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Suzuki's type characterizations of completeness for partial metric spaces and fixed points for partially ordered metric spaces

Daniela Paesano, Pasquale Vetro *,[1]

Università degli Studi di Palermo, Dipartimento di Matematica e Informatica, Via Archirafi, 34, 90123 Palermo, Italy

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Recently, Suzuki [T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc. 136 (2008) 1861–1869] proved a fixed point theorem that is a generalization of the Banach contraction principle and characterizes the metric completeness. In this paper we prove an analogous fixed point result for a self-mapping on a partial metric space or on a partially ordered metric space. Our results on partially ordered metric spaces generalize and extend some recent results of Ran and Reurings [A.C.M. Ran, M.C. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2004) 1435–1443], Nieto and Rodríguez-López [J.J. Nieto, R. Rodríguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order 22 (2005) 223–239]. We deduce, also, common fixed point results for two self-mappings. Moreover, using our results, we obtain a characterization of partial metric 0-completeness in terms of fixed point theory. This result extends Suzuki's characterization of metric completeness.

1. Introduction

In 1994 Matthews [4] introduced the notion of a partial metric space as a part of the study of denotational semantics of data for networks, showing that the contraction mapping principle can be generalized to the partial metric context for applications in program verification.

Recently, Romaguera [9] proved that a partial metric space (X, p) is 0-complete if and only if every Caristi type mapping on X has a fixed point. The result of Romaguera extended Kirk's [3] characterization of metric completeness to a kind of complete partial metric spaces.

In 2008, Suzuki obtained the following interesting fixed point theorem.

Theorem 1. *([10, Theorem 2]) Let (X, d) be a complete metric space and let T be a self-mapping on X. Define a nonincreasing function θ from $[0, 1)$ onto $(1/2, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leqslant r \leqslant (\sqrt{5}-1)/2, \\ (1-r)r^{-2} & \text{if } (\sqrt{5}-1)/2 < r < 2^{-1/2}, \\ (1+r)^{-1} & \text{if } 2^{-1/2} \leqslant r < 1. \end{cases}$$

* Corresponding author.
E-mail address: vetro@math.unipa.it (P. Vetro).

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Assume that there exists $r \in [0,1)$ such that

$$\theta(r)d(x,Tx) \leqslant d(x,y) \quad \textit{implies} \quad d(Tx,Ty) \leqslant rd(x,y)$$

for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover $\lim_{n\to+\infty} T^n x = z$ for all $x \in X$.

In this paper, we prove a fixed point result of Suzuki type for self-mappings on partial metric spaces or on partially ordered metric spaces. Our results on partially ordered metric spaces generalize and extend some recent results of Ran and Reurings [8] and Nieto and Rodríguez-López [5]. For other references on partially ordered metric spaces, see Altun and Durmaz [1]. We deduce, also, common fixed point results for two self-mappings. Moreover, using our result we obtain a characterization of partial metric 0-completeness in terms of fixed point theory. This result extend Suzuki's [10] characterization of metric completeness.

2. Preliminaries

First, we recall some definitions and some properties of partial metric spaces that can be found in [4,6,7,9,11]. A partial metric on a non-empty set X is a function $p: X \times X \to [0,+\infty)$ such that for all $x, y, z \in X$:

(p_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(p_2) $p(x,x) \leqslant p(x,y)$,
(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,y) \leqslant p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a non-empty set and p is a partial metric on X. It is clear that, if $p(x,y) = 0$, then from (p_1) and (p_2) it follows that $x = y$. But if $x = y$, $p(x,y)$ may not be 0. A basic example of a partial metric space is the pair $([0,+\infty), p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in [0,+\infty)$. Other examples of partial metric spaces which are interesting from a computational point of view can be found in [4].

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon)\colon x \in X,\ \varepsilon > 0\}$, where

$$B_p(x,\varepsilon) = \{y \in X\colon\ p(x,y) < p(x,x) + \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$.

Let (X,p) be a partial metric space. A sequence $\{x_n\}$ in (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to+\infty} p(x,x_n)$.

A sequence $\{x_n\}$ in (X,p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to+\infty} p(x_n,x_m)$.

A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to+\infty} p(x_n,x_m)$.

A sequence $\{x_n\}$ in (X,p) is called 0-Cauchy if $\lim_{n,m\to+\infty} p(x_n,x_m) = 0$. We say that (X,p) is 0-complete if every 0-Cauchy sequence in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = 0$.

On the other hand, the partial metric space $(\mathbb{Q} \cap [0,+\infty), p)$, where $\mathbb{Q}$ denotes the set of rational numbers and the partial metric p is given by $p(x,y) = \max\{x,y\}$, provides an example of a 0-complete partial metric space which is not complete.

Lemma 1. *Let (X,p) be a partial metric space and $\{x_n\} \subset X$ a 0-Cauchy sequence. The sequence $\{p(x,x_n)\}$ is Cauchy in $\mathbb{R}$ for all $x \in X$.*

Proof. For each $n, m \in \mathbb{N}$, by the triangle inequality, we have

$$p(x,x_n) \leqslant p(x,x_m) + p(x_m,x_n) - p(x_m,x_m), \qquad p(x,x_m) \leqslant p(x,x_n) + p(x_n,x_m) - p(x_n,x_n)$$

and hence $p(x_n,x_m) \geqslant p(x,x_n) - p(x,x_m)$ and $p(x_n,x_m) \geqslant p(x,x_m) - p(x,x_n)$. It implies that

$$\left|p(x,x_n) - p(x,x_m)\right| \leqslant p(x_n,x_m),$$

and so $\{p(x,x_n)\}$ is Cauchy in $\mathbb{R}$, since $\lim_{n,m\to+\infty} p(x_n,x_m) = 0$. □

Lemma 2. *Let (X,p) be a partial metric space and $\{x_n\} \subset X$. If $\lim_{n\to+\infty} x_n = x \in X$ and $p(x,x) = 0$, then $\lim_{n\to+\infty} p(x_n,z) = p(x,z)$ for all $z \in X$.*

Proof. By the triangle inequality

$$p(x,z) - p(x_n,x) \leqslant p(x_n,z) \leqslant p(x,z) + p(x_n,x)$$

and hence $\lim_{n\to+\infty} p(x_n,z) = p(x,z)$. □

Let X be a non-empty set and $T, f : X \to X$. The mappings T, f are said to be weakly compatible if they commute at their coincidence points (i.e. $Tfx = fTx$ whenever $Tx = fx$). A point $y \in X$ is called point of coincidence of T and f if there exists a point $x \in X$ such that $y = Tx = fx$.

Lemma 3. *Let X be a non-empty set and the mappings $T, f : X \to X$ have a unique point of coincidence v in X. If T and f are weakly compatible, then T and f have a unique common fixed point.*

Proof. Since v is a point of coincidence of T and f. Therefore, $v = fu = Tu$ for some $u \in X$. By weakly compatibility of T and f we have

$$Tv = Tfu = fTu = fv.$$

It implies that $Tv = fv = w$ (say). Then w is a point of coincidence of T and f. Therefore, $v = w$ by uniqueness. Thus v is a unique common fixed point of T and f. □

Let X be a non-empty set. If (X, d) is a metric space and $(X, \preccurlyeq)$ is partially ordered, then $(X, d, \preccurlyeq)$ is called a partially ordered metric space. $x, y \in X$ are called comparable if $x \preccurlyeq y$ or $y \preccurlyeq x$ holds. Let $f, T : X \to X$ be two mappings, T is said to be f-increasing if $fx \prec fy$ implies $Tx \prec Ty$ for all $x, y \in X$. If f is the identity mapping on X, then T is increasing. If $Tx \preccurlyeq Ty$ whenever $x, y \in X$ and $x \preccurlyeq y$ then T is said to be nondecreasing.

3. Fixed point theorems in partial metric spaces

In this section, we prove the following theorem, which is extension of Theorem 1.

Theorem 2. *Let (X, p) be a 0-complete partial metric space and T be a self-mapping on X. Define a nonincreasing function $\theta : [0, 1) \to (1/2, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leqslant r \leqslant (\sqrt{5}-1)/2, \\ (1-r)r^{-2} & \text{if } (\sqrt{5}-1)/2 < r < 2^{-1/2}, \\ (1+r)^{-1} & \text{if } 2^{-1/2} \leqslant r < 1. \end{cases}$$

Assume that there exists $r \in [0, 1)$ such that

$$\theta(r)p(x, Tx) \leqslant p(x, y) \quad \text{implies} \quad p(Tx, Ty) \leqslant rp(x, y) \tag{1}$$

for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover $\lim_{n\to+\infty} T^n x = z$ for all $x \in X$.

Proof. Since $\theta(r) \leqslant 1$, then $\theta(r)p(x, Tx) \leqslant p(x, Tx)$ for every $x \in X$. By hypothesis,

$$p\big(Tx, T^2x\big) \leqslant rp(x, Tx) \tag{2}$$

for every $x \in X$. Now, we fix $y \in X$ and define a sequence $\{y_n\}$ in X by $y_n = T^n y$. From (2), we deduce that $p(y_n, y_{n+1}) \leqslant r^n p(y, Ty)$ and hence

$$p(y_n, y_m) \leqslant \frac{r^n}{1-r} p(y, Ty)$$

for all $n, m \in \mathbb{N}$ with $m > n$. It implies that $\lim_{n,m\to+\infty} p(y_n, y_m) = 0$ and so $\{y_n\}$ is a 0-Cauchy sequence. Since X is 0-complete, $\{y_n\}$ converges to some point $z \in X$, that is

$$\lim_{n\to+\infty} p(y_n, z) = p(z, z) = 0.$$

We show that, for all $x \in X \setminus \{z\}$, we have

$$p(Tx, z) \leqslant rp(x, z). \tag{3}$$

Fix $x \in X \setminus \{z\}$. Since $p(x, z) > 0$ and $\lim_{n\to+\infty} p(y_n, z) = 0$, there exists $\nu \in \mathbb{N}$ such that $p(y_n, z) \leqslant p(x, z)/3$ for all $n \geqslant \nu$.

From $p(x, z) \leqslant p(x, y_n) + p(y_n, z) - p(y_n, y_n) \leqslant p(x, y_n) + p(y_n, z)$ and $p(y_n, y_{n+1}) \leqslant p(y_n, z) + p(y_{n+1}, z)$, for all $n \geqslant \nu$, we obtain

$$\begin{aligned} \theta(r)p(y_n, Ty_n) &\leqslant p(y_n, Ty_n) = p(y_n, y_{n+1}) \\ &\leqslant p(y_n, z) + p(y_{n+1}, z) \\ &\leqslant (2/3)p(x, z) = p(x, z) - p(x, z)/3 \\ &\leqslant p(x, z) - p(y_n, z) \leqslant p(y_n, x). \end{aligned}$$

By (1), it follows that

$$p(y_{n+1}, Tx) \leqslant rp(y_n, x) \quad \text{for all } n \geqslant \nu. \tag{4}$$

By Lemma 2, $\lim_{n\to+\infty} p(y_{n+1}, Tx) = p(z, Tx)$ and $\lim_{n\to+\infty} p(y_n, x) = p(z, x)$. From (4), as $n \to +\infty$, we deduce that (3) holds.

Now, we show that there exists $j \in \mathbb{N}$ such that $T^j(z) = z$. Arguing by contradiction, we assume that $T^j z \neq z$ for all $j \in \mathbb{N}$. Then (3) yields

$$p\big(T^{j+1}z, z\big) \leqslant r^j p(Tz, z), \quad \text{for all } j \in \mathbb{N}. \tag{5}$$

We consider the following three cases:

(i) $0 \leqslant r \leqslant (\sqrt{5}-1)/2$;
(ii) $(\sqrt{5}-1)/2 < r < 2^{-1/2}$;
(iii) $2^{-1/2} \leqslant r < 1$.

In the case (i), we note that $r^2 + r \leqslant 1$ and $2r^2 < 1$. Now, we assume $p(T^2z, z) < p(T^2z, T^3z)$, then by (2), we have

$$\begin{aligned} p(z, Tz) &\leqslant p\big(z, T^2z\big) + p\big(Tz, T^2z\big) \\ &< p\big(T^2z, T^3z\big) + p\big(Tz, T^2z\big) \\ &\leqslant r^2 p(z, Tz) + rp(z, Tz) \leqslant p(z, Tz). \end{aligned}$$

This is a contradiction. So we have

$$p\big(T^2z, z\big) \geqslant p\big(T^2z, T^3z\big) = \theta(r)p\big(T^2z, T^3z\big)$$

and by (1), we obtain $p(T^3z, Tz) \leqslant rp(T^2z, z)$. Now, by (5), we deduce that

$$\begin{aligned} p(z, Tz) &\leqslant p\big(z, T^3z\big) + p\big(Tz, T^3z\big) \\ &\leqslant r^2 p(z, Tz) + rp\big(T^2z, z\big) \\ &\leqslant r^2 p(z, Tz) + r^2 p(Tz, z) = 2r^2 p(z, Tz) \\ &< p(z, Tz), \end{aligned}$$

that is a contradiction.

In the case (ii), we note that $2r^2 < 1$. If we assume $p(T^2z, z) < \theta(r)p(T^2z, T^3z)$, in view of (2), then we have

$$\begin{aligned} p(z, Tz) &\leqslant p\big(z, T^2z\big) + p\big(Tz, T^2z\big) \\ &< \theta(r)p\big(T^2z, T^3z\big) + p\big(Tz, T^2z\big) \\ &\leqslant \theta(r)r^2 p(z, Tz) + rp(z, Tz) \\ &= \big((1-r)r^{-2}r^2 + r\big)p(z, Tz) = p(z, Tz). \end{aligned}$$

This is a contradiction, and so

$$p\big(T^2z, z\big) \geqslant \theta(r)p\big(T^2z, T^3z\big).$$

As in the previous case, we can derive

$$p(z, Tz) \leqslant 2r^2 p(z, Tz) < p(z, Tz),$$

that is a contradiction.

In the case (iii), we note that for $x, y \in X$, either

$$\theta(r)p(x, Tx) \leqslant p(x, y) \quad \text{or} \quad \theta(r)p\big(Tx, T^2x\big) \leqslant p(Tx, y)$$

holds. On the contrary, if

$$\theta(r)p(x, Tx) > p(x, y) \quad \text{and} \quad \theta(r)p\big(Tx, T^2x\big) > p(Tx, y)$$

then we have

$$\begin{aligned} p(x, Tx) &\leqslant p(x, y) + p(Tx, y) \\ &< \theta(r)\big(p(x, Tx) + p\big(Tx, T^2x\big)\big) \\ &\leqslant \theta(r)\big(p\big(x, T(x)\big) + rp(x, Tx)\big) \\ &= (1+r)^{-1}(1+r)p(x, Tx) = p(x, Tx). \end{aligned}$$

This is again a contradiction. Now, since either

$$\theta(r)p(y_{2n}, y_{2n+1}) \leqslant p(y_{2n}, z) \quad \text{or} \quad \theta(r)p(y_{2n+1}, y_{2n+2}) \leqslant p(y_{2n+1}, z)$$

holds for every $n \in \mathbb{N}$, by (1), it follows that either

$$p(y_{2n+1}, Tz) \leqslant rp(y_{2n}, z) \quad \text{or} \quad p(y_{2n+2}, Tz) \leqslant rp(y_{2n+1}, z)$$

holds for every $n \in \mathbb{N}$.

Moreover, since $\lim_{n\to+\infty} p(y_n, z) = 0$, we deduce that either

$$\lim_{n\to+\infty} p(y_{2n+1}, Tz) = 0 = p(Tz, Tz) \quad \text{or}$$
$$\lim_{n\to+\infty} p(y_{2n+2}, Tz) = 0 = p(Tz, Tz).$$

Consequently, there exists a subsequence of $\{y_n\}$ which converges to Tz, This implies $Tz = z$, which is a contradiction. Therefore in all the cases, there exists $j \in \mathbb{N}$ such that $T^j z = z$. Since $\{T^n z\}$ is a 0-Cauchy sequence, we obtain $z = Tz$. If not, that is if $z \neq Tz$, from $p(T^{nj}z, T^{nj+1}z) = p(z, Tz)$ for all $n \in \mathbb{N}$ it follows that $\{T^n z\}$ is not a 0-Cauchy sequence. That is, z is a fixed point of T. The uniqueness of the fixed point follows easily from (3). □

Example 1. Let $X = [0, 1]$ and $p(x, y) = \max\{x, y\}$, then it is clear that (X, p) is a complete partial metric space. Fix $r \in [0, 1)$, and define $T : X \to X$ by

$$Tx = \begin{cases} 0 & \text{if } x = 0, \\ \frac{r}{2n} - r\frac{2n-1}{2n}(2nx - 1) & \text{if } \frac{1}{2n} \leqslant x \leqslant \frac{1}{2n-1}, \\ \frac{r}{2n} + r\frac{2n+1}{2n}(2nx - 1) & \text{if } \frac{1}{2n+1} \leqslant x \leqslant \frac{1}{2n}. \end{cases}$$

For all $x, y \in X$, we have

$$p(Tx, Ty) = \max\{Tx, Ty\} \leqslant \max\{rx, ry\} = r\max\{x, y\} = rp(x, y),$$

which ensures that condition (1) of Theorem 2 is satisfied and so T has a unique fixed point in X.

Now, if we consider the complete metric space (X, d), where $d(x, y) = |x - y|$ for all $x, y \in X$, we cannot use Theorem 1 to deduce that T has a unique fixed point. In fact, if for odd $n > 1$ we choose $x = \frac{1}{2n-1}$ and $y = \frac{1}{n-1}$, we have

$$d(x, Tx) = \frac{1}{2n-1} \leqslant \frac{n}{(n-1)(2n-1)} = d(x, y),$$

but

$$d(Tx, Ty) = \frac{r}{n-1} \leqslant d(x, y) = \frac{n}{(n-1)(2n-1)} \leqslant \frac{3}{5(n-1)}$$

is not satisfied if $r \geqslant 3/5$. Consequently, the contractive condition of Theorem 1 is not satisfied if $r \geqslant 3/5$.

4. Fixed point theorems in partially ordered metric spaces

In this section, we prove fixed point results in partially ordered metric spaces by using Suzuki's contractive condition of Theorem 1.

Theorem 3. *Let $(X, d, \preccurlyeq)$ complete partially ordered metric space. Let $T : X \to X$ be an increasing mapping with respect to $\preccurlyeq$. Define $\theta : [0, 1) \to (1/2, 1]$ as in Theorem 1 and assume that there exists $r \in [0, 1)$ such that*

$$\theta(r)d(x, Tx) \leqslant d(x, y) \quad \textit{implies} \quad d(Tx, Ty) \leqslant rd(x, y), \tag{6}$$

for all comparable $x, y \in X$. If the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$,*
(ii) *for an increasing sequence $\{x_n\} \subset X$ converging to $x \in X$ we have $x_n \prec x$ for all n,*
(iii) *for two nondecreasing sequence $\{x_n\}, \{y_n\} \subset X$ such that $x_n \preccurlyeq y_n$, $\lim_{n\to+\infty} x_n = x$ and $\lim_{n\to+\infty} y_n = y$ we have $x \preccurlyeq y$,*

then T has a fixed point in X. Moreover, the fixed point of T is unique if:

(iv) *for all $x, y \in X$ that are not comparable there exists $v \in X$ comparable with x and y.*

Proof. Let $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$. If $x_0 = Tx_0$, then the result is proved. Hence, we suppose $x_0 \prec Tx_0$. Define a sequence $\{x_n\} \subset X$ such that $x_{n+1} = T^{n+1}x_0 = Tx_n$, for all $n \geqslant 0$. Since T is increasing and $x_0 \prec Tx_0$, therefore

$$x_0 \prec x_1 = Tx_0 \prec T^2x_0 = x_2 \prec T^3x_0 = x_3$$

and, continuing this process, we have $x_0 \prec x_1 \prec \cdots \prec x_n \prec \cdots$, that is $\{x_n\}$ is an increasing sequence. Note that $\theta(r)d(x_{n-1}, Tx_{n-1}) \leqslant d(x_{n-1}, Tx_{n-1})$ holds for all $n > 0$. Since x_{n-1} and Tx_{n-1} are comparable for all $n > 0$, by condition (6), we have $d(Tx_{n-1}, T^2x_{n-1}) \leqslant rd(x_{n-1}, Tx_{n-1})$. Iterating this process, it follows that

$$d\big(Tx_{n-1}, T^2x_{n-1}\big) \leqslant rd(x_{n-1}, Tx_{n-1}) \leqslant \cdots \leqslant r^n d(x_0, x_1) \quad \text{for all } n > 0.$$

So, for any positive integers m and n, with $m > n$, we obtain

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \\ &\leqslant \big(r^n + \cdots + r^{m-1}\big)\, d(x_0, x_1) \\ &\leqslant \frac{r^n}{1-r}\, d(x_0, x_1), \end{aligned}$$

which implies that $\{x_n\}$ is a Cauchy sequence. By the hypothesis that (X, d) is complete, there exists $z \in X$ such that $\lim_{n\to+\infty} x_n = z$.

First, we show that there exists $j \in \mathbb{N}$ such that $T^jz = z$. Arguing by contradiction, we assume that $T^jz \neq z$, for all $j \in \mathbb{N}$. Note that, by condition (ii), $\{x_n\}$ increasing implies $x_n \prec z$ for all $n \geqslant 0$. Since T is increasing, we have that $x_{n+1} = Tx_n \prec Tz$, for all $n \geqslant 0$. Taking the limit as $n \to +\infty$, by (iii), we obtain that $z \prec Tz$ which implies that $\{T^nz\}$ is an increasing sequence. So, as we have shown for $\{x_n\}$, also $\{T^nz\}$ is a Cauchy sequence. Note, T^jz is comparable with x_n, for all $j, n \geqslant 0$.

Now, we prove that $d(T^{j+1}z, z) \leqslant rd(T^jz, z)$ for all $j \in \mathbb{N}$. Since $d(T^jz, z) > 0$ and $\lim_{n\to+\infty} d(x_n, z) = 0$, there exists $\nu \in \mathbb{N}$ such that $d(x_n, z) \leqslant d(T^jz, z)/3$ for all $n \geqslant \nu$.

We have

$$\begin{aligned} \theta(r)d(x_n, Tx_n) &\leqslant d(x_n, Tx_n) = d(x_n, x_{n+1}) \\ &\leqslant d(x_n, z) + d(x_{n+1}, z) \\ &\leqslant (2/3)d\big(T^jz, z\big) = d\big(T^jz, z\big) - d\big(T^jz, z\big)/3 \\ &\leqslant d\big(T^jz, z\big) - d(x_n, z) \leqslant d\big(x_n, T^jz\big). \end{aligned}$$

By (6), it follows that

$$d\big(x_{n+1}, T^{j+1}z\big) \leqslant rd\big(x_n, T^jz\big) \quad \text{for all } n \geqslant \nu. \tag{7}$$

From (7), as $n \to +\infty$, we deduce that $d(T^{j+1}z, z) \leqslant rd(T^jz, z)$ for all $j \in \mathbb{N}$. This implies that (5) holds, that is

$$d\big(T^{j+1}z, z\big) \leqslant rd\big(T^jz, z\big) \leqslant \cdots \leqslant r^jd(Tz, z) \quad \text{for all } j \in \mathbb{N}.$$

Moreover, since $\theta(r)d(z, Tz) \leqslant d(z, Tz)$ implies $d(Tz, T^2z) \leqslant r\,d(z, Tz)$, we derive that (2) holds with $x = z$.

Proceeding as in the proof of Theorem 2, by considering the following three cases:

1) $0 \leqslant r \leqslant (\sqrt{5} - 1)/2$;
2) $(\sqrt{5} - 1)/2 < r < 2^{-1/2}$;
3) $2^{-1/2} \leqslant r < 1$;

we obtain that there exists $j \in \mathbb{N}$ such that $T^jz = z$. It implies that $Tz = z$, since $\{T^nz\}$ is a Cauchy sequence.

Now, we prove the uniqueness of the fixed point. Suppose there exists z^* such that $Tz^* = z^*$, with $z \neq z^*$. We have two possible cases:

Case 1) z and z^* are comparable, by (6) with $x = z$ and $y = z^*$, we obtain that $d(z, z^*) = d(Tz, Tz^*) \leqslant rd(z, z^*)$, which is a contradiction. Hence $z = z^*$.

Case 2) z and z^* are not comparable and there exists x comparable with z and z^*. First we note that for each $x \in X$ comparable with z, by (6) with $x = T^{n-1}z$ and $y = T^{n-1}x$, we derive that

$$d\big(T^nz, T^nx\big) \leqslant rd\big(T^{n-1}z, T^{n-1}x\big) \quad \text{for all } n > 0.$$

Consequently, $d(T^nz, T^nx) \leqslant r^nd(z, x)$ for all $n > 0$ and hence $\lim_{n\to+\infty} d(T^nz, T^nx) = 0$. Similarly $\lim_{n\to+\infty} d(T^nz^*, T^nx) = 0$. From

$$0 < d\big(z, z^*\big) = d\big(T^nz, T^nz^*\big) \leqslant d\big(T^nz, T^nx\big) + d\big(T^nz^*, T^nx\big)$$

as $n \to +\infty$, we deduce that $0 < d(z, z^*) \leqslant 0$, which is again a contradiction. Thus $z = z^*$. $\square$

Example 2. Let $X=[0,+\infty)$ and $d(x,y)=|x-y|$, then it is clear that (X,d) is a complete metric space. We can define a partial order on X as follows:

$$x \preccurlyeq y \quad \text{if } x=y \quad \text{or} \quad x,y\in[0,1] \text{ and } x\leqslant y.$$

Then $(X,d,\preccurlyeq)$ is a complete partially ordered metric space. Fix $r\in[0,1)$. Define an increasing function $T:X\to X$ by $Tx=rx^2/2$.

For all $x,y\in X$, with $x\preccurlyeq y$, we have

$$d(Tx,Ty)=\frac{r}{2}\bigl(y^2-x^2\bigr)\leqslant rd(x,y),$$

which ensures that the contractive condition (6) is satisfied. Also, conditions (i)–(iii) of Theorem 3 are satisfied and so T has a fixed point in X.

Now, if we consider the complete metric space (X,d), we cannot use the Theorem 1 to deduce that T has a unique fixed point. In fact, if we choose $x=1$ and $y\geqslant\frac{2}{r}$, we have

$$d(x,Tx)=1-\frac{r}{2}\leqslant\frac{2}{r}-1\leqslant d(x,y),$$

but

$$d(Tx,Ty)=\frac{r}{2}\bigl(y^2-1\bigr)\geqslant d(x,y).$$

This shows that the contractive condition of Theorem 1 is not satisfied.

Now, let $T:X\to X$ and define

$$M(x,y)=\max\left\{d(x,y),d(y,Ty),\frac{1}{2}\bigl[d(x,Ty)+d(y,Tx)\bigr]\right\}$$

and

$$M_1(x,y)=\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}\bigl[d(x,Ty)+d(y,Tx)\bigr]\right\}.$$

Theorem 4. *Let $(X,d,\preccurlyeq)$ complete partially ordered metric space. Let $T:X\to X$ be a continuous mapping which is increasing with respect to $\preccurlyeq$. Assume that there exists $r\in[0,1)$ such that*

$$d(x,Tx)\leqslant d(x,y) \quad \textit{implies} \quad d(Tx,Ty)\leqslant rM(x,y), \tag{8}$$

for all comparable $x,y\in X$. If there exists $x_0\in X$ such that $x_0\preccurlyeq Tx_0$, then T has a fixed point in X.

Proof. Let $x_0\in X$ such that $x_0\preccurlyeq Tx_0$. If $x_0=Tx_0$, then the result is proved. Hence, we suppose $x_0\prec Tx_0$. As in the proof of Theorem 3, we deduce that the sequence $\{x_n\}$, where $x_{n+1}=T^{n+1}x_0=Tx_n$ for all $n\geqslant 0$, is increasing. Note that $d(x_{n-1},Tx_{n-1})\leqslant d(x_{n-1},Tx_{n-1})$ holds for all $n>0$. Since x_{n-1} and Tx_{n-1} are comparable for all $n>0$, by condition (8), we have

$$d(x_n,x_{n+1})=d\bigl(Tx_{n-1},T^2x_{n-1}\bigr)\leqslant rM(x_{n-1},Tx_{n-1})=rM(x_{n-1},x_n),$$

where

$$\begin{aligned}M(x_{n-1},x_n)&=\max\left\{d(x_{n-1},x_n),d(x_n,x_{n+1}),\frac{1}{2}\bigl[d(x_{n-1},x_{n+1})+d(x_n,x_n)\bigr]\right\}\\&=\max\bigl\{d(x_{n-1},x_n),d(x_n,x_{n+1})\bigr\}.\end{aligned}$$

Since $M(x_{n-1},x_n)=d(x_n,x_{n+1})$ give a contradiction, then $M(x_{n-1},x_n)=d(x_{n-1},x_n)$. It follows that

$$d(x_n,x_{n+1})\leqslant rd(x_{n-1},x_n)\leqslant\cdots\leqslant r^nd(x_0,x_1) \quad \text{for all } n>0.$$

So, for any positive integers m and n, with $m>n$, we obtain

$$d(x_n,x_m)\leqslant\frac{r^n}{1-r}d(x_0,x_1),$$

which implies that $\{x_n\}$ is a Cauchy sequence. By the hypothesis that (X,d) is complete, there exists $z\in X$ such that $\lim_{n\to+\infty}x_n=z$. Since T is continuous, from $x_{n+1}=Tx_n$ as $n\to+\infty$, it follows that $Tz=z$. $\square$

From Theorem 4 we obtain the following corollary.

Corollary 1. *Let $(X, d, \preccurlyeq)$ complete partially ordered metric space. Let $T : X \to X$ be a continuous mapping which is increasing with respect to $\preccurlyeq$. Assume that there exists $r \in [0, 1)$ such that*

$$d(x, Tx) \leqslant d(x, y) \quad \textit{implies} \quad d(Tx, Ty) \leqslant rM_1(x, y), \tag{9}$$

for all comparable $x, y \in X$. If there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$, then T has a fixed point in X.

5. Common fixed point results

In this section we use the following lemma, that is a consequence of the axiom of choice, to obtain common fixed point results for two self-mappings defined on a partial metric space or a partially ordered metric space.

Lemma 4. *([2, Lemma 2.1]) Let X be a non-empty set and $f : X \to X$ a mapping. Then there exists a subset $E \subset X$ such that $fE = fX$ and $f : E \to X$ is one-to-one.*

Theorem 5. *Let (X, p) be a partial metric space and let $T, f : X \to X$ be such that $TX \subset fX$. Define $\theta : [0, 1) \to (1/2, 1]$ as in Theorem 1 and assume that there exists $r \in [0, 1)$ such that, for all $x, y \in X$,*

$$\theta(r)p(fx, Tx) \leqslant p(fx, fy) \quad \textit{implies} \quad p(Tx, Ty) \leqslant rp(fx, fy). \tag{10}$$

Assume also that T and f are weakly compatible. If fX is a 0-complete subspace of X, then T and f have a unique common fixed point.

Proof. By Lemma 4, there exists $E \subset X$ such that $fE = fX$ and $f : E \to X$ is one-to-one. Define

$$S : fE \to fE \quad \text{by } Sfx = Tx \quad \text{for all } fx \in fE.$$

Since f is one-to-one on E, S is well defined. Note that, for all $fx, fy \in fE$,

$$\theta(r)p(fx, Sfx) \leqslant p(fx, fy) \quad \text{implies} \quad p(Sfx, Sfy) \leqslant rp(fx, fy).$$

By Theorem 2, as fE is 0-complete, S has a unique fixed point on fE, say fz. Then, $fz \in X$ is a point of coincidence of T and f, that is $Tz = fz$. Now, by (10), T and f have a unique point of coincidence. Since T and f are weakly compatible, by Lemma 3, we deduce that fz is the unique common fixed point of T and f. □

Theorem 6. *Let $(X, \preccurlyeq)$ be a partially ordered set and let $T, f : X \to X$ be such that T is an f-increasing mapping with $TX \subset fX$. Suppose that there exists a metric d on X such that fX is a complete subset of X. Define $\theta : [0, 1) \to (1/2, 1]$ as in Theorem 1 and assume that there exists $r \in [0, 1)$ such that*

$$\theta(r)d(fx, Tx) \leqslant d(fx, fy) \quad \textit{implies} \quad d(Tx, Ty) \leqslant rd(fx, fy) \tag{11}$$

for all comparable $fx, fy \in X$. If the following conditions hold:

(i) *there exists $x_0 \in X$ such that $fx_0 \preccurlyeq Tx_0$;*
(ii) *for an increasing sequence $\{x_n\} \subset X$ converging to $x \in X$ we have $x_n \prec x$ for all n;*
(iii) *for two nondecreasing sequence $\{x_n\}, \{y_n\} \subset X$ such that $x_n \preccurlyeq y_n$, $\lim_{n\to+\infty} x_n = x$ and $\lim_{n\to+\infty} y_n = y$, we have $x \preccurlyeq y$;*
(iv) *the set of points of coincidence of T and f, say $PC(T, f)$, is totally ordered and T and f are weakly compatible;*

then T and f have a unique common fixed point.

Proof. By Lemma 4, there exists $E \subset X$ such that $fE = fX$ and $f : E \to X$ is one-to-one. Define

$$S : fE \to fE \quad \text{by } Sfx = Tx \quad \text{for all } fx \in fE.$$

Since f is one-to-one on E, S is well defined. Note that, for all comparable $fx, fy \in fE$,

$$\theta(r)d(fx, Sfx) \leqslant d(fx, fy) \quad \text{implies} \quad d(Sfx, Sfy) \leqslant rd(fx, fy).$$

Since T is f-increasing, we have that S is increasing. In fact, $fx \prec fy \Rightarrow Tx \prec Ty$ and hence $Sfx = Tx \prec Ty = Sfy$. Since fE is complete, by Theorem 3, S has a fixed point on fE, say fz. Then, $fz \in fE$ is a point of coincidence of T and f, that is $Tz = Sfz = fz$.

Now, we prove that T and f have a unique point of coincidence. Let $fw \in PC(T, f)$ with $fw \neq fz$, then

$$\theta(r)d(fz, Tz) \leqslant d(fz, fw)$$

and, by (11), we have $d(Tw, Tz) \leqslant rd(fw, fz)$, that is $d(fw, fz) \leqslant rd(fw, fz)$, which is a contradiction. It follows that $fw = fz$. Since T and f are weakly compatible, by Lemma 3, we deduce that fz is the unique common fixed point of T and f. □

6. Completeness in partial metric spaces and fixed points

In this section we characterize those partial metric spaces for which every Suzuki mapping has a fixed point in the style of Suzuki's characterization of metric completeness. This will be done by means of the notion of 0-complete partial metric space which was introduced by Romaguera in [9].

Theorem 7. *Let (X, p) be a partial metric space and define a function θ as in Theorem 1. For $r \in [0, 1)$ and $\eta \in (0, \theta(r)]$, let $A_{r,\eta}$ be the family of mappings T on X satisfying the following*:

(a) *For all $x, y \in X$*

$$\eta p(x, Tx) \leqslant p(x, y) \quad \textit{implies} \quad p(Tx, Ty) \leqslant rp(x, y).$$

Let $B_{r,\eta}$ be the family of mappings T on X satisfying (a) and the following:

(b) *TX is countable infinite*;
(c) *Every subset of TX is closed.*

Then the following are equivalent;

(i) *(X, p) is 0-complete*;
(ii) *Every mapping $T \in A_{r,\theta(r)}$ has a fixed point for all $r \in [0, 1)$*;
(iii) *There exist $r \in (0, 1)$ and $\eta \in (0, \theta(r)]$ such that every mapping $T \in B_{r,\eta}$ has a fixed point.*

Proof. By Theorem 2, (i) implies (ii). Since $B_{r,\eta} \subset A_{r,\theta(r)}$ for $r \in [0, 1)$ and $\eta \in (0, \theta(r)]$, (ii) implies (iii). Let us prove that (iii) implies (i). We assume (iii). Arguing by contradiction, we also assume that (X, p) is not 0-complete, that is, there exists a 0-Cauchy sequence $\{y_n\}$ which does not converge.

Define a function from X into $[0, +\infty)$ by $fx = \lim_{n\to+\infty} p(x, y_n)$ for $x \in X$. Since, by Lemma 1, the sequence $\{p(x, y_n)\}$ is Cauchy in $\mathbb{R}$ for all $x \in X$, then the function f is well defined. The following are obvious:

(P1) $fx > 0$ for all $x \in X$.
In fact, if $fx = \lim_{n\to+\infty} p(x, y_n) = 0$ then $p(x, x) = 0$ and $\lim_{n\to+\infty} y_n = x$, which is a contradiction.
(P2) $\lim_{n\to+\infty} fy_n = \lim_{n,m\to+\infty} p(y_n, y_m) = 0$.
(P3) $fx - fy \leqslant p(x, y) \leqslant fx + fy$ for all $x, y \in X$.
It follows from
- $p(x, y) \leqslant p(x, y_n) + p(y_n, y) - p(y_n, y_n)$;
- $p(x, y_n) \leqslant p(x, y) + p(y, y_n) - p(y, y)$.

Now, by (P1) and (P2), there exists $\nu : X \to \mathbb{N}$ such that $fy_{\nu(x)} \leqslant \frac{\eta r}{3+\eta r} fx$ for each $x \in X$. Define $T : X \to X$, by $Tx = y_{\nu(x)}$. Then it is obvious that

$$fTx \leqslant \frac{\eta r}{3 + \eta r} fx \quad \text{and} \quad Tx \in \{y_n \colon n \in \mathbb{N}\},$$

for all $x \in X$. T does not have a fixed point, since $Tx \neq x$ for all $x \in X$ because $fTx < fx$. From $TX \subset \{y_n \colon n \in \mathbb{N}\}$, it follows that (b) holds. Also, it is not difficult to prove (c). Let us prove (a). Fix $x, y \in X$ with $\eta p(x, Tx) \leqslant p(x, y)$. In the case where $fy > 2fx$, by (P3), we have

$$\begin{aligned} p(Tx, Ty) &\leqslant fTx + fTy \leqslant \frac{\eta r}{3 + \eta r}(fx + fy) \\ &\leqslant \frac{r}{3}(fx + fy) \leqslant \frac{r}{3}(fx + fy) + \frac{2r}{3}(fy - 2fx) \\ &= r(fy - fx) \leqslant rp(x, y). \end{aligned}$$

In the other case, where $fy \leqslant 2fx$, we have

$$p(x, y) \geqslant \eta p(x, Tx) \geqslant \eta(fx - fTx) \geqslant \eta\left(1 - \frac{\eta r}{3 + \eta r}\right) fx = \frac{3\eta}{3 + \eta r} fx$$

and hence

$$p(Tx, Ty) \leqslant fTx + fTy \leqslant \frac{\eta r}{3 + \eta r}(fx + fy) < \frac{3\eta r}{3 + \eta r} fx \leqslant rp(x, y).$$

Therefore we have shown (a), that is, $T \in B_{r,\eta}$. By (iii), T has a fixed point which yields a contradiction. Hence we obtain that X is 0-complete. This completes the proof. □

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