On Generalized Set-Valued Variational Inclusions

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1. INTRODUCTION

In 1998, using the concept and technique of resolvent operators, Noor [6] introduced and studied a new class of variational inclusions, which is called the generalized set-valued variational inclusion, and constructed a new iterative algorithm to solve variational inclusions. The author proved the convergence of the iterative sequences generated by this algorithm under the assumption that $T : H \rightarrow C(H)$ (where $C(H)$ denotes the family of all nonempty subsets of $H$) is $H$-Lipschitz continuous and strongly monotone with respect to the first argument of a mapping $N(\cdot, \cdot)$ (see Theorem 3.2 in [6]). However, we can show that the monotone mapping $N(T(\cdot), \cdot)$ cannot be multivalued. Indeed, the mapping $T$ in [6, Theorem 3.2] is a single-valued mapping. One of the purposes of this paper is to prove this result. Then, for solving a class of the generalized variational inclusions for set-valued mappings without compactness and monotonicity, we construct a new iterative algorithm and prove the convergence of the iterative sequence defined by the algorithm. Finally, for solving a class of the generalized single-valued variational inclusions, we study a new iterative algorithm, which is called the perturbed Ishikawa iterative process.
2. PRELIMINARIES

Let $H$ be a real Hilbert space endowed with a norm $\| \cdot \|$ and inner product $(\cdot, \cdot)$ and $CB(H)$ be a family of nonempty bounded closed subsets of $H$. Let $T, V : H \rightarrow CB(H)$ be the multivalued operators and $g : H \rightarrow H$ be a single-valued operator.

For a given maximal monotone operator $A : H \rightarrow H$ and nonlinear operator $N(\cdot, \cdot) : H \times H \rightarrow H$, we consider the problem of finding $u \in H, w \in Tu, y \in Vu$ such that

$$0 = N(w, y) + A(g(u)). \quad (2.1)$$

The problem (2.1) is called the general set-valued variational inclusion (see [6]).

If $T, V$ are the single-valued mappings, the problem (2.1) can be replaced to finding $u \in H$ such that

$$0 = N(Tu, Vu) + A(g(u)). \quad (2.2)$$

If $A$ is a maximal monotone operator on $H$, then for a constant $\rho > 0$, the resolvent operator associated with $A$ is defined by

$$J_A(u) = (I + \rho A)^{-1}(u) \quad \text{for all } u \in H,$$

where $I$ is the identity operator. It is also known that the operator $A$ is maximal monotone if and only if the resolvent operator $J_A$ is defined everywhere on the space [4]. Furthermore the resolvent operator $J_A$ is single-valued and nonexpansive.

**Definition 2.1** [6]. For all $u_1, u_2 \in H$, the set-valued operator $T : H \rightarrow CB(H)$ is said to be strongly montone with respect to the first argument of the operator $N(\cdot, \cdot)$, if there exists a constant $\alpha > 0$ such that

$$(N(w_1, \cdot) - N(w_2, \cdot), u_1 - u_2) \geq \alpha \|u_1 - u_2\|^2 \quad \text{for all } w_1 \in Tu_1, w_2 \in Tu_2.$$  

In relation to problem (2.1), we consider the problem of finding $z, u \in H, y \in Vu$ such that

$$N(w, y) + \rho^{-1}R_Az = 0. \quad (2.3)$$

Here $\rho > 0$ is a constant and $R_A = I - J_A$. Equations of type (2.3) are called the resolvent equations.

**Lemma 2.1** [6]. The function $(u, w, y)$ is a solution of (2.1) if and only if $(u, w, y)$ satisfies the relation

$$g(u) = J_A[g(u) - \rho N(w, y)]. \quad (2.4)$$
**Lemma 2.2** [6]. The variational inclusions (2.1) have a solution \( u \in H, w \in Tu, y \in Vu \), if and only if the resolvent equations (2.3) have a solution \( z, u \in H, w \in Tu, y \in Vu \), where

\[
g(u) = J_A z \quad (2.5)
\]

and

\[
z = g(u) - \rho N(w, y). \quad (2.6)
\]

The resolvent equations (2.3) can be written as

\[R_A z = -\rho N(w, y)\]

which implies that

\[
z = J_A z - \rho N(w, y) = g(w) - \rho N(w, y), \quad \text{using (2.5)}.
\]

This fixed-point formulation allows us to suggest the following iterative method.

**Algorithm 2.1.** For given \( z_0 \in H \), we take \( u_0 \in H \) such that

\[g(u_0) = J_A z_0.\]

Let \( w_0 \in Tu_0, y_0 \in Vu_0 \), and \( z_1 = g(u_0) - \rho N(w_0, y_0) \).

For \( z_1 \), we take \( u_1 \) such that \( g(u_1) = J_A z_1 \). Then, by Nadler [5], there exist \( w_1 \in Tu_1 \) such that

\[
\|w_1 - w_0\| \leq (1 + 1)H(Tu_1, Tu_0),
\]

\[
\|y_1 - y_0\| \leq (1 + 1)H(Vu_1, Vu_0),
\]

where \( H(\cdot, \cdot) \) is the Hausdorff metric on \( CB(H) \). Let

\[
z_2 = g(u_1) - \rho N(w_1, y_1).
\]

By induction, we can obtain sequences \( \{z_n\}, \{u_n\}, \{w_n\}, \) and \( \{y_n\} \) as

\[
g(u_n) = J_A z_n
\]

\[
w_n \in Tu_n : \|w_{n+1} - w_n\| \leq \left(1 + \frac{1}{1+n}\right)H(Tu_{n+1}, Tu_n)
\]

\[
y_n \in Vu_n : \|y_{n+1} - y_n\| \leq \left(1 + \frac{1}{1+n}\right)H(Vu_{n+1}, Vu_n)
\]

\[
z_{n+1} = g(u_n) - \rho N(w_n, y_n).
\]

The following algorithm is called the perturbed Ishikawa iterative process.
Algorithm 2.2. For single-valued operators $T, V, g : H \to H$, define $Q : H \to H$ by

$$Qu = u - g(u) + J_A[g(u) - \rho N(Tu, Vu)].$$

For given $u_0 \in H$, the Ishikawa iterative scheme with error [4] is defined by

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n Q u_n + e_n,$$
$$u_n = (1 - \beta_n)u_n + \beta_n Q u_n + f_n,$$

where $\{e_n\}, \{f_n\}$ are two summable sequences in $H$, and $\{\alpha_n\}, \{\beta_n\}$ are two real sequences in $[0, 1]$ satisfying

$$\beta_n \leq \alpha_n, \quad \sum \alpha_n = \infty, \quad \sum \alpha_n^2 < +\infty.$$

We recall the following results that will be needed later on.

Lemma 2.3 [2]. Let the single-valued operator $A : H \to H$ be maximal strongly monotone with constant $\alpha > 0$. Then the resolvent operator $J_A = (I + \rho A)^{-1}$ is Lipschitz continuous with constant $1/(1 + \rho \alpha)$ where $\rho > 0$ is a constant.

Lemma 2.4 [3]. Suppose $X$ is an arbitrary real Banach space and $T : X \to X$ is a strongly accretive and Lipschitz continuous operator. For a fixed $f \in X$, define $S : X \to X$ by $Sx = f + x - Tx$ for each $x \in X$. Let $\{f_n\}, \{e_n\}$ be two summable sequences in $X$, and $\{\alpha_n\}, \{\beta_n\}$ be two real sequences satisfying

(i) $0 \leq \beta_n \leq \alpha_n < 1$,
(ii) $\sum \alpha_n = +\infty, \sum \alpha_n^2 < +\infty$.

Then for any $x_0 \in X$, the iteration sequence $\{x_n\}$ in $X$ is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S y_n + e_n,$$
$$y_n = (1 - \beta_n)x_n + \beta_n S x_n + f_n,$$

which converges strongly to the unique solution $x^*$ of the equation $Tx = f$.

Remark 2.1. If $X$ is a Hilbert space, the accretive operator in Lemma 2.4 is a monotone operator.
3. MAIN RESULTS

THEOREM 3.1. Let the operator $N(\cdot, \cdot)$ be Lipschitz continuous with constant $\beta > 0$ with respect to the first argument. If $T$ is $H$-Lipschitz continuous with constant $\mu > 0$ and monotone with respect to the first argument of the operator $N(\cdot, \cdot)$ and, for each fixed $k \in H$, $\text{int} \, D(N(T(\cdot), k)) \neq \emptyset$, then $N(T(\cdot), k)$ cannot be multivalued in $\text{int} \, D(N(T(\cdot), k))$.

Proof. We first note that the operator $N(T(\cdot), \cdot)$ is Lipschitz continuous with constant $\beta, \mu > 0$ with respect to the first argument by assumptions.

Assume now that $N(T(\cdot), k)$ is a multivalued monotone mapping for some $k \in H$. Then exist $u_0 \in \text{int} \, D(N(T(\cdot), k))$ and $v_1, v_2 \in N(Tu_0, k)$ such that $v_1 \neq v_2$. So there exist $w_1, w_2 \in Tu_0$ such that $v_1 = N(w_1, k), v_2 = N(w_2, k)$. Since $N(T(\cdot), k)$ is Lipschitz continuous, for $\epsilon = \|v_1 - v_2\|$ we can take $\delta = \epsilon/4B\mu$, so that for any $u_1 \in H$, when $\|u_1 - u_0\| < \delta$, we have

$$d(v_1, N(Tu_1, k)) \leq H(N(Tu_0, k), N(Tu_1, k)) \leq \beta\mu\|u_0 - u_1\| < \epsilon/4,$$

where $d(v_1, N(Tu_1, k)) = \inf\{\|u_1 - z\| : z \in N(Tu_1, k)\}$. Therefore there exists $v(u_1) \in N(Tu_1, k)$ such that

$$\|v(u_1) - v_1\| < \epsilon/2. \quad (3.1)$$

By monotonicity of $T$ with respect to $(N(\cdot, k))$, we have

$$v(u_1) - v_1, u_1 - u_0 \geq 0 \quad (3.2)$$

$$v(u_1) - v_2, u_1 - u_0 \geq 0. \quad (3.3)$$

Now, we take $0 < \delta_0 < \delta$ and $u_1 = \delta_0(v_2 - v_1)/\|v_2 - v_1\| + u_0$. Then $\|u_1 - u_0\| = \delta_0 < \delta$. Thus, we can choose the number $v(u_1) \in N(Tu_1, \cdot)$ as an inequality in (3.2) and (3.3). Using (3.2), (3.3), and $v_1 - v_2 = (\|v_2 - v_1\|/\delta_0)(u_1 - u_0)$, we obtain

$$v(u_1) - v_1, v_2 - v_1 \geq 0 \quad (3.4)$$

$$v(u_1) - v_2, v_2 - v_1 \geq 0. \quad (3.5)$$

By (3.5), we have

$$(v(u_1) - v_1, v_2 - v_1) \geq (v_2 - v_1, v_2 - v_1) = \|v_2 - v_1\|^2. \quad (3.6)$$

From (3.4) and (3.6), we get $2(v(u_1) - v_1, v_2 - v_1) \geq \|v_1 - v_2\|^2$. Hence

$$\|v(u_1) - v_1\| \geq \|v_1 - v_2\|/2 = \epsilon/2. \quad (3.7)$$

Thus, we obtain a contradiction to (3.1). Hence $N(T(\cdot), k)$ cannot be a multivalued mapping.
Remark 3.1. If the operator T in [1, Theorem 3.2] is single-valued, then Theorem 3.2 in [1] remains valid.

Remark 3.2. According to Theorem 3.1 and its proof, the Lipschitz continuous set-valued operator cannot be monotone. Therefore, from the assumptions of Theorem 4.1 of [7], Theorem 4.1 of [10], Theorem 3.5 of [11], and Theorems 3.4 and 3.5 of [1], we know that all the set-valued monotone mappings in these theorems are single-valued mappings indeed.

Theorem 3.2. Let the operator \( N(\cdot, \cdot) \) be Lipschitz continuous with constant \( \beta > 0 \) with respect to the first argument and Lipschitz continuous with constant \( \eta > 0 \) with respect to the second argument. Let the single-valued operator \( g : H \to H \) be strongly monotone with constant \( \sigma > 0 \) and Lipschitz continuous with constant \( \delta > 0 \). Assume that \( V : H \to CB(H) \) is \( H \)-Lipschitz continuous with constant \( \xi > 0 \), and \( T : H \to CB(H) \) is \( H \)-Lipschitz continuous with constant \( \mu > 0 \). If \( A \) is a maximal strongly monotone operator with constant \( \alpha > 0 \) and either (i)

\[
\rho \frac{2k\alpha - s}{2\alpha(s - k\alpha)} \leq \frac{\sqrt{4k\alpha(s - k\alpha) + (s - 2k\alpha)^2}}{2\alpha(s - k\alpha)} = \frac{s}{2\alpha(s - k\alpha)}, \quad s > k\alpha, 
\]

or (ii) \( s = \eta \xi + \beta \mu, k = 1 - \sqrt{1 - 2\sigma + \delta^2} \in (0, 1) \), then there exist \( u, z \in H, w \in Tu, y \in Vu \) satisfying the set-valued resolvent inclusion (2.3) and (2.6), and the sequences \( \{z_n\}, \{u_n\}, \{w_n\}, \) and \( \{y_n\} \) generated by Algorithm 2.1 converge, respectively, to \( z, u, w, \) and \( y \) strongly in \( H \).

Proof. From Algorithm 2.1, we have

\[
\|z_{n+1} - z_n\| = \|g(u_n) - g(u_{n-1}) - \rho N(w_n, y_n) + \rho N(w_{n-1}, y_{n-1})\| \\
\leq \|g(u_n) - g(u_{n-1})\| + \rho\|N(w_n, y_n) - N(w_{n-1}, y_{n-1})\| \\
\leq \rho\|N(w_n, y_n) - N(w_{n-1}, y_{n-1})\|. 
\]

By (2.5) and Lemma 2.3, we obtain

\[
\|g(u_n) - g(u_{n-1})\| = \|J_A z_n - J_A z_{n-1}\| \\
\leq \frac{1}{1 + \rho\alpha}\|z_n - z_{n-1}\|. 
\]

From the assumptions of \( T \) and \( V \), we get

\[
\|N(w_n, y_{n-1}) - N(w_{n-1}, y_{n-1})\| \leq \eta \xi (1 + 1/n)\|u_n - u_{n-1}\|, \quad (3.10) \\
\|N(w_n, y_{n-1}) - N(w_{n-1}, y_{n-1})\| \leq \beta \mu (1 + 1/n)\|u_n - u_{n-1}\|. \quad (3.11) 
\]
Also, using the strong monotonicity and Lipschitz continuity of the operator $g$ and (2.5), we find that

$$\|u_n - u_{n-1}\| = \|u_n - u_{n-1} - (g(u_n) - g(u_{n-1})) + J_A z_n - J_A z_{n-1}\|$$

$$\leq \|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\| + \frac{1}{(1 + \rho \alpha)} \|z_n - z_{n-1}\|$$

$$\leq \sqrt{1 - 2\sigma + \delta^2} \|u_n - u_{n-1}\| + \frac{1}{(1 + \rho \alpha)} \|z_n - z_{n-1}\|,$$

which implies that

$$\|u_n - u_{n-1}\| \leq \frac{1}{(1 - \sqrt{1 - 2\sigma + \delta^2})(1 + \rho \alpha)} \|z_n - z_{n-1}\|. \quad (3.12)$$

Combining (3.8)–(3.12), we obtain

$$\|z_{n+1} - z_n\| \leq \frac{1}{(1 - \sqrt{1 - 2\sigma + \delta^2})(1 + \rho \alpha)} \times [1/(1 + \rho \alpha) + \rho(\eta \xi + \beta \mu)(1 + 1/n)] \|z_n - z_{n-1}\|$$

$$= \theta_n \|z_n - z_{n-1}\|, \quad (3.13)$$

where

$$\theta = \frac{1}{(1 - \sqrt{1 - 2\sigma + \delta^2})(1 + \rho \alpha)} \times [1/(1 + \rho \alpha) + \rho(\eta \xi + \beta \mu)(1 + 1/n)].$$

Let

$$\theta_n = \frac{1}{(1 - \sqrt{1 - 2\sigma + \delta^2})(1 + \rho \alpha)} [1/(1 + \rho \alpha) + \rho(\eta \xi + \beta \mu)].$$

We know that $\theta_n \downarrow \theta$. It follows from (i) or (ii) that $\theta \in (0, 1)$. Hence $0 \leq \theta_n < 1$ for $n$ sufficiently large. Therefore, (3.13) implies that $\{z_n\}$ is a Cauchy sequence in $H$ and we can suppose that $z_n \to z \in H$. From (3.12), we know that the sequence $\{u_n\}$ is a Cauchy sequence in $H$, that is, there exists $u \in H$ with $u_n \to u$.

Now we prove that $w_n \to w \in Tu$ and $y_n \to y \in Vu$. In fact, it follows from Algorithm 2.1 that

$$\|w_n - w_{n-1}\| \leq \left(1 + \frac{1}{n}\right) \mu \|u_n - u_{n-1}\|,$$

$$\|y_n - y_{n-1}\| \leq \left(1 + \frac{1}{n}\right) \xi \|u_n - u_{n-1}\|,$$
that is, \( \{w_n\} \) and \( \{y_n\} \) are also Cauchy sequences in \( H \). Let \( w_n \to w \in H \), \( y_n \to y \in H \). By using the continuity of the operators \( T, N, g, V, J_A \), and Algorithm 2.1, we have

\[
z = g(u) - \rho N(w, y) = J_A z - \rho N(w, y) \in H.
\]

Now we show that \( y \in Vu \). In fact,

\[
d(y, Vu) = \inf \{ \|y - z\| : z \in Vu \}
\leq \|y - y_n\| + d(y_n, Vu)
\leq \|y - y_n\| + H(Vu_n, Vu)
\leq \|y - y_n\| + \xi \|u_n - u\| \to 0.
\]

Hence, \( y \in Vu \). Similarly, \( w \in Tu \). This completes the proof of Theorem 3.1.

Remark 3.3. Theorem 3.2 is a correction and improvement of Theorem 3.2 in [6].

**Theorem 3.3.** Let the operator \( N(\cdot, \cdot) \) be Lipschitz continuous with constant \( \beta > 0 \) with respect to the first argument and Lipschitz continuous with constant \( \eta > 0 \) with respect to the second argument. Let the single-valued operator \( g : H \to H \) be strongly monotone with constant \( \sigma > 0 \) and Lipschitz continuous with constant \( \delta > 0 \). Assume that mapping \( T : H \to H \) is single-valued Lipschitz continuous with constant \( \mu > 0 \) and strongly monotone with constant \( \alpha > 0 \) with respect to first argument of \( N(\cdot, \cdot) \). Let the operator \( V : H \to H \) be single-valued Lipschitz continuous with constant \( \xi > 0 \). If the following conditions hold,

\[
|\rho - \frac{\alpha - k \eta \xi}{\beta^2 \mu^2 - \eta^2 \xi^2}| < \sqrt{\frac{1}{(\alpha - k \eta \xi)^2} - (\beta^2 \mu^2 - \eta^2 \xi^2)(1 - k^2)}
\]

\[
\beta^2 \mu^2 - \eta^2 \xi^2
\]

\[
\alpha > k \eta \xi + \sqrt{(\beta^2 \mu^2 - \eta^2 \xi^2)(1 - k^2)}
\]

\[0 < \sigma < 1, \quad \rho \eta \xi < k, \quad k = \sigma - \sqrt{1 - 2 \sigma + \delta^2}.
\]

then the sequence \( \{u_n\} \) generated by Algorithm 2.2 converges to the unique solution of problem (2.2).

Proof. It follows from Lemma 2.2 that the problem (2.2) is equivalent to the resolvent equation

\[
g(u) = J_A [g(u) - \rho N(Tu, Vu)],
\]

that is,

\[
0 = g(u) - J_A [g(u) - \rho N(Tu, Vu)] = Q(u).
\]
According to Lemma 2.3, we need only prove that the operator $Q : H \rightarrow H$ is strongly monotone with constant $\theta \in (0, 1)$ and Lipschitz continuous. From the assumption of the Theorem, it is easy to see that the operator $Q$ is Lipschitz continuous.

Let $u, v \in H$. Then by the strong monotonicity and Lipschitz continuity of $g$ and $N(T(\cdot), \cdot)$, we have

\[
\|u - v - g(u) + g(v)\|^2 \leq (1 - 2\sigma + \delta^2)\|u - v\|^2
\]
\[
\|u - v - \rho N(Tu, Vu) + \rho N(Tv, Vu)\|^2 \leq (1 - 2\rho\sigma + \rho^2\beta^2\mu^2)\|u - v\|^2.
\]

By the Lipschitz continuity of $V$, we also have

\[
\|N(Tv, Vu) - N(Tv, \hat{V}u)\| \leq \eta\|u - v\|.
\]

Thus,

\[
(Qu - Qv, u - v) = (g(u) - g(v), u - v) - (J_A[g(u) - \rho N(Tu, Vu)] - J_A[g(v) - \rho N(Tv, \hat{V}u)], u - v)
\]
\[
\geq \sigma\|u - v\|^2 - \|J_A[g(u) - \rho N(Tu, Vu)] - J_A[g(u) - \rho N(Tu, Vu)]\|\|u - v\|
\]
\[
\geq \sigma\|u - v\|^2 - \|g(u) - g(v) - \rho N(Tu, Vu)\|\|u - v\|
\]
\[
\geq \sigma\|u - v\|^2 - \|u - v - g(u) + g(v)\| - \|u - v - \rho N(Tu, Vu) + \rho N(Tv, Vu)\|
\]
\[
\geq \sigma\|u - v\|^2 - \|u - v - g(u) + g(v)\| - \|u - v - \rho N(Tu, Vu) + \rho N(Tv, Vu)\|
\]
\[
\geq \left(\sigma - \sqrt{1 - 2\sigma + \delta^2} - \sqrt{1 - 2\rho\sigma + \rho^2\beta^2\mu^2}
\right)
\]
\[
- \rho\eta\xi\|u - v\|^2
\]
\[
= \left(k - \sqrt{1 - 2\rho\sigma + \rho^2\beta^2\mu^2 - \rho\eta\xi}\right)\|u - v\|^2
\]
\[
= \theta\|u - v\|^2,
\]

where \(\theta = k - \sqrt{1 - 2\rho\sigma + \rho^2\eta^2\xi^2 - \rho\eta\xi}\). Therefore, $Q : H \rightarrow H$ is a strongly monotone operator with constant $\theta$ and from (3.14)–(3.16), it follows that $\theta \in (0, 1)$. Thus, Lemma 2.4 is applicable where $f = 0$ and the proof is complete. \[\]
REFERENCES


