from Spec A into GF(3). Thus each element of A is a function $x : \mathrm{Spec}\, A \to \mathrm{GF}(3)$ defined by $x(P) = x/1$, or what is the same thing, $x(P) = x + P$, the equivalence class of x modulo P. Now $x(P) \in \{0, 1, -1\}$, $V(x - i) = \{P \in \mathrm{Spec}\, A \mid x - i \in P\}$ is closed for $i = 0, 1, -1$, and hence $V(x)$, $V(x-1)$ and $V(x+1)$ partition Spec A into open and closed subsets on which the function x is constant. Conversely, if V_1, V_2, V_3 is a partition of Spec A by open subsets, it is well known that V_1, V_2, V_3 correspond to orthogonal idempotents e_1, e_2, e_3 in A such that $e_1 + e_2 + e_3 = 1$ and $V_i = V(1 - e_i)$, $i = 1, 2, 3$, so that as functions on Spec A, e_1, e_2, e_3 are simply the characteristic functions of V_1, V_2, V_3 respectively. Then if $a, b, c \in \{0, 1, -1\}$, $x = ae_1 + be_2 + ce_3 \in A$ and, as a function on Spec A, x is constant on each of V_1, V_2, V_3. Consequently, A consists precisely of those functions $f : \mathrm{Spec}\, A \to \mathrm{GF}(3)$ which are locally constant, i.e., constant on open sets.

It is immediate from the preceding paragraph that SL(2, A) can be embedded in $\prod_{P \in \mathrm{Spec}\, A} \mathrm{SL}(2, 3)$ as the subgroup of functions $T : \mathrm{Spec}(A) \to \mathrm{SL}(2, 3)$ which are locally constant.

The structure of $G = \mathrm{SL}(2, 3)$ is well known. It is of order 24 and exponent 12. The commutator subgroup is the unique subgroup of order 8 and is isomorphic to the quaternion group. In fact, we may take $i = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, $j = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ and $k = ij$. The elements of order 4 are $\pm i$, $\pm j$ and $\pm k$, and are all conjugate in G.

Note that $H(\mathrm{GF}(3)) = [G, G] = \{T \in G \mid T^4 = I\}$ and that $[G, H(\mathrm{GF}(3))] = \{\pm I\}$, the only other normal subgroup of G.

From these observations about SL(2, 3) and the fact that if A is locally GF(3), SL(2, A) is the group of locally constant functions from Spec A to SL(2, 3), the following lemma is evident:

Lemma 5.2. *Let A be a ring locally isomorphic to* GF(3).

(i) *The exponent of* SL(2, A) *is* 12.

(ii) *If* $T \in \mathrm{SL}(2, A)$, *then* $T^2 = I$ *iff* T *is scalar.*
(iii) *If* J *is an ideal in* A, *then*

$$H(J) = \{T \in \mathrm{SL}(2, A) \mid T \equiv I \bmod J \text{ and } T^4 = I\}.$$

Definition 5.3. Let A be locally GF(3) and let J be an ideal in A. Then we let

$$Q(J) = \{T \in H(J) \mid \text{for every } P \in \mathrm{Spec}(A),\ T(P) \neq -I\}.$$

Lemma 5.4. *If A is locally* GF(3) *and N is a subgroup of* $\mathrm{SL}(2, A)$ *with* $Q(J) \subseteq N$, *then* $H(J) \subseteq N$ *(i.e.,* $Q(J)$ *generates* $H(J)$*).*

Proof. Suppose $T \in H(J)$. Then $T(P)^4 = I$ for every $P \in \mathrm{Spec}(A)$. Now $U = \{P \mid T(P) = -I\}$ is open. If U is empty, $T \in Q(J)$. If not, the function S which is $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ on U and I elsewhere is in $Q(J)$. Hence $T = S^2 \cdot (S^2T)$ and our proof is complete since S, $S^2T \in Q(J)$. □

Lemma 5.5. *Let A be locally* GF(3) *and suppose* $T \in Q(A)$. *If* $l(T) = e$, *then T is conjugate to* R_e *where* $R_e(P) = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ *if* $e(P) = 1$ *and* $R_e(P) = I$ *if* $e(P) = 0$. *Consequently, any two elements of* $Q(A)$ *of the same level are conjugate.*

Proof. For each element X of order 4 in $\mathrm{SL}(2, 3)$ let $\rho(X)$ be an element of $\mathrm{SL}(2, 3)$ with $\rho(X)^{-1}X\rho(X) = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Let $T \in Q(A)$ with $l(T) = e$. Define S on $\mathrm{Spec}(A)$ by $S(P) = I$ if $e(P) = 0$ and $S(P) = \rho(T(P))$ if $e(P) = 1$. Now S is locally constant and $S^{-1}TS = R_e$ as claimed. □

Theorem 5.6. *Suppose A is locally* GF(3). *Let N be a subgroup of* $\mathrm{SL}(2, A)$ *with* $l(N) = J$. *Then N is a normal subgroup of* $\mathrm{SL}(2, A)$ *if and only if* $H(J) \subseteq N$.

Proof. Since $[\mathrm{SL}(2, A), L(2, A; J)] = H(J)$, any subgroup of level J containing $H(J)$ is normal.

Now suppose N is normal in $\mathrm{SL}(2, A)$. Let $S \in Q(J)$ with $l(S) = e$. By Lemma 3.4 there exists a commutator $[X, Y]$ with X elementary, $Y \in N$ and $l([X, Y]) = e$. Now $[X(P), Y(P)] \neq -I$ for any P since $X(P)$ is elementary. However, $[X(P), Y(P)]^4 = I$ for every P and hence $[X, Y] \in Q(J)$ and is conjugate to S by Lemma 5.5. Therefore $S \in N$, $Q(J) \subseteq N$, and thus by Lemma 5.4, $H(J) \subseteq N$. □

6. The main theorems

In this section we present the main structure theorems for normal subgroups of $\mathrm{SL}(2, A)$, Theorems 6.10–6.12. Fundamental to these theorems are the containment relations $\mathrm{SL}(2, A; \mathrm{vn}(J)) \subseteq H(J)$, and $H(J) \subseteq N$, where N is a subgroup of

SL(2, A) and $J = l(N)$. We begin by exploring the circumstances under which these relations hold.

Lemma 6.1. *Let J be an ideal in a commutative ring A and let N be a subgroup of* SL(2, A) *of level J normalized by $E(2, A)$. Then there exists an ideal J_0 with* SL(2, A; J_0) $\subseteq N$ *and J/J_0 von Neumann regular if and only if* SL(2, A; vn(J)) $\subseteq N$.

Proof. Since $J/\mathrm{vn}(J)$ is von Neumann regular by Lemma 3.9, we may take $J_0 = \mathrm{vn}(J)$ when given $\mathrm{SL}(2, A; \mathrm{vn}(J)) \subseteq N$.

Given J_0 with $\mathrm{SL}(2, A; J_0) \subseteq N$ and J/J_0 von Neumann regular, there exists $J_1 \supseteq J_0$ maximal with these properties by Lemma 1.5.

Theorem 3.8 and Lemma 3.10 then imply $\mathrm{vn}(J) \subseteq J_1$, completing our proof. □

Theorem 6.2. *Let J be an ideal in a commutative ring A and let N be a subgroup of* SL(2, A) *with $l(N) = J$. Suppose N contains* SL(2, A; J_0) *for some ideal J_0 such that J/J_0 is a von Neumann regular ideal in A/J_0. Then the following are equivalent*:

(i) *N is normalized by $E(2, A)$*;

(ii) $H(J) \subseteq N$;

(iii) *N is normal in* SL(2, A).

Proof. Clearly (ii) implies (i) and (iii) implies (i).

Assume now that $E(2, A)$ normalizes N. Lemma 6.1 implies that $\mathrm{SL}(2, A; \mathrm{vn}(J)) \subseteq N$. Set $A' = A/\mathrm{vn}(J)$, $J' = J/\mathrm{vn}(J)$, and let N' be the natural image of N in SL(2, A').

Now J' is von Neumann regular and hence N' is normal in SL(2, A') by Theorem 3.5. Since N is the pre-image of N' in SL(2, A), N is normal in SL(2, A). We have established that (i) implies (iii).

We now complete the proof by showing that (i) implies (ii). Let T be an arbitrary element of $H(J')$ and let $l(T) = e$. By Lemma 3.10, $\mathrm{vn}(J') = 0$, and by Lemma 3.9, $eA' = (3e)A' + (4e)A'$ where $3e$ and $4e$ are orthogonal idempotents.

Since $3eT \in H(3eJ')$ (where $J'e$ is considered as an ideal in $A'e$), $3eT \in [E(2, 3eA'), 3eN']$ by Theorem 4.4 and so $3eT + (1 - 3e)I \in N'$.

Since $4eT \in H(4eJ')$, $4eT \in [E(2, 4eA'), 4eN']$ by Theorem 5.6, and so $4eT + (1 - 4e)I \in N'$. But $T = (3eT + (1 - 3e)I)(4eT + (1 - 4e)I)$ and thus $T \in N'$. Since $H(J') \subseteq N'$, $H(J) \subseteq N$ and the proof is complete. □

Corollary 6.3. *If J is a von Neumann regular ideal in a commutative ring A and N is a normal subgroup of* SL(2, A) *with $l(N) = J$, then* $[\mathrm{SL}(2, A), N] = H(J)$.

Proof. The group $M = [\mathrm{SL}(2, A), N]$ is normal in SL(2, A) and $l(M) = J$ by

Lemma 3.4. By Theorem 6.2, $H(J) \subseteq M$. But $M \subseteq [\mathrm{SL}(2, A), L(2, A; J)] = H(J)$ by Theorem 3.5, and thus $M = H(J)$. □

Corollary 6.4. *Let J be an ideal in a commutative ring A such that J satisfies $\mathrm{SR}_2(A, J)$ and is 2-divisible. Suppose N is a subgroup of* $\mathrm{SL}(2, A)$ *with $l(N) = J$. Then the following are equivalent*:

(i) *N is normalized by $E(2, A)$*;

(ii) *$H(J) \subseteq N$*;

(iii) *N is normal in* $\mathrm{SL}(2, A)$.

Proof. By Lemma 1.9, there is a largest ideal J_0 in A with $\mathrm{SL}(2, A; J_0) \subseteq N$. If (i) holds, then by Lemma 3.3, J/J_0 is von Neumann regular. That (ii) and (iii) hold now follows from Theorem 6.2. If (ii) or (iii) holds, then (i) holds trivially. □

If J is an ideal in a commutative ring A, then the level of $H(J)$ is easily seen to be J by substituting elementary matrices for $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in (1). By Lemma 6.1, $\mathrm{SL}(2, A, \mathrm{vn}(J)) \subseteq H(J)$ if and only if there exists an ideal J_0 with J/J_0 von Neumann regular and $\mathrm{SL}(2, A; J_0) \subseteq H(J)$.

We may take $J_0 = 0$ when J is von Neumann regular, and by Lemma 3.3 with $N = H(J)$ in Reduction hypothesis 2.1, J_0 exists when J is 2-divisible and satisfies $\mathrm{SR}_2(A, J)$. It would be interesting to know for what rings and ideals J, $\mathrm{SL}(2, A; \mathrm{vn}(J)) \subseteq H(J)$.

In the next theorem we characterize $H(J)$ when $\mathrm{SL}(2, A; \mathrm{vn}(J)) \subseteq H(J)$.

Theorem 6.5. *Let J be an ideal in a commutative ring A and suppose that* $\mathrm{SL}(2, A; \mathrm{vn}(J)) \subseteq H(J)$. *Then for any $T \in \mathrm{SL}(2, A)$, $T \in H(J)$ if and only if*

(i) $T^4 \equiv I \bmod (3J + \mathrm{vn}(J))$;

(ii) $T^3 \equiv I \bmod (2J + \mathrm{vn}(J))$.

Proof. Consider the natural homomorphism to $A/\mathrm{vn}(J)$ and the map induced on $\mathrm{SL}(2, A)$. By Lemma 3.9, $J/\mathrm{vn}(J)$ is von Neumann regular so that Lemma 3.6 implies that the image of $H(J)$ is $H(J/\mathrm{vn}(J))$. Since $\mathrm{SL}(2, A, \mathrm{vn}(J)) \subseteq H(J)$, the pre-image of $H(J/\mathrm{vn}(J))$ is $H(J)$. Therefore, it suffices to work $\bmod\, \mathrm{vn}(J)$, and by Lemma 3.10 this amounts to assuming $\mathrm{vn}(J) = 0$.

Note that all matrices T in this argument are in $\mathrm{SL}(2, A; J)$ either because they are in $H(J)$ or by (ii) and (iii).

Since $\mathrm{vn}(J) = 0$, Lemma 3.9 tells us that J is von Neumann regular. Let $T \in \mathrm{SL}(2, A; J)$ and suppose $l(T) = e$. Consider the ring Ae. By Lemma 3.9, $Ae = 3eA + 4eA$ with $6e = 0$, so that

$$\mathrm{SL}(2, Ae) = \mathrm{SL}(2, Ae; 3eA)\, \mathrm{SL}(2, Ae; 4eA)\,.$$

Therefore

$$\begin{aligned} H(Ae) &= [\mathrm{SL}(2, Ae), \mathrm{SL}(2, Ae)] \\ &= [\mathrm{SL}(2, 3eA), \mathrm{SL}(2, 3eA)][\mathrm{SL}(2, 4eA), \mathrm{SL}(2, 4eA)] \\ &= H(4eA)H(3eA)\,. \end{aligned}$$

Now $T \in H(J)$ if and only if $Te \in H(Ae)$ since $T = Te + (1-e)I$. But $Te \in H(Ae)$ if and only if $3eT \in H(3eA)$ and $4eT \in H(4eA)$.

By Lemma 3.9, $3eA$ is a Boolean ring so that $3eT \in H(3eA)$ if and only if $(3eT)^3 = 3eI$ by Lemma 4.3.

Lemma 3.9 also tells us that $4eA$ is locally GF(3), so $4eT \in H(4eA)$ if and only if $(4eT)^4 = 4eI$ by Lemma 5.2.

Since $Te = 4eT + 3eT$ it follows that $Te \in H(Ae)$ if and only if $(Te)^4 \equiv I \bmod 3eA$ and $(Te)^3 \equiv I \bmod 2eA$.

This completes the proof. □

We remark that for an arbitrary commutative ring A and ideal J we may work $\bmod \mathrm{vn}(J)$ and thereby learn that (i) and (ii) are always necessary for membership in $H(J)$.

These conditions are easy to verify and useful, as the next theorem will show.

Lemma 6.6. *Let J be an ideal in a commutative ring and let $T = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2, A)$ with a invertible in A and $c \in J$. If $\mathrm{SL}(2, A; \mathrm{vn}(cA)) \subseteq H(J)$, then $T = RS$ with $R \in H(J)$ and S upper triangular.*

Proof. This proof is simply a matter of writing down appropriate matrices. They are

$$R = \begin{bmatrix} 1 & ac \\ a^{-1}c & 1 + c^2 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} a & b - \delta c \\ 0 & a^{-1}\delta \end{bmatrix},$$

where $\delta = \det(T)$. Obviously $T = RS$. It remains to verify that $R \in H(J)$. Computing $\bmod \mathrm{vn}(cA)$ we get

$$R^2 = \begin{bmatrix} 1 + c^2 & 3ac \\ 3a^{-1}c & 1 + 4c^2 \end{bmatrix} \quad \text{and} \quad R^3 \equiv \begin{bmatrix} 1 + 4c^2 & 8ac \\ 8a^{-1}c & 1 + 12c^2 \end{bmatrix}.$$

Now R^2 is scalar and $\bmod 3cA$ and hence $R^4 \equiv I \bmod 3cA$. We have also $R^3 \equiv I \bmod 2cA$, and thus we see that $R \in H(J)$ by Theorem 6.5. □

The next two lemmas, which play a role in the structure theorems, provide interesting insight into the nature of the ideal $\mathrm{vn}(J)$.

Lemma 6.7. *Let J be an ideal in a commutative ring A. Then*

(i) *If $x \in A$ and $x^3 - x \in J$, then $x^3 - x \in \mathrm{vn}(J)$;*

(ii) *If $u \in A$ is a unit with $u^2 \equiv 1 \bmod J$, then $u^2 \equiv 1 \bmod \mathrm{vn}(J)$; and*

(iii) *If* $\mathrm{SL}(2, A; \mathrm{vn}(J)) \subseteq H(J)$, *then* $\left[\begin{smallmatrix} u^{-1} & 0 \\ 0 & u \end{smallmatrix}\right] \in H(J)$ *for any unit* u *in* A *with* $u \equiv 1 \bmod J$.

Proof. It suffices to work mod $\mathrm{vn}(J)$, and so we assume $\mathrm{vn}(J) = 0$. By Lemma 3.9, $(x^3 - x)A$ has an idempotent generator e. Now $x^3 - x = (x^3 - x)(1 - e) + (x^3 - x)e = 0 + (xe)^3 - xe = 0$. This proves (i). Since $u^2 \equiv 1 \bmod J$, $u^3 - u \in J$, and hence $u^3 = u$ and $u^2 = 1$, establishing (ii).

Now we apply Theorem 6.5. Let u be any unit in A with $u \equiv 1 \bmod J$. Since $u^2 \equiv 1 \bmod J$, $u^2 = 1$ by (ii). Let $u = 1 + t$. Then $t^2 = -2t$. Therefore

$$u^3 - 1 = t^3 + 3t^2 + 3t = (t^3 - t) + 3t^2 + 4t = 3t^2 + 4t = -2t .$$

Since $u^2 = 1$, $u^4 = 1$ and $\left[\begin{smallmatrix} u^{-1} & 0 \\ 0 & u \end{smallmatrix}\right] \in H(J)$ by Theorem 6.5. □

Lemma 6.8. *Let J be an ideal in a commutative ring A. If a triangular matrix $T \in H(J)$, then T is congruent to a scalar matrix* $\bmod \mathrm{vn}(J)$.

Proof. Since $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] \in E(2, A)$ it normalizes $H(J)$, and hence it suffices to consider an upper triangular matrix $T = \left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]$ in $H(J)$. We work mod $\mathrm{vn}(J)$ and show T is scalar. We use the remarks after Theorem 6.5 that the conditions given there are necessary.

Now $a^2 - 1 \equiv 0 \bmod J$ and hence $a^2 \equiv 1 \bmod \mathrm{vn}(J)$ by Lemma 6.7. Therefore $a \equiv d \bmod \mathrm{vn}(J)$. We have $T^3 \equiv \left[\begin{smallmatrix} a & 3b \\ 0 & a \end{smallmatrix}\right] \equiv I \bmod (2J + \mathrm{vn}(J))$ by Theorem 6.5(ii), giving $b \in 2J + \mathrm{vn}(J)$. Also, $T^4 \equiv \left[\begin{smallmatrix} 1 & 4ab \\ 0 & 1 \end{smallmatrix}\right] \equiv I \bmod (3J + \mathrm{vn}(J))$ by Theorem 6.5(i), giving $b \in 3J + \mathrm{vn}(J)$. Since $6J \subseteq \mathrm{vn}(J)$ by Lemma 3.9, $b \in \mathrm{vn}(J)$, completing the proof. □

The largest possible normal subgroup $\mathrm{SL}(2, A)$ having level ideal J is $L(2, A; J)$. The next theorem asserts that, with the appropriate hypotheses, the structure theorems hold for $L(2, A; J)$. This paves the way for the full theorems.

Theorem 6.9. *Let J be an ideal in a commutative ring A. Suppose that for every $a \in A$, $b \in J$ with $(a, b)A = A$, there exists $x \in A$ with $a + bx$ a unit in A. Suppose* $\mathrm{SL}(2, A; \mathrm{vn}(J)) \subseteq H(J)$. *Then $L(2, A; J) = H(J)U(L(2, A; J))$.*

Proof. Let $T = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in L(2, A; J)$. By hypothesis there exists $x \in A$ with $a + cx = u$ a unit in A. By (3), $X^{-1}TX = T' = \left[\begin{smallmatrix} u & * \\ * & * \end{smallmatrix}\right]$ where $X = \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]$. By Lemma 6.6 there is an upper triangular matrix S with $T' = RS$ and $R \in H(J)$. Now $T = (XRX^{-1})(XSX^{-1})$, which places T in $H(J)U(L(2, A; J))$ and completes the proof since T was arbitrary. □

We have finally arrived at the main theorems. The first one is, in essence, a description of the normal subgroups of SL(2, A) for A a von Neumann regular ring.

Theorem 6.10. *Let J be a von Neumann regular ideal in a commutative ring A, and let N be a subgroup of* SL(2, A) *with $l(N) = J$. Then N is normal in* SL(2, A) *if and only if $H(J) \subseteq N$. If N is a normal subgroup of* SL(2, A), *then $N = H(J)U(N)$.*

Proof. If $(a, b)A = A$ with b in J, and e is the idempotent generator of bA, then $a(1-e) + e = a + (1-a)e$ is a unit so that the hypothesis of Theorem 6.9 is satisfied. The theorem is obvious from Theorems 6.2 and 6.9. □

Theorem 6.11. *Let A be an* SR_2*-ring with $\frac{1}{2} \in A$, and suppose N is a subgroup of* SL(2, A) *with $l(N) = J$. Then N is normal in* SL(2, A) *if and only if $H(J) \subseteq N$. If N is normal, then $N = H(J)U(N)$.*

Proof. The theorem is an obvious consequence of Corollary 6.4 and Theorem 6.9. □

Theorem 6.12. *Let A be an* SR_2*-ring with $\frac{1}{2} \in A$. Then there is a* 1-1 *correspondence between normal subgroups of* SL(2, A) *and triples (J, P, G) where J is an ideal in A, P is an additive subgroup of J containing* vn(J), *and G is a subgroup of the group of units of A such that $\{u \mid u$ is a unit in A and $u \equiv 1 \bmod J\} \subseteq G \subseteq \{u \mid u$ is a unit in A and $u^2 \equiv 1 \bmod J\}$.*

Proof. We know by Theorem 6.11 that if N is a normal subgroup of SL(2, A), then $N = H(J)U(N)$ where $J = l(N)$.

First we show that $U(N) = D(N)E(N)$ where $E(N)$ denotes the set of elementary matrices in $U(N)$. Suppose $T = \left[\begin{smallmatrix} u^{-1} & b \\ 0 & u \end{smallmatrix}\right] \in N$. Since $u^2 \equiv 1 \bmod J$, $T \equiv \left[\begin{smallmatrix} u & b \\ 0 & u \end{smallmatrix}\right] \bmod \mathrm{vn}(J)$ by Lemma 6.7. Since 2 is invertible, $3J \equiv 0 \bmod \mathrm{vn}(J)$ by Lemma 3.9. Therefore $T^3 \equiv \left[\begin{smallmatrix} u & 0 \\ 0 & u \end{smallmatrix}\right] \bmod \mathrm{vn}(J)$. Since $\left[\begin{smallmatrix} u^{-1} & 0 \\ 0 & u \end{smallmatrix}\right] \equiv \left[\begin{smallmatrix} u & 0 \\ 0 & u \end{smallmatrix}\right] \bmod \mathrm{vn}(J)$ and $\mathrm{SL}(2, A, \mathrm{vn}(J)) \subseteq N$, $\left[\begin{smallmatrix} u^{-1} & 0 \\ 0 & u \end{smallmatrix}\right] \in N$ and so does $\left[\begin{smallmatrix} 1 & ub \\ 0 & 1 \end{smallmatrix}\right]$, since $T = \left[\begin{smallmatrix} u^{-1} & 0 \\ 0 & u \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & ub \\ 0 & 1 \end{smallmatrix}\right]$. We have $U(N) = D(N)E(N)$ as claimed.

We now define a map from normal subgroups to triples by $N \to (J, P, G)$ where $J = l(N)$, $P = \{x \mid \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right] \in N\}$, and $G = \{u \mid \left[\begin{smallmatrix} u^{-1} & 0 \\ 0 & u \end{smallmatrix}\right] \in N\}$. Since $\mathrm{SL}(2, A, \mathrm{vn}(J)) \subseteq N$, $\mathrm{vn}(J) \subseteq P$. Since $l(N) = J$, $u^2 \equiv 1 \bmod J$ for every u in G, and by Lemma 6.7, $u \in G$ if $u \equiv 1 \bmod J$.

Since J and $U(N)$ determine N, the map is injective. It remains to show that it is surjective. Let (J, P, G) be a triple satisfying the hypotheses and set $M = H(J)U$ where $U = \{\left[\begin{smallmatrix} u^{-1} & x \\ 0 & u \end{smallmatrix}\right] \mid u \in G$ and $x \in u^{-1}P\}$. Using the hypotheses on P and G, and Lemma 6.7, one sees that U is a subgroup of SL(2, A). By Theorem 6.11, M is a normal subgroup of level J. If $U = U(M)$, then $M \to (J, P, G)$, and we are done.

Obviously $U \subseteq U(M)$. If $T \in U(M)$, there exists $X \in U$ with $S = TX^{-1} \in H(J)$ since $M = H(J)U$. Now $S = \left[\begin{smallmatrix} v^{-1} & t \\ 0 & v \end{smallmatrix}\right]$ and $v \equiv 1 \bmod J$, so $v \in G$. By Lemma 6.8, $t \in \mathrm{vn}(J)$, so that $vt \in P$, and this shows that $S \in U$. This completes the proof. □

Just as the correspondence in Theorem 6.12 was derived from Theorem 6.11, one can derive a similar correspondence from Theorem 6.10. In fact, if A is von Neumann regular, one gets a correspondence between subgroups N of $\mathrm{SL}(2, A)$ and 5-tuples consisting of the level ideal J, two additive subgroups P_1, P_2 of J, a group G of units of A (congruent to 1 mod J), and a homomorphism relating G to P_2.

Theorem 6.13. *Suppose that A is a commutative ring, N is a normal subgroup of* $\mathrm{SL}(2, A)$, *and $l(N) = J$ is 2-divisible and contained in the Jacobson radical of A. Then $N = \mathrm{SL}(2, A; J)D(N)$.*

Proof. Since J is contained in the Jacobson radical, J satisfies $\mathrm{SR}_2(A, J)$. By Lemma 3.9, $J/\mathrm{vn}(J)$ is von Neumann regular, and hence $\mathrm{vn}(J) = J$, since $J/\mathrm{vn}(J)$ is in the radical of $A/\mathrm{vn}(J)$ and has no nonzero idempotents. By Lemma 3.3 and Lemma 6.1, $\mathrm{SL}(2, A; J) = \mathrm{SL}(2, A; \mathrm{vn}(J)) \subseteq N$. Let $T \in N$. Then $T \equiv \left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right] \bmod J$, for some a in A. Since J is contained in the Jacobson radical, a is a unit in A so that $\left[\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right] \in N$. The theorem is apparent. □

Note that this theorem applies to any commutative ring in which 2 is a unit.

7. Normal subgroups of GL(2, A)

In this section we exploit the results of Section 6 in order to exhibit the structure of normal subgroups of $\mathrm{GL}(2, A)$ for A an SR_2-ring with $\frac{1}{2} \in A$ or $\frac{1}{6} \in A$. As noted in the introduction, the descriptions given here are analogous to those obtained by other authors for $\mathrm{GL}(n, A)$, $n \geq 3$, except that $H(J)$ replaces $E(n, A; J)$.

Lemma 7.1. *Let A be an* SR_2*-ring with $\frac{1}{2} \in A$. If M is a normal subgroup of* $\mathrm{GL}(2, A)$ *containing* $\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$, *then M contains* $\mathrm{SL}(2, A; Ab)$.

Proof. Let N be the smallest normal subgroup of $\mathrm{GL}(2, A)$ containing $\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$. Clearly $N \subseteq \mathrm{SL}(2, A; Ab)$ and $l(N) = bA$. By Theorem 6.9, $H(bA) \subseteq N$. We will show $N = \mathrm{SL}(2, A; Ab)$.

Let $A' = A/\mathrm{vn}(bA)$ and let b' be the image of b under the natural map. Using Lemma 3.10 we have that $(b')^2 = e$ is an idempotent with $A'e$ locally isomorphic to $\mathrm{GF}(3)$. Now N' is certainly contained in $\mathrm{SL}(2, A'; eA')$ and is normalized by

GL(2, A'; eA'). Hence, if we show the smallest normal subgroup of GL(2, eA') containing $\left[\begin{smallmatrix}1 & b'\\ 0 & 1\end{smallmatrix}\right]$ is SL(2, eA'), we will have $N' = \mathrm{SL}(2, A', eA')$.

Now b' is a unit in eA' and for every unit u in eA' any normal subgroup containing $\left[\begin{smallmatrix}1 & b'\\ 0 & 1\end{smallmatrix}\right]$ contains

$$\begin{bmatrix}1 & ub'\\ 0 & 1\end{bmatrix} = \begin{bmatrix}u & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & b'\\ 0 & 1\end{bmatrix}\begin{bmatrix}u^{-1} & 0\\ 0 & 1\end{bmatrix}.$$

By Lemma 2.4, these units generate $A'e$ as an additive group. Hence a normal subgroup of SL(2, $A'e$) containing $\left[\begin{smallmatrix}1 & b'\\ 0 & 1\end{smallmatrix}\right]$ contains $E(2, A'e) = \mathrm{SL}(2, A'e)$. Therefore $N' = \mathrm{SL}(2, A', eA')$ and $N = \mathrm{SL}(2, A, bA)$. □

Theorem 7.2. *Let A be an* SR_2*-ring with $\frac{1}{2} \in A$, and let M be a subgroup of* GL(2, A) *with $l(M) = J$. Then M is normal in* GL(2, A) *if and only if $H(J) \subseteq M$ and there exists an ideal $J_0 \subseteq J$ such that $l(D(M)) \subseteq J_0$ and $M = H(J)\,\mathrm{SL}(2, A; J_0)D(M)$.*

Proof. Since $H(J)$ and $\mathrm{SL}(2, A; J_0)$ are normal subgroups of GL(2, A), and $D(M)$ consists of scalar matrices mod $\mathrm{SL}(2, A; J_0)$, the conditions of the theorem are certainly sufficient.

On the other hand, suppose M is normal. Let $T = \left[\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right] \in M$ with $\det(T) = \delta$, and let u be a unit in A. Then

$$X = \left[\begin{bmatrix}u & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}a & b\\ c & d\end{bmatrix}\right] = \delta^{-1}\begin{bmatrix}\delta + bc(1-u^{-1}) & bd(1-u^{-1})\\ ac(1-u) & \delta + bc(1-u)\end{bmatrix}. \tag{5}$$

Now $X \in M \cap \mathrm{SL}(2, A) = N$. Applying (4) we can take d to be a unit, while $u = \frac{1}{2}$ will give us $-bd \in l(N)$ and so $b \in l(N)$. By Lemma 1.4, $l(N) = l(M) = J$ so that by Theorem 6.9, $H(J) \subseteq N$.

Let J_0 be the largest ideal so that $\mathrm{SL}(2, A; J_0) \subseteq N$. By Lemma 1.1, $l(D(M)) \subseteq J_0$. We now show that $N = H(J)\,\mathrm{SL}(2, A; J_0)D(N)$.

By Theorem 6.9, it suffices to show that if $S = \left[\begin{smallmatrix}d^{-1} & b\\ 0 & d\end{smallmatrix}\right] \in N$, then $S \in H(J)\,\mathrm{SL}(2, A; J_0)D(N)$. Applying (5) to S with $u = \frac{1}{2}$ we get $X = \left[\begin{smallmatrix}1 & -bd\\ 0 & 1\end{smallmatrix}\right] \in N$. By Lemma 7.1, $b \in J_0$. Then $\left[\begin{smallmatrix}d^{-1} & 0\\ 0 & d\end{smallmatrix}\right] = SX \in D(N)$ and $S \in \mathrm{SL}(2, A; J_0)D(N)$.

Now let $T = \left[\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right]$ be an arbitrary element of M with $\det(T) = \delta$. Since A is an SR_2-ring, there exists $y \in A$ with $a + cy = v$, a unit in A. By (3), T has a conjugate $T' = \left[\begin{smallmatrix}v & *\\ * & *\end{smallmatrix}\right]$. By Lemma 6.6, $T' = RS$ with $R \in H(J)$ and $S = \left[\begin{smallmatrix}v & t\\ 0 & v^{-1}\delta\end{smallmatrix}\right]$. We again apply (5) to S with $u = \frac{1}{2}$ and get $\left[\begin{smallmatrix}1 & -tv^{-1}\delta\\ 0 & 1\end{smallmatrix}\right]$ in M and $t \in J_0$ by Lemma 7.1. Since $t \in J_0$, S is certainly in $\mathrm{SL}(2, A; J_0)D(M)$ and so $T' \in H(J)\,\mathrm{SL}(2, A; J_0)D(M)$. Now $T^{-1}T' \in N$, and hence $T \in H(J)\,\mathrm{SL}(2, A; J_0)D(M)$, completing our proof. □

Theorem 7.3. *Let A be an* SR_2*-ring with $\frac{1}{6} \in A$, let M be a subgroup of* GL(2, A)

with $l(M)=J$. *Then* M *is normal in* $\mathrm{GL}(2, A)$ *if and only if* $M = \mathrm{SL}(2, A; J)D(M)$.

Proof. This follows immediately from Theorems 2.6 and 7.2. □

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