An expression of the general common least-squares solution to a pair of matrix equations with applications

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ABSTRACT

In this paper, an explicit representation of the general common least-squares solution to the pair of matrix equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$ is obtained. Furthermore, we use this result to determine the condition for the existence of a Hermitian least-squares solution to the matrix equation $AXB = C$, and the expression of the general Hermitian least-squares solution is also given. Special attention is paid to consider the existence of Hermitian $[1, i]$-inverses of $A, i = 3, 4$, and the representations of the Hermitian generalized inverses are presented. Finally, a numerical example is given to illustrate our result.

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1. Introduction

Throughout this paper, we denote the set of all $m \times n$ complex matrices by $C^{m \times n}$. For any $A \in C^{m \times n}$, its rank, conjugate transpose, the Moore–Penrose inverse $[1]$ and Hermitian $[1, i]$-inverse will be denoted by $r(A), A^*, A^+$ and $A^{[1,i]}$, $i = 3, 4$, respectively. $R_A$ and $L_A$ stand for the two orthogonal projectors $R_A = I_m - AA^+$ and $L_A = I_n - A^+A$ induced by $A$.

Recently, research on linear matrix equations has received more and more attention and has had lots of nice results. For example, Mitra $[2,3]$ has provided conditions for the existence of a solution and a representation of the general common solution to the pair of individually consistent matrix equations

$$A_1XB_1 = C_1 \quad \text{and} \quad A_2XB_2 = C_2,$$

where $A_1 \in C^{m \times n}, B_1 \in C^{p \times k}, C_1 \in C^{m \times k}, A_2 \in C^{l \times n}, B_2 \in C^{p \times q}, C_2 \in C^{l \times q}$ are known and $X \in C^{n \times p}$ is unknown.

Conditions for the existence of a common solution to the system (1.1) have also been considered by many other authors $[4–7]$. Navarra et al. $[8]$ gave new necessary and sufficient conditions for the existence of a common solution to (1.1) and derived a new representation of the general common solution to these two equations. Chu $[9]$ presented a numerical algorithm for the common solution to system (1.1). Peng et al. $[10]$ derived an iterative method for symmetric solutions and optimal approximation solution of the system of matrix equations (1.1).

However, most of the previous works were focused on the solvable Eqs. (1.1). If the equations are unsolvable, an approximate solution that is often used, especially in statistical applications, is the least-squares solution. A common least-squares solution $X$ to the matrix equations (1.1) means that $X$ is not only the least-squares solution to matrix equation $A_1XB_1 = C_1$ but also to matrix equation $A_2XB_2 = C_2$. Liu $[11]$ established the condition for the matrix equations (1.1) to have a common least-squares solution by using the extremal ranks, however, the general common least-squares solution was not given.

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Let
\[ AXB = C, \]  
(1.2)
where \( A, B \) and \( C \) are given matrices over the complex field, and \( X \) is unknown. Several authors have studied the Eq. (1.2). For instance, Mitra [12] considered solutions with fixed ranks for (1.2). Khatri and Mitra [13] gave a necessary and sufficient condition of (1.2) to have a Hermitian solution. Peng [14] presented an iterative method for the least-squares symmetric solution of (1.2). Navarra et al. [8] determined a new condition for the existence of a Hermitian solution and a new representation of the general Hermitian solution to Eq. (1.2). Specially, the general Hermitian solution to equation \( AXA^* = B \) was considered in [15].

Motivated by previous works, in this paper, in order to derive a representation of the common least-squares solution to system (1.1), we adopt another method to establish the condition for the existence of a common least-squares solution, and then give its expression. As for its applications, we use our new result to determine condition for the existence of a Hermitian least-squares solution and a representation of the general Hermitian least-squares solution to the matrix equation (1.2). Special attention is paid to consider the existence of Hermitian \([i, i]\)-inverses, \( i = 3, 4 \), and their general representations.

In our development we will need the following lemmas to derive our results.

Wang [16] presented some necessary and sufficient conditions for the existence of a solution to the system \( A_1X_1B_1 + A_2X_2B_2 = C \) over the real quaternion algebra, and gave an expression of the general solution when the solvability conditions are satisfied. Obviously, these results are also true over the complex field.

**Lemma 1.1.** Let \( A_1 \in C^{m \times n}, B_1 \in C^{p \times k}, A_2 \in C^{m \times l}, B_2 \in C^{q \times k} \) and \( C \in C^{m \times k} \) be known and \( X_1 \in C^{n \times p}, X_2 \in C^{l \times q} \) unknown; \( M = R_A A_2, N = B_2 L_B, S = A_2 L_M \). Then the following statements are equivalent:

(1) The system \( A_1X_1B_1 + A_2X_2B_2 = C \) is solvable;

(2) The following rank equalities are satisfied
\[
\begin{align*}
& \quad r \left( \begin{array}{cc}
A_1 & C \\
0 & B_2 \\
\end{array} \right) = r \left( \begin{array}{cc}
A_1 & 0 \\
0 & B_2 \\
\end{array} \right), \\
& \quad r \left( \begin{array}{cc}
A_2 & C \\
0 & B_1 \\
\end{array} \right) = r \left( \begin{array}{cc}
A_2 & 0 \\
0 & B_1 \\
\end{array} \right), \\
& \quad r \left( \begin{array}{cc}
C & A_1 \\
A_2 & C \\
\end{array} \right) = r \left( \begin{array}{cc}
A_1 & A_2 \\
A_1 & A_2 \\
\end{array} \right), \\
& \quad r \left( \begin{array}{cc}
B_1 & C \\
B_2 & C \\
\end{array} \right) = r \left( \begin{array}{cc}
B_1 & C \\
B_2 & C \\
\end{array} \right).
\end{align*}
\]

In this case, the general solution of (1.3) can be expressed as
\[ X_1 = A_1^+CB_1^+ - A_1^+A_2M^+R_A CB_1^+ - A_1^+SA_2N^+B_2B_1^+ - A_1^+SVR_NB_2B_1^+ + L_A U + Z R_{B_1}, \]
\[ X_2 = M^+R_A CB_1^+ + L_M S^+SA_2N^+CL_B N^+ + L_M (V - S^+SVNN^+) + W R_{B_2}, \]
where \( U, V, W \) and \( Z \) are arbitrary matrices over a complex field with appropriate sizes.

**Lemma 1.2.** Let \( A \in C^{m \times n}, B \in C^{p \times k} \) and \( C \in C^{m \times k} \) be known matrices. Then \( X \in C^{n \times p} \) is a least-squares solution to the matrix equation \( AXB = C \) if and only if
\[ AXB = AA^+CB^+B \quad \text{or} \quad A^*AXB^* = A^*CB^* \]
and a representation of the general least-squares solution is
\[ X = A^+CB^+ + L_A V + W R_B, \]
where \( V, W \in C^{n \times p} \) are arbitrary matrices.

The following rank equalities for block matrices can be found in [17].

**Lemma 1.3.** Let \( A \in C^{m \times n}, B \in C^{m \times k}, C \in C^{l \times n} \) and \( D \in C^{l \times k} \). Then
\[
\begin{align*}
\quad r \left( \begin{array}{cc}
A & B \\
\end{array} \right) &= r(A) + r([I - AA^+]B), \\
\quad r \left( \begin{array}{cc}
A & \\
C \\
\end{array} \right) &= r(A) + r(C(I - A^+A)), \\
\quad r(D - CA^+B) &= r \left[ \begin{array}{cc}
A^*A^+ & A^*B \\
CA^* & D \\
\end{array} \right] - r(A).
\end{align*}
\]
2. Main results

In this section, our purpose is to establish the condition for the existence of a common least-squares solution to system (1.1), and then give the general common least-squares solution.

Theorem 2.1. Let

\[ P_1 = A_2L_1, \quad Q_1 = R_1B_2, \quad M_1 = R_1A_2, \quad E_1 = A_2A_2^+ C_2B_2^+ B_2 - A_2A_1^+ C_1B_1^+ B_2. \]
\[ P_2 = A_1L_2, \quad Q_2 = R_2B_1, \quad M_2 = R_2A_1, \quad E_2 = A_1A_1^+ C_1B_1^+ B_1 - A_1A_2^+ C_2B_2^+ B_1. \]

Then, the matrix equations (1.1) have a common least-squares solution if and only if

\[
r \begin{pmatrix}
-A_1^+ C_1 B_1^+ & 0 & A_1^+ A_2 \\
0 & A_2^+ C_2 B_2^+ & A_2^+ A_2 \\
B_1^+ B_1 & B_2^+ B_2 & 0
\end{pmatrix} = r \begin{pmatrix}
A_1 \\
A_2 \\
B_1^+ B_1 \\
B_2^+ B_2
\end{pmatrix}.
\]

(2.1)

In this case, the general common least-squares solution can be expressed as

\[
X_1 = A_1^+ C_1 B_1^+ + P_1^+ E_1 B_1^+ - P_1^+ A_2 M_1^+ R_1 E_1 B_1^+ + M_1^+ R_1 E_1 Q_1^+
\]
\[
- P_1^+ A_2 L_1 V_1 Q_1 B_2^+ + L_1 V_1 R_2 B_1 + L_1 L_1 U_1 + L_1 Z_1 R_2 + W_1 R_1 R_1,
\]

or

\[
X_2 = A_2^+ C_2 B_2^+ + P_2^+ E_2 B_2^+ - P_2^+ A_1 M_2^+ R_2 E_2 B_2^+ + M_2^+ R_2 E_2 Q_2^+
\]
\[
- P_2^+ A_1 L_2 V_2 Q_2 B_1^+ + L_2 V_2 R_1 B_2 + L_2 L_2 U_2 + L_2 Z_2 R_1 + W_2 R_2 R_2,
\]

where \( U_i, V_i, W_i \) and \( Z_i, i = 1, 2 \), are arbitrary matrices with appropriate sizes.

Proof. In view of Lemma 1.2, the general expression of the least-squares solution to \( A_1 X_1 B_1 = C_1 \) can be written as

\[
X_1 = A_1^+ C_1 B_1^+ + L_1 \tilde{V} + \tilde{W} R_1,
\]

(2.4)

where \( \tilde{V}, \tilde{W} \in \mathbb{C}^{n \times p} \) are arbitrary matrices.

The pair of matrix equations (1.1) has a common least-squares solution if and only if there exist some \( \tilde{V} \) and \( \tilde{W} \) such that \( X_1 \) is a least-squares solution to system \( A_2 X_2 B_2 = C_2 \), i.e.,

\[
A_2^2 A_2 L_1 \tilde{V} B_2 B_2^* + A_2^2 A_2 \tilde{W} R_1 B_2 B_2^* - A_2^2 A_2^+ A_2 C_1 B_1^+ B_2 B_2^* = A_2^2 C_2 B_2^* - A_2^2 A_2^+ A_2 C_1 B_1^+ B_2 B_2^*
\]

(2.5)

is solvable.

On account of Lemma 1.1, (2.5) is solvable if and only if the following rank equalities are satisfied

\[
r \begin{pmatrix}
A_2^2 A_2 L_1 & 0 \\
0 & A_2^2 C_2 B_2^* - A_2^2 A_2 A_2^+ C_1 B_1^+ B_2 B_2^*
\end{pmatrix} = r \begin{pmatrix}
A_2^2 A_2 L_1 & 0 \\
0 & R_1 B_2 B_2^*
\end{pmatrix},
\]

(2.6)

\[
r \begin{pmatrix}
A_2^2 A_2 & 0 \\
0 & A_2^2 C_2 B_2^* - A_2^2 A_2 A_2^+ C_1 B_1^+ B_2 B_2^*
\end{pmatrix} = r \begin{pmatrix}
A_2^2 A_2 & 0 \\
0 & B_2 B_2^*
\end{pmatrix},
\]

(2.7)

\[
r \begin{pmatrix}
A_2^2 C_2 B_2^* - A_2^2 A_2 A_2^+ C_1 B_1^+ B_2 B_2^* & A_2^2 A_2 L_1 \\
0 & A_2^2 A_2^+ 
\end{pmatrix} = r \begin{pmatrix}
A_2^2 A_2 L_1 & A_2^2 A_2 \\
0 & A_2^2 A_2^+
\end{pmatrix},
\]

(2.8)

\[
r \begin{pmatrix}
B_2 B_2^* & A_2^2 A_2 A_2^+ C_1 B_1^+ B_2 B_2^* \\
0 & R_1 B_2 B_2^*
\end{pmatrix} = r \begin{pmatrix}
B_2 B_2^* & A_2^2 A_2 A_2^+ C_1 B_1^+ B_2 B_2^* \\
0 & R_1 B_2 B_2^*
\end{pmatrix}.
\]

(2.9)

Note that the equalities (2.7)–(2.9) are obvious. Applying Lemma 1.3 shows that (2.6) is equivalent to

\[
r \begin{pmatrix}
0 & A_2^2 C_2 B_2^* - A_2^2 A_2 A_2^+ C_1 B_1^+ B_2 B_2^* \\
B_1 & 0
\end{pmatrix} = r \begin{pmatrix}
A_1 \\
A_2 \\
B_1 \\
B_2
\end{pmatrix}.
\]

(2.10)
It is not difficult to find, by Lemma 1.3 and block elementary operations

\[
\begin{pmatrix}
0 & 0 & A_1 \\
0 & A_2^*C_2B_2 - A_2^*A_1^*C_1B_1^*B_2B_2^* & A_2^*A_2 \\
B_1 & B_2B_2^* & 0
\end{pmatrix}
\]

\[
= r \begin{pmatrix}
0 & 0 & A_1^* \\
0 & A_2^*C_2B_2 - A_2^*A_2^* & 0 \\
B_1 & B_2B_2^* & 0
\end{pmatrix}
\]

\[
= r \begin{pmatrix}
0 & 0 & A_1^* \\
0 & A_2^*C_2B_2 - A_2^*A_2^* & 0 \\
B_1 & B_2B_2^* & 0
\end{pmatrix}
\]

\[
= r \begin{pmatrix}
0 & 0 & A_1^* \\
0 & A_2^*C_2B_2 - A_2^*A_2^* & 0 \\
B_1 & B_2B_2^* & 0
\end{pmatrix}
\]

\[
= r \begin{pmatrix}
0 & 0 & A_1^* \\
0 & A_2^*C_2B_2 - A_2^*A_2^* & 0 \\
B_1 & B_2B_2^* & 0
\end{pmatrix}
\]

\[
= r \begin{pmatrix}
0 & 0 & A_1^* \\
0 & A_2^*C_2B_2 - A_2^*A_2^* & 0 \\
B_1 & B_2B_2^* & 0
\end{pmatrix}
\]

\[
= r \begin{pmatrix}
0 & 0 & A_1^* \\
0 & A_2^*C_2B_2 - A_2^*A_2^* & 0 \\
B_1 & B_2B_2^* & 0
\end{pmatrix}
\]

Combining (2.10) and (2.11) yields (2.1).

Note that, system (2.5) is equivalent to

\[
A_2L_1\tilde{V}B_2 + A_2\tilde{W}R_8B_2 = A_2A_2^+C_2B_2^+B_2 - A_2A_1^+C_1B_1^+B_2 = E_1.
\]

(2.12)

According to Lemma 1.1, the general solution of system (2.12) is

\[
\tilde{V} = P_1^+E_1B_2^+ - P_1^+A_2M_1^+R_8E_1B_2^+ - P_1^+A_2L_1V_1Q_1B_2^+ + L_1U_1 + Z_1R_8,
\]

(2.13)

\[
\tilde{W} = M_1^+R_8E_1Q_1^+ + L_1V_1 + W_1R_8,
\]

(2.14)

where \(U_1, V_1, W_1, Z_1\) are arbitrary matrices with appropriate sizes.

Substituting (2.13) and (2.14) into (2.4) gives (2.2).

Similarly, the expression (2.3) can be established. \(\square\)

Although the expressions of \(X_1\) and \(X_2\) are different, under the condition (2.1), the sets \(\{X_1\}\) and \(\{X_2\}\) are equivalent.

**Corollary 2.1.** Suppose that the two matrix equations in (1.1) are solvable, respectively. And let

\[
P_1 = A_2L_1, \quad Q_1 = R_8B_2, \quad M_1 = R_8A_2, \quad E_1 = C_2 - A_2A_1^+C_1B_1^+B_2,
\]

\[
P_2 = A_1L_1, \quad Q_2 = R_8B_2, \quad M_2 = R_8A_1, \quad E_2 = C_1 - A_1A_2^+C_2B_2^+B_2.
\]

Then, the matrix equations (1.1) have a common solution if and only if

\[
\begin{pmatrix}
-C_1 & 0 & A_1 \\
0 & C_2 & A_2 \\
B_1 & B_2 & 0
\end{pmatrix}
\]

\[
= r \begin{pmatrix}
A_1 \\
A_2 \\
B_1 & B_2
\end{pmatrix}
\]

In this case, the general common least-squares solution can be expressed as

\[
X_1 = A_1^+C_1B_1^+ + P_1^+E_1B_2^+ - P_1^+A_2M_1^+R_8E_1B_2^+ + M_1^+R_8E_1Q_1^+ + P_1^+A_2L_1V_1Q_1B_2^+ + L_1V_1R_8B_1 + L_1L_1U_1 + L_1Z_1R_8B_2 + W_1R_8R_8B_1,
\]

or

\[
X_2 = A_2^+C_2B_2^+ + P_2^+E_1B_2^+ - P_2^+A_1M_2^+R_8E_1B_2^+ + M_2^+R_8E_2Q_2^+ + P_2^+A_1L_1V_1Q_1B_2^+ + L_1L_1V_1R_8B_1 + L_1L_1U_2 + L_1Z_1R_8B_2 + W_2R_8R_8B_2,
\]

where \(U_i, V_i, W_i\) and \(Z_i, i = 1, 2\), are arbitrary matrices with appropriate sizes.
3. Hermitian least-squares solution and Hermitian generalized inverse

In this section, the condition for existence a Hermitian least-squares solution to Eq. (1.2) is established, and the expression of the general Hermitian least-squares solution is presented. Based on the result, we consider the existence of Hermitian \{1, i\}-inverses, i = 3, 4, and the general representations.

For further discussion, we will use the following preliminary result, which is stated informally in [3].

**Lemma 3.1.** Let $A \in C^{p \times q}$, $B \in C^{q \times p}$ and $C \in C^{p \times p}$ be known matrices. The matrix equation (1.2) has a Hermitian solution if and only if the pair of matrix equations

$$AXB = C \quad \text{and} \quad B^*XA^* = C^*$$

have a common solution. Provided a Hermitian solution exists, a representation of the general Hermitian solution to (1.2) is of the form

$$X_i = \frac{X + X^*}{2},$$

where $X$ is the representation of the general common solution to Eqs. (3.1).

Our representation of the general Hermitian least-squares solution to (1.2) is based on the following corollary.

**Corollary 3.1.** Let $A \in C^{p \times q}$, $B \in C^{q \times p}$ and $C \in C^{p \times p}$ be known matrices. The matrix equation (1.2) has a Hermitian least-squares solution if and only if the pair of matrix equations

$$AXB = C \quad \text{and} \quad B^*XA^* = C^*$$

have a common least-squares solution. In this case, a representation of the general Hermitian least-squares solution to (1.2) is of the form

$$X_i = \frac{X + X^*}{2},$$

where $X$ is the representation of the general common least-squares solution to Eqs. (3.1).

**Proof.** In view of Lemma 1.2, if $X$ is a Hermitian least-squares solution to Eq. (1.2), then

$$A^*AXBB^* = A^*CB^* \quad \text{and} \quad A^*AX^*BB^* = A^*CB^*.$$  \hfill (3.2)

The latter equation is equivalent to $BB^*XA^*A = BC^*A$, which implies $X$ is also a Hermitian least-squares solution to equation $B^*XA^* = C^*$.

Conversely, if the pair of Eqs. (3.1) have a common least-squares solution $X$, namely,

$$A^*AXBB^* = A^*CB^* \quad \text{and} \quad BB^*XA^*A = BC^*A,$$

which are equivalent to

$$A^*AXBB^* = A^*CB^* \quad \text{and} \quad A^*AX^*BB^* = A^*CB^*.$$

This means that both $X$ and $X^*$ are the least-squares solution to Eq. (1.2). Therefore, the Eq. (1.2) has a Hermitian least-squares solution.

Let $X$ be the common solution (i.e., the common least-squares solution) to Eqs. (3.1). From Lemma 3.1, we know that $X = (X + X^*)/2$ is the general Hermitian least-squares solution to (1.2). \hfill \Box

**Theorem 3.1.** Let $A \in C^{p \times q}$, $B \in C^{q \times p}$ and $C \in C^{p \times p}$ be known matrices. The matrix equation (1.2) has a Hermitian least-squares solution if and only if

$$r \begin{pmatrix} -A^*CB^* & 0 & A^*A \\ 0 & BC^*A & BB^* \\ BB^* & A^*A & 0 \end{pmatrix} = 2r \begin{pmatrix} A^* & B \end{pmatrix}. \hfill (3.3)$$

In this case, the general Hermitian least-squares solution can be expressed as

$$X_i = \frac{X + X^*}{2},$$

where

$$X = A^*CB^* + P^*E(A^*)^* - P^*B^*M^*P_0E(A^*)^* + M^*P_0EQ^*$$

$$- P^*B^*L_0VQ(A^*)^* + L_0VQ_0R_0 + L_0VQ_0U + L_0VQ_0Z + WR_0R_0,$$

and $P = B^*L_0$. $Q = R_0A^*$, $M = R_0B^*$, $E = B^*BC^*AA^* - B^*A^*CB^*A^*$, $U$, $V$, $W$ and $Z$ are arbitrary matrices with appropriate sizes.
Proof. The necessary and sufficient condition for the existence of Hermitian least-squares solution follows from (2.1) and Corollary 3.1. The representation of the general Hermitian least-squares solution can be obtained from (2.2) and Corollary 3.1. \square

By Theorem 3.1, we derive the following results.

**Corollary 3.2.** Let \( A \in \mathbb{C}^{p\times q}, B \in \mathbb{C}^{q\times p} \) and \( C \in \mathbb{C}^{p\times p} \) be known matrices. The consistent matrix equation (1.2) has a Hermitian solution if and only if

\[
r \begin{pmatrix} -C & 0 & A \\ 0 & C^* & B^* \\ B & A^* & 0 \end{pmatrix} = 2r \begin{pmatrix} A^* & B \end{pmatrix}.
\]

In this case, the general Hermitian solution can be expressed as

\[
X = \frac{X + X^*}{2},
\]

where

\[
X = A^+CB^+ + P^+E(A^+) + P^+B^M^+R_pE(A^+)^* + M^+R_pEQ^+ - P^+B^L_pVQ(A^+) + L_pVR_p + L_pU + L_pZL_A + WR_pR_p,
\]

and \( P = B^*L_A, Q = R_pA^+, M = R_pB^*, E = C^* - B^*A^+CB^+A^+, U, V, W \) and \( Z \) are arbitrary matrices with appropriate sizes.

**Corollary 3.3** ([18]). Let \( A, B \in \mathbb{C}^{p\times p} \) be known matrices, such that \( B^+ = B \). Then the matrix equation \( AXA^+ = B \) has a Hermitian least-squares solution, and the general Hermitian least-squares solution can be expressed as

\[
X = A^+B(A^+)^* + L_AVL_A + L_AW + W^*L_A
\]

\[
= A^+B(A^+)^* + L_AZ + Z^*L_A,
\]

where \( W, Z \in \mathbb{C}^{p\times p} \) are arbitrary matrices, and \( V \in \mathbb{C}^{p\times p} \) is a Hermitian matrix.

Next, we use the previous results to determine the conditions for the existence of Hermitian \([1, i]\)-inverses, \( i = 3, 4 \), and their general expressions. Recall that, for every \( A \in \mathbb{C}^{n\times n}, X \in A^{[1, 3]} \) if and only if \( X \) is a least-squares solution of

\[
AX = I_n.
\]

According to Theorem 3.1, there exists a Hermitian \([1, 3]\)-inverse, if and only if the Eq. (3.4) has a Hermitian least-squares solution. And the solution will be given in the following.

**Theorem 3.2.** Let \( A \in \mathbb{C}^{n\times n} \). There exists a Hermitian \([1, 3]\)-inverse of \( A \) if and only if

\[
(A^*)^2A = A^*A^2, \quad \text{or} \quad AA^+A^* = A^2.
\]

In this case, the general Hermitian \([1, 3]\)-inverse of \( A \) can be expressed as

\[
A^{(1, 3)}_h = A^+ + (A^+)^* - \frac{1}{2}A^+(A + A^*)(A^+) + \frac{1}{2}L_AHL_A,
\]

where \( H \) is an arbitrary Hermitian matrix over a complex field.

**Proof.** The first condition in (3.5) follows from (3.3), and the equivalence of the two conditions in (3.5) is obvious. Moreover, the general Hermitian \([1, 3]\)-inverse of \( A \) can be expressed as

\[
A^{(1, 3)}_h = \frac{X + X^*}{2},
\]

where

\[
X = A^+ + L_A(A^+) + L_AZL_A.
\]

Hence

\[
A^{(1, 3)}_h = A^+ + (A^+)^* - \frac{1}{2}A^+(A + A^*)(A^+) + \frac{1}{2}L_A(Z + Z^*)L_A
\]

\[
= A^+ + (A^+)^* - \frac{1}{2}A^+(A + A^*)(A^+) + L_AH_L_A.
\]

The proof is completed. \square
For the Hermitian \{1, 4\}-inverse we have the following result.

**Corollary 3.4.** Let \( A \in \mathbb{C}^{n \times n} \). There exists a Hermitian \{1, 4\}-inverse of \( A \) if and only if
\[
A^2 A^+ = A(A^+)^2, \quad \text{or} \quad A^+ A^+ A = A^2.
\]
In this case, the general Hermitian \{1, 4\}-inverse of \( A \) can be expressed as
\[
A^{(1,4)}_h = A^+ + (A^+)^* - \frac{1}{2} (A^+)^*(A + A^+)^* A^+ + R_H A,
\]
where \( H \) is an arbitrary Hermitian matrix over a complex field.

**Proof.** Taking the conjugate transpose of (3.4), we get
\[
(A^*)^* A^* = I_n. \tag{3.6}
\]
The set of Hermitian least-squares solutions of Eq. (3.6) is \( \{ (A^*)^{(1,3)}_h \} \), which coincides with \( \{ A^{(1,4)}_h \} \). According to Theorem 3.2, this corollary can be easily obtained. \( \square \)

Specially, if \( A \) a Hermitian matrix, then Theorem 3.2 and Corollary 3.3 reduce to the following statements.

**Corollary 3.5.** Let \( A \in \mathbb{C}^{n \times n} \). If \( A \) is a Hermitian matrix. Then, the general Hermitian \{1, 3\}-inverse and Hermitian \{1, 4\}-inverse of \( A \) can be expressed as
\[
A^{(1,3)}_h = A^{(1,4)}_h = A^+ + L_A H A = A^+ + R_H A,
\]
where \( H \) is an arbitrary Hermitian matrix over a complex field.

4. Example

In this section, we present a numerical example to demonstrate Theorem 3.2.
Consider the matrix
\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & -1 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

It is easy to verify that the matrix \( A \) satisfies \( (A^+)^2 A = A^+ A^2 \). According to Theorem 3.1, the existence of a Hermitian \{1, 3\}-inverse of \( A \) is obvious. A simple computation shows that
\[
A^+ = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0.5 & 0 & 0 \\
0 & 0.5 & 0 & 0 \\
0 & 1 & -1 & 0
\end{bmatrix}.
\]
Hence,
\[
\{ A^{(1,3)}_h \} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0.75 + a & 0.25 - a & 0 \\
0 & 0.25 - a & -0.25 + a & -1 \\
0 & 1 & -1 & 0
\end{bmatrix} a \in \mathbb{C}.
\]

5. Conclusion

We have obtained an explicit representation of the general common least-squares solution to the Eqs. (1.1). Using the representation, we have determined the condition for the existence of a Hermitian least-squares solution to the matrix equation \( A X B = C \) and a representation of the general Hermitian least-squares solution. Finally, the existence of Hermitian \{1, i\}-inverses of \( A, i = 3, 4 \), and the representations of the Hermitian generalized inverses are presented.

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