Discrete Mathematics 61 (1986) 119–132
North-Holland

ON THE COVERING OF PAIRS BY QUADRUPLES

A.M. ASSAF*
Technion–Israel Institute of Technology, Haifa, Israel

Received 4 March 1985

Let X be a finite set of size v, let λ be a positive integer, and $\alpha(4, \lambda, v)$ be the minimum number of quadruples such that each pair of elements of X is contained in at least λ of them. Mills [5, 6] has determined $\alpha(4, 1, v)$ for all v. In this paper $\alpha(4, \lambda, v)$ is determined for all v and $\lambda > 1$.

1. Introduction

1.1. *Designs*

Let X be a finite set of points and let $\mathcal{B} = \{B_i : i \in I\}$ be a family of—not necessarily distinct—subsets B_i (called blocks) of X. The pair $(X, \mathcal{B})$ is called a hypergraph. When certain regularity conditions are imposed, the resultant object is called a design.

1.2. *Balanced incomplete block designs (BIBD)*

Let $v \geqslant k \geqslant 2$ and λ be positive integers. A design $(X, \mathcal{B})$ is called a balanced incomplete block design (BIBD) $B[k, \lambda; v]$ if

(i) $|X| = v$;
(ii) the blocks are of size k;
(iii) every 2-set $\{x, y\} \subset X$ is contained in exactly λ blocks of $\mathcal{B}$.

A well-known result states that a necessary condition for the existence of a BIBD $B[k, \lambda; v]$ is that $\lambda(v-1) \equiv 0 \pmod{k-1}$ and $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$.

We shall use the following

Theorem 1.1 (H. Hanani [2]). *Let λ and $v \geqslant 4$ be positive integers. A necessary and sufficient condition for the existence of a BIBD $B[4, \lambda; v]$ is that $\lambda(v-1) \equiv 0$ (mod 3) and $\lambda v(v-1) \equiv 0$ (mod 12).*

1.3. *Group divisible designs*

We shall consider designs of the form $(X, \mathcal{G}, \mathcal{P})$, where X is a finite set of points, $\mathcal{G}$ is a parallel class of subsets of X called groups and $\mathcal{P}$ is a family of subsets of X called (proper) blocks.

* This paper is a part of a Ph.D. thesis written at the Technion–Israel Institute of Technology, Under the supervision of Professor H. Hanani.

Let m, k, λ and v be positive integers. A design $(X, \mathcal{G}, \mathcal{P})$ is a group divisible design $\mathrm{GD}[k, \lambda, m; v]$ if

(i) $|X| = v$;
(ii) $|G_i| = m$ for every $G_i \in \mathcal{G}$;
(iii) $|B_j| = k$ for every $B_j \in \mathcal{P}$;
(iv) $|G_i \cap B_j| \leq 1$ for every $G_i \in \mathcal{G}$ and every $B_j \in \mathcal{P}$;
(v) every 2-subset $\{x, y\} \subset X$ such that x and y belong to distinct groups; is contained in exactly λ blocks of $\mathcal{P}$.

A necessary condition for the existence of a group divisible design $\mathrm{GD}[k, \lambda, m; v]$ is that $v \equiv 0 \pmod m$, $\lambda(v-m) \equiv 0 \pmod{k-1}$, $\lambda v(v-m) \equiv 0 \pmod{k(k-1)}$ and $v \geq km$ or $v = m$.

We shall use the following

Theorem 1.2 (Brouwer–Hanani–Schrijver [1]). *Let m, λ and v be positive integers. A necessary and sufficient condition for the existence of a group divisible design $\mathrm{GD}[4, \lambda, m; v]$ is that the design is not $\mathrm{GD}[4, 1, 2; 8]$ and not $\mathrm{GD}[4, 1, 6; 24]$ and that $v \equiv 0 \pmod m$, $\lambda(v-m) \equiv 0 \pmod 3$, $\lambda v(v-m) \equiv 0 \pmod{12}$ and $v \geq 4m$ or $v = m$.*

1.4. *Covering and packing designs*

A design $(X, \mathcal{B})$ is called a covering design $\mathrm{AD}[k, \lambda, v; b]$ (or, respectively, a packing design $\mathrm{SD}[k, \lambda, v; b]$) if

(i) $|X| = v$;
(ii) the blocks are of size k;
(iii) $|\mathcal{B}| = b$;
(iv) every 2-subset $\{x, y\} \subset X$ is included in at least (at most) λ blocks of $\mathcal{B}$.

Naturally, we are interested in covering designs having a minimal number of blocks and, conversely, in packing designs with the maximal number of blocks.

Denote by $\alpha(k, \lambda, v)$ the smallest number b of blocks for which $\mathrm{AD}[k, \lambda, v; b]$ exists and by $\sigma(k, \lambda, v)$ the greatest value of b for which $\mathrm{SD}[k, \lambda, v; b]$ exists. Clearly

$$\sigma(k, \lambda, v) \leq \lambda v(v-1)/k(k-1) \leq \alpha(k, \lambda, v)$$

and the equality sign on both sides holds if and only if a BIBD $B[k, \lambda; v]$ exists.

Schonheim [7] introduced the notation

$$\phi(k, \lambda, v) = \left\lceil \frac{v}{k} \left\lceil \frac{v-1}{k-1} \lambda \right\rceil \right\rceil, \qquad \psi(k, \lambda, v) = \left\lfloor \frac{v}{k} \left\lfloor \frac{v-1}{k-1} \lambda \right\rfloor \right\rfloor,$$

where $\lceil x \rceil$ is the smallest and $\lfloor x \rfloor$ the largest integer satisfying $\lfloor x \rfloor \leq x \leq \lceil x \rceil$ and proved

Theorem 1.3 (Schonheim [7]). *For every positive integer k, λ and $v \geq k$*

$$\sigma(k, \lambda, v) \leq \psi(k, \lambda, v) \leq \lambda v(v-1)/k(k-1) \leq \phi(k, \lambda, v) \leq \alpha(k, \lambda, v).$$

A design $(X, \mathscr{B})$ with $X' \subset X$ is called an almost covering design AD*$[k, \lambda, v(t); b]$ if

(i) $|X| = v$;

(ii) the blocks are of size k;

(iii) $|\mathscr{B}| = b$;

(iv) $|X'| = t$

(v) every pairset $\{x, y\} \subset X$ such that $\{x, y\} \not\subset X'$ is included in at least λ blocks of $\mathscr{B}$;

(vi) no pairset $\{x, y\} \subset X'$ is included in any block of $\mathscr{B}$.

For $k = 3$ and for every λ and v Hanani [2, p. 367] determined $\alpha(3, \lambda, v)$ and $\sigma(3, \lambda, v)$.

2. Covering designs with $k = 4$

In the case $k = 4$ and $\lambda = 1$ Mills [5, 6] proved

Theorem 2.1. *For every positive integer $v \geq 4$*

$$\alpha(4, 1, v) = \phi(4, 1, v), \quad v \neq 7, 9, 10, 19$$

$$\alpha(4, 1, 7) = \phi(4, 1, 7) + 1,$$

$$\alpha(4, 1, 9) = \phi(4, 1, 9) + 1,$$

$$\alpha(4, 1, 10) = \phi(4, 1, 10) + 1,$$

$$\alpha(4, 1, 19) = \phi(4, 1, 19) + 2,$$

We shall obtain analogous results for $\lambda > 1$. For our proofs we need the following results of Hanani and Mills.

Lemma 2.1 (Hanani [2]). *For given integers k, λ and m let* GD$[k, \lambda, m, mn]$ *exist for every integer $n \geq k$. If in addition for $u = m + t$ $(0 \leq t \leq m)$ both designs*

$$\text{AD}[k, \lambda, u, (\lambda u^2 + a_1 u + a_0)/k(k-1)]$$

and

$$\text{AD}^*[k, \lambda, u(t), m(\lambda(u+t) + a_1)/k(k-1)]$$

exist, then

$$\alpha(k, \lambda, v) \leq (\lambda v^2 + a_1 v + a_0)/k(k-1),$$

for every $v = mn + t$.

Lemma 2.2 (Mills [6, p. 139]). *Let X be a set of order $4w+u$ where $w \equiv 0$ or $1 \pmod 4$ and $0<u<w$. Then there exists a collection $\mathcal{D}$ of w^2+5 subsets of X such that*:

(i) *the collection $\mathcal{D}$ covers all pairs of X exactly once*;

(ii) *$\mathcal{D}$ consists of $w(w-u)$ sets of order 4, wu sets of order 5, four sets of order w and one set of order u.*

Theorem 2.2 (Mills [5, p. 161]). *If $v \equiv 7$ or $10 \pmod{12}$ and if $\mathcal{C}$ is a collection of $\phi(4, 1, v)$ quadruples that covers all pairs of a set X of order v, then there is one pair which occurs four times in these quadruples, while all other pairs occur exactly once.*

3. Constructions

We proceed to prove the following

Theorem. *For every positive integer $\lambda > 1$ and $v \geqslant 4$, $\alpha(4, \lambda, v) = \phi(4, \lambda, v)$ holds.*

For all values of v for which $\alpha(4, 1, v) = \phi(4, 1, v)$ it is sufficient to prove, by Theorem 1.1, our theorem for $2 \leqslant \lambda \leqslant 5$. For the few cases that $\alpha(4, 1, v) > \phi(4, 1, v)$, i.e., $v = 7, 9, 10, 19$ we have to prove also that $\alpha(4, 7, v) = \phi(4, 7, v)$.

First we need the following two lemmas.

Lemma 3.1. *For $v=6$ and $\lambda=3$ there exists* AD[4, 3, 6; 8].

Proof. Let $X = Z_6$, then the blocks are

$$\langle 0,1,4,5\rangle \quad \langle 1,3,4,5\rangle \quad \langle 0,3,4,5\rangle \quad \langle 0,1,2,3\rangle$$
$$\langle 2,3,4,5\rangle \quad \langle 0,2,4,5\rangle \quad \langle 1,2,4,5\rangle \quad \langle 0,1,2,3\rangle$$

Lemma 3.2. *for $v=7$ and $\lambda=3$ there exists* AD[4, 3, 7; 11].

Proof. Let $X = Z_7$, then the blocks are

$$\langle 0,1,2,4\rangle \quad \langle 1,2,3,4\rangle \quad \langle 2,4,5,6\rangle \quad \langle 0,2,3,6\rangle$$
$$\langle 1,2,5,6\rangle \quad \langle 0,1,3,4\rangle \quad \langle 1,3,5,6\rangle \quad \langle 0,4,5,6\rangle$$
$$\langle 3,4,5,6\rangle \quad \langle 0,2,3,5\rangle \quad \langle 0,1,5,6\rangle$$

In order to solve our problem, we divide it into several cases.

3.1. $v \equiv 1$ *or* 4 (mod 12)

In this case $\alpha(4, \lambda, v) = \phi(4, \lambda, v)$ follows from Theorem 1.1.

3.2. $v \equiv 7$ *or* 10 (mod 12)

In this case $\alpha(4, \lambda, v) = \phi(4, \lambda, v)$.

Proof. For $\lambda = 2$, we have $B[4, 2; v]$.

For $\lambda = 3$, the blocks of $AD[4, 3, v; b]$ are the blocks of $B[4, 2; v]$ with the blocks of $AD[4, 1, v; b]$ by Theorem 2.1. This method does not work for $v = 7$ and 10. For $v = 7$ see Lemma 3.2, for $v = 10$, [4], let $X = \{a, b\} \cup (Z_4 \times Z_2)$, then the required blocks are:

$\langle a, b, (0, 0), (0, 1)\rangle$ (mod(4, −)) $\langle (0, 1), (1, 0), (2, 0), (3, 0)\rangle$ (mod(4, −))

$\langle a, (0, 0), (0, 1), (1, 1)\rangle$ (mod(4, −)) $\langle (0, 0), (1, 0), (0, 1), (2, 1)\rangle$ (mod(4, −))

$\langle b, (0, 0), (2, 1), (3, 1)\rangle$ (mod(4, −)) $\langle (0, 1), (1, 1), (2, 1), (3, 1)\rangle$

$\langle a, b, (0, 0), (2, 0)\rangle$ $(+(i, -)\ i = 0, 1)$

For $v = 19$, let $X = Z_{13} \cup \{a, b, c, d, e, f\}$, then the blocks that cover each pair of X at least three times are

$\langle 0, 1, 6, a\rangle$ (mod 13) $\langle 0, 3, 7, c\rangle$ (mod 13) $\langle 0, 2, 6, e\rangle$ (mod 13)

$\langle 0, 2, 5, b\rangle$ (mod 13) $\langle 0, 4, 5, d\rangle$ (mod 13) $\langle 0, 1, 3, f\rangle$ (mod 13)

$\langle a, b, c, d, e, f\rangle$

For the last blocks apply Lemma 3.1.

For $\lambda = 4$ it follows from Theorem 1.1.

For $\lambda = 5$, then the blocks of $AD[4, 5, v; b]$ are the blocks of $B[4, 2; v]$ with the blocks of $AD[4, 3, v; b]$.

3.3. $v \equiv 0$ (mod 12)

In this case $\alpha(4, \lambda, v) = \phi(4, \lambda, v)$.

Proof. For $\lambda = 2$, the blocks of $AD[4, 2, v; b]$ are the blocks of $AD[4, 1, v; b]$, taken twice.

For $\lambda = 3$, it follows from Theorem 1.1.

For $\lambda = 4$, then the blocks of $AD[4, 4, v; b]$ are the blocks of $B[4, 3; v]$ with the blocks of $AD[4, 1, v; b]$.

For $\lambda = 5$, then the blocks of $AD[4, 5, v; b]$ are the blocks of $B[4, 3; v]$ with the blocks of $AD[4, 2, v; b]$.

3.4. $v \equiv 2 \pmod{12}$

In this case $\alpha(4, \lambda, v) = \phi(4, \lambda, v)$.

Proof. (1) For $\lambda = 2$, apply Lemma 2.1 with $m = 12$ and $t = 2$. According to this lemma it is sufficient to prove the existence of AD*[4, 2, 14(2); 31], AD[4, 2, 14; 32], AD[4, 2, 26; 111] and AD[4, 2, 38; 238].

For AD*[4, 2, 14(2); 31], take the blocks of B[4, 1; 13], and further, take the blocks of AD[4, 1, 15; 19] as they appeared in Mills' paper [5, p. 71] with the following changes: we interchange 1 with 13, 2 with 14 and 3 with 15; then we eliminate the block $\langle 13, 14, 15 \rangle$ and in all the remaining blocks we change 15 to 14.

The blocks of AD[4, 2, 14; 32] can be constructed in the following way:

(a) taking the blocks of B[4, 1; 13];

(b) taking the blocks of AD[4, 1, 15; 19] as above by interchanging 15 with 7, and then changing 15 to 14 in all the blocks except the block $\langle 6, 15, 10, 14 \rangle$ from which we drop only the point 15.

For AD[4, 2, 26; 111] we take the blocks of B[4, 1; 25] and then we take the blocks of AD[4, 1, 27; 61] as given by Mills [5, p. 71]. What is essential in this construction is that we have the block $\langle (2, 2), (2, 3), (2, 6), (2, 7) \rangle$ which contains the pairs $((2, 2), (2, 6))$ and $((2, 3), (2, 7))$, which appear once more elsewhere, so we can drop the point $(2, 7)$ from this block and in the remaining blocks we change $(2, 7)$ to $(2, 6)$.

For AD[4, 2, 38; 238]: since in the construction of AD[4, 1, 39; 127], [5, p. 72] we have the block $\langle (2, 1), (2, 2), (2, 7), (2, 8) \rangle$ which contains two pairs that appear once more, we can apply the method as above to get the blocks of AD(4, 2, 38; 238].

(2) For $\lambda = 3$, again apply Lemma 2.1 with $m = 12$ and $t = 2$. Accordingly, it is sufficient to prove the existence of AD*[4, 3, 14(2); 45], AD[4, 3, 14; 46], AD[4, 3, 26; 163] and AD[4, 3, 38; 352].

For AD*[4, 3, 14(2); 45] we take the blocks of B[4, 2; 13] and the blocks of B[4, 1; 16], dropping the block $\{13, 14, 15, 16\rangle$ and changing in the remaining blocks both points 15 and 16 to 14.

For AD[4, 3, 14; 46], [4] let $X = Z_{12} \cup \{a, b\}$ and take the blocks

$$\langle 0, 1, 3, 7 \rangle \pmod{12}, \quad \langle a, b, 0, 6 \rangle \ (+i, i \in Z_6), \quad \langle b, 1, 2, 6 \rangle \ (+2i, i \in Z_6),$$
$$\langle 0, 2, 3, 5 \rangle \pmod{12}, \quad \langle a, 0, 1, 5 \rangle \ (+2i, i \in Z_6), \quad \langle a, 0, 4, 8 \rangle \ (+2i, i \in Z_2),$$
$$\langle b, 1, 5, 9 \rangle \ (+2i, i \in Z_2).$$

For AD[4, 3, 26; 163], let $X = Z_{19} \cup \{a, b, c, d, e, f, g\}$ then the blocks are

$$\langle 0, 4, 7, 9 \rangle \pmod{19}, \quad \langle 0, 5, 8, c \rangle \pmod{19}, \quad \langle 0, 5, 7, f \rangle \pmod{19},$$
$$\langle 0, 6, 8, a \rangle \pmod{19}, \quad \langle 0, 4, 10, d \rangle \pmod{19}, \quad \langle 0, 3, 4, g \rangle \pmod{19},$$
$$\langle 0, 1, 7, b \rangle \pmod{19}, \quad \{0, 1, 9, e \rangle \pmod{19}, \quad \langle a, b, c, d, e, f, g \rangle,$$

and for the last block apply Lemma 3.2.

For AD[4, 3, 38; 352], apply Lemma 2.2: $38 = 4 \cdot 8 + 6$ where $8 \equiv 0 \pmod 4$ and $6 < 8$.

According to this lemma, there is a covering of the pairs of the 38 points such that each pair is contained in exactly one block. The blocks are of sizes 4, 5, 8 and one block of size 6. On the blocks of sizes 4, 5, 8 we construct BIBD with $\lambda = 3$, and for the block of size 6 we use Lemma 3.1.

(3) For $\lambda = 4$, first we give the construction of AD[4, 1, v; b], where $v \equiv 2 \pmod{12}$. Take the blocks of $B[4, 1; v-1]$, then divide the $v-1$ points into triples and to each triple add the point v. Since $v - 1 \equiv 1 \pmod{12}$, then when dividing the $(v-1)$ points to triples, there will be a point $(v-1)$ left out and when adding the point v to this point, we will have a block of order two.

Now the construction of AD[4, 4, v; b], $v \equiv 2 \pmod{12}$ can be done by taking the blocks of AD[4, 3, v; b] which has a pair that occurs six times, say $(v-1, v)$ and the blocks of AD[4, 1, v; b] which has a block of order two $\langle v-1, v\rangle$ and drop this block. The remaining blocks are the blocks of AD[4, 4, v; b].

(4) For $\lambda = 5$, apply Lemma 2.1 with $m = 12$, $t = 2$ and $\lambda = 5$. It is sufficient to construct AD*[4, 5, 14(2); 76], AD[4, 5, 14; 77], AD[4, 5, 26; 273] and AD[4, 5, 38; 589].

For AD*[4, 5, 14(2); 76] we take the blocks of AD*[4, 3, 14(2); 45] and AD*[4, 2, 14(2); 31].

For AD[4, 5, 14; 77] we take the blocks of AD[4, 3, 14; 46], $B[4, 1; 13]$ and the following blocks

$\langle 3, 4, 5, 12\rangle$	$\langle 1, 8, 9, 13\rangle$	$\langle 1, 3, 7, 10\rangle$
$\langle 6, 7, 8, 12\rangle$	$\langle 0, 3, 9, 13\rangle$	$\langle 2, 3, 8, 11\rangle$
$\langle 9, 10, 11, 12\rangle$	$\langle 0, 7, 11, 13\rangle$	$\langle 2, 4, 7, 9\rangle$
$\langle 0, 1, 2, 12\rangle$	$\langle 1, 4, 11, 13\rangle$	$\langle 0, 4, 8, 10\rangle$
$\langle 3, 4, 6, 13\rangle$	$\langle 5, 7, 8, 13\rangle$	$\langle 1, 5, 6, 9\rangle$
$\langle 2, 5, 10, 13\rangle$	$\langle 2, 6, 10, 13\rangle$	$\langle 0, 5, 6, 11\rangle$

For AD[4, 5, 26; 273], apply Lemma 2.1 with $m = 6$, $t = 2$ and $\lambda = 5$. According to this lemma we have to prove the existence of AD*[4, 5, 8(2); 23] and AD[4, 5, 8; 24].

For AD*[4, 5, 8(2); 23] let $X = Z_6 \cup \{a, b\}$, then the blocks are

$\langle 0, 1, 3, a\rangle \pmod 6$,	$\langle 3, 4, 5, a\rangle$,	$\langle 0, 1, 5, b\rangle$,
$\langle 0, 1, 4, b\rangle \pmod 6$,	$\langle 1, 2, 3, b\rangle$,	$\langle 0, 1, 3, 4\rangle$,
$\langle 0, 2, 4, a\rangle$,	$\langle 0, 4, 5, b\rangle$,	$\langle 1, 2, 4, 5\rangle$,
$\langle 1, 3, 5, a\rangle$,	$\langle 2, 3, 4, b\rangle$,	$\langle 0, 2, 3, 5\rangle$,
$\langle 0, 1, 2, a\rangle$,		

For AD[4, 5, 8; 24], let $X = Z_8$, then the blocks are

$$\langle 0, 1, 3, 4\rangle \pmod 8, \quad \langle 0, 2, 4, 5\rangle \pmod 8, \quad \langle 0, 1, 2, 4\rangle \pmod 8.$$

For AD[4, 5, 38; 589], again apply Lemma 2.1 with $m = 6$, $t = 2$ and $\lambda = 5$. According to this lemma we have to prove the existence of AD*[4, 5, 8(2); 23] and AD[4, 5, 8; 24] which we have done above.

3.5. $v \equiv 3 \pmod{12}$

(1) For $\lambda = 2$, then the blocks of AD[4, 2, v; b] are the blocks of AD[4, 1, v; b] taken twice.

(2) For $\lambda = 3$, the blocks of AD[4, 3, v; b] for every $v \equiv 3 \pmod{12}$ can be constructed in the following way:

(a) take the blocks of $B[4, 1; v-2]$;

(b) take the blocks of $B[4, 1; v+1]$ with assumption that the points $v-1$, v and $v+1$ are not included in one block, hence, there are two blocks $\langle a, b, v; v+1\rangle$ and $\langle c, d, v-1; v+1\rangle$. The point $v+1$ we change to v in all the blocks of $B[4, 1; v+1]$ except the block $\langle a, b, v, v+1\rangle$. In this block we change $(v+1)$ to $(v-1)$;

(c) again take the same blocks of $B[4, 1; v+1]$, interchange $v-1 \leftrightarrow v$. Now in the block $\langle a, b, v-1, v+1\rangle$ we change $v+1$ to v and in all other blocks of $B[4, 1; v+1]$ we change $v+1$ to $v-1$.

(3) For $\lambda = 4$, the blocks of AD[4, 4, v; b] are the blocks of AD[4, 3, v; b] and AD[4, 1, v; b].

(4) For $\lambda = 5$, apply Lemma 2.1 with $m = 12$, $t = 2$ and $\lambda = 5$. According to this lemma we have to prove the existence of AD*[4, 5, 15(3); 87], AD[4, 5, 15; 90], AD[4, 5, 27; 297], and AD[4, 5, 39; 624].

For AD*[4, 5, 15(3); 87], take the blocks of AD*[4, 3, 15(3); 51] and the blocks of AD*[4, 2, 15(3); 36]. The blocks of AD*[4, 3, 15(3); 51] can be constructed by

(a) taking the blocks of $B[4, 1; 13]$;

(b) taking the blocks of $B[4, 1; 16]$ twice, without the blocks $\langle 13, 14, 15, 16\rangle$, in the first by changing 16 to 14, and secondly by changing 16 to 15. The blocks of AD*[4, 2, 15(3); 36] can be constructed by

(a) taking the blocks of $B[4, 1; 16]$ without the block $\langle 13, 14, 15, 16\rangle$;

(b) by taking the blocks of AD[4, 1, 14; 18], [6, p. 139] without the block $\langle 13, 14\rangle$.

For AD[4, 5, v; b], $v = 15, 27, 39$:

(a) take the blocks of AD[4, 3, v; b]. According to this construction there is a pair which appears six times. Assume this pair is (1, 2):

(b) take the blocks of AD[4, 1, v; b] $v = 15, 27, 39$ given by Mills [5, p. 71]. This construction contains a triple; let us assume the triple is $\langle 1, 2, 3\rangle$ which we drop;

(c) again we take the blocks of AD[4, 1, v; b], $v = 15, 27, 39$. Here we assume the triple to be $\langle 1, 2, 4\rangle$. To this triple we add the point 3, so we get back the two pairs (1, 3), (2, 3) which we lost in (b). This gives the construction of AD[4, 5, v; b], $v \equiv 3 \pmod{12}$.

3.6. $v \equiv 5 \pmod{12}$

(1) For $\lambda = 2$, the blocks of AD[$4, 2, v; b$] can be constructed in the following way:

(a) We take the blocks of AD[$4, 1, v+2; b$] given by Mills. According to this construction there is exactly one pair which occurs exactly four times [6, p. 161]. Let us assume this pair is $(v-1, v)$. We can also assume that we have the block $\langle v-1, v, v+1, v+2\rangle$; we drop this block and in the remaining blocks of AD[$4, 1, v+2; b$] we change $v+2$ to v and change $v+1$ to $v-1$.

(b) We take the blocks of AD[$4, 1, v-2; b$].

The above construction does not work for $v = 5, 17$. For $v = 5$ and $\lambda = 2$ let $X = Z_5$, then the blocks are $\langle 0, 1, 2, 3\rangle$ $(+i, i \in Z_4)$.

The blocks of AD[4, 2, 17; 47] can be achieved in the following way:

(a) take the blocks of B[4, 1; 16];

(b) take the blocks of AD[4, 1, 18; 27] given by Mills [5, p. 67].

According to this construction there is a block which contains two pairs which have appeared. Assume this block is $\langle 5, 18, 6, 17\rangle$ and the two pairs which have appeared are $(5, 18)$ and $(6, 17)$. From this block we drop the point 18 and in the other blocks we change 18 to 17.

(2) For $\lambda = 3$, there exists B[4, 3; 2].

(3) For $\lambda = 4$, the blocks of AD[$4, 4, v; b$] are the blocks of AD[$4, 3, v; b$] with the blocks of AD[$4, 1, v; b$].

(4) For $\lambda = 5$, the blocks of AD[$4, 5, v; b$] are the blocks of AD[$4, 3, v; b$] with the blocks of AD[$4, 2, v; b$].

3.7. $v \equiv 6 \pmod{12}$

(1) For $\lambda = 2$, the blocks of AD[$4, 2, v; b$] are the blocks of AD[$4, 1, v; b$] each block taken twice.

(2) For $\lambda = 3$, we have $\alpha(4, 3, v) = \phi(4, 3, v)$. To prove this we distinguish two cases:

Case A. $v \equiv 6$ or $42 \pmod{48}$; then let $v = 4w + 6$ where $w \equiv 0$ or $1 \pmod 4$; then by Lemma 2.2 there exists a collection of $w^2 + 5$ blocks of sizes 4, 5, w and one block of size 6.

On these blocks we construct covering with $\lambda = 3$. For $v = 4$, 5 and w we have balanced incomplete block design. For $v = 6$ see Lemma 3.1.

Case B. $v \equiv 18$ or $30 \pmod{48}$, $v \neq 18, 30$ and 66; then let $v \equiv 4w + 14$ where $w \equiv 0$ or $1 \pmod 4$. Again apply Lemma 2.2 and the proof of Case B is exactly the same as Case A. Remember that we have constructed AD[4, 3, 14; 46].

For AD[4, 3, 18; 77], [4], let $X = \{a, b\} \cup ((\{\infty\} \cup Z_3) \times Z_4)$ then the blocks are

$$\begin{array}{ll}
\langle a, b, (0,0), (0,2)\rangle & (\bmod(3,-)),\\
\langle a, (\infty,0), (0,0), (1,1)\rangle & (\bmod(3,4)),\\
\langle (0,0), (0,1), (1,0), (1,2)\rangle & (\bmod(3,4)),\\
\langle (\infty,0), (\infty,1), (\infty,2), (\infty,3)\rangle & (3 \text{ times}),\\
\langle (\infty,0), (0,2), (0,3), (1,1)\rangle & (\bmod(3,4)\text{twice}),\\
\langle (\infty,0), (0,0), (1,0), (2,0)\rangle & (\bmod(-,4)\text{twice}),\\
\langle a, b, (0,1), (0,3)\rangle & (\bmod(3,-)),\\
\langle b, (\infty,0), (0,2), (1,3)\rangle & (\bmod(3,4)).
\end{array}$$

For AD[4, 3, 30; 218] let $X = Z_{23} \cup \{a, b, c, d, e, f, g\}$, then the required blocks are

$$\begin{array}{llll}
\langle 0,2,8,11\rangle & (\bmod 23), & \langle 0,10,11,d\rangle & (\bmod 23),\\
\langle 0,3,8,10\rangle & (\bmod 23), & \langle 0,8,11,e\rangle & (\bmod 23),\\
\langle 0,4,9,a\rangle & (\bmod 23), & \langle 0,7,9,f\rangle & (\bmod 23),\\
\langle 0,6,7,b\rangle & (\bmod 23), & \langle 0,4,5,g\rangle & (\bmod 23),\\
\langle 0,4,10,c\rangle & (\bmod 23), & \langle a,b,c,d,e,f,g\rangle. &
\end{array}$$

For the last block see Lemma 3.2.

In order to construct AD[4, 3, 66; 1073] we need the concept of a resolvable design. A resolvable design RB(4, λ; v) is a balanced incomplete block design $B(4, \lambda; v)$ the blocks of which can be partitioned into parallel classes. Hanani, Ray-Chandhuri and Wilson [3] proved that for every $v \equiv 4 \pmod{12}$ there exists RB(4, 1; v).

Now for AD[4, 3, 66; 1073] take the blocks of RB[4, 1; 52) and from the blocks of RB(4, 1; 52) take fourteen parallel classes, and to each class we add a point. In this way we add fourteen distinct points. On the blocks of size 5 we construct a covering with $\lambda = 3$, on the block of size 14 we construct AD[4, 3, 14, 46], and the remaining blocks of RB(4, 1; 52) take each block three times.

(3) For $\lambda = 4$, the blocks of AD[4, 4, v; b] are the blocks of AD[4, 3, v; b] and the blocks of AD[4, 1, v; b].

(4) For $\lambda = 5$, the blocks of AD[4, 5, v; b] are the blocks of AD[4, 3, v; b] and the blocks of AD[4, 2, v; b].

3.8. $v \equiv 8 \pmod{12}$

(1) $\lambda = 2$, then the blocks of AD[4, 2, v; b] can be constructed in the following way:

(a) take the blocks of AD[4, 1, $v-2$; b] on the set $X = \{3, 4, \ldots, v\}$;

(b) take the blocks of AD[4, 1, $v+2$; b] on the set $A = \{1, 2, \ldots, v+1, v+$

2} with the following changes: Since $v+2 \equiv 10 \pmod{12}$ we have a pair which occurs exactly four times. Let this pair be $(1,2)$. Assume also that we have the block $\langle 1, 2, v+1, v+2 \rangle$. We drop this block and in the other blocks we change $v+1$ to 1 and $v+2$ to 2.

The above construction does not work for $v=8$. Hence for AD[4, 2, 8; 10] let $X = Z_8$, then the blocks are:

$$\langle 1,2,3,4\rangle \quad \langle 0,1,6,7\rangle \quad \langle 1,3,5,7\rangle \quad \langle 0,2,4,7\rangle \quad \langle 2,5,6,7\rangle$$
$$\langle 3,4,6,7\rangle \quad \langle 1,4,5,6\rangle \quad \langle 0,1,2,5\rangle \quad \langle 0,3,4,5\rangle \quad \langle 0,2,3,6\rangle.$$

(2) For $\lambda = 3$, we have $B[4, 3; v]$.

(3) For $\lambda = 4$, the blocks of AD$[4, 4, v; b]$ are the blocks of $B[4, 3; v]$ and the blocks of AD$[4, 1, v; b]$.

(4) For $\lambda = 5$, then the blocks of AD$[4, 5, v; b]$ are the blocks of $B[4, 3; v]$ and the blocks of AD$[4, 2, v; b]$.

3.9. $v \equiv 9 \pmod{12}$

(1) For $\lambda = 2$, the blocks of AD$[4, 2, v; b]$ are the blocks of AD$[4, 1, v; b]$, each block taken twice.

(2) For $\lambda = 3$, we have a $B[4, 3; v]$.

(3) For $\lambda = 4$, the blocks of AD$[4, 4, v; b]$ are the blocks of $B[4, 3; v]$ together with the blocks of AD$[4, 1, v; b]$.

(4) For $\lambda = 5$, the blocks of AD$[4, 5, v; b]$ are the blocks of $B[4, 3; v]$ and AD$[4, 2, v; b]$.

The above construction does not work for $v = 9$, $\lambda = 2, 4$. For AD[4, 2, 9; 14] the 12 quadruples

$$\langle A,B,D,E\rangle \quad \langle A,B,G,H\rangle \quad \langle D,E,G,H\rangle \quad \langle A,C,E,I\rangle$$
$$\langle A,C,D,F\rangle \quad \langle A,C,G,I\rangle \quad \langle D,F,G,I\rangle \quad \langle B,E,F,G\rangle$$
$$\langle B,C,E,F\rangle \quad \langle B,C,H,I\rangle \quad \langle E,F,H,I\rangle \quad \langle C,D,G,H\rangle$$

and the two triples $\langle A, F, H\rangle$; $\langle B, D, I\rangle$ cover all pairs twice. This construction was given by Mills.

For AD[4, 4, 9; 25], [4], let $X = \{a\} \cup (Z_4 \times Z_2)$ then the blocks are:

$$\begin{array}{ll}
\langle a, (0,0), (1,0), (2,0)\rangle & (\text{mod}(4,-)), \\
\langle a, (0,0), (0,1), (1,1)\rangle & (\text{mod}(4,-)), \\
\langle a, (0,0), (2,1), (3,1)\rangle & (\text{mod}(4,-)), \\
\langle (0,0), (1,0), (0,1), (2,1)\rangle & (\text{mod}(4,-)\text{twice}), \\
\langle (0,0), (2,0), (0,1), (1,1)\rangle & (\text{mod}(4,-)), \\
\langle (0,1), (1,1), (2,1), (3,1)\rangle. &
\end{array}$$

3.10. $v \equiv 11 \pmod{12}$

(1) For $\lambda = 2$, the blocks of AD$[4, 2, v; b]$ can be constructed in the following way:

(a) Take the blocks of AD$[4, 1, v+1; b]$ and assume they contain the block $\langle x, y, v, v+1\rangle$ $(x, y < v-2)$, and the pair $(v, v+1)$ is contained in exactly one block.

(b) take the blocks of AD$[4, 1, v-1; b]$. Since $v-1 \equiv 10 \pmod{12}$ we have a pair which occurs exactly four times and all the other pairs occur exactly once. Assume we have the block $\langle x, y, v-2, v-1\rangle$ and the pair $(v-2, v-1)$ occurs four times. Now we change the point $v+1$ to $v-1$ in the block $\langle x, y, v, v+1\rangle$ and in the block $\langle x, y, v-2, v-1\rangle$ we change $v-1$ to v. In the remaining blocks of AD$[4, 1, v+1; b]$ we change $v+1$ to v.

For AD$[4, 2, 11; 20]$ let $X = Z_{10} \cup \{\infty\}$ then the required blocks are:

$$\langle \infty, 0, 1, 4\rangle \pmod{10}, \qquad \langle 0, 1, 3, 5\rangle \pmod{10}.$$

(2) For $\lambda = 3$ we distinguish two cases:

Case A. $v \equiv 11$ or $23 \pmod{48}$, then $\alpha(4, 3, v) = \phi(4, 3, v)$.

Proof. Set $v = 4w + 7$, then $w \equiv 0$ or $1 \pmod 4$ and $7 < w$ for every $v > 23$. According to Lemma 2.2 there are $w^2 + 4$ blocks of order 4, 5, w and one block of order 7. On these blocks we construct covering with $\lambda = 3$. It is clear that for $v = 4$, 5 and w we have balanced incomplete block design. For $v = 7$, $\lambda = 3$ see Lemma 3.2.

For $v = 11$ and $\lambda = 3$, let $X = \{a, b\} \cup ((\{\infty\} \cup Z_2) \times Z_3)$, then the required blocks are [4]:

$$\begin{array}{llll}
\langle a, b, (\infty, 0), (\infty, 1)\rangle & (\mathrm{mod}(-, 3)), & \langle b, (0, 0), (0, 1), (0, 2)\rangle & (\mathrm{mod}(2, -)), \\
\langle a, b, (0, 0), (1, 0)\rangle & (\mathrm{mod}(-, 3)), & \langle (\infty, 0), (\infty, 1), (0, 1), (1, 2)\rangle & (\mathrm{mod}(-, 3)), \\
\langle a, (\infty, 0), (0, 0), (1, 0)\rangle & (\mathrm{mod}(-, 3)), & \langle (\infty, 0), (\infty, 1), (0, 2), (1, 1)\rangle & (\mathrm{mod}(-, 3)), \\
\langle b, (\infty, 0), (0, 2), (1, 2)\rangle & (\mathrm{mod}(-, 3)), & \langle (\infty, 0), (0, 0), (0, 1), (1, 2)\rangle & (\mathrm{mod}(-, 3)), \\
\langle a, (0, 0), (0, 1), (0, 2)\rangle & (\mathrm{mod}(2, -)), & \langle (\infty, 0), (0, 2), (1, 0), (1, 1)\rangle & (\mathrm{mod}(-, 3)).
\end{array}$$

For $v = 23$ and $\lambda = 3$, let $X = Z_{17} \cup \{a, b, c, d, e, f\}$. Then the required blocks are:

$$\begin{array}{llll}
\langle 0, 1, 3, 8\rangle & \pmod{17}, & \langle 0, 5, 8, d\rangle & \pmod{17}, \\
\langle 0, 4, 6, a\rangle & \pmod{17}, & \langle 0, 4, 6, e\rangle & \pmod{17}, \\
\langle 0, 7, 8, b\rangle & \pmod{17}, & \langle 0, 5, 6, f\rangle & \pmod{17}, \\
\langle 0, 3, 7, c\rangle & \pmod{17}, & \langle a, b, c, d, e, f\rangle. &
\end{array}$$

For the last block see Lemma 3.1.

Case B. $v \equiv 35$ or $47 \pmod{48}$, then $\alpha(4, 3, v) = \phi(4, 3, v)$.

Proof. Set $v = 4w + 15$, then $w \equiv 0$ or 1 (mod 4) and for $v > 71$ we have $w > 15$. Apply Lemma 2.2 we have $w^2 + 4$ blocks of order 4, 5, w and one block of order 15. On these blocks we construct covering with $\lambda = 3$.

For $v = 35$, $\lambda = 3$ let $X = Z_{29} \cup \{a, b, c, d, e, f\}$, then the required blocks are

$\langle 0, 2, 6, 13\rangle$	(mod 29),	$\langle 0, 8, 12, a\rangle$	(mod 29),	$\langle 0, 3, 10, e\rangle$	(mod 29),
$\langle 0, 1, 11, 14\rangle$	(mod 29),	$\langle 0, 10, 13, b\rangle$	(mod 29),	$\langle 0, 11, 12, f\rangle$	(mod 29),
$\langle 0, 5, 12, 14\rangle$	(mod 29),	$\langle 0, 4, 9, c\rangle$	(mod 29),	$\langle a, b, c, d, e, f\rangle$.	
$\langle 0, 8, 9, 14\rangle$	(mod 29),	$\langle 0, 6, 8, d\rangle$	(mod 29),		

For the last block apply Lemma 3.1.

For $v = 47$, $\lambda = 3$ let $X = Z_{41} \cup \{a, b, c, d, e, f\}$ the required blocks are

$\langle 0,9,17,20\rangle$	(mod 41),	$\langle 0,4,14,15\rangle$	(mod 41),	$\langle 0,15,17,c\rangle$	(mod 41),
$\langle 0,7,12,16\rangle$	(mod 41),	$\langle a,b,c,d,e,f\rangle$,		$\langle 0,7,21,d\rangle$	(mod 41),
$\langle 0,6,16,18\rangle$	(mod 41),	$\langle 0,6,15,19\rangle$	(mod 41),	$\langle 0,8,20,e\rangle$	(mod 41),
$\langle 0,5,13,19\rangle$	(mod 41),	$\langle 0,16,19,a\rangle$	(mod 41),	$\langle 0,2,3,f\rangle$	(mod 41).
$\langle 0,1,11,18\rangle$	(mod 41),	$\langle 0,13,18,b\rangle$	(mod 41),		

For the block of size 6 apply Lemma 3.1.

(3) For $\lambda = 4$, then the blocks of AD[4, 4, v; b] are the blocks of AD[4, 3, v; b] and the blocks of AD[4, 1, v; b].

(4) For $\lambda = 5$ we have $\alpha(4, 5, v) = \phi(4, 5, v)$.

The construction of AD[4, 5, v; b] can be done in the following way:

(a) Take the blocks of SD[4, 3, v; b] where $v \equiv 11$ (mod 12) which can be constructed by taking the blocks of B[4, 2; $v - 1$] and the blocks of B[4, 1; $v + 2$], dropping the block $\langle v-1, v, v+1, v+2\rangle$ and changing both the points $v + 1$ and $v + 2$ to v. According to this construction there is a pair which is not included in any block of SD[4, 3, v; b]. Assume this pair is $(v-2, v-1)$.

(b) Take the blocks of AD[4, 1, $v - 1$; b]. Since $v - 1 \equiv 10$ (mod 12) we have one pair which occurs exactly four times. Assume this pair is $(v-2, v-1)$. Assume also that we have the block $\langle 1, 2, v-2, v-1\rangle$. In this block we change $v - 2$ to v.

(c) Take the blocks of AD[4, 1, $v + 1$; b] and assume that the pair $(v-2, v-1)$ occurs twice. Assume also that we have the block $\{1, 2, v, v+1\rangle$, and assume the pair $(v, v+1)$ occurs only once. In this block we change $v + 1$ to $v - 2$. In the other blocks of AD[4, 1, $v + 1$; b] we change $v + 1$ to v.

For AD[4, 5, 11; 47], [4], let $X = \{a, b, c\} \cup Z_8$ then the required blocks are

$\langle a, b, c\rangle$,		$\langle b, c, 0, 4\rangle$	$(+i, i \in Z_4)$,	$\langle a, 0, 1, 3\rangle$	(mod 8),
$\langle a, b, 0, 1\rangle$	$(+2i, i \in Z_4)$,	$\langle 0, 2, 4, 6\rangle$	$(+i, i \in Z_2)$,	$\langle b, 0, 1, 3\rangle$	(mod 8),
$\langle a, c, 1, 2\rangle$	$(+2i, i \in Z_4)$,	$\langle 0, 1, 3, 5\rangle$	(mod 8),	$\langle c, 0, 1, 4\rangle$	(mod 8).

In order to complete the proof of our theorem we have to show that $\alpha(4, 7, v) = \phi(4, 7, v)$ for $v = 7, 9, 10, 19$. For $v = 7, 10, 19$ the blocks of AD$[4, 7, v; b]$ are the blocks of AD$[4, 3, v; b]$ and the blocks of $B[4, 4; v]$.

For AD$[4, 7, 9; 43]$, take the blocks of AD$[4, 4, 9; 25]$ and the blocks of $B[4, 3; 9]$. □

References

[1] A.E. Brouwer, A. Schrijver and H. Hanani, Group divisible designs with block-size four, Discrete Math. 20 (1977) 1–10.
[2] H. Hanani, Balanced incomplete block designs and related designs, Discrete Math. 11 (1975) 225–369.
[3] H. Hanani, D.K. Ray-Chaudhuri and R.M. Wilson, On resolvable designs, Discrete Math. 3 (1972) 343–357.
[4] A. Hartman, On small packing and covering designs with block size 4, submitted.
[5] W.H. Mills, On the covering of pairs by quadruples I, J. Combin. Theory Ser. A 13 (1972) 55–78.
[6] W.H. Mills, On the covering of pairs by quadruples II, J. Combin. Theory Ser. A 15 (1973) 138–166.
[7] J. Schonheim, On covering, Pacific J. Math. 14 (1964) 1405–1411.