ITERATED DOUBLE COVERS AND CONNECTED COMPONENTS OF MODULI SPACES

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1. INTRODUCTION

Let $S$ be a smooth minimal surface of general type over the field of complex numbers and denote by $\mathcal{M}(S)$ the coarse moduli space of surfaces of general type homeomorphic to $S$, $\mathcal{M}(S)$ is a quasi-projective variety by Gieseker theorem [1].

Since $K^2 > 0$, the divisibility $r(S)$ of the canonical class $k_S = c_1(K_S) \in H^2(S, \mathbb{Z})$ is well defined, i.e.

$$r(S) = \max \{ r \in \mathbb{N} | r^{-1}c_1(S) \in H^2(S, \mathbb{Z}) \}.$$

$r(S)$ is a positive integer which is invariant under deformation and the set $\mathcal{M}_d(S) = \{ [S'] \in \mathcal{M}(S) | r(S') = r(S) \}$ is a subvariety of $\mathcal{M}(S)$ and the number of connected components of $\mathcal{M}_d(S)$ is bounded by a function $\delta$ of the numerical invariants $K^2_S, \chi(C_S)$.

It is known that $\delta$ is not bounded [2]. Here we prove that "in general" $\delta$ takes quite large values, more precisely we have

**Theorem A.** For every real number $4 \leq \beta \leq 8$ there exists a sequence $S_n$ of simply connected surfaces of general type such that:

(a) $y_n = K^2_{S_n}, x_n = \chi(C_{S_n}) \rightarrow \infty$ as $n \rightarrow \infty$.

(b) $\lim \limits_{n \rightarrow \infty} (y_n/x_n) = \beta$.

(c) $\delta(S_n) \geq y_n^{(1/5) \log y_n}$ (here $\delta(S_n)$ is the number of connected components of $\mathcal{M}_d(S_n)$).

Note that if $S_1, S_2$ are simply connected minimal surfaces of general type with $\chi > 1$ belonging to the same $\mathcal{M}_d$ then, according to the Freedman theorem [3] and the results of Wall about automorphisms of unimodular quadratic forms [4], there exists a homeomorphism between $S_1$ and $S_2$ which preserves the canonical classes.

After the recent work of Seiberg and Witten [5] it is known that, up to sign, the canonical class $k$ is a differentiable invariant of minimal surfaces of general type and it is possible that surfaces belonging to the same $\mathcal{M}_d$ have the same Donaldson polynomials.

We note moreover that the lower bound we achieve is considerably greater than the previous bounds (in [2, 6, 7] bounds of type $\delta \geq \log \log (K^2)$ have been proved) and in particular we prove the impossibility of a polynomial upper bound of $\delta$. (Actually we have upper bounds which grow exponentially on $K^2$, for example, for the moduli space of regular surfaces we have $\delta \leq c y^{7/2} (y = K^2), c = constant$ [8].)

**Theorem A** relies on the explicit description of the connected components in the moduli space of a wide class of surfaces of general type whose Chern numbers spread in all the region $\frac{1}{4} c_2, c_1^2 \leq c_2^2 \leq 2c_2$. 

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1. A finite map between normal algebraic surfaces \( p : X \to Y \) is called a **simple iterated double cover** associated to a sequence of line bundles \( L_1, \ldots, L_n \in \text{Pic}(Y) \) if the following conditions hold:

1. There exist \( n + 1 \) normal surfaces \( X = X_0, \ldots, X_n = Y \) and \( n \) flat double covers \( \pi_i : X_{i-1} \to X_i \) such that \( p = \pi_n \circ \cdots \circ \pi_1 \).
2. If \( p_i : X_i \to Y \) is the composition of \( \pi_j \) with \( j > i \) then we have for every \( i = 1, \ldots, n \) the eigensheaves decomposition \( \pi_i \ast \mathcal{O}_{X_{i-1}} = \mathcal{O}_{X_i} \otimes \mathcal{O}_Y(-L_i) \).

For any sequence \( L_1, \ldots, L_n \in \text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \) define \( N(L_1, \ldots, L_n) \) as the image in the moduli space of the set of surfaces of general type whose canonical model is a simple iterated double cover of \( \mathbb{P}^1 \times \mathbb{P}^1 \) associated to \( L_1, \ldots, L_n \).

The main theme of this paper is to determine sufficient conditions on the sequence \( L_1, \ldots, L_n \) in such a way that the set \( N(L_1, \ldots, L_n) \) has "good" properties; the conditions we find are summarized in the following definition:

**Definition C.** A sequence \( L_1, \ldots, L_n, L_i = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a_i, b_i), n > 2 \) of line bundles on \( \mathbb{P}^1 \times \mathbb{P}^1 \) is called a **good sequence** if it satisfies the following conditions.

1. \( a_i, b_i \geq 3 \) for every \( i = 1, \ldots, n \).
2. \( \max_{j<i} \min(2a_i - a_j, 2b_i - b_j) < 0 \).
3. \( a_i \geq b_i + 2, \quad b_{i-1} \geq a_{i-1} + 2 \).
4. \( a_i, b_i \) are even for \( i = 2, \ldots, n \).
5. \( \max_{i<n} 2a_i - a_{i+1} 2b_i - b_{i+1} \geq 2 \).

The main result we prove is (Theorems 5.1, 5.2 and 5.7):

**Theorem D.** Let \( L_1, \ldots, L_n \) be a good sequence in sense of Definition C, then:

1. \( N(L_1, \ldots, L_n) \) is a nonempty connected component of the moduli space.
2. \( N(L_1, \ldots, L_n) \) is reduced, irreducible and unirational. (For (a) and (b) the condition C5 is not necessary.)
3. The generic \( [S] \in N(L_1, \ldots, L_n) \) has \( \text{Aut}(S) = \mathbb{Z}/2\mathbb{Z} \).
4. If \( M_1, \ldots, M_m \) is another good sequence and \( N(L_1, \ldots, L_n) = N(M_1, \ldots, M_m) \) then \( n = m \) and \( L_i = M_i \) for every \( i = 1, \ldots, n \).

Simple iterated double covers of \( \mathbb{P}^1 \times \mathbb{P}^1 \) associated to good sequences are simply connected (because of (C1), according to [9, Theorem 1.8]) and by Freedman's result [3] two of them are homeomorphic if and only if they have the same invariants \( K^2, \chi \) and \( r \) mod 2.

It is clear that the proof of Theorem A reduces to counting the number of good sequences giving the same invariants \( K^2, \chi \) and \( r \).

Theorem D gives us some new interesting examples of homeomorphic but not deformation equivalent surfaces of general type.

**Example E.** Two deformation inequivalent surfaces \( S_1, S_2 \) are homeomorphic with the same divisibility which are double covers of the same surface \( S_0 \).

Define \( S_0 \subset \mathbb{P}^1 \times \mathbb{P}^1 \) a simple iterated double cover associated to \( L_1 = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(8, 12), L_2 = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(8, 4) \); by adjunction formula \( K_{S_0} = p^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(14, 14) \).

Let \( a \neq b \) be an integer \( \geq 17 \) and let \( D_1 \in |p^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2a, 2b)|, D_2 \in |p^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2b, 2a)| \) be two smooth divisors, the double cover \( S_1, S_2 \) of \( S_0 \) with branching divisors \( D_1, D_2, \ldots \)
respectively, have the required properties. Note that $D_1^2 = D_2^2$, $K_{S_0}$, $D_1 = K_{S_0}$, $D_2$ and $D_1$, $D_2$ have the same genus.

It is worth mentioning here another interesting fact (Corollary 5.8), if

$$X = X_0 \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_n} X_n = \mathbb{P}^1 \times \mathbb{P}^1$$

is a simple iterated double cover associated to a good sequence then the surfaces $X_1$ and the map $\pi_1$ (and then by induction $X_i$ and $\pi_i$ for all $i = 1, \ldots, n$) are uniquely determined by $X$.

In fact, assume for simplicity that $[X] \in N(L_1, \ldots, L_n)$ is generic, then by Theorem D(c) $X$ has only a nontrivial automorphism $\tau$ and then $X_1$ is the quotient $X/\tau$.

Using the same idea we prove D(d) as a consequence of D(a), D(b) and D(c).

Every simple iterated double cover $X$ associated to $L_1, \ldots, L_n \in \text{Pic}(Y)$ can be embedded in the total space of the vector bundle $V = L_1 \oplus \cdots \oplus L_n \rightarrow Y$, e.g. in the case $n = 2$ the equations of $X$ are

$$z_1^2 = f_1 + z_2g_1, \quad z_2^2 = f_2$$

with $z_i \in H^0(V, p^*L_i)$ the tautological section, $f_i \in H^0(Y, 2L_i)$ and $g_1 \in H^0(Y, 2L_1 - L_2)$.

Thus simple iterated double covers are naturally parametrized by a Zariski open subset of a finite dimensional vector space and then the proof of the openness of $N(L_1, \ldots, L_n)$ reduces to showing the surjectivity of a Kodaira-Spencer map.

In order to prove the closure of $N(L_1, \ldots, L_n)$ in the moduli space we must show that if a 1-parameter family of simple iterated double covers degenerates to a surface of general type $X_0$, then $[X_0] \in N(L_1, \ldots, L_n)$.

Here the main trouble is to prove that the flatness of all covering maps is preserved under specialization. Section 4 is devoted to proving this fact under some special and at first sight very strange assumption (e.g. (C4)). The key result is the classification of involutions acting on smoothing of rational double points (Proposition 4.2); from this it follows that if a family of smooth double covers $X_t \rightarrow Y_t$, $t \in \Delta^*$ degenerates to a nonflat double cover $X_0 \rightarrow Y_0$ and $X_0$ has at most rational double points then $Y_0$ has at least one cyclic singularity at $y_0$ and the Milnor fibre $F_t$ of the smoothing $(Y, y_0) \rightarrow (\Delta, 0)$ has the canonical class in $H^2(F_t, \mathbb{Z})$ not divisible by 2. In particular, if $r(Y_t)$ is even then the inclusion $F_t \subset Y_t$ gives a contradiction.

The proof of D(c) (Section 5) uses a degeneration argument.

Analog of Definition C and Theorem D for simple iterated double cover of $\mathbb{P}^2$ appear in [10].

2. PRELIMINARIES AND CONVENTIONS

All varieties we consider are over the ground field $\mathbb{C}$. If $Y$ is a normal irreducible proper algebraic variety we denote by $\text{Pic}(Y)$ the Picard group of line bundles on $Y$, by $\Omega_Y^1$ the sheaf of Kähler differentials and by $K_Y$ the Weil canonical divisor of $Y$. For every coherent sheaf $\mathcal{F}$ on $Y$ we denote by $h^0(\mathcal{F}, y)$ the dimension of the $\mathbb{C}$-vector space $H^0(Y, \mathcal{F})$.

We denote by $\mathcal{M}$ the coarse moduli space of surfaces of general type, $\mathcal{M}$ is the disjoint countable union of the quasiprojective varieties $[1]

$$\mathcal{M}_{x, y} = \{[S] \in \mathcal{M} | S \text{ minimal}, K_S^2 = y, \chi(\mathcal{O}_S) = x\}.$$  

We recall that the complex analytic germ of $\mathcal{M}$ at a point $[S]$ is isomorphic to $\text{Def}(X)/\text{Aut}(X)$ where $X$ is the canonical model of $S$, $\text{Def}(X)$ is the base space of the Kuranishi family of $X$ and $\text{Aut}(X)$ is the finite [11] group of biregular automorphisms of $X$.  

Let $f : X \to Y$ be a morphism between complex algebraic varieties. If $\mathcal{F}$ is an $\mathcal{O}_X$-module and $\mathcal{G}$ is an $\mathcal{O}_Y$-module the natural sheaf morphisms $f_* f^* \mathcal{G} \to f_* \mathcal{F} \to \mathcal{F}$ induce isomorphisms $\text{Hom}_{\mathcal{O}_X}(f^* \mathcal{G}, \mathcal{F}) \simeq \text{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_* \mathcal{F})$ and $\text{Hom}_{\mathcal{O}_X}(f^* \mathcal{G}, \mathcal{F}) = \text{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_* \mathcal{F})$ (cf. [12, p. 110]).

**Lemma 2.1.** In the notation above assume $\mathcal{F}, \mathcal{G}$ are coherent:

(a) If $f$ is flat (i.e. $f^*$ is an exact functor) then there exists a convergent spectral sequence of vector spaces

$$E_2^{p,q} = \text{Ext}^p_{\mathcal{O}_X}(\mathcal{G}, R^q f_* \mathcal{F}) \Rightarrow \text{Ext}^q_{\mathcal{O}_Y}(f^* \mathcal{G}, \mathcal{F}).$$

(b) If $f$ is finite then there exists a convergent spectral sequence of $\mathcal{O}_Y$-modules

$$E_2^{p,q} = f_* \mathcal{E}xt^p_{\mathcal{O}_X}(L^q f^* \mathcal{G}, \mathcal{F}) \Rightarrow \mathcal{E}xt^q_{\mathcal{O}_Y}(\mathcal{G}, f_* \mathcal{F}).$$

(c) If $f$ is finite flat then for every $i \geq 0$ we have

$$\text{Ext}^i_{\mathcal{O}_X}(f^* \mathcal{G}, \mathcal{F}) = \text{Ext}^i_{\mathcal{O}_Y}(\mathcal{G}, f_* \mathcal{F}), \quad f_* \mathcal{E}xt^i_{\mathcal{O}_X}(f^* \mathcal{G}, \mathcal{F}) = \mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{G}, f_* \mathcal{F}).$$

**Proof.** (a) Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module, from the exactness of the functor $f^*$ and formula $\text{Hom}_{\mathcal{O}_X}(f^* \mathcal{G}, \mathcal{F}) = \text{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_* \mathcal{F})$ it follows that the direct image $f_* \mathcal{I}$ is an injective $\mathcal{O}_Y$-module.

The functor $\mathcal{F} \to \text{Hom}_{\mathcal{O}_X}(f^* \mathcal{G}, \mathcal{F})$ is the composition of $\mathcal{F} \to f_* \mathcal{F} \to \text{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_* \mathcal{F})$ and the sequence in (a) is the Grothendieck spectral sequence associated to this composition.

(b) The proof is similar to (a), we only recall that since $f$ is finite $f_*$ is an exact functor from coherent sheaves on $X$ to coherent sheaves on $Y$ and the $\mathcal{E}xt$ can be computed applying the contravariant Hom to locally free resolutions (cf. [12, III.6.5]).

(c) Is an obvious consequence of (a) and (b).}

Throughout this paper by a tower of height $n$ we shall mean the data of $n + 1$ irreducible algebraic varieties of the same dimension $X_0, \ldots, X_n$ and $n$ finite flat morphisms $\pi_i : X_{i-1} \to X_i$.

A tower is smooth (resp. normal) if every $X_i$ is smooth (resp. normal).

A deformation of the tower $(X_i, \pi_i)$ parametrized by a germ of complex space $(S, 0)$ is a commutative diagram

$$
\begin{array}{c}
X_0 \xrightarrow{\pi_1} X_1 \to \cdots \to X_n \to 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
\tilde{X}_0 \xrightarrow{\tilde{\pi}_1} \tilde{X}_1 \to \cdots \to \tilde{X}_n \to S
\end{array}
$$

such that for every $i = 0, \ldots, n$ the induced diagram

$$
\begin{array}{c}
X_i \to 0 \\
\downarrow \quad \downarrow \\
\tilde{X}_i \to S
\end{array}
$$

is a deformation of $X_i$ parametrized by $S$. Note that for tower of height 1 this is the usual definition of deformations of maps [13].

Denote by $\text{Def}(X_i, \pi_i)$ the functor of isomorphism classes of deformations of the tower $(X_i, \pi_i)$ and, for $j = 0, \ldots, n$, by $r_i : \text{Def}(X_i, \pi_i) \to \text{Def}(X_j)$ the induced morphism of functors.
Let now $\pi: X \to Y$ be a finite flat map between irreducible reduced algebraic varieties. By Lemma 1.1 we have an isomorphism

$$\Phi: \text{Ext}^1_c(\pi^*\Omega^1_Y, \mathcal{O}_X) \cong \text{Ext}^1_c(\Omega^1_X, \pi_*\mathcal{O}_X)$$

and the natural maps $\pi^*\Omega^1_Y \to \Omega^1_X$, $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ induce maps of Ext groups

$$\text{Ext}^1_c(\Omega^1_X, \mathcal{O}_X) \xrightarrow{\alpha} \text{Ext}^1_c(\pi^*\Omega^1_Y, \mathcal{O}_X)$$

where if $e \in \text{Ext}^1_c(\Omega^1_Y, \mathcal{O}_Y)$ is the isomorphism class of the extension

$$0 \to \mathcal{O}_Y \to E \to \Omega^1_Y \to 0$$

then $\Phi^{-1}\beta(e)$ is the isomorphism class of the extension

$$0 \to \mathcal{O}_X = \pi^*\mathcal{O}_Y \to \pi^*E \to \pi^*\Omega^1_Y \to 0.$$ 

The maps $\alpha$ and $\Phi^{-1}\beta$ have an interesting interpretation in terms of obstruction to deforming the map $\pi$.

We recall that if $Z$ is a reduced variety and $T^1_Z$ is the vector space of deformation of $Z$ over the double point $D = \text{Spec}(\mathbb{C}[t]/(t^2))$ there exists an isomorphism $T^1_Z = \text{Ext}^1_c(\Omega^1_Z, \mathcal{O}_Z)$ which to the deformation $Z \subset \tilde{Z} \to D$ associates the extension

$$0 \to \mathcal{O}_Z \to \Omega^1_Z \otimes \mathcal{O}_Z \to \Omega^1_Z \to 0.$$ 

If $T^1_Z$ is the space of first order deformations of the map $\pi$ then there exists a commutative diagram

$$\begin{array}{ccc}
T^1_Z & \xrightarrow{\Phi} & T^1_X = \text{Ext}^1_c(\Omega^1_X, \mathcal{O}_X) \\
\downarrow r^* & & \downarrow \Phi \\
T^1_Y = \text{Ext}^1_c(\Omega^1_Y, \mathcal{O}_Y) & \xrightarrow{\text{Ob}^{-1}\beta} & \text{Ext}^1_c(\pi^*\Omega^1_Y, \mathcal{O}_X)
\end{array}$$

where $r^*$ and $r_Y$ are the natural forgetting maps. In fact if $\tilde{X} \to \tilde{Y}$ is a deformation of $\pi$ over the double point then by local flatness criterion [14, Theorem 22.3] it is easy to see that it is flat and the relation $a^{-1}(\pi) = \Phi^{-1}\beta(r_Y(\pi))$ follows from the following commutative diagram

$$\begin{array}{ccc}
0 \to \mathcal{O}_X = \pi^*\mathcal{O}_Y \to \pi^*\Omega^1_Y \otimes \mathcal{O}_X \to \pi^*\Omega^1_Y \to 0 \\
\downarrow & & \downarrow \\
0 \to \mathcal{O}_X \to \Omega^1_X \otimes \mathcal{O}_X \to \Omega^1_X \to 0.
\end{array}$$

Sometimes, especially in Section 3, if $(S, 0)$ is a germ of complex vector space we consider $(S, 0)$ as a covariant functor from the category of local artinian $\mathbb{C}$-algebras to the category of sets defined in the following way:

$$(S, 0)(A) = \{\text{morphisms } \varphi: (\text{Spec} A, 0) \to (S, 0)\}$$

where $0 \in \text{Spec} A$ is the closed point. It is well known that $(S, 0)$ is smooth as a germ if and only if it is smooth as a functor. (See [15] for the definition and first properties of a smooth functor.)

3. DEFORMATIONS

From now on by a surface we mean a complex projective surface. Let $X$ be a normal surface and let $\pi: X \to Y$ be the quotient of $X$ by an involution $\tau$. 

Lemma 3.1. In the above notation the following conditions are equivalent:

(i) \( \pi \) is flat.

(ii) There exists a line bundle \( \pi: L \rightarrow Y \) and a section \( f \in H^0(Y, 2L) \) such that the pair \( X, \tau \) is isomorphic to the subvariety of \( L \) defined by the equation \( z^2 = f, z \in H^0(L, \pi^*L) \) is the tautological section, and the involution is obtained by multiplication of \(-1\) in the fibres of \( L \).

(iii) The fixed subvariety \( R = \text{Fix}(\tau) \) is a Cartier divisor.

Moreover if \( X \) is smooth then \( \pi \) is flat if and only if \( Y \) is smooth.

Proof. The proof is standard, we give a sketch.

(i) \( \Rightarrow \) (ii) If \( \pi \) is flat then the group \( G = \{1, \tau\} \) acts on the rank 2 locally free sheaf \( \pi_*O_X \) and yields a character decomposition \( \pi_*O_X = \sigma_Y \oplus \sigma_Y(-L) \) for some \( L \in \text{Pic}(Y) \). \( X \) depends only on the \( \sigma_Y \) algebra structure of \( \pi_*O_X \) which is uniquely determined by a map \( f: \sigma_Y(-2L) \rightarrow \sigma_Y, f \in H^0(Y, 2L) \).

(ii) \( \Rightarrow \) (iii) is clear since \( R \) is the divisor of a section of \( \pi^*L \).

(iii) \( \Rightarrow \) (i) Let \( p \) be a fixed point of \( \tau \), then \( G \) acts on the local \( \mathbb{C} \)-algebra \( B = O_{x,p} \). Let \( A = B^G \) be the subring of invariant functions and let \( I \) be the ideal of \( R \). By definition, \( I \) is the ideal of \( B \) generated by \( \pi_f - f, \) all \( f \in B \).

If \( I \) is a principal ideal, it is easy to see using Nakayama's lemma that there exists a generator \( h \) of \( I \) such that \( \tau h = -h \) and then \( B \) is a free \( A \)-module generated by \( 1, h \).

If \( X \) is smooth, by (i) \( \Leftrightarrow \) (iii) it follows that \( \pi \) is flat if and only if \( \tau \) has no isolated fixed point, i.e. if and only if \( Y \) is smooth. (Note that if \( Y \) is smooth then \( \pi \) is always flat.) \[ \square \]

In this section, we investigate the deformations of \( X \) under the hypothesis that \( \pi \) is flat. Unfortunately, in general, if \( Y \) is singular \( \pi \) is not flat (consider for example the quotient of \( \mathbb{P}^2 \) by the involution \( \tau(x_0, x_1, x_2) = (-x_0, x_1, x_2) \)) but such a restriction will be sufficient for our later use.

Consider thus \( X = L \rightarrow Y \) defined by the equation \( z^2 = f(y) \). Denote \( D = \text{div}(f) \subset Y \), \( R = \text{div}(z) \subset X \).

Note that \( \pi^*D = 2R \) and \( X \) is normal if and only if \( Y \) is normal and \( D \) is reduced. If \( K_X, K_Y \) are the Weil canonical divisors of \( X \) and \( Y \), respectively, we have the adjunction formula \( K_X = \pi^*K_Y + R \), which follows from the usual Hurwitz formula for smooth varieties and from the reflexivity of canonical sheaves on normal varieties. In particular, if \( Y \) is Gorenstein then \( X \) is also Gorenstein.

Let \( \tilde{X} \) be the variety defined in \( L \times H^0(Y, D) \) by

\[ \tilde{X} = \{(z, y, h) | z^2 = f(y) + h(y)\} \cdot \]

Clearly, \( \tilde{X} \) is a double flat cover of \( Y \times H^0(Y, D) \), hence the second projection \( \tilde{X} \rightarrow H^0(Y, D) \) is flat and defines a map of functors \( \text{Nat}_{\pi}: (H^0(Y, D), 0) \rightarrow \text{Def}(X) \).

Definition 3.2. The image of the map \( \text{Nat}_{\pi} \) is called the set of natural deformations of \( X \) associated to \( \pi \).

Proposition 3.3. In the above notation let \( X \rightarrow Y \) be a deformation of the map \( \pi \) parametrized by a smooth germ \( (H, 0) \) and let \( r_X: (H, 0) \rightarrow \text{Def}(X) \) and \( r_Y: (H, 0) \rightarrow \text{Def}(Y) \) be
the induced maps. Assume:

(i) \( \tau_Y \) is smooth.
(ii) The image of \( r_X \) contains the natural deformations.
(iii) \( \text{Ext}_r^1(O_Y^1, - L) = 0, \quad H^1(O_X) = 0. \)

Then \( \dim T^1_X = \dim T^1_Y + h^0(O_Y(D)) + h^0(\theta_Y) - h^0(\theta_Y) - h^0(\theta_Y - L) - 1 \) and the map \( r_X \) is smooth.

We prove this proposition after presenting some lemmas.

**Lemma 3.4.** There exists an exact sequence of \( O_X \)-modules

\[
0 \to \pi^*\Omega^1_Y \to \Omega^1_X \to O_X(-R) \to 0. \tag{1}
\]

**Proof.** Let \( i: X \to L \) be the inclusion as in Lemma 2.1, since \( L \supset Y \) is locally a product there exists an obvious inclusion of sheaves \( \pi^{-1}\Omega^1_Y \subset i^{-1}\Omega^1_L \), tensoring with the flat module \( O_X \) we get an injection \( \pi^*\Omega^1_Y \to \Omega^1_L \otimes O_X \).

The sheaf \( \Omega^1_{L/Y} \) is clearly locally free and it is the \( O_L \) dual of the sheaf of vertical vector fields and therefore it is naturally isomorphic to \( \pi^*(\sigma) \).

We have the following first and second exact sequences of differentials

\[
0 \to \pi^*\Omega^1_Y \to \Omega^1_L \otimes O_X \to \Omega^1_L \otimes O_X \to 0
\]

\[
0 \to \pi^*\Omega^1_Y \to \Omega^1_L \otimes O_X \to \Omega^1_L \otimes O_X \to 0 \tag{2}
\]

and (1) is obtained by applying the snake lemma to

\[
0 \to O_X(-\pi*D) \to \Omega^1_L \otimes O_X \to \Omega^1_L \to 0
\]

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\downarrow \quad \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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Thus in order to prove the lemma it is enough to show that $y \cdot \pi^*$ is surjective.

Since $R$ is a locally principal divisor in the normal surface $X$ we have [14, Section 18; 11] $\text{Ext}^1 (\Omega^{-1} (R), \mathcal{O}_X) = H^0 (\mathcal{O}(2R))$ and since $\pi_* \mathcal{O}_X = \mathcal{O}_D$, $H^0 (\mathcal{O}(2R)) = H^0 (\mathcal{O}(D))$ and the restriction map $H^0 (\mathcal{O}(D)) \to H^0 (\mathcal{O}(D))$ is surjective if $H^1 (\mathcal{O}_Y) = 0$.

**Proof of Proposition 2.3.** We have a commutative diagram

$$
\begin{array}{ccc}
T_{\alpha} H & \xrightarrow{dr_x} & \text{Ext}^1_k (\Omega_X, \mathcal{O}_X) \\
\downarrow{dr_x} & & \downarrow{s} \\
\text{Ext}^1_{\mathcal{O}_Y} (\Omega_Y, \mathcal{O}_Y) & \xrightarrow{\Phi^{-1} \beta} & \text{Ext}^1_{\mathcal{O}_Y} (\pi^* \Omega_Y, \mathcal{O}_X).
\end{array}
$$

By Lemma 2.1 and hypothesis (iii) the map $\Phi^{-1} \beta$ is bijective. The kernel of $\alpha$ is the set of natural deformations and by (ii) is contained in the image of $dr_x$. It is now trivial to observe that $dr_x$ surjective implies $dr_Y$ surjective and since $H$ is smooth this is sufficient to prove that $r_x$ is smooth and $\dim T^x = \dim T^x + \dim \text{Im} e$.

With a more complicated proof it is possible to prove Proposition 3.3 without assuming $H$ to be smooth (cf. [10]).

**Definition 3.6.** (a) A normal tower $(X_i, \pi_i)$ of height $n$ is said to be *simple* if for every $i$, $\pi_i : X_{i-1} \to X_i$ is a flat double cover and there exist line bundles $L_1, \ldots, L_n \in \text{Pic} (X_n)$ such that $\pi_{i+1}^* \mathcal{O}_{X_{i+1}} = \mathcal{O}_{X_i} \oplus p_i^* (-L_i)$ where $p_i$ is the composition of $\pi_j$ for $j > i$.

(b) If $(X_0, \pi_0, L_0)$ is a simple tower we call the surface $X = X_0$ a *simple iterated double cover* of $Y$ associated to $L_1, \ldots, L_n \in \text{Pic} (Y)$ and the involution $\tau : X \to X$ such that $X/\tau = X_1$ the trivial involution.

Clearly the trivial involution depends on the simple tower and in general $X$ does not determine $\tau$.

It is important to observe that if $(X_n, \pi_n, L_n)$ is a smooth simple tower and $\text{Pic} (X_n)$ is without torsion then the maps $p_i^* : \text{Pic} (X_n) \to \text{Pic} (X_i)$ are injective and the line bundles $L_1, \ldots, L_n$ are uniquely determined by the maps $\pi_1, \ldots, \pi_n$.

**Theorem 3.7.** Let $(X_n, \pi_n, L_n)$ be a simple tower of height $n$ and let $(H, 0)$ be a smooth germ parametrizing a deformation of the tower. Denote $X = X_0$, $Y = X_n$ and let $r_i : (H, 0) \to \text{Def} (X_i)$ be the induced maps. Assume:

(i) $H^1 (\mathcal{O}_Y) = 0$.

(ii) $r_n : (H, 0) \to \text{Def} (Y)$ is smooth.

(iii) The natural deformations of $\pi_{i+1} : X_i \to X_{i+1} are contained in the image of $r_i$.

(iv) For every sequence $1 < j_1 < j_2 < \cdots < j_k < n$, $h > 0$

$$
\text{Ext}^1_k (\Omega_Y, \sum_{s=1}^h \mathcal{O}_{L_{j_s}}) = 0, \quad H^1 (\sum_{s=1}^h \mathcal{O}_{L_{j_s}}) = 0.
$$

(v) For every $i \in \{2, \ldots, n\}$ and for every subset $\{j_1, \ldots, j_h\} \subset \{1, \ldots, i-1, i+1, \ldots, n\}$ with $h > 0$ and $j_1 < i$

$$
H^0 (\sum_{s=1}^h \mathcal{O}_{L_{j_s}}) = 0.
$$

Then $r_0 : H \to \text{Def} (X)$ is smooth.
Note. If $H^0(L_i) \neq 0$ for every $i$ then condition (v) is equivalent to (vi) for every $j < i$ $H^0(Y, 2L_i - L_j) = 0$.

Proof. Induction on $n$, for $n = 1$ is just Proposition 3.3.

Assuming the theorem to be true for towers of height $n - 1$ it suffices to prove that conditions (i)-(v) hold for the surface $Z = X_{n-1}$ and the line bundles $M_i = \pi_*^*L_i$, $i = 1, \ldots, n - 1$.

The only nontrivial condition to check is the part of (iv) concerning Ext. Let $R \subset Z$, $D \subset Y$ be, respectively, the ramification and branching divisors of $\pi_*$. Applying $\text{Hom}_{\mathcal{O}_Y}(-, \sum_{s=1}^{n-1} M_s)$ to the exact sequence

$$0 \to \pi_*^*\Omega_Y^1 \to \Omega_{L_n}^1 \to \mathcal{O}_R(- R) \to 0$$

we get

$$H^0\left(\mathcal{O}_Y\left(2L_n - \sum_{s=1}^{n-1} L_s\right)\right) = \text{Ext}^1_{\mathcal{O}_Y}\left(\mathcal{O}_R(- R), \sum_{s=1}^{n-1} M_s\right) \to \text{Ext}^1_{\mathcal{O}_Y}\left(\Omega_{L_n}^1, \sum_{s=1}^{n-1} M_s\right)$$

$$\to \text{Ext}^1_{\mathcal{O}_Y}\left(\pi_*^*\Omega_Y^1, \sum_{s=1}^{n-1} M_s\right) = \text{Ext}^1_{\mathcal{O}_Y}\left(\Omega_Y^1, \sum_{s=1}^{n-1} L_s\right) \oplus \text{Ext}^1_{\mathcal{O}_Y}\left(\Omega_{L_n}^1, \sum_{s=1}^{n-1} L_s\right)$$

and the vector space on the left belongs to the exact sequence

$$H^0\left(\mathcal{O}_Y\left(2L_n - \sum_{s=1}^{n-1} L_s\right)\right) \to H^0\left(\mathcal{O}_D\left(2L_n - \sum_{s=1}^{n-1} L_s\right)\right) \to H^1\left(\mathcal{O}_Y\left(\sum_{s=1}^{n-1} L_s\right)\right).$$

COROLLARY 3.8. Let $Y$ be a rigid (i.e. $T^Y = 0$) normal surface and let $X$ be a simple iterated double cover of $Y$ associated to $L_1, \ldots, L_n \in \text{Pic}(Y)$. If conditions (i), (iv) and (v) of Theorem 3.7 are satisfied then Def($X$) is smooth.

Proof. $X$ is the top of a simple tower $(X_n, \pi_n, L_i)$ of height $n$; thus, according to Theorem 3.7 it is enough to show the existence of a smooth family of deformations of the tower satisfying conditions 3.7(ii) and (iii).

By Lemma 3.1 applied $n$ times we can embed $X$ in the vector bundle $V = L_1 \oplus \cdots \oplus L_n \to Y$ by the equations

$$z_i^t = f_i, \quad i = 1, \ldots, n$$

where $z_i : V \to \mathbb{P}^1L_i$ is a tautological section and $f_i = H^0(X_i, \mathbb{P}^1L_i)$ where $X_i$ is the surface in $L_{i+1} \oplus \cdots \oplus L_n$ of equations $z_j^t = f_j, j > i$ and $\pi_i$ is the restriction to $X_i$, the natural projection $L_1 \oplus \cdots \oplus L_n \to L_{i+1} \oplus \cdots \oplus L_n$. Note that there exists a natural identification of vector spaces

$$H^0(X_i, \mathbb{P}^1L_i) = \bigoplus_{h=0}^{n-i} \sum_{\{j_1, \ldots, j_n\} \in \{i+1, \ldots, n\}} z_{j_1} \cdots z_{j_n}H^0(Y, 2L_i - \sum_{j} L_{j}).$$

Take $H = \bigoplus_{i=1}^n H^0(X_i, \mathbb{P}^1L_i)$ and the map $H \to \text{Def}(X_n, \pi_n)$ is given by

$$(h_1, \ldots, h_n) \to X' = \{z_i^2 = f_i + h_i\}. \quad (*)$$

Clearly, $H \to \text{Def}(Y) = 0$ is smooth and the image of $r_i$ contains the natural deformations of each $\pi_i$. \qed
The deformations of $X$ defined by the equation $(\ast)$ are called \textit{natural deformations} of $X$ associated to the simple tower $(X, \pi_i, L_i)$. Note that the trivial involution $\tau: z_1 \to -z_1$ extends to every natural deformation of the tower, therefore if the family of natural deformations is complete (e.g. Corollary 3.8) then $\tau$ acts trivially on $T_X$.

Example 3.9. If $Y = \mathbb{P}^2$ and $\deg L_i = a_i$ then the hypotheses of Corollary 3.8 are satisfied if, for every $i$, $a_i \geq 4$ and $a_i > 2a_{i+1}$.

As in the introduction define $N(L_1, \ldots, L_n)$ the subset of moduli space of surfaces of general type whose canonical model is a simple iterated double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ associated to $L_1, \ldots, L_n$ and by $N_0(L_1, \ldots, L_n)$ the subset of $N(L_1, \ldots, L_n)$ of surfaces whose canonical model is nonsingular. it is clear that $N_0$ is an open subset of $N$.

\textbf{Corollary 3.10.} If $Y = \mathbb{P}^1 \times \mathbb{P}^1$, $L_i = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a_i, b_i)$ with $a_i, b_i \geq 3$ and for every $j < i$ min$(2a_i - a_j, 2b_i - b_j) < 0$ then $N(L_1, \ldots, L_n)$ and $N_0(L_1, \ldots, L_n)$ are open subsets of the moduli space $\mathcal{M}$.

\textit{Proof.} Take $[S] \in N(L_1, \ldots, L_n)$ and let $(X_i, \pi_i, L_i)$ be a tower with bottom $\mathbb{P}^1 \times \mathbb{P}^1$ and top the canonical model $X$ of $S$.

It is easy to show that $L_1, \ldots, L_n$ satisfy the conditions of Corollary 3.8 and then we have a surjective map of germs of complex spaces $(H, 0) \to (\text{Def}(X), 0) \to (\mathcal{M}, [S])$ where $H$ is the parameter space of natural deformations associated to the tower. The result now follows immediately since by the explicit construction of natural deformations the image of $(H, 0)$ is contained in $(N(L_1, \ldots, L_n), [S])$. \hfill $\Box$

\section{4. DEGENERATIONS}

Let $f: X \to \Lambda = \{ t \in \mathbb{C} : |t| < 1 \}$ be a proper flat family of normal projective surfaces and let $\tau: X \to X$ be an involution preserving $f$. Let $X_0 \to Y_0$ be the projection to quotient and assume that $\tau: X_0 \to Y_0$ is flat for every $t \neq 0$.

In general $\pi_0: X_0 \to Y_0$ is not flat; this section is almost entirely devoted to proving the following theorem which gives a sufficient condition for the map $\pi_0$ to be flat.

\textbf{Theorem 4.1.} In the above situation suppose that:

(i) $X_0, Y_0$ are smooth surfaces for $t \neq 0$.
(ii) $X_0$ has at most rational double points (RDPs) as singularities
(iii) The divisibility of the canonical class of $Y_0$ is even for $t \neq 0$.

Then $Y_0$ has at most RDPs and the map $\pi: X \to Y$ is flat.

Since flatness is a local property we need to investigate the quotient of smoothing of RDP. In the next proposition, by a cyclic singularity of type $\frac{1}{n}(a, b)$ we shall mean the quotient of $\mathbb{C}^2$ by the cyclic linear group of order $n$ generated by diag $(e^a, e^b)$ where $e$ is a primitive $n$th root of 1. For generalities about the Milnor fibre of a smoothing see for example [17, Section 1].

\textbf{Proposition 4.2.} Let $f: (X, 0) \to (\mathbb{C}, 0)$ be a smoothing of a RDP $X_0$ and let $f': (Y, 0) \to \Delta$ be the quotient of $(X, 0)$ by an involution $\tau$ preserving $f$. 

Suppose that $(Y, 0)$ is a smoothing of the singularity $(Y_0, 0) = (X_0, 0)/\pi$ and let $F_i \subseteq Y_i$ be the associated Milnor fibre. Then either one of the following possibilities holds:

(i) $Y_0$ is a RDP and the quotient projection $\pi: (X, 0) \to (Y, 0)$ is flat.
(ii) $Y_0$ is cyclic of type $\frac{1}{2}(1, 2d - 1)$ and the intersection form on $H_2(F, \mathbb{Z})$ is odd and negative definite.
(iii) $f'$ is a $\mathbb{Q}$-Gorenstein smoothing of the cyclic singularity of type $\frac{1}{2}(1, 2d - 1)$, the torsion subgroup of $H^2(F, \mathbb{Z})$ has order 2 and is generated by the canonical class.

Proof of Theorem 4.1. It is enough to prove that the map $Y \to \Delta$ cannot be locally of type (ii) or (iii) described in Proposition 4.2. Let $p \in Y$ be a singular point; $(Y, p)$ cannot be of type (ii) above since the inclusion $F_i \subset Y$ induces an isometry $H_2(F, \mathbb{Z}) \to H_2(Y, \mathbb{Z})$ with respect to the intersection forms and the intersection form of $Y_i$ is even by Wu's formula.

If $(Y, p)$ is of type (iii) above and if $r: H^2(Y, \mathbb{Z}) \to H^2(F, \mathbb{Z})$ is the natural restriction then $r(c_1(K_Y))$ generates the torsion subgroup of $H^2(F)$ which is $\mathbb{Z}/2\mathbb{Z}$ but this gives a contradiction since $c_1(K_X)$ is 2-divisible.

From this proof it is clear that the condition $r(Y)$ even is essential in order to have $Y_0$ with most rational double points. Moreover, it is not difficult to construct examples where the divisibility is odd and $Y_0$ has singularities of type 4.2 (iii) (e.g. canonical coverings of some $\mathbb{Q}$-Gorenstein degeneration of the projective plane [10, 18]).

Our strategy of proof of Proposition 4.2 divides into two steps. The first step is the classification of all conjugacy classes of involutions acting on a RDP; this computation has already been done by Catanese and the result is illustrated in Tables 1 and 2.

Corollary 4.3. Let $X \to Y$ be the flat double cover of normal surfaces. If $X$ is smooth then $Y$ is smooth. If $X$ has at most RDPs then $Y$ has at most RDPs.

Proof. According to Table 2 the only involutions whose fixed locus is a Cartier divisor are exactly of types (a) and (f).

<table>
<thead>
<tr>
<th>Table 1. Equations of RDPs in $\mathbb{C}^3$</th>
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<tr>
<td>$E_6$</td>
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<td>$E_7$</td>
</tr>
<tr>
<td>$E_8$</td>
</tr>
<tr>
<td>$D_n, n \geq 4$</td>
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<tr>
<td>$A_n$</td>
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<tr>
<td>Smooth</td>
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<tr>
<th>Table 2. [19, Theorem 2.1] Conjugacy classes of involutions acting on the RDPs defined as in Table 1</th>
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<tbody>
<tr>
<td>(a)</td>
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<td>(b)</td>
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<td>(d)</td>
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<td>(e)</td>
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<td>(f)</td>
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We also need the following result proved in [19, Theorem 1.2]:

If $G$ is a finite group of automorphisms of rational double points of type $E_7$ or $E_8$ then $G$ is cyclic.

The second step in the proof of Proposition 4.2 is to give a (very rough) classification of the smoothing of the involutions of Table 2 according to the following definition.

Definition 4.4. Let $(X_0, 0)$ be a singularity and $g_0 \in Aut(X_0, 0)$. A smoothing of $g_0$ is the data of a smoothing $(X, 0) \to (C, 0)$ of $(X_0, 0)$ and an automorphism $g$ of $(X, 0)$ preserving the map $t$ such that $g_0$ is the restriction of $g$ to $X_0$ and the quotient $(Y, 0) = (X/g, 0) \to (C, 0)$ is smoothing of $(X_0/g_0)$.

The following Cartan-type lemma will be very useful for our purposes.

Lemma 4.5. Let $(X, 0) \to (C, 0)$ be a morphism of germs of analytic singularities and let $G \subset Aut(X, 0)$ be a finite subgroup preserving $t$.
Assume $G$ acts linearly on a finite dimensional $C$-vector space $V$ and let $i_0: (X_0, 0) \to (V, 0)$ be a $G$-embedding, then there exists a $G$-embedding $i: (X, 0) \to (V \times C, 0)$ extending $i_0$ and such that $t = p \circ i$ where $p$ is the projection on the second factor.

Proof (sketch). We can assume without loss of generality that $V$ is $G$-isomorphic to $(m_0/m_0^2)$, the Zariski tangent space of $X_0$ at $0$.
If $z_1, \ldots, z_n$ is a basis of $V$ and $i_0^*: C\{z_1, \ldots, z_n\} \to \mathcal{O}_{X_0}$ then since $G$ is finite and $char C = 0$ there exists a $G$-lifting of $i_0^*$, say $\eta^*: C\{z_1, \ldots, z_n\} \to \mathcal{O}_Y$.
It is now easy to prove that the map $i: (X, 0) \to (V \times C, 0)$ associate to $i^*: C\{z_1, \ldots, z_n, t\} \to \mathcal{O}_X i^*(t) = t, i^*(z_i) = \eta^*(z_i)$ is the desired embedding.

Lemma 4.6. The involutions of types (b) and (d) are not smoothable.

Proof. There are several cases to investigate; here we make only a particular case for illustrating the idea, for the other cases the proof is similar.
Let $X_0 = D$ and $\tau$ involution of type (b) and assume that the action of $\tau$ extends to a smoothing $(X, 0) \to (C, 0)$. By Lemma 4.5 we can assume that $(X, 0)$ is defined in $C^4$ by the equation

$$z^2 + x(y^2 + x^{n-2}) + t\varphi(x, y, z, t) = 0.$$

$\tau(x, y, z, t) = (x, -y, -z, t)$ and $\varphi$ is $\tau$-invariant.
The fixed locus of $\tau$ is the germ of curve of equation $x^{n-1} + t\varphi(x, 0, 0, t) = 0$ contained in the plane $y = z = 0$ and then for $|t| < 1, \tau$ has a finite number of fixed points on $X$, and the quotient $X/\tau$ is singular.

Lemma 4.7. Let $(X, 0) \to (C, 0)$ be a smoothing of a RDP and let $\tau$ be an involution of $(X, 0)$ preserving $t$. If $\tau$ is of type (a) or (f) then $X_0/\tau$ is a RDP and the projection to $(Y, 0) = (X/\tau)$ is flat.

Proof. In case (a), by Lemma 4.5, we can assume $(X, 0) \subset (C^4, 0)$ defined by the equation

$$f(x, y^2, z) + t\varphi(x, y^2, z, t) = 0.$$
and $\tau(x, y, z, t) = (x, -y, z, t)$. Thus the equation of $(Y, 0)$ is

$$f(x, s, z) + t\varphi(x, s, z, t) = 0$$

and $(X, 0)$ is defined in $(Y \times \mathbb{C}, 0)$ by the equation $y^2 = s$. The case of involution of case (f) is similar.

**Proof of Proposition 4.2.** By Lemma 4.6 the restriction of $\tau$ to $X_0$ can be only of type (a), (c), (e) and (f).

In cases (a) and (f) by Lemma 4.7 the situation of Proposition 4.2(i) holds.

In case (c) by [19, Theorems 2.4 and 3.1] the situation of Proposition 4.2(ii) holds.

Since $Y - \{0\}$ is smooth $\tau$ must act freely on $X - \{0\}$ and then $Y$ is Q-Gorenstein of order 2. The statement about the Milnor fibre is proved in [18, Proposition 13].

**Lemma 4.8.** Let $X \to \Delta$ be a proper flat family of normal surfaces and let $\mathcal{L}$ be a line bundle on $X$.

(i) If $\mathcal{L}_t = \mathcal{L} \otimes \mathcal{O}_{X_t}$ is trivial for every $t \neq 0$ then $\mathcal{L}$ is trivial.

(ii) If $\mathcal{L}_0 = \mathcal{L} \otimes \mathcal{O}_{X_0}$ is trivial, $h^1(\mathcal{O}_{X_0}) = 0$ and $X_t$ is smooth for $t \neq 0$ then $\mathcal{L}$ is trivial.

**Proof.** The first part is well known (cf. [12, p. 291, ex. 12.4]).

If $h^1(\mathcal{O}_{X_0}) = 0$ then by semicontinuity and base change $H^1(\mathcal{O}_X) = 0$. According to [20, 1.8.8] $X_0$ is a deformation retract of some open neighbourhood, therefore if $X_t$ is smooth for $t \neq 0$ then the restriction map $H^2(X, \mathbb{Z}) \to H^2(X_0, \mathbb{Z})$ is bijective. From the exponential sequence it follows that the restriction map $Pic(X) \to Pic(X_0)$ is injective (cf. [18, Lemma 2]).

**Corollary 4.9.** In the situation at the beginning of Section 4, assume that $X_t$ is smooth for $t \neq 0$, $X_0$ has most RDPs and $Y_t = \mathbb{P}^1 \times \mathbb{P}^1$ for $t \neq 0$.

If, for $t \neq 0$, $\pi_0^* \mathcal{O}_{X_0} = \mathcal{O}_{Y_t} \otimes \mathcal{O}_{Y_t}(a, b)$ with $a \neq b$ (this condition is independent of the particular isomorphism from $Y_t$ to $\mathbb{P}^1 \times \mathbb{P}^1$) then $Y_0$ is a Segre–Hirzebruch surface $\mathbb{F}_{2k}$.

**Proof.** By Theorem 4.1, $Y_0$ has most RDPs and the map $\pi : X \to Y$ is a flat double cover and we have $\pi_* \mathcal{O}_X = \mathcal{O}_Y \otimes \mathcal{L} \mathcal{L}$ line bundle.

If $Y_0$ is smooth then it is well known that it is a surface $\mathbb{F}_{2k}$ for some $k \geq 0$. If $Y_0$ is singular its minimal resolution of singularities is $\mathbb{F}_2$ (this follows from Brieskorn–Tyurina theory on simultaneous resolution [21]) and $Y_0$ is the irreducible singular quadric in $\mathbb{P}^3$ whose Picard group is generated by the hyperplane section $\mathcal{O}_{Y_0}(1)$.

But if $\mathcal{L}_0 = n \cdot \mathcal{O}_{Y_0}(1)$ then $\mathcal{L}_t = \mathcal{O}_{Y_t}(n, n)$ contrary to the assumption.

**Theorem 4.10.** Let $f : X \to \Delta$ be a proper flat map from a normal 3-dimensional complex space $X$ to the unit disk such that:

1. $X_0$ has at most rational double points as singularities.
2. $f : X^* \to \Delta^* = \Delta - \{0\}$ is a family of iterated smooth double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ associated to line bundles $L_1, \ldots, L_n \in Pic(\mathbb{P}^1 \times \mathbb{P}^1)$.
3. $L_i = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a_i, b_i)$ with $a_i, b_i \geq 3, a_i \geq b_i + 2$ and $a_0, b_1$ even for $i = 2, \ldots, n$.

Then if $f' : Z \to \Delta$ is the relative canonical model of $X$ there exists a factorization of $f'$ $Z \to Y \to \Delta$ such that $\pi$ is finite flat, $\pi_t : Z_t \to Y_t$ is an iterated flat double cover for every $t$, $Y_t = \mathbb{P}^1 \times \mathbb{P}^1$ for $t \neq 0$ and $Y_0 = \mathbb{F}_{2k}$. 
Proof. By induction on $n$.

Case $n = 1$. The action of the involution $\tau$ on $X^*$ extends to a biregular action on $Z$ (cf. [22, Theorem 1.8]) and taking quotient we have a factorization $Z \to Y = Z/\tau \to \Delta$ where $Y_t = \mathbb{P}^1 \times \mathbb{P}^1$ for $t \neq 0$. The result follows from Corollary 4.9.

Case $n > 1$. As in case $n = 1$ there exists an involution acting on $Z$ preserving fibres and a factorization

$$Z \xrightarrow{\pi_2} V = Z/\tau \to \Delta$$

where, for $t \neq 0$, $V_t$ is a smooth iterated double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ associated to line bundles $L_2, \ldots, L_n$. By the adjunction formula, the divisibility of the canonical class of $V_t$ is even and by Theorem 4.1 $\pi_1$ is flat and $V_0$ has at most rational double points.

By induction we have then a factorization

$$Z \xrightarrow{\pi} V \xrightarrow{\delta} W \xrightarrow{\pi_2} Y \to \Delta$$

where $W$ is the relative canonical model of $V$. Then we complete the proof by proving that $\delta$ is an isomorphism.

By normality of $V_0$ and $W_0$ the fibres of $\delta$ are connected. Assume that there exists an irreducible curve $C \subset V_0$ contracted by $\delta$ and let $D \subset Z_0$ be the strict transform of $C$.

Since $\pi_1$ is flat we have $\pi_1_*\mathcal{O}_Z = \mathcal{O}_Y \oplus M$ for a line bundle $M$ such that for $t \neq 0$, $M_t = \delta^*\pi_2^*L_1$. By Lemma 4.8 if $\mathcal{L}$ is the line bundle on $Y$ such that $\mathcal{L}_t = L_1$ then $M = \delta^*\pi_2^*\mathcal{L}$ and $C \cdot M = 0$. Using the adjunction formula $K_{Z_0} - \pi_1^*(K_{V_0} + M_0)$ and $D \cdot K_{Z_0} = 0$, which is impossible since $K_{Z_0}$ is ample.

Proposition 4.11. In the same hypotheses of Theorem 4.10 if in addition $n \geq 2$ and $2b_{n-1} > b_n + 2$ then $Y_0 = \mathbb{F}_{2k}$ with

$$k \leq \max \left( \frac{a_{n-1}}{b_{n-1} - 1}, \frac{2a_{n-1} - a_n}{2b_{n-1} - b_n - 2} \right).$$

In particular, if $b_{n-1} \geq a_{n-1} + 2$ then $Y_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

Proof. Without loss of generality we can assume $n = 2$ and $k > 0$.

Let $\sigma_0, F$ be the standard basis of $\text{Pic}(\mathbb{F}_{2k})$ ($\sigma_0^2 = 2k, F^2 = 0, F \cdot \sigma_0 = 1$) and let $\sigma_\infty \in \mathbb{F}_{2k}$ be the "section at infinity" (i.e. the unique effective divisor linearly equivalent to $\sigma_0 - 2kF$). We recall that for an effective divisor $D \sim \sigma_\infty + bF$ if $b < -2k$ then $2\sigma_\infty \subset D$ and in particular $D$ is not reduced.

In our situation, we have two line bundles $L_1, L_2$ on $\mathbb{F}_{2k}$ such that $Z_0$ is isomorphic to a surface in $L_1 \oplus L_2$ defined by the equations

$$\begin{cases}
z^2 = f, & f \in H^0(2L_2) \\
w^2 = g + zh, & g \in H^0(2L_1), h \in H^0(2L_1 - L_2).
\end{cases}$$

Since $L_i$ deforms to the line bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a_i, b_i)$ we have

(1) \[ L_1 = b_1\sigma_0 + (a_1 - b_1k)F \] \[ L_2 = b_2\sigma_0 + (a_2 - b_2k)F \]

or

(2) \[ L_1 = a_1\sigma_0 + (b_1 - a_1k)F \] \[ L_2 = a_2\sigma_0 + (b_2 - a_2k)F. \]

Since $a_2 \geq h_2 + 2$ and the divisor of $f$ is reduced we must be in case (1). Moreover, since $2\sigma_\infty$ is not contained in both the divisors of $g$ and $h$ we have

$$2(a_1 - b_1k) \geq -2k \text{ or } (2a_1 - a_2) - (2b_1 - b_2)k \geq -2k.$$
5. AUTOMORPHISMS

**Theorem 5.1.** Let $L_1, \ldots, L_n$ be fixed line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$. If $n \geq 2$, $a_i, b_i \geq 3$, $a_i, b_i$ even for $i \geq 2$, $a_n > b_n + 2$, $b_{n-1} > a_{n-1} + 2$ and

$$\max_{i<j} \min(2a_i - a_j, 2b_i - b_j) < 0$$

then $N(L_1, \ldots, L_n)$ is a connected component of the moduli space $\mathcal{M}$, irreducible and unirational.

**Proof.** By Corollary 3.10 it is enough to prove that $N(L_1, \ldots, L_n)$ contains the closure of $N_0(L_1, \ldots, L_n)$ in $\mathcal{M}$, but this is a consequence of Theorem 4.10 and Proposition 4.11. 

Here we want to study the group of automorphisms of the generic element of $N(L_1, \ldots, L_n)$. Clearly, if $[S] \in N(L_1, \ldots, L_n)$ then there exists at least one involution acting on the canonical model of $X$ and then $\text{Aut}(S)$ always contains a subgroup of order 2. Our main result is the following.

**Theorem 5.2.** If $L_1, \ldots, L_n$ is a good sequence (in the sense of Definition C) of line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ then there exists a nonempty Zariski open subset $U \subset N(L_1, \ldots, L_n)$ such that for every $[S] \in U$ $\text{Aut}(S)$ has order exactly 2.

We prove this theorem later on, after some preparatory material. The first lemma is the particular case $n = 1$ of Theorem 5.2.

**Lemma 5.3.** If $a, b \geq 3$ then for generic $f \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, (2a, 2b))$ the only nontrivial automorphism of the surface $S$ of the equation $z^2 = f$ is the involution $\tau : z \mapsto -z$.

**Proof.** For generic $f$ the divisor $D = \text{div}(f)$ is a smooth curve and there does not exist any nontrivial automorphism $h$ of $\mathbb{P}^1 \times \mathbb{P}^1$ such that $h(D) = D$.

The divisor $R = \text{div}(z) \subset S$ is the set of critical points of the canonical map and then for every $g \in \text{Aut}(S)$ $g(R) = R$ and for every $p \in R$ $g^{-1} \tau g(p) = p$ and since the stabilizer of $R$ is cyclic $[9, \text{Proposition 1.11}] g^{-1} \tau g = \tau$. Thus $g$ induces the identity on $S/\tau$ and then $g = \text{Id}$ or $g = \tau$.

**Lemma 5.4.** Let $S$ be a surface of general type and assume that its canonical model $X$ has at least one rational double point of type $E_7$ or $E_8$ at a point $p$.

Then there exists at most one involution $\tau$ of $X$ such that $\tau(p) = p$.

**Proof.** Let $G \subset \text{Aut}(X) = \text{Aut}(S)$ be the subgroup generated by the involutions leaving $p$ fixed, since $\text{Aut}(S)$ is finite $[11]$ $G$ is finite and by $[19, \text{Theorem 1.2}]$, $G$ is cyclic.

**Lemma 5.5.** Let $X \to Y$ be a double cover with $X$ a canonical model of a surface of general type and $Y$ smooth.

If $X$ has at least one rational double point of type $E_7$ or $E_8$ then every automorphism of $X$ commutes with the trivial involution $\tau$.

**Proof.** Let $\{p_1, \ldots, p_s\}$ be the (nonempty) set of singular points of $X$ which are RDPs of type $E_7$ or $E_8$. Since $Y$ is smooth $p_1, \ldots, p_s$ belong to the fixed locus of $\tau$ and therefore for every $g \in \text{Aut}(X)$ and every $i = 1, \ldots, s$, $g^{-1} \tau g(p_i) = p_i$ and by Lemma 5.4, $g \tau = \tau g$. 

Lemma 5.6. If $L_1, \ldots, L_n$ is a good sequence of line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ then there exists an iterated flat double cover

$$p : X \rightarrow X_1 \rightarrow \cdots \rightarrow X_n = \mathbb{P}^1 \times \mathbb{P}^1$$

associated to $L_1, \ldots, L_n$ such that $X_1$ is smooth and $X$ has exactly $2^{n-2}$ rational double points of type $E_8$.

Proof. We look for a surface $X$ given by equations

$$
\begin{align*}
    z_1^2 &= f_1 + z_2^2 h_1 \\
    z_2^2 &= f_2 \\
    \vdots \\
    z_n^2 &= f_n
\end{align*}
$$

with $f_i \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, 2L_i)$ and $h_1 \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, 2L_1 - L_2)$. We first fix $f_2, \ldots, f_n$ such that the divisors $D_i = d_iu(j_{D_i})$ and the surface $X_1 = \{z_2 = f_0, i > 1\}$ are smooth.

Take $u \in D_2 = \bigcup_{i > 2} D_i$ and $l \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, (1, 1))$ such that $E = \text{div}(l)$ is the tangent line of $D_2$ at $u$ and fix $h_1 = f_i^u$ with $k(u) \neq 0$.

We now claim that for generic $f_i \in H^0(\mathcal{M}_n^2, \mathcal{O}_{\mathbb{P}^1, \mathbb{P}^1}(2L_1))$ (here $\mathcal{M}_n \subset \mathcal{O}_{\mathbb{P}^1, \mathbb{P}^1}$ is the ideal sheaf of $\{u\}$) the surface $X$ has the required properties.

By Bertini's theorem for generic $f_i$ the surface $X$ is smooth outside $p^{-1}(u)$ and $\partial^3 f_i/\partial x^3 \neq 0$ where $x, y$ are local coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$ at $u$ such that $y = f_2$.

If $v \in p^{-1}(u)$ then $x, y$ are local coordinates of $X_2$ at $v$ and the local equation of $X$ is

$$
\begin{align*}
    z_2^2 &= f_1(x, y) + z_2(a y^2 + h(x, y)) \\
    z_3^2 &= y
\end{align*}
$$

with $a \neq 0$ and $h \in \mathcal{M}^3$. We can rewrite the equation as

$$
    z_1^2 = x^3 e(x, z_2) + x^2 z_2^3 \phi_1(z_2) + x z_2^3 \phi_2(z_2) + z_2^3 \phi_3(z_2)
$$

with $e(0, 0) \neq 0$ and $\phi_i(0) \neq 0$. By the computation of [20, pp. 63–64] it follows that this is the equation of a rational double point of type $E_8$.

Proof of Theorem 5.2. We prove the theorem by induction on $n$. The case $n = 1$ is proved in Lemma 5.3; thus we can assume that there exists a nonempty Zariski open subset $V \subset N(L_1, \ldots, L_n)$ such that for $[S] \in V$, $\text{Aut}(S) = \mathbb{Z}/2\mathbb{Z}$.

For every finite group $G$ define

$$
    N^G = \{[S] \in N(L_1, \ldots, L_n)|G \text{ is isomorphic to a subgroup of } \text{Aut}(S)\}.
$$

By [22, Theorem 1.8] $N^G$ is closed in $N = N(L_1, \ldots, L_n)$ and since $K_2^G$ is constant on $N$, $N^G = \emptyset$ if $\text{ord}(G) > 0$ [23,24]. Clearly, $U$ is the complement of the union of $N^G$s for $\text{ord}(G) > 2$, so we only need to show that $U \neq \emptyset$.

For a fixed integer $m \geq 5$ and for every group $G$ we may write [22, Proof of Theorem 1.8] $N^G$ as a finite union of closed subset $N^{G,q}$ where $q$ belong to a (finite) set of representatives of isomorphism classes of faithful representation $G \subset GL(P_m(S), \mathbb{C})$ and $N^{G,q}$ is the intersection of $N$ with the image of the natural map $H^0 \rightarrow \mathcal{M}$ where $H^0$ is the Hilbert scheme of the $q$-invariant $m$-canonical images of surfaces of general type in $\mathbb{P}^{m-1}$.

Assume that for some $G, q, N^{G,q} = N$ and let $X \rightarrow Z = X/\tau \rightarrow \Delta$ be a family of flat iterated double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ with $X_0$ as in Lemma 5.6 and $Z \in V \subset N(L_2, \ldots, L_n)$ for $\tau \neq 0$. 
After a possible change of base $\Delta \xrightarrow{\tau} \Delta$ the group $G$ acts on $X$ preserving fibres and let $g \in G$. By Lemma 5.5, $g$ commutes with $\tau$ on the central fibre and by continuity of eigenvalues $g t = t g$ on $X$ and $g$ induces an automorphism $g'$ on $Z_t$ for every $t \in \Delta$.

The equation for $X$, as a double cover of $\mathbb{P}^1$, is $z_t = f_t + z_t h_t$ with $f_t, h_t, \text{Aut}(Z_t)$-invariant and $\text{div}(f_t) \neq \text{div}(z_t h_t)$ then $g'$ must be the identity and $g = 1, \tau$, in particular $\text{ord}(G) \leq 2$.

**Corollary 5.7.** Let $L_1, \ldots, L_n, M_1, \ldots, M_m$ be two good sequences of line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ with $L_1, \ldots, L_n$ good and $M_1, \ldots, M_m$ satisfying conditions (C1) and (C2).

If $N(L_1, \ldots, L_n) \cap N(M_1, \ldots, M_m) = \emptyset$ then $n = m$ and $L_i = f^* M_i$ for every $i$ and some $f \in \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$.

**Proof.** By Corollary 3.10 and Theorem 5.1, $N(M_1, \ldots, M_m)$ is an open subset of $N(L_1, \ldots, L_n)$. By Theorem 5.2 applied to the good sequence $L_1, \ldots, L_n$ there exists an iterated smooth double cover $\pi: X_0 \rightarrow \cdots \rightarrow X_n = \mathbb{P}^1 \times \mathbb{P}^1$ with $[X] \in N(M_1, \ldots, M_m)$ such that for every $i < n$, $\text{Aut}(X_i) = \{1, \tau_i\}$ and $X_{i+1} = X_i/\tau_i$.

Since $X_i$ is of general type for every $i < n$ we must have $n = m$.

Moreover, we have already seen that the sequence $L_i$ is uniquely determined by the maps $\pi_i: X_{i-1} \rightarrow X_i$ and then up to automorphisms $L_i = M_i$ for every $i$.

**Corollary 5.8.** Let $X = X_0 \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_n} \mathbb{P}^1 \times \mathbb{P}^1$ be a simple iterated double cover associated to a good sequence such that $X$ has most rational double points as singularities. Then the trivial involution of the tower (Definition 3.6) is completely determined by the surface $X$.

**Proof.** Let $\nu: \text{Aut}(X) \rightarrow \text{Aut}(T_X)$ be the homomorphism induced by the natural action of $\text{Aut}(X)$ in the space of first order deformations and denote $G = \ker \nu$. Since $\tau \in G$ it is enough to prove that $G = \mathbb{Z}/2\mathbb{Z}$.

$\text{Aut}(X)$ is finite and there exists the universal deformation of $X$ (15.3.12) $f: \tilde{X} \rightarrow (S, 0)$.

Moreover, there exists a natural action of $\text{Aut}(X)$ on the germ $(S, 0)$ and we have $(\mathcal{M}, [X]) = (S, 0)/\text{Aut}(X)$.

By Cartan's lemma, $G$ acts trivially on $(S, 0)$, the action of $G$ on $X$ extends to an action on every fibre of $f$, and the result follows from Theorem 5.2.

6. **INVIARNTS AND A LOWER BOUND FOR THE NUMBER OF CONNECTED COMPONENTS**

We begin with a general formula for the computation of Chern numbers of simple iterated double covers. For this it is convenient to introduce for every algebraic surface $S$ its index $I_S = K_S^2 - 8\chi(0_S)$.

**Lemma 6.1.** Let $p: X \rightarrow Y$ be a smooth simple iterated double cover associated to a sequence $L_1, \ldots, L_n \in \text{Pic}(Y)$ Then:

(a) $K_Y^2 = 2^a(K_Y + \sum_{i=1}^n L_i)^2$

(b) $I_X = 2^a(I_Y - \sum_{i=1}^n L_i^2)$. 

Proof. (a) is a simple application of the Hurwitz formula, the details are left to the reader. We prove (b) by induction on \( n \), the formula being trivially true for \( n = 0 \).

Assume \( n > 0 \) and consider a factorization

\[
p: X \longrightarrow Z \rightarrow Y
\]

with \( q \), a simple iterated double cover associated to \( L_2, \ldots, L_n \) and \( \pi_*\mathcal{O}_X = \mathcal{O}_Z \oplus \mathcal{O}_Z(-q^*L_1) \). Thus

\[
K_X^2 = 2(K_Z + q^*L_1)^2, \quad \chi(\mathcal{O}_X) = \chi(\mathcal{O}_Z) + \chi(-q^*L_1) = 2\chi(\mathcal{O}_Z) + \frac{1}{2}q^*L_1(K_Z + q^*L_1)
\]

and then \( I_X = 2I_Z - 2(q^*L_1)^2 = 2I_Z - 2aL_1^2 \).

For a smooth simple iterated double cover \( p: X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \) associated to the sequence \( L_1, \ldots, L_m \), \( L_i = O(a_i, b_i) \) with \( a_i, b_i \geq 3 \) we have:

\[
\begin{align*}
\pi_i(X) &= 0 \quad [9, \text{Theorem 1.8}], \\
K_i^2 &= 2^n + (\sum a_i - 2)(\sum b_i - 2), \\
\chi(\mathcal{O}_X) &= 1 + h^0(K_X) = 1 + \sum_{i=1}^n \sum_{a_i + \cdots + a_n - 1} (b_i + \cdots + b_n - 1), \\
r(X) &= \max \{ r \in \mathbb{N} | r^2 - c_i(X) \in H^2(X, \mathbb{Z}) \} = \text{G.C.D.}(\sum a_i - 2, \sum b_i - 2) \quad [7].
\end{align*}
\]

Remark 6.2. If \( a_i = a = \text{constant} \) then \( K^2, \chi \) and \( r \) depend only on \( n, a \) and \( T = \sum b_i \). In fact, according to Lemma 6.1, we have:

\[
K_i^2 = 2^{n+1}(na - 2)(T - 2), \quad r = \text{G.C.D.}(na - 2, T - 2), \quad \chi = \frac{1}{2}K_i^2 + 2^{n-1}aT.
\]

It is well known that the intersection form of any simply connected algebraic surface different from \( \mathbb{C}P^2 \) is indefinite and it is classified by rank, signature and type. Moreover, if we restrict for simplicity to simply connected surfaces of general type, rank, signature and type of the intersection form depend only on \( K^2, \chi \) and \( r \mod 2 \).

According to Freedman, results on the topology of four-manifolds \([3, 1.5 + \text{addendum}]\), for every bijective isometry \( \phi: H^2(S, \mathbb{Z}) \rightarrow H^2(S', \mathbb{Z}) \) with \( S, S' \) simply connected algebraic surfaces, there exists a preserving orientation homeomorphism between \( S \) and \( S' \) inducing \( \phi \).

In general, given a unimodular quadratic form of rank \( b \) and signature \( \sigma \) over an integral lattice \( \Lambda \), a primitive vector \( v \in \Lambda \) is said to be of characteristic type if \( v \cdot x \equiv x^2 \pmod{2} \) for every \( x \in \Lambda \), otherwise it is said to be of ordinary type. Note that if the quadratic form is even then every primitive vector is of ordinary type.

A theorem of Wall \([4]\) states that if \( b - |\sigma| \geq 4 \) then the group of isometric automorphisms of \( \Lambda \) acts transitively on the set of primitive vectors of fixed norm and type. If \( \Lambda = H^2(S, \mathbb{Z}) \), \( S \) a simply connected compact complex surface the condition \( b - |\sigma| \geq 4 \) is equivalent to \( \chi(\mathcal{O}_S) > 1 \) and the primitive root of \( k_S \) is characteristic if and only if \( r(S) \) is odd.

Conclusion: There exists a homeomorphism \( f: S \rightarrow S' \) between simply connected algebraic surfaces with \( \chi > 1 \) matching up the canonical classes if and only if \( S, S' \) have the same invariants \( K^2, \chi, r \) or equivalently if and only if \( S' \in \mathcal{M}_d(S) \).

Proof of Theorem A. We keep the notation used in the statement of Theorem A. We first set \( T_n = 8.3^n \) and we choose a sequence of integers \( d_n \) such that

\[
\begin{align*}
(1) \quad d_n &\leq d_n \leq n^2, \\
(2) \quad \lim d_n / (\gamma_n + 1) = (8/\beta) - 1 \quad \text{where} \quad \gamma_n = d_n/(6n - 2).
\end{align*}
\]
Let \( q_n \) be the cardinality of the set
\[
Q_n = \left\{ \text{good sequences } L_1, \ldots, L_n \mid I_2 = \mathcal{O}_{P^1 \times P^1}(6, h_i), \sum_{i=1}^{n} h_i = T_n \right\}
\]

The second step is to choose for every \( n \) an iterated smooth double cover \( X_n \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \) associated to an element of \( Q_n \).

By adjunction formula, Corollary 5.7 and Remark 6.2, we have:
\[
K_{X_n} = \pi^* \mathcal{O}_{P^1 \times P^1}(6n - 2, T_n - 2), \quad \delta(X_n) \geq q_n
\]
\[
\lim_{n \to \infty} x_n = 1, \quad \text{where } x_n = \frac{8y(C_{X_n})}{K_{X_n}^2}.
\]

The last step is to define \( S_n \) as a smooth double cover of \( X_n \) associated to the line bundle
\[
M_n = \pi^* \mathcal{O}_{P^1 \times P^1}(d, nT_n - 2).\]

It is clear that for every \( (L_1, \ldots, L_n) \in Q_n \) the sequence \( M_n \) is good and the invariants of \( S_n \) are independent of the particular choice of \( L_1, \ldots, L_n \).

In fact an easy calculation shows
\[
y_n = K_{X_n}^2 = 2(1 + y_n)(1 + n)K_{X_n}^2
\]
\[
\frac{8x_n}{y_n} = \frac{8y(C_{S_n})}{y_n} = 1 + \frac{m_n + x_n - 1}{(1 + y_n)(1 + n)}.
\]

Therefore we have \( \delta(S_n) \geq q_n \) and \( \lim(y_n/x_n) = \beta \).

**Claim.** \( q_n \geq 3^{4(n-1)^2} \).

**Proof of Claim.** We have an injective map \( \phi : P_n \rightarrow Q_n \) where
\[
P_n = \left\{ (c_1, \ldots, c_n) \in \mathbb{N}^{n-1} \mid c_n = 2, c_i \leq 3^n, c_i > 2c_{i+1} \right\}
\]
and \( \phi(c_1, \ldots, c_n) = (L_1, \ldots, L_n) \) where \( L_i = \mathcal{O}_{P^1 \times P^1}(6, 2c_i) \) for \( i \geq 2 \), \( L_1 = \mathcal{O}_{P^1 \times P^1}(6, T_n - 2\sum_{i>1} c_i) \).

If \( p_n \) is the cardinality of \( P_n \) we have \( p_2 = 1 \) and for \( n \geq 3 \)
\[
q_n \geq p_n \geq 3^{n-1}p_{n-1} \geq 3^{(n-1) + (n-2) + \cdots + 2} = 3^{\frac{4n(n-1)}{2}} \geq 3^{4(n-1)^2}. \quad \square
\]

Note that \( y_n \leq Cn^3 6^{n-1} \) where \( C > 0 \) is a constant independent of \( n \) and since \( \log_3 6 \leq 5/3 \) we have for \( n \gg 0 \), \( y_n \leq 3^{\frac{5(n-1)}{3}} \) and then
\[
\delta(S_n) \geq q_n \geq y_n^{\frac{5}{6} \log y_n} \geq y_n^{\frac{1}{3} \log y_n}. \quad \square
\]

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