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On the constants for multiplication in Sobolev spaces

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Abstract

For $n > d/2$, the Sobolev (Bessel potential) space $H^n(\mathbf{R}^d, \mathbf{C})$ is known to be a Banach algebra with its standard norm $\| \; \|_n$ and the pointwise product; so, there is a best constant K_{nd} such that $\|fg\|_n \leqslant K_{nd}\|f\|_n\|g\|_n$ for all f, g in this space. In this paper we derive upper and lower bounds for these constants, for any dimension d and any (possibly noninteger) $n \in (d/2, +\infty)$. Our analysis also includes the limit cases $n \to (d/2)^+$ and $n \to +\infty$, for which asymptotic formulas are presented. Both in these limit cases and for intermediate values of n, the lower bounds are fairly close to the upper bounds. Numerical tables are given for $d = 1, 2, 3, 4$, where the lower bounds are always between 75 and 88% of the upper bounds.

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1. Introduction

The theory of Sobolev spaces contains a lot of inequalities which involve real constants; often, the classical arguments employed to prove these inequalities allow to infer the existence of such constants, but are unsuitable to evaluate them accurately. On the other hand, a precise knowledge of these constants is desirable for several reasons: apart from the intrinsic interest of the problem, there are many applications where a fully quantitative analysis relies on these numbers.

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The inequality analyzed in this paper refers to the pointwise multiplication in $H^n(\mathbf{R}^d, \mathbf{C})$ for any $n > d/2$. We are interested in the best constant K_{nd} such that

$$\|fg\|_n \leqslant K_{nd}\|f\|_n\|g\|_n$$

for all $f, g \in H^n(\mathbf{R}^d, \mathbf{C})$, where $\| \ \|_n$ is the standard norm of this space (see Eqs. (1.2), (1.3) later on in this Introduction, and Eq. (2.1) in the next section).

The constants K_{nd} are relevant in relation to PDEs with polynomial nonlinearities, since they allow precise estimates on certain approximation methods and on blow up phenomena. To cite only one example, we refer to the semilinear heat equation in one space dimension discussed in [10]; here, an estimate on K_{11} has been employed to compute the error of the Galerkin approximate solutions, and the blow up times for certain initial data.

Evaluating K_{nd} for arbitrary n and d is a nontrivial task. For example, let the problem be formulated in the variational language: maximize $\|fg\|_n$ with the constraints $\|f\|_n = \|g\|_n = 1$; if n is integer one can write the corresponding Euler–Lagrange equations, but these form a cubic system of PDEs of order $2n$ for f and g.

Due to the difficulty of the problem, one could be satisfied even if, in spite of the exact value of K_{nd}, one had sufficiently close lower and upper bounds for it. Such bounds are proposed in this paper, for any integer d and (possibly noninteger) $n \in (d/2, +\infty)$. Our upper bounds depend on an accurate use of the Fourier transform and of the convolution: the conclusion of this analysis is an inequality

$$K_{nd} \leqslant K^+_{nd},$$

where K^+_{nd} is the sup on $[0, +\infty)$ of a function of hypergeometric type. This sup is easily evaluated, analytically in certain cases and numerically otherwise.

The lower bounds we propose follow directly from the inequality that defines K_{nd}, choosing for f and g appropriate trial functions: these often depend on one or two real parameters, so one gets the highest lower bound from the chosen functions maximizing with respect to the parameters. In any case, this procedure gives inequalities of the form

$$K^-_{nd} \leqslant K_{nd},$$

where K^-_{nd} depends on the trial functions: we will consider two specific choices, giving rise to what we call the "Bessel" or "Fourier" lower bounds. Both types of bounds are expressible via special functions of hypergeometric type, or by one-dimensional integrals which are easily computed numerically. For given values of n and d, the best available estimate from below for K_{nd} is obtained choosing for K^-_{nd} the highest between the Bessel and the Fourier bounds. For certain values of n only one kind of lower bound is easily computed, so one must be content with it. Our investigation also includes the limit cases $n \to (d/2)^+$ and $n \to +\infty$; the second limit requires the asymptotic analysis of certain integrals, which is performed via the Laplace method. To give an idea of our results, we summarize some of them.

(i) For $n \to (d/2)^+$,

$$K^+_{nd} = \frac{M_d}{\sqrt{n - d/2}}\left[1 + O\left(n - \frac{d}{2}\right)\right],$$

where M_d is an explicitly given constant (see the next section, Eq. (2.6)); on the other hand, denoting with K_{nd}^- a conveniently chosen Bessel lower bound, one finds

$$K_{nd}^- = \sqrt{\frac{2}{3}} \frac{M_d}{\sqrt{n - d/2}} \left[1 + O\left(n - \frac{d}{2}\right)\right];$$

so, in this limit $K_{nd}^+/K_{nd}^- \to \sqrt{2/3} > 0.816$.

(ii) For $n \to +\infty$,

$$K_{nd}^+ = T_d \frac{(2/\sqrt{3})^n}{n^{d/4}} \left[1 + O\left(\frac{1}{n}\right)\right],$$

with T_d another explicitly given constant (see Eq. (2.7)). On the other hand, denoting with K_{nd}^- an appropriate Fourier lower bound, one finds

$$K_{nd}^- = \frac{(5/3)^{1/2}}{7^{1/4}} T_d \frac{(2/\sqrt{3})^n}{n^{d/4}} \left[1 + O\left(\frac{1}{n}\right)\right];$$

thus, $K_{nd}^+/K_{nd}^- \to (5/3)^{1/2} 7^{-1/4} > 0.793$.

(iii) For $d = 1, 2, 3, 4$ we have explored the whole interval $n \in (d/2, +\infty)$, choosing for each K_{nd}^- the most convenient Bessel or Fourier lower bound and comparing it with the upper bound K_{nd}^+; for the sample values of n we have considered, K_{nd}^+/K_{nd}^- ranges between 0.750 and 0.880. A table of these upper and lower bounds is reported in the paper.

(iv) As previously said, K_{nd}^+ is the sup of a hypergeometric-like function. Even though this is easily computed numerically, to avoid this burden one can use a majorant $K_{nd}^{++} \geqslant K_{nd}^+$. We define K_{nd}^{++} using only elementary functions of n; this bound reproduces correctly the asymptotic behavior of K_{nd}^+ for $n \to (d/2)^+$, $n \to +\infty$, and for $1 \leqslant d \leqslant 7$ is very close to it on the whole range $(d/2, +\infty)$.

At the end of this Introduction we will give some details on the organization of the paper. Before presenting this, we insert a few comments on some related literature.

Connections with previous works. In our paper [8], we estimated the constants for more general inequalities related to multiplication in Sobolev spaces; in particular, we discussed the constants K_{nad} in the "tame" (or "Nash–Moser") inequality

$$\|fg\|_n \leqslant K_{nad} \max\left(\|f\|_n \|g\|_a, \|f\|_a \|g\|_n\right)$$

for $d/2 < a \leqslant n$ and $f, g \in H^n(\mathbf{R}^d, \mathbf{C})$; here $\| \ \|_a$ is the norm of $H^a(\mathbf{R}^d, \mathbf{C})$. (The cited work is partly related to the previous one [7], and to the subsequent one [9] on the tame functional calculus in Sobolev spaces.) In the special case $n = a$, the inequality written above coincides with the inequality of the present paper.

For arbitrary d, a, n, in [8] we derived upper and lower bounds for K_{nad}. The lower bounds were of the Bessel and Fourier types also considered here (with no analysis of the limit $n \to (d/2)^+$, and a discussion of the limit a fixed, $n \to +\infty$, of course different from the present limit $n \to +\infty$; some explicit formulas of [8] for these lower bounds are replaced here with equivalent, but simpler versions, and we also give some new formula).

The upper bounds for K_{nad} were obtained by a different method than the present one for K_{nd}; furthermore, if the upper estimates of [8] are applied with $n = a$ they are found to be rougher than the present ones on K_{nd}.

The method we use here to get the upper bounds refines an idea which appeared in [13] in relation to the multiplication in the space $H^n(\mathbf{T}, \mathbf{C})$, where $\mathbf{T} := \mathbf{R}/(2\pi\mathbf{Z})$ is the one-dimensional torus. The author of [13] was not interested in a precise estimate of the constant for multiplication, so he inserted in his argument some majorization which, although unnecessary, simplified the proof of the convergence of a series; the upper bound on the constant for the multiplication in $H^n(\mathbf{T}, \mathbf{C})$ arising from this simplification behaves like const.$\times 2^n$ for large n (see p. 294 of the cited paper). Here we replace the one-dimensional torus with $\mathbf{R}^d$, and the Fourier series with the d-dimensional Fourier transform. The literal translation of the technique of [13] in our framework would give again an upper bound for K_{nd} behaving like 2^n for $n \to +\infty$; on the contrary, here we use only the strictly necessary majorizations and finally obtain the bound K^+_{nd} involving a hypergeometric function, which as explained behaves like $(2/\sqrt{3})^n n^{-d/4}$ for $n \to +\infty$ and is accurate for small n as well.

Organization of the paper. In Section 2 we state precisely all the results about the previously mentioned upper and lower bounds for K_{nd}. As a preparation for the proofs, in Section 3 we write a list of known identities frequently cited in the sequel, on the following subjects: radial integrals, radial Fourier transforms, hypergeometric functions, integrals with three Bessel functions and the asymptotics of Laplace integrals (the last two topics are also treated in Appendices A and B). In Section 4 we prove all statements about the upper bounds K^+_{nd}. In Sections 5 and 6 we prove all the results about the Bessel and Fourier lower bounds, respectively.

In the remaining part of this Introduction, we fix some notations and definitions employed throughout the paper.

Basic notations on $\mathbf{R}^d$ and Fourier transforms. We consider an arbitrary space dimension d; the running variable in $\mathbf{R}^d$ is $x = (x_1, \ldots, x_d)$, and $k = (k_1, \ldots, k_d)$ when $\mathbf{R}^d$ is interpreted as the "wave vector" space of the Fourier transform. We write $\bullet$ and $|\ |$ for the inner product and the Euclidean norm of $\mathbf{R}^d$ (so that $|x| = \sqrt{x_1{}^2 + \cdots + x_d{}^2}$, $|k| = \sqrt{k_1{}^2 + \cdots + k_d{}^2}$, $k \bullet x = k_1 x_1 + \cdots + k_d x_d$).

We denote with $\mathcal{F}, \mathcal{F}^{-1} : S'(\mathbf{R}^d, \mathbf{C}) \to S'(\mathbf{R}^d, \mathbf{C})$ the Fourier transform of tempered distributions and its inverse, choosing normalizations so that (for f in $L^1(\mathbf{R}^d, \mathbf{C})$) it is $\mathcal{F}f(k) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} dx\, e^{-ik\bullet x} f(x)$. The restriction of $\mathcal{F}$ to $L^2(\mathbf{R}^d, \mathbf{C})$, with the standard inner product and the associated norm $\|\ \|_{L^2}$, is a Hilbertian isomorphism.

Sobolev spaces. For real $n \geqslant 0$, let us introduce the operators

$$S'(\mathbf{R}^d, \mathbf{C}) \to S'(\mathbf{R}^d, \mathbf{C}), \qquad g \mapsto \sqrt{1-\Delta}^{\,n} g := \mathcal{F}^{-1}\Big(\sqrt{1+|\mathrm{k}|^2}^{\,n} \mathcal{F}g\Big) \tag{1.1}$$

where $\sqrt{1+|\mathrm{k}|^2}^{\,n}$ means the function $k \in \mathbf{R}^d \mapsto \sqrt{1+|k|^2}^{\,n}$. The nth order Sobolev (or Bessel potential [3]) space of L^2 type and its norm are

$$\begin{aligned} H^n(\mathbf{R}^d, \mathbf{C}) &:= \big\{ f \in S'(\mathbf{R}^d, \mathbf{C}) \mid \sqrt{1-\Delta}^{\,n} f \in L^2(\mathbf{R}^d, \mathbf{C}) \big\} \\ &= \big\{ f \in S'(\mathbf{R}^d, \mathbf{C}) \mid \sqrt{1+|\mathrm{k}|^2}^{\,n} \mathcal{F}f \in L^2(\mathbf{R}^d, \mathbf{C}) \big\}, \end{aligned} \tag{1.2}$$

$$\|f\|_n := \left\|\sqrt{1-\Delta}^{\,n} f\right\|_{L^2} = \left\|\sqrt{1+|\mathrm{k}|^2}^{\,n}\,\mathcal{F}f\right\|_{L^2}. \tag{1.3}$$

For n integer, these definitions imply

$$H^n(\mathbf{R}^d,\mathbf{C}) = \{f \in S'(\mathbf{R}^d,\mathbf{C}) \mid \nabla^m f \in L^2(\mathbf{R}^d, \otimes^m \mathbf{C}^d)\ \forall m \in \{0,\dots,n\}\} \tag{1.4}$$

where

$$\nabla^m f := (\partial_{\lambda_1,\dots,\lambda_m} f)_{(\lambda_1,\dots,\lambda_m)\in\{1,\dots,d\}^m} \tag{1.5}$$

and ∂_{λ_i} is the distributional derivative with respect to the coordinate x_{λ_i}. The statement $\nabla^m f \in L^2(\mathbf{R}^d, \otimes^m \mathbf{C}^d)$ means that

$$+\infty > \sum_{\lambda_1,\dots,\lambda_m=1,\dots d} \int_{\mathbf{R}^d} dx \left|(\partial_{\lambda_1,\dots,\lambda_m} f)(x)\right|^2 := \left\|\nabla^m f\right\|^2_{L^2}, \tag{1.6}$$

and the norm (1.3) can be written as

$$\|f\|_n = \sqrt{\sum_{m=0}^{n} \binom{n}{m} \left\|\nabla^m f\right\|^2_{L^2}}. \tag{1.7}$$

Other notations. Some useful functions. The Pochhammer symbol of $a \in \mathbf{R}$, $\ell \in \mathbf{N}$, is

$$(a)_\ell := a(a+1)\dots(a+\ell-1). \tag{1.8}$$

The semifactorial of an odd $m \in \mathbf{N}$ is

$$m!! := 1.3\dots(m-2)m, \tag{1.9}$$

and we also define $(-1)!! := 1$. We refer to [1,5,14] as our standards for special functions. In this paper, we frequently use the Gamma function and its logarithmic derivative $\psi(w) := \Gamma'(w)/\Gamma(w)$; for future reference, we write here their properties more frequently employed in the sequel. These are: the shift formulas

$$\Gamma(w+1) = w\Gamma(w), \tag{1.10}$$

$$\psi(w+1) = \psi(w) + \frac{1}{w}; \tag{1.11}$$

the special values

$$\Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(1) = 1, \quad \psi(1/2) = -\gamma_E - 2\log 2, \quad \psi(1) = -\gamma_E \tag{1.12}$$

(with γ_E the Euler–Mascheroni constant); the duplication formula

$$\Gamma(2w) = \frac{2^{2w-1}}{\sqrt{\pi}}\Gamma(w+1/2)\Gamma(w); \tag{1.13}$$

the identity

$$\int_0^{+\infty} du \frac{u^{\sigma-1}}{(1+u)^\gamma} = \frac{\Gamma(\sigma)\Gamma(\gamma-\sigma)}{\Gamma(\gamma)} \quad \text{for } \gamma > \sigma > 0. \tag{1.14}$$

Another function of which we make wide use is the Gaussian hypergeometric function ${}_2F_1(\alpha, \beta, \gamma; w) \equiv F(\alpha, \beta, \gamma; w)$. We are especially interested in the function

$$F_{nd} : [0, +\infty) \to (0, +\infty), \qquad u \mapsto F_{nd}(u) := F\left(2n - \frac{d}{2}, n, n + \frac{1}{2}; -u\right), \tag{1.15}$$

$$d \in \mathbf{N} \setminus \{0\},\ n \in (d/2, +\infty).$$

This function has the equivalent representation

$$F_{nd}(u) = \frac{1}{(1+u)^n} F\left(n, \frac{d}{2} + \frac{1}{2} - n, n + \frac{1}{2}; \frac{u}{1+u}\right), \tag{1.16}$$

following from a familiar Kummer transformation (see Section 3, where we return to some statements appearing here); we also mention the special case

$$F_{nd}(u) = \sum_{\ell=0}^{n-d/2-1/2} \frac{(n)_\ell (d/2 + 1/2 - n)_\ell}{(n+1/2)_\ell \ell!} \frac{u^\ell}{(1+u)^{n+\ell}} \quad \text{for } n - \frac{d}{2} - \frac{1}{2} \in \mathbf{N}. \tag{1.17}$$

As usual with Sobolev spaces, a central role in our considerations is played by the functions

$$G_{nd} : \mathbf{R}^d \to \mathbf{C}, \qquad k \mapsto G_{nd}(k) := \frac{1}{(1+|k|^2)^n}; \tag{1.18}$$

$$g_{nd} : \mathbf{R}^d \to \mathbf{C}, \qquad g_{nd} := \mathcal{F}^{-1} G_{nd}. \tag{1.19}$$

It is clear that $g_{nd} \in H^n(\mathbf{R}^d, \mathbf{C})$ if $n > d/2$; explicitly, one has [3,6]

$$g_{nd}(x) = \frac{|x|^{n-d/2}}{2^{n-1}\Gamma(n)} K_{n-d/2}(|x|) \tag{1.20}$$

for $x \in \mathbf{R}^d$; here K_ν are the modified Bessel functions of the second kind, or Macdonald functions.

2. Description of the main results

Let $d \in \mathbf{N} \setminus \{0\}$. For (integer or noninteger) $n > d/2$, the space $H^n(\mathbf{R}^d, \mathbf{C})$ is known to be a Banach algebra under the pointwise multiplication: see, e.g., [2].

2.1. Definition. For $n > d/2$, we put

$$K_{nd} := \min\{K \geqslant 0 \mid \|fg\|_n \leqslant K\|f\|_n\|g\|_n \text{ for all } f, g \in H^n(\mathbf{R}^d, \mathbf{C})\} \tag{2.1}$$

and refer to this as the best (or sharp) constant for the multiplication in $H^n(\mathbf{R}^d, \mathbf{C})$.

In the sequel we present our upper and lower bounds on K_{nd}.

Upper bounds on K_{nd}. These are given by the following proposition, to be proved in Section 4.

2.2. Proposition. (i) *For all $n > d/2$,*

$$K_{nd} \leqslant K_{nd}^+ := \sqrt{\sup_{u\in[0,+\infty)} \mathcal{S}_{nd}(u)}. \tag{2.2}$$

$$\mathcal{S}_{nd} : [0, +\infty) \to (0, +\infty), \qquad \mathcal{S}_{nd}(u) := \frac{\Gamma(2n - d/2)}{(4\pi)^{d/2}\Gamma(2n)}(1 + 4u)^n F_{nd}(u), \tag{2.3}$$

with F_{nd} as in Eq. (1.15) *or* (1.16). *$\mathcal{S}_{nd}$ is bounded, and its boundary values for $u = 0$, $u \to +\infty$ are*

$$\mathcal{S}_{nd}(0) = \frac{\Gamma(2n - d/2)}{(4\pi)^{d/2}\Gamma(2n)}, \qquad \mathcal{S}_{nd}(+\infty) = \frac{\Gamma(n + 1 - d/2)}{2^{d-1}\pi^{d/2}(n - d/2)\Gamma(n)}. \tag{2.4}$$

(ii) *For $d/2 < n \leqslant d/2 + 1/2$ the function $\mathcal{S}_{nd}$ is increasing, so that*

$$K_{nd}^+ = \sqrt{\mathcal{S}_{nd}(+\infty)} = \frac{1}{2^{d/2-1/2}\pi^{d/4}}\sqrt{\frac{\Gamma(n + 1 - d/2)}{(n - d/2)\Gamma(n)}}. \tag{2.5}$$

For fixed d and $n \to (d/2)^+$, this implies

$$K_{nd}^+ = \frac{M_d}{\sqrt{n - d/2}}\left[1 + O\left(n - \frac{d}{2}\right)\right], \qquad M_d := \frac{1}{2^{d/2-1/2}\pi^{d/4}\sqrt{\Gamma(d/2)}}. \tag{2.6}$$

(iii) *For fixed d and $n \to +\infty$,*

$$K_{nd}^+ = \sqrt{\mathcal{S}_{nd}\left(\frac{1}{2}\right)}\left[1 + O\left(\frac{1}{n}\right)\right] = T_d\frac{(2/\sqrt{3})^n}{n^{d/4}}\left[1 + O\left(\frac{1}{n}\right)\right], \quad T_d := \frac{3^{d/4+1/4}}{2^d\pi^{d/4}}. \tag{2.7}$$

Of course, in Eqs. (2.4) and (2.5) we could write $\Gamma(n + 1 - d/2)/(n - d/2) = \Gamma(n - d/2)$; the expression in the left-hand side has been preferred to handle the limit $n \to (d/2)^+$. Similar choices have been made for other formulas in the sequel.

"Bessel" lower bounds on K_{nd}. The general method to obtain lower bounds on this constant is based on the obvious inequality

$$K_{nd} \geqslant \frac{\|fg\|_n}{\|f\|_n \|g\|_n} \tag{2.8}$$

for all nonzero $f, g \in H^n(\mathbf{R}^d, \mathbf{C})$; this gives a lower bound for any pair of "trial functions" f, g. Inspired by [8], we choose for f and g the function

$$g_{\lambda nd}(x) := g_{nd}(\lambda x) \tag{2.9}$$

where $\lambda \in (0, +\infty)$ is a parameter and g_{nd} is defined by Eq. (1.19). By comparison with that equation, we find

$$g_{\lambda nd} = \mathcal{F}^{-1} G_{\lambda nd}, \qquad G_{\lambda nd}(k) := \frac{1}{\lambda^d (1 + |k|^2/\lambda^2)^n}. \tag{2.10}$$

To give a lower bound for K_{nd} in terms of these functions simply amounts to compute $\|g_{\lambda nd}\|_n$, $\|g^2_{\lambda nd}\|_n$. These norms were already calculated in [8]; here we give them in a more simple and complete form, and add an analysis of the limit case when n is close to $d/2$. All these facts are described by the forthcoming proposition, to be proved in Section 5.

2.3. Proposition. (i) *For all $n > d/2$ and $\lambda > 0$,*

$$K_{nd} \geqslant \mathcal{K}^B_{nd}(\lambda) := \frac{\|g^2_{\lambda nd}\|_n}{\|g_{\lambda nd}\|^2_n}, \tag{2.11}$$

whence

$$K_{nd} \geqslant K^B_{nd} := \sup_{\lambda > 0} \mathcal{K}^B_{nd}(\lambda). \tag{2.12}$$

The norms in Eq. (2.11) *are given by*

$$\|g_{\lambda nd}\|^2_n = \frac{\pi^{d/2} \Gamma(n + 1 - d/2)}{(n - d/2)\Gamma(n)\lambda^d} F\left(-n, \frac{d}{2}, n; 1 - \lambda^2\right); \tag{2.13}$$

$$\|g_{\lambda nd}\|^2_n = \frac{\pi^{d/2}}{\Gamma(d/2)\Gamma(2n)\lambda^d} \times \sum_{\ell=0}^{n} \binom{n}{\ell} \Gamma(\ell + d/2)\Gamma(2n - d/2 - \ell)\lambda^{2\ell} \quad \textit{for } n \textit{ integer}; \tag{2.14}$$

$$\left\|g^2_{\lambda nd}\right\|^2_n = \frac{\pi^{d/2}\Gamma^2(2n - d/2)}{\Gamma(d/2)\Gamma^2(2n)\lambda^d} \int_0^{+\infty} du\, u^{d/2-1}\left(1 + 4\lambda^2 u\right)^n F^2_{nd}(u), \tag{2.15}$$

with F_{nd} as in Eqs. (1.15)–(1.16);

$$\left\|g^2_{\lambda nd}\right\|^2_n = \frac{\pi^{d/2}\Gamma^2(2n-d/2)}{\Gamma(d/2)\Gamma^2(2n)\lambda^d}\sum_{\ell,m=0}^{n-d/2-1/2}\frac{(n)_\ell(d/2+1/2-n)_\ell}{(n+1/2)_\ell\,\ell!}\,\frac{(n)_m(d/2+1/2-n)_m}{(n+1/2)_m\,m!}$$
$$\times\frac{\Gamma(d/2+\ell+m)\Gamma(n-d/2)}{\Gamma(n+\ell+m)}F\left(-n,\frac{d}{2}+\ell+m,n+\ell+m;1-4\lambda^2\right)$$
$$\text{for } n-\tfrac{d}{2}-\tfrac{1}{2} \text{ integer.} \tag{2.16}$$

(ii) *Let $d/2 < n \leqslant d/2+1/2$. Then, for all $\lambda > 0$,*

$$\left\|g^2_{\lambda nd}\right\|^2 \geqslant \mathcal{G}_{nd}(\lambda), \tag{2.17}$$

so that

$$\mathcal{K}^B_{nd}(\lambda) \geqslant \mathcal{K}^{BB}_{nd}(\lambda) := \frac{\sqrt{\mathcal{G}_{nd}(\lambda)}}{\|g_{\lambda nd}\|^2}, \tag{2.18}$$

$$K^B_{nd} \geqslant K^{BB}_{nd} := \sup_{\lambda>0}\mathcal{K}^{BB}_{nd}(\lambda). \tag{2.19}$$

Here:

$$\mathcal{G}_{nd}(\lambda) := \frac{\pi^{d/2}\Gamma^2(2n-d/2)}{(n-d/2)^3\Gamma^2(2n)\lambda^d}\left[P^2_{nd}\frac{\Gamma(n+1-d/2)}{\Gamma(n)}F\left(-n,\frac{d}{2},n;1-4\lambda^2\right)\right.$$
$$-P_{nd}Q_{nd}\frac{\Gamma(2n+1-d)}{\Gamma(2n-d/2)}F\left(-n,\frac{d}{2},2n-\frac{d}{2};1-4\lambda^2\right)$$
$$\left.+q^2_{nd}\frac{\Gamma(3n+1-3d/2)}{3\Gamma(3n-d)}F\left(-n,\frac{d}{2},3n-d;1-4\lambda^2\right)\right]; \tag{2.20}$$

$$P_{nd} := \frac{\Gamma(n+1/2)\Gamma(n+1-d/2)}{\sqrt{\pi}\,\Gamma(2n-d/2)}, \tag{2.21}$$

$$Q_{nd} := \frac{\Gamma(n+1/2)\Gamma(d/2+1-n)}{\Gamma(n)\Gamma(1/2+d/2-n)}, \qquad q_{nd} := \begin{cases} Q_{nd} & \text{if } P_{nd}\geqslant Q_{nd},\\ P_{nd}-(n-d/2) & \text{if } P_{nd}<Q_{nd}.\end{cases}$$

(*In the above definition of Q_{nd} one should intend $\Gamma(0) := \infty$, so that $Q_{nd} = 0$ for $n = d/2+1/2$.*) *For any fixed d, λ and for $n \to (d/2)^+$,*

$$\mathcal{K}^{BB}_{nd}(\lambda) = \sqrt{\frac{2}{3}}\frac{M_d}{\sqrt{n-d/2}}\left[1+O\left(n-\frac{d}{2}\right)\right], \tag{2.22}$$

with M_d as in the asymptotic expression (2.6) *for the upper bound K^+_{nd}* (*note that $\sqrt{2/3} > 0.816$*).

As clarified in the sequel, the Bessel lower bounds are less interesting for large n; therefore, it is not worth to determine their asymptotics for $n \to +\infty$.

"Fourier" lower bounds on K_{nd}. Another choice for the trial functions amounts to choose for f and g the function

$$f_{p\sigma d}(x) := e^{ipx_1} e^{-(\sigma/2)|x|^2} \tag{2.23}$$

where the "Fourier character" $x \to e^{ipx_1}$ is regularized at infinity by a Gaussian factor (we take this hint from [8], but we develop it in a different way).

As we will see, this choice is especially interesting for large n. The Sobolev norm of any order n of this function can be expressed using the modified Bessel function of the first kind I_ν, the Pochhammer symbol (1.8) and the semifactorial (1.9). Our results on the Fourier lower bounds are contained in the forthcoming proposition, to be proved in Section 6.

2.4. Proposition. (i) *Let $n > d/2$. For all $p, \sigma > 0$,*

$$K_{nd} \geqslant \mathcal{K}^F_{nd}(p,\sigma) := \frac{\| f_{2p,2\sigma,d} \|_n}{\| f_{p\sigma d} \|_n^2}; \tag{2.24}$$

hence

$$K_{nd} \geqslant K^F_{nd} := \sup_{p,\sigma>0} \mathcal{K}^F_{nd}(p,\sigma). \tag{2.25}$$

For all $p, \sigma > 0$,

$$\| f_{p\sigma d} \|_n^2 = \frac{2\pi^{d/2}}{\sigma^{d/2+1} p^{d/2-1}} \int_0^{+\infty} d\rho\, \rho^{d/2} (1+\rho^2)^n e^{-\frac{\rho^2+p^2}{\sigma}} I_{d/2-1}\left(\frac{2p}{\sigma}\rho\right); \tag{2.26}$$

in particular, for n integer,

$$\| f_{p\sigma d} \|_n^2 = \pi^{d/2} \sum_{\ell=0}^{n} \sum_{j=0}^{\ell} \sum_{g=0}^{j} \binom{n}{\ell} \binom{\ell}{j} \binom{2j}{2g} \frac{(2g-1)!!}{2^g} \times (d/2 - 1/2)_{\ell-j}\, p^{2j-2g} \sigma^{\ell+g-j-d/2}. \tag{2.27}$$

(ii) *Fix the attention on the "special" lower bound*

$$K^{FF}_{nd} := \mathcal{K}^F_{nd}\left(p = \frac{1}{2\sqrt{2}}, \sigma = \frac{3}{4n}\right); \tag{2.28}$$

then

$$K^{FF}_{nd} = \frac{(5/3)^{1/2}}{7^{1/4}} T_d \frac{(2/\sqrt{3})^n}{n^{d/4}} \left[1 + O\left(\frac{1}{n}\right)\right] \quad \textit{for } n \to +\infty, \tag{2.29}$$

with T_d as in the asymptotic formula (2.7) *for the upper bound* (*note that* $(5/3)^{1/2}/7^{1/4} > 0.793$).

Remark. The result (2.29) depends on the asymptotic analysis of a Laplace integral. The values for (p, σ) in Eq. (2.28) have been chosen because they simplify this analysis, and give rise to the term $(2/\sqrt{3})^n n^{-d/4}$ also appearing in the asymptotics (2.7) for the upper bound. One could discuss the asymptotics of $\mathcal{K}^F_{nd}(p, \sigma = c/n)$ for arbitrary choices of p and c in $(0, +\infty)$; however, this generalization complicates the implementation of the Laplace method and, in comparison with (2.29), yields no sensible increase of the dominant term.

Table of the upper and lower bounds on K_{nd} for $d = 1, 2, 3, 4$ and some test values of n. This is Table 1, which has been constructed using the upper bounds K^+_{nd} given by Proposition 2.2, and choosing conveniently one of the lower bounds K^B_{nd}, K^{BB}_{nd}, K^F_{nd}, K^{FF}_{nd} in Propositions 2.3 and 2.4; the chosen lower bound is generally indicated with K^-_{nd}, and its type is specified within the table. We have chosen the values of n within a very wide range, from $d/2 + 10^{-4}$ to $d/2 + 120$; for a better appreciation of the discrepancy between the upper and lower bounds, instead of K^-_{nd} we have reported the ratio K^-_{nd}/K^+_{nd}.

To compute K^+_{nd}, we must find the sup of the function $\mathcal{S}_{nd}$ in Proposition 2.2, which is given explicitly by item (ii) of the same proposition for $d/2 < n \leqslant d/2 + 1/2$, and must be computed directly from the function $\mathcal{S}_{nd}$ in the other cases; we have done this numerically in most cases, and sometimes analytically: some examples are given in Section 4. For large n, the numerical search for the maximum of $\mathcal{S}_{nd}$ has been done starting from $u = 1/2$, as suggested by item (iii) of Proposition 2.2.

Table 1
Bounds $K^-_{nd} \leqslant K_{nd} \leqslant K^+_{nd}$ for $d = 1, 2, 3, 4$ and $n - d/2 = 10^{-4}, 10^{-2}, 10^{-1}, 1/4, 1/2, 1, 3/2, 3, 6, 15, 30, 60, 120$ (the symbol $^-$ stands for one of the types BB, B, F, FF, indicated below)

$d = 1$

n	$\frac{1}{2}+10^{-4}$	$\frac{1}{2}+10^{-2}$	$\frac{1}{2}+10^{-1}$	3/4	1	3/2	2	7/2	13/2	31/2	61/2	121/2	241/2
K^+_{nd}	56.5	5.69	1.90	1.30	1.00	0.852	0.814	0.834	1.07	3.09	22.4	1410	6.63×10^6
$\frac{K^-_{nd}}{K^+_{nd}}$	0.816 (BB)	0.818 (BB)	0.824 (BB)	0.834 (B)	0.842 (B)	0.810 (B)	0.777 (B)	0.766 (F)	0.787 (F)	0.794 (F)	0.794 (F)	0.789 (FF)	0.791 (FF)

$d = 2$

n	$1+10^{-4}$	$1+10^{-2}$	$1+10^{-1}$	5/4	3/2	2	5/2	4	7	16	31	61	121
K^+_{nd}	39.9	3.99	1.27	0.798	0.565	0.428	0.378	0.332	0.361	0.831	5.08	269	1.07×10^6
$\frac{K^-_{nd}}{K^+_{nd}}$	0.816 (BB)	0.817 (BB)	0.826 (BB)	0.844 (B)	0.865 (B)	0.842 (B)	0.811 (B)	0.752 (F)	0.772 (F)	0.788 (F)	0.794 (F)	0.786 (FF)	0.789 (FF)

$d = 3$

n	$\frac{3}{2}+10^{-4}$	$\frac{3}{2}+10^{-2}$	$\frac{3}{2}+10^{-1}$	7/4	2	5/2	3	9/2	15/2	33/2	63/2	123/2	243/2
K^+_{nd}	22.6	2.25	0.692	0.421	0.283	0.198	0.164	0.128	0.120	0.223	1.15	51.2	1.71×10^5
$\frac{K^-_{nd}}{K^+_{nd}}$	0.816 (BB)	0.817 (BB)	0.826 (BB)	0.847 (B)	0.875 (B)	0.858 (B)	0.830 (B)	0.763 (B)	0.759 (F)	0.781 (F)	0.788 (F)	0.782 (FF)	0.787 (FF)

$d = 4$

n	$2+10^{-4}$	$2+10^{-2}$	$2+10^{-1}$	9/4	5/2	3	7/2	5	8	17	32	62	122
K^+_{nd}	11.3	1.12	0.340	0.202	0.130	0.0857	0.0678	0.0473	0.0389	0.0590	0.259	9.72	2.73×10^4
$\frac{K^-_{nd}}{K^+_{nd}}$	0.816 (BB)	0.817 (BB)	0.826 (BB)	0.849 (B)	0.880 (B)	0.867 (B)	0.842 (B)	0.779 (B)	0.750 (F)	0.775 (F)	0.785 (F)	0.778 (FF)	0.785 (FF)

Concerning K^-_{nd}, we have always chosen for it the most convenient between the lower bounds in Propositions 2.3 and 2.4 (i.e., the highest one or, in some limit cases, the most easily computable).

As for the Bessel lower bounds, for n sufficiently distant from $d/2$ we have computed numerically the function $\lambda \to \mathcal{K}^B_{nd}(\lambda)$ and its maximum K^B_{nd}. For n very close to $d/2$, this computation is very difficult because the integrals in $\mathcal{K}^B_{nd}$ converge too slowly; in this case, we have turned the attention to the function $\lambda \to \mathcal{K}^{BB}_{nd}(\lambda)$ and estimated numerically its maximum K^{BB}_{nd}.

Concerning the Fourier lower bounds, for n not very large we have determined K^F_{nd} maximizing numerically the function $(p,\sigma) \to \mathcal{K}^F_{nd}(p,\sigma)$; for very large n, we have turned the attention to the bound K^{FF}_{nd} which is easily computed numerically.

The Bessel lower bounds are generally higher than the Fourier ones for small n; the contrary happens for large n.

A more accurate $n \to (d/2)^+$ asymptotics for K^+_{nd}. This is introduced for the reasons explained in the next paragraph. For the sake of brevity, let us put

$$n_d := n - \frac{d}{2}; \tag{2.30}$$

in place of Eq. (2.6), we propose a higher order expansion

$$K^+_{nd} = \frac{M_d}{\sqrt{n_d}}\left[1 - N_d n_d + O\left(n_d^2\right)\right], \qquad N_d := \frac{\psi(d/2) + \gamma_E}{2}. \tag{2.31}$$

This is derived from the explicit expression (2.5) of K^+_{nd}, inserting therein the expansions

$$\Gamma(1+n_d) = \Gamma(1) + \Gamma'(1)n_d + O\left(n_d^2\right) = 1 - \gamma_E n_d + O\left(n_d^2\right), \tag{2.32}$$

$$\Gamma(n) = \Gamma\left(\frac{d}{2}\right) + \Gamma'\left(\frac{d}{2}\right)n_d + O\left(n_d^2\right) = \Gamma\left(\frac{d}{2}\right)\left[1 + \psi\left(\frac{d}{2}\right)n_d + O\left(n_d^2\right)\right]$$

(recall that $\Gamma'(w) = \Gamma(w)\psi(w)$, and use Eq. (1.12)).

"Elementary" upper bounds K^{++}_{nd}. The results (2.31), (2.7) on the asymptotics of K^+_{nd} in the limits $n \to (d/2)^+$, $n \to +\infty$ suggest a way to build new majorants

$$K^{++}_{nd} \geqslant K^+_{nd} \geqslant K_{nd}, \quad n \in (d/2, +\infty), \tag{2.33}$$

that are presented hereafter. Even though less precise than the $^+$ upper bounds, the $^{++}$ bounds have the advantage of being elementary functions of n; we will show that they are very close to the $^+$ bounds on the whole interval $(d/2, +\infty)$ up to $d = 7$, and fairly close to them up to $d = 10$. For any d, the elementary $^{++}$ bounds reproduce the asymptotics (2.31), (2.7) of the $^+$ bounds at the leading order.

In order to construct K^{++}_{nd}, we first define a function $n \in (d/2, +\infty) \mapsto z_{nd}$ through the equation

$$K^+_{nd} = \frac{(2/\sqrt{3})^n}{n^{d/4}}\left[\left(\frac{3d}{8}\right)^{d/4}\frac{M_d}{\sqrt{n_d}}\left(1 - \frac{n_d}{n}\right)^{3/2}(1 + V_d n_d) + T_d\left(\frac{n_d}{n}\right)^{3/2} + z_{nd}\frac{n_d}{n^2}\right],$$

$$V_d := \log\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2} + \frac{3}{d} - N_d, \quad n_d \text{ as in (2.30).} \tag{2.34}$$

This equation is easily solved for z_{nd}. From the explicit expression for z_{nd} and from the asymptotics (2.31) (2.7), one gets

$$z_{nd} = O(\sqrt{n_d}) \quad \text{for } n \to (d/2)^+, \qquad z_{nd} = O(1) \quad \text{for } n \to +\infty; \tag{2.35}$$

the coefficient V_d is defined as above just in order to give the first one of these relations.

On account of Eqs. (2.35), for fixed d the function $n \to z_{nd}$ is bounded on the interval $(d/2, +\infty)$; this ensures the finiteness of

$$Z_d := \sup_{n \in (d/2, +\infty)} z_{nd}. \tag{2.36}$$

Now, putting

$$K_{nd}^{++} := \frac{(2/\sqrt{3})^n}{n^{d/4}} \left[\left(\frac{3d}{8}\right)^{d/4} \frac{M_d}{\sqrt{n_d}} \left(1 - \frac{n_d}{n}\right)^{3/2} (1 + V_d n_d) + T_d \left(\frac{n_d}{n}\right)^{3/2} + Z_d \frac{n_d}{n^2} \right], \tag{2.37}$$

we see from (2.34) that $K_{nd}^{+} \leqslant K_{nd}^{++}$. From Eqs. (2.37) and (2.31), (2.7), we also infer

$$\frac{K_{nd}^{++}}{K_{nd}^{+}} = 1 + O(n_d) \quad \text{for } n \to (d/2)^+, \qquad \frac{K_{nd}^{++}}{K_{nd}^{+}} = 1 + O\left(\frac{1}{n}\right) \quad \text{for } n \to +\infty. \tag{2.38}$$

The forthcoming Table 2 reports, for $1 \leqslant d \leqslant 10$, the numerical values of the constants Z_d in Eq. (2.36) and of the quantities

$$\Theta_d := \sup_{n \in (d/2, +\infty)} \frac{K_{nd}^{++}}{K_{nd}^{+}}. \tag{2.39}$$

The table has been constructed in this way. First of all, for each d in the above range the function $n \mapsto z_{nd}$ defined by (2.34) has been plotted (expressing z_{nd} in terms of K_{nd}^{+} and evaluating the latter numerically); from the graph of $n \mapsto z_{nd}$, the sup Z_d has been evaluated. Secondly, for the same values of d the ratio K_{nd}^{++}/K_{nd}^{+} has been plotted as a function of n, and its sup Θ_d has been evaluated from the graph.

Table 2
Constants Z_d and Θ_d (for the elementary upper bounds K_{nd}^{++})

d	1	2	3	4	5	6	7	8	9	10
Z_d	0	0.00925	0.0458	0.0782	0.105	0.122	0.128	0.125	0.115	0.102
Θ_d	1.041	1.039	1.044	1.044	1.044	1.044	1.049	1.105	1.197	1.363

3. Some background

In this section we review some known facts, frequently cited in the rest of the paper to prove the statements of Section 2.

Some d-dimensional integrals. We frequently need to compute integrals of functions on $\mathbf{R}^d$ which depend only on the radius $|\ |$ (radially symmetric functions), or on the radius and one angle. In this case, we use the formulas

$$\int_{\mathbf{R}^d} dx\,\varphi\big(|x|\big) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^{+\infty} dr\, r^{d-1}\varphi(r); \tag{3.1}$$

$$\int_{\mathbf{R}^d} dx\,\chi\big(|x|, \eta \bullet x\big) = \frac{2\pi^{d/2-1/2}}{\Gamma(d/2-1/2)} \times \int_0^{+\infty} dr\, r^{d-1} \int_0^{\pi} d\theta\, \sin\theta^{d-2}\chi(r, r\cos\theta)$$
$$\big(d \geqslant 2;\ \eta \in \mathbf{R}^d,\ |\eta| = 1\big), \tag{3.2}$$

holding for all (sufficiently regular) complex valued functions φ on $(0, +\infty)$ and χ on $(0, +\infty) \times (0, \pi)$. (When writing the analogous formulas for integrals on the "wave vector" space $(\mathbf{R}^d, dk)$, the radius r will be renamed ρ.)

Radial Fourier transforms. Consider two (sufficiently regular) radially symmetric functions

$$f : \mathbf{R}^d \to \mathbf{C},\quad x \to f(x) = \varphi\big(|x|\big), \qquad F : \mathbf{R}^d \to \mathbf{C},\quad k \to F(k) = \Phi\big(|k|\big); \tag{3.3}$$

the Fourier and inverse Fourier transforms $\mathcal{F}f$, $\mathcal{F}^{-1}F$ are also radially symmetric, and given by [4]

$$(\mathcal{F}f)(k) = \frac{1}{|k|^{d/2-1}} \int_0^{+\infty} dr\, r^{d/2} J_{d/2-1}\big(|k|r\big)\varphi(r), \tag{3.4}$$

$$\big(\mathcal{F}^{-1}F\big)(x) = \frac{1}{|x|^{d/2-1}} \int_0^{+\infty} d\rho\, \rho^{d/2} J_{d/2-1}\big(|x|\rho\big)\Phi(\rho), \tag{3.5}$$

where J_ν are the Bessel functions of the first kind. As anticipated, the latter formula allows to infer Eq. (1.20) of the Introduction; in this case, Eq. (3.5) is applied with $\Phi(\rho) = 1/(1+\rho^2)^n$ and the corresponding integral over ρ is given in [14, p. 434].

Hypergeometric function. As anticipated, in this paper we use extensively the function $F(\alpha,\beta,\gamma;w)$; we are always interested in real values of the parameters α,β,γ and of the argument w. For future citation, we report here some properties of F. First of all, we cite: the symmetry property

$$F(\alpha,\beta,\gamma;w)=F(\beta,\alpha,\gamma;w); \tag{3.6}$$

the special values

$$F(\alpha,\beta,\gamma;0)=1, \tag{3.7}$$

$$F(\alpha,\beta,\gamma;1)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \quad \text{for } \gamma>\alpha+\beta,\ \gamma\neq 0,-1,-2,\ldots;$$

the particular cases

$$F(\alpha,\beta,\beta;w)=(1-w)^{-\alpha}, \tag{3.8}$$

$$F(\alpha,-m,\gamma;w)=\sum_{\ell=0}^{m}\frac{(\alpha)_\ell(-m)_\ell}{(\gamma)_\ell}\frac{w^\ell}{\ell!} \quad \text{for } m\in\mathbf{N}. \tag{3.9}$$

Secondly, we recall that

$$F(\alpha,\beta,\gamma;w)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 ds\, s^{\beta-1}(1-s)^{\gamma-\beta-1}(1-ws)^{-\alpha}>0 \tag{3.10}$$

$$\text{for } \gamma>\beta>0,\ w<1,$$

$$F(\alpha,\beta,\gamma;1-w)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^{+\infty} du\, u^{\beta-1}(1+u)^{\alpha-\gamma}(1+wu)^{-\alpha}>0 \tag{3.11}$$

$$\text{for } \gamma>\beta>0,\ w>0$$

((3.11) follows from (3.10) with a change of variable $s=u/(1+u)$).

Thirdly, we mention the differentiation formula

$$\frac{d}{dw}F(\alpha,\beta,\gamma;w)=\frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1;w); \tag{3.12}$$

this formula, combined with the positivity statement in (3.10), implies

$$\frac{d}{dw}F(\alpha,\beta,\gamma;w)>0 \quad \text{for } \alpha>0,\ \gamma>\beta>0,\ w<1. \tag{3.13}$$

Finally, we recall the Kummer transformations

$$F(\alpha,\beta,\gamma;w) = \frac{1}{(1-w)^\beta} F\left(\beta, \gamma-\alpha, \gamma; \frac{w}{w-1}\right), \tag{3.14}$$

$$F(\alpha,\beta,\gamma;w) = (1-w)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma; w); \tag{3.15}$$

the first one allows to pass from the form (1.15) to the form (1.16) for F_{nd}. The positivity of F_{nd} is granted by (3.10). The expression (1.17) of F_{nd} for $n-d/2-1/2$ integer follows from (1.16) and (3.9).

An integral involving Bessel functions. In Section 4 we will use the integral

$$I_{\mu\nu}(h) := \int_0^{+\infty} dr\, r^{\mu+\nu+1} J_\mu(hr) K^2_{\nu/2}(r) \quad (\mu > -1, \nu > 0, h > 0), \tag{3.16}$$

involving a Bessel function of the first kind J_μ and the square of a Macdonald function $K_{\nu/2}$. This is given by

$$I_{\mu\nu}(h) = \frac{\sqrt{\pi}\,\Gamma(\mu+\nu+1)\Gamma(\mu+\nu/2+1)}{2^{\mu+2}\Gamma(\mu+\nu/2+3/2)} h^\mu \times F\left(\mu+\nu+1, \mu+\nu/2+1, \mu+\nu/2+3/2; -h^2/4\right). \tag{3.17}$$

The above result is probably known, but it is not easy to trace it in the most common tables on integrals of Bessel functions; for this reason, the proof of (3.17) is given in Appendix A.

Laplace integrals. The classical theory of these integrals is widely employed in this paper, to discuss the $n \to +\infty$ asymptotics of our bounds on K_{nd}.

By a *standard* Laplace integral, we mean an integral depending on a parameter n, of the form

$$L(n) := \int_0^b dt\, \vartheta(t) e^{-n\varphi(t)}, \tag{3.18}$$

under the following assumptions:

$$0 < n_0 < n < +\infty, \quad 0 < b \leqslant +\infty; \tag{3.19}$$

$$\varphi \in C^1\big((0,b), \mathbf{R}\big), \qquad \varphi'(t) > 0 \quad \forall t \in (0,b), \qquad \lim_{t\to 0^+} \varphi(t) = 0,$$

$$\vartheta \in C\big((0,b), \mathbf{R}\big), \qquad \int_0^b dt\, \left|\vartheta(t)\right| e^{-n\varphi(t)} < +\infty \quad \text{for all } n \text{ as above.}$$

Here and in the sequel, $'$ is the derivative; we shall also put

$$\xi := \frac{\vartheta}{\varphi'} \in C\big((0,b),\mathbf{R}\big). \tag{3.20}$$

The Laplace method gives the $n \to +\infty$ asymptotics of $L(n)$, using the idea that the major contributions to this integral should come from the regions close to the minimum point of φ, i.e., to $t = 0$. (In certain cases, this asymptotics gives a fairly good approximation of $L(n)$ also for non-large values of n.) The asymptotic behavior of $L(n)$ is described by the following proposition (see, e.g., [11]; for uniformity of language, the proof is reviewed in Appendix B).

3.1. Proposition. *Suppose that conditions* (3.19) *hold, and that*

$$\xi(t) = \sum_{i=0}^{\ell-1} P_i \varphi(t)^{\alpha_i - 1} + O\big(\varphi(t)^{\alpha_\ell - 1}\big) \quad \textit{for } t \to 0^+, \tag{3.21}$$

where $\ell \in \{1,2,\ldots\}$, $P_1,\ldots,P_{\ell-1} \in \mathbf{R}$, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_\ell$. *Then*

$$L(n) = \sum_{i=0}^{\ell-1} P_i \frac{\Gamma(\alpha_i)}{n^{\alpha_i}} + O\left(\frac{1}{n^{\alpha_\ell}}\right) \quad \textit{for } n \to +\infty. \tag{3.22}$$

More on Laplace integrals. By a *general* Laplace integral, we mean an integral depending on a parameter n of the form

$$\Lambda(n) := \int_a^c ds\, \Theta(s) e^{-n\Phi(s)}, \tag{3.23}$$

where

$$0 < n_0 < n < +\infty, \quad -\infty \leqslant a < c \leqslant +\infty, \qquad \Phi \in C^1\big((a,c),\mathbf{R}\big), \tag{3.24}$$

$$\Theta \in C\big((a,c),\mathbf{R}\big), \qquad \int_a^c ds\, \big|\Theta(s)\big| e^{-n\Phi(s)} < +\infty \quad \text{for all } n \text{ as above.}$$

Under suitable conditions on Φ, $\Lambda(n)$ can be expressed in terms of one or more standard Laplace integrals. As a first example, suppose

$$a > -\infty, \qquad \Phi'(s) > 0 \quad \text{for all } s \in (a,c), \qquad \Phi(a) := \lim_{s\to a^+} \Phi(s) > -\infty \tag{3.25}$$

(the limit certainly exists by the monotonicity of Φ, but it could be $-\infty$); then

$$\Lambda(n) = e^{-n\Phi(a)} L(n), \tag{3.26}$$

$L(n)$ as in (3.18) with $b := c - a$, $\varphi(t) := \Phi(a+t) - \Phi(a)$, $\vartheta(t) := \Theta(a+t)$.

Similarly, if

$$c < +\infty, \qquad \Phi'(s) < 0 \quad \text{for all } s \in (a, c), \qquad \Phi(c) := \lim_{s \to c^-} \Phi(s) > -\infty, \tag{3.27}$$

we can write

$$\Lambda(n) = e^{-n\Phi(c)} L(n), \tag{3.28}$$

$L(n)$ as in (3.18) with $b := c - a$, $\varphi(t) := \Phi(c - t) - \Phi(c)$, $\vartheta(t) := \Theta(c - t)$.

As a final example, suppose

$$\Phi'(s) \lesseqgtr 0 \quad \text{for } s \lesseqgtr h \quad \big(h \in (a, c)\big); \tag{3.29}$$

then we can write

$$\Lambda(n) = e^{-n\Phi(h)} \big[L^-(n) + L^+(n)\big], \tag{3.30}$$

$$L^{\mp}(n) := \int_0^{b^{\mp}} dt\, \vartheta^{\mp}(t) e^{-n\varphi^{\mp}(t)}, \quad b^- := h - a,\ b^+ := c - h,$$

$$\varphi^{\mp}(t) := \Phi(h \mp t) - \Phi(h), \qquad \vartheta^{\mp}(t) := \Theta(h \mp t) \quad \text{for } t \in (0, b^{\mp}),$$

and $L^{\mp}(n)$ are standard Laplace integrals.

In all the previous examples, after re-expressing $\Lambda(n)$ in terms of standard Laplace integrals one should expand in powers of φ or $\varphi^{\mp}$ the functions $\xi := \theta/\varphi'$ or $\xi^{\mp} := \theta^{\mp}/\varphi^{\mp\prime}$. Assuming sufficient smoothness for Θ and Φ, the coefficients of these expansions can be expressed directly in terms of the derivatives of Θ and Φ at $s = a, c$ or h, respectively [11]. In the third example, where Φ has its minimum at an inner point h of (a, c), there is typically an alternation of equal and opposite coefficients in the expansions of ξ^- and ξ^+; this yields some cancellation effects in the expansion of $L^-(n) + L^+(n)$.

4. Proofs for the upper bounds on K_{nd}

Let us write $F * G$ for the convolution of two (sufficiently regular) complex functions F, G on $\mathbf{R}^d$, given by

$$(F * G)(k) := \int_{\mathbf{R}^d} dh\, F(k - h) G(h). \tag{4.1}$$

We have

$$\mathcal{F}(fg) = \frac{1}{(2\pi)^{d/2}} \mathcal{F}f * \mathcal{F}g \tag{4.2}$$

for all sufficiently regular functions f and g on $\mathbf{R}^d$ (and in particular, for f, g as in the forthcoming lemma).

4.1. Lemma. *For all $n > d/2$,*

$$K_{nd} \leqslant \sqrt{\sup_{k\in\mathbf{R}^d} \mathcal{S}_{nd}(k)}, \tag{4.3}$$

where

$$\mathcal{S}_{nd}(k) := \frac{(1+|k|^2)^n}{(2\pi)^d}(G_{nd} * G_{nd})(k) \tag{4.4}$$

and $G_{nd}(k) := 1/(1+|k|^2)^n$ for all $k \in \mathbf{R}^d$, as in Eq. (1.18).

Proof. Consider any two functions $f, g \in H^n(\mathbf{R}^d, \mathbf{C})$. Then

$$\|fg\|_n^2 = \int_{\mathbf{R}^d} dk\big(1+k^2\big)^n \big|\mathcal{F}(fg)(k)\big|^2 = \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} dk\big(1+k^2\big)^n \big|(\mathcal{F}f * \mathcal{F}g)(k)\big|^2. \tag{4.5}$$

On the other hand, by making explicit the convolution we find

$$\begin{aligned}(\mathcal{F}f * \mathcal{F}g)(k) &= \int_{\mathbf{R}^d} dh\, \mathcal{F}f(k-h)\mathcal{F}g(h)\\ &= \int_{\mathbf{R}^d} dh \frac{1}{\sqrt{1+|k-h|^2}^{\,n}\sqrt{1+|h|^2}^{\,n}}\\ &\quad \times \Big(\sqrt{1+|k-h|^2}^{\,n}\mathcal{F}f(k-h)\sqrt{1+|h|^2}^{\,n}\mathcal{F}g(h)\Big).\end{aligned} \tag{4.6}$$

Now, Hölder's inequality $|\int dh\, U(h)V(h)|^2 \leqslant (\int dh\, |U(h)|^2)(\int dh\, |V(h)|^2)$ gives

$$\big|(\mathcal{F}f * \mathcal{F}g)(k)\big|^2 \leqslant C_{nd}(k)P(k), \tag{4.7}$$

$$C_{nd}(k) := \int_{\mathbf{R}^d} \frac{dh}{(1+|k-h|^2)^n(1+|h|^2)^n} = (G_{nd} * G_{nd})(k),$$

$$P(k) := \int_{\mathbf{R}^d} dh\, \big(1+|k-h|^2\big)^n \big|\mathcal{F}f(k-h)\big|^2 \big(1+|h|^2\big)^n \big|\mathcal{F}g(h)\big|^2.$$

Inserting (4.7) into Eq. (4.5) we get

$$\begin{aligned}\|fg\|_n^2 &\leqslant \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} dk\big(1+|k|^2\big)^n C_{nd}(k)P(k)\\ &\leqslant \Big(\sup_{k\in\mathbf{R}^d}\frac{(1+|k|^2)^n}{(2\pi)^d}C_{nd}(k)\Big)\int_{\mathbf{R}^d} dk\, P(k) = \Big(\sup_{k\in\mathbf{R}^d}\mathcal{S}_{nd}(k)\Big)\int_{\mathbf{R}^d} dk\, P(k).\end{aligned} \tag{4.8}$$

But

$$\int_{\mathbf{R}^d} dk\, P(k)$$

$$= \Bigl(\int_{\mathbf{R}^d} dk\bigl(1+|k|^2\bigr)^n \bigl|\mathcal{F}f(k)\bigr|^2\Bigr)\Bigl(\int_{\mathbf{R}^d} dh\bigl(1+|h|^2\bigr)^n \bigl|\mathcal{F}g(h)\bigr|^2\Bigr) = \|f\|_n^2 \|g\|_n^2, \tag{4.9}$$

so we are led to the thesis. □

4.2. Lemma. *For $n > d/2$ and $k \in \mathbf{R}^d$,*

$$\mathcal{S}_{nd}(k) = \mathscr{S}_{nd}\left(\frac{|k|^2}{4}\right), \tag{4.10}$$

where $\mathscr{S}_{nd}$ is the function in Eq. (2.3) *of Proposition* 2.2.

Proof. Let us recall that G_{nd} is the Fourier transform of the function g_{nd}, already considered in Eqs. (1.19), (1.20). We have

$$\mathcal{S}_{nd}(k) = \frac{(1+|k|^2)^n}{(2\pi)^d}(\mathcal{F}g_{nd} * \mathcal{F}g_{nd})(k) = \frac{(1+|k|^2)^n}{(2\pi)^{d/2}}\bigl(\mathcal{F}g_{nd}^2\bigr)(k). \tag{4.11}$$

But g_{nd}^2 is a radially symmetric function, whose explicit expression in terms of the Macdonald function is given by (1.20). We insert this expression in formula (3.4) for the radially symmetric Fourier transform and obtain

$$\bigl(\mathcal{F}g_{nd}^2\bigr)(k) = \frac{1}{2^{2n-2}\Gamma^2(n)|k|^{d/2-1}} \int_0^{+\infty} dr\, r^{2n-d/2} J_{d/2-1}\bigl(|k|r\bigr) K_{n-d/2}^2(r); \tag{4.12}$$

the last integral is computed via Eq. (3.17), and the final result is

$$\bigl(\mathcal{F}g_{nd}^2\bigr)(k) = \frac{\Gamma(2n-d/2)}{2^{d/2}\Gamma(2n)} F_{nd}\left(\frac{|k|^2}{4}\right), \tag{4.13}$$

with F_{nd} as in (1.15) or (1.16) (to obtain this, one also uses Eq. (1.13) for Γ). Inserting (4.13) into (4.11) we get the thesis. □

Proof of Proposition 2.2, item (i). Lemmas 4.1 and 4.2 give immediately the bound (2.2) for K_{nd}, with $\mathscr{S}$ as in Eq. (2.3).

We now pass to the boundary values of the function $\mathscr{S}_{nd}$ for $u = 0$ and $u \to +\infty$. To determine $\mathscr{S}_{nd}(0)$, use either Eq. (1.15) or Eq. (1.16), together with Eq. (3.7); the result agrees with Eq. (2.4).

To determine $\lim_{u\to+\infty} \mathscr{S}_{nd}(u)$ we use Eq. (1.16), the limits

$$\left(\frac{1+4u}{1+u}\right)^n \to 4^n, \qquad \frac{u}{u+1} \to 1 \quad \text{for } u \to +\infty, \tag{4.14}$$

and Eq. (3.7); these relations imply

$$\mathcal{S}_{nd}(+\infty) = \frac{2^{2n-d}}{\pi^{d/2+1/2}} \frac{\Gamma(n+1/2)\Gamma(n-d/2)}{\Gamma(2n)} = \frac{\Gamma(n-d/2)}{2^{d-1}\pi^{d/2}\Gamma(n)}, \tag{4.15}$$

where the last equality follows from (1.13). This gives the expression in (2.4) after using (1.10) with $w = n - d/2$.

Of course, the continuity of $\mathcal{S}_{nd}$ on $[0, +\infty)$ and the finiteness of its $u \to +\infty$ limit ensure that $\mathcal{S}_{nd}$ is bounded on its domain. □

Proof of Proposition 2.2, item (ii).

Step 1. The function $\mathcal{S}_{nd}$ is increasing if $d/2 < n \leqslant d/2 + 1/2$.

To prove this, we use Eqs. (2.3), (1.16) and the following remarks:

(a) the functions $u \in [0, +\infty) \to (1+4u)/(1+u) \in [1, 4)$ and $u \in [0, +\infty) \to u/(1+u) \in [0, 1)$ are increasing;

(b) the function $w \in (-\infty, 1) \to F(n, d/2 + 1/2 - n, n + 1/2; w)$ is increasing for $d/2 < n < d/2 + 1/2$, due to (3.13); in the limit case $n = d/2 + 1/2$, this function equals 1 everywhere (by (3.9), with $m = 0$).

Of course, the fact that $\mathcal{S}_{nd}$ is increasing implies $\sup_{[0,+\infty)} \mathcal{S}_{nd} = \mathcal{S}_{nd}(+\infty)$, and this fact, with Eq. (2.4), yields Eq. (2.5).

Step 2. The asymptotics (2.6) *of K^+_{nd} for $n \to (d/2)^+$.*

This is evident from (2.5). □

Now we must prove item (iii) of the same proposition, concerning the $n \to +\infty$ behavior of K^+_{nd}; a fairly long series of lemmas will be established to this purpose. A main point in this argument is the integral representation, coming from Eqs. (2.3), (1.15) and (3.10),

$$\mathcal{S}_{nd}(u) = \frac{\Gamma(2n-d/2)\Gamma(n+1/2)}{2^d \pi^{d/2+1/2}\Gamma(n)\Gamma(2n)} \mathcal{C}_{nd}(u), \tag{4.16}$$

$$\mathcal{C}_{nd}(u) := (1+4u)^n \int_0^1 ds \frac{s^{n-1}}{\sqrt{1-s}(1+us)^{2n-d/2}}.$$

For future convenience, we write

$$\mathcal{C}_{nd}(u) = \mathcal{A}_{nd}(u) + \mathcal{B}_{nd}(u), \tag{4.17}$$

$$\mathcal{A}_{nd}(u) := (1+4u)^n \int_{1/4}^1 ds \frac{s^{n-1}}{\sqrt{1-s}(1+us)^{2n-d/2}}, \tag{4.18}$$

$$\mathcal{B}_{nd}(u) := (1+4u)^n \int_0^{1/4} ds \frac{s^{n-1}}{\sqrt{1-s}(1+us)^{2n-d/2}}.$$

4.3. Lemma. *Define*

$$B_{nd} := \sup_{u\in[0,+\infty)} \mathcal{B}_{nd}(u); \tag{4.19}$$

then, for fixed d and $n \to +\infty$,

$$B_{nd} = O\left(\frac{1}{\sqrt{n}}\left(\frac{9}{8}\right)^n\right). \tag{4.20}$$

Proof. We will estimate $\mathcal{B}_{nd}(u)$ with different methods for $u \in [0, 2]$ and $u \in (2, +\infty)$, respectively.

Let $0 \leqslant u \leqslant 2$; we re-express the definition of $\mathcal{B}_{nd}(u)$ as

$$\mathcal{B}_{nd}(u) = (1+4u)^n \int_0^{1/4} ds \frac{s^{d/4-1}}{\sqrt{1-s}}\left(\frac{s}{(1+us)^2}\right)^{n-d/4}. \tag{4.21}$$

The function $s \to s/(1+us)^2$ is increasing for $0 \leqslant s < 1/u$; but $1/u > 1/4$, so the maximum of this function for $0 \leqslant s \leqslant 1/4$ is attained at $s = 1/4$. From here one gets

$$\begin{aligned}\mathcal{B}_{nd}(u) &\leqslant (1+4u)^n \int_0^{1/4} ds \frac{s^{d/4-1}}{\sqrt{1-s}}\left(\frac{1/4}{(1+u/4)^2}\right)^{n-d/4}\\ &= (1+4u)^{d/4}\left(\frac{1+4u}{(2+u/2)^2}\right)^{n-d/4} \int_0^{1/4} ds \frac{s^{d/4-1}}{\sqrt{1-s}}.\end{aligned} \tag{4.22}$$

On the other hand, the function $u \to (1+4u)/(2+u/2)^2$ is increasing for $0 \leqslant u \leqslant 2$, and equals 1 when $u = 2$; from here and from $(1+4u)^{d/4} \leqslant 9^{d/4}$ one easily obtains

$$\sup_{u\in[0,2]} \mathcal{B}_{nd}(u) \leqslant C_d \quad \text{for all } n > d/2, \qquad C_d := 9^{d/4} \int_0^{1/4} ds \frac{s^{d/4-1}}{\sqrt{1-s}}. \tag{4.23}$$

We pass to bind $\mathcal{B}_{nd}$ for $u \in (2, +\infty)$. Returning to Eq. (4.18), we write

$$\begin{aligned}\mathcal{B}_{nd}(u) &\leqslant \frac{2}{\sqrt{3}}(1+4u)^n \int_0^{1/4} ds \frac{s^{n-1}}{(1+us)^{2n-d/2}}\\ &= \frac{2}{\sqrt{3}} \frac{(1+4u)^n}{u^n} \int_0^{u/4} dq \frac{q^{n-1}}{(1+q)^{2n-d/2}},\end{aligned} \tag{4.24}$$

where the first inequality follows from $1/\sqrt{1-s} \leqslant 2/\sqrt{3}$ for $0 \leqslant s \leqslant 1/4$, and the subsequent equality is obtained putting $s = q/u$. On the other hand, $(1+4u)/u < 9/2$ for $u > 2$ and $\int_0^{u/4} < \int_0^{+\infty}$ on positive functions, so

$$\sup_{u\in(2,+\infty)} \mathcal{B}_{nd}(u) \leqslant \frac{2}{\sqrt{3}}\left(\frac{9}{2}\right)^n \int_0^{+\infty} dq \frac{q^{n-1}}{(1+q)^{2n-d/2}}$$
$$= \frac{2}{\sqrt{3}}\left(\frac{9}{2}\right)^n \frac{\Gamma(n-d/2)\Gamma(n)}{\Gamma(2n-d/2)} \tag{4.25}$$

(recall Eq. (1.14)). We now apply the duplication formula (1.13) with $w = n - d/4$; this gives

$$\sup_{u\in(2,+\infty)} \mathcal{B}_{nd}(u) \leqslant \sqrt{\frac{\pi}{3}}\, 2^{2+d/2}\left(\frac{9}{8}\right)^n \frac{\Gamma(n-d/2)}{\Gamma(n-d/4)} \frac{\Gamma(n)}{\Gamma(n-d/4+1/2)}. \tag{4.26}$$

Putting together Eqs. (4.23), (4.26) we get

$$B_{nd} \leqslant \max\left(C_d, \sqrt{\frac{\pi}{3}}\, 2^{2+d/2}\left(\frac{9}{8}\right)^n \frac{\Gamma(n-d/2)}{\Gamma(n-d/4)} \frac{\Gamma(n)}{\Gamma(n-d/4+1/2)}\right) \tag{4.27}$$

for all $n > d/2$. As a final step, we recall that [11, p. 119]

$$\frac{\Gamma(w+a)}{\Gamma(w+b)} = w^{a-b}\left[1 + O\left(\frac{1}{w}\right)\right] \qquad \text{for fixed } a, b \in \mathbf{R} \text{ and } w \to +\infty; \tag{4.28}$$

this implies, for $n \to +\infty$,

$$\frac{\Gamma(n-d/2)}{\Gamma(n-d/4)} \frac{\Gamma(n)}{\Gamma(n-d/4+1/2)} = n^{-1/2}\left[1 + O\left(\frac{1}{n}\right)\right] \tag{4.29}$$

and Eqs. (4.27), (4.29) yield the thesis (4.20). □

4.4. Lemma. *For all $n > d/2$ one has*

$$\sup_{u\in[0,+\infty)} \mathcal{A}_{nd}(u) \leqslant A_{nd}, \tag{4.30}$$

$$A_{nd} := 2^{2n-d/2} \frac{(1-d/2n)^{n-d/2}}{(1-d/4n)^{2n-d/2}} \int_{1/4}^{1} ds \frac{1}{s\sqrt{1-s}(4-s)^{n-d/2}}.$$

For fixed d and $n \to +\infty$,

$$A_{nd} = \sqrt{\pi} \frac{3^{d/2+1/2}}{2^{d/2}\sqrt{n}} \left(\frac{4}{3}\right)^n \left[1 + O\left(\frac{1}{n}\right)\right]. \tag{4.31}$$

Proof. *Step 1. The bound* (4.30).

The definition of $\mathcal{A}_{nd}$ implies

$$\sup_{u\in[0,+\infty)} \mathcal{A}_{nd}(u) \leqslant \int_{1/4}^{1} ds \frac{s^{n-1}}{\sqrt{1-s}} H_{nd}(s), \tag{4.32}$$

$$H_{nd}(s) := \sup_{u\in[0,+\infty)} \frac{(1+4u)^n}{(1+su)^{2n-d/2}}.$$

For $s \in (1/4, 1)$, the function $u \in [0, +\infty) \to (1 + 4u)^n/(1 + su)^{2n-d/2}$ attains its maximum when u equals

$$u_{nd}(s) := \frac{8n + (d - 4n)s}{4(2n - d)s}. \tag{4.33}$$

Thus

$$H_{nd}(s) = \frac{(1+4u)^n}{(1+su)^{2n-d/2}}\bigg|_{u=u_{nd}(s)} = \frac{(1-\frac{d}{2n})^{n-d/2}}{(1-\frac{d}{4n})^{2n-d/2}} \frac{2^{2n-d/2}}{s^n(4-s)^{n-d/2}}, \tag{4.34}$$

and inserting this equation into (4.32) one gets the thesis (4.30).

Step 2. The asymptotics (4.31).

We re-express Eq. (4.30) for A_{nd} as

$$A_{nd} = 2^{2n-d/2} U_{nd} \int_{1/4}^{1} ds\, \Theta(s) e^{-(n-d/2)\Phi(s)}, \tag{4.35}$$

$$U_{nd} := \frac{(1-\frac{d}{2n})^{n-d/2}}{(1-\frac{d}{4n})^{2n-d/2}}, \qquad \Theta(s) := \frac{1}{s\sqrt{1-s}}, \qquad \Phi(s) := \log(4-s).$$

In this representation we recognize a Laplace integral in the parameter $n - d/2$; we have $\Phi'(s) < 0$ for all $s \in (1/4, 1)$, $\Phi(1) = \log 3$, and the scheme of Eqs. (3.23)–(3.28) suggests to rephrase Eq. (4.35) as

$$A_{nd} = 2^{2n-d/2} U_{nd} e^{-(n-d/2)\Phi(1)} L\left(n - \frac{d}{2}\right) = \left(\frac{3}{2}\right)^{d/2} \left(\frac{4}{3}\right)^n U_{nd} L\left(n - \frac{d}{2}\right),$$

$$L(m) := \int_0^{3/4} dt\, \vartheta(t) e^{-m\varphi(t)}, \tag{4.36}$$

$$\vartheta(t) := \Theta(1-t) = \frac{1}{\sqrt{t}(1-t)}, \qquad \varphi(t) := \Phi(1-t) - \Phi(1) = \log\left(1 + \frac{t}{3}\right).$$

The last integral has the standard Laplace form (3.18), and the framework of Proposition 3.1 prescribes to analyze it introducing the function

$$\xi(t) := \frac{\vartheta(t)}{\varphi'(t)} = \frac{3+t}{(1-t)\sqrt{t}}. \tag{4.37}$$

For $t \to 0^+$, one has

$$\varphi(t) = \frac{t}{3} + O\big(t^2\big), \qquad t = 3\varphi(t) + O\big(\varphi(t)^2\big),$$

$$\xi(t) = \frac{3}{\sqrt{t}} + O(\sqrt{t}) = \frac{\sqrt{3}}{\sqrt{\varphi(t)}} + O\Big(\sqrt{\varphi(t)}\Big). \tag{4.38}$$

Now, application of Proposition 3.1 to the last relation (4.38) gives

$$L(m) = \frac{\sqrt{3\pi}}{\sqrt{m}} + O\left(\frac{1}{m^{3/2}}\right) \quad \text{for } m \to +\infty. \tag{4.39}$$

On the other hand (taking the logarithm and expanding),

$$U_{nd} = 1 + O\left(\frac{1}{n}\right) \quad \text{for } n \to +\infty; \tag{4.40}$$

inserting Eqs. (4.39), (4.40) into (4.36), one easily derives the thesis (4.31) . □

4.5. Lemma. *For fixed d and $n \to +\infty$,*

$$\mathcal{C}_{nd}\left(\frac{1}{2}\right) = \sqrt{\pi}\,\frac{3^{d/2+1/2}}{2^{d/2}\sqrt{n}}\left(\frac{4}{3}\right)^n\left[1 + O\left(\frac{1}{n}\right)\right] \tag{4.41}$$

(*note that the right-hand sides of this equation and* (4.31) *coincide*).

Proof. The definition (4.16) gives

$$\mathcal{C}_{nd}\left(\frac{1}{2}\right) = 3^n \int_0^1 ds\, \frac{s^{n-1}}{\sqrt{1-s}(1+s/2)^{2n-d/2}} = 3^n \int_0^1 ds\, \Theta_d(s) e^{-(n-d/4)\Phi(s)}, \tag{4.42}$$

$$\Theta_d(s) := \frac{s^{d/4-1}}{\sqrt{1-s}}, \qquad \Phi(s) := 2\log(1+s/2) - \log s.$$

We have again a Laplace integral, with parameter $n - d/4$; one finds $\Phi'(s) < 0$ for all $s \in (0,1)$, $\Phi(1) = 2\log(3/2)$ and referring again to the scheme (3.23)–(3.28) we re-express (4.42) as

$$\mathcal{C}_{nd}\left(\frac{1}{2}\right) = 3^n e^{-(n-d/4)\Phi(1)} L_d\left(n - \frac{d}{4}\right) = \left(\frac{3}{2}\right)^{d/2}\left(\frac{4}{3}\right)^n L_d\left(n - \frac{d}{4}\right), \tag{4.43}$$

$$L_d(m) := \int_0^1 dt\, \vartheta_d(t) e^{-m\varphi(t)},$$

$$\vartheta_d(t) := \Theta_d(1-t) = \frac{(1-t)^{d/4-1}}{\sqrt{t}}, \quad \varphi(t) := \Phi(1-t) - \Phi(1) = 2\log\left(1-\frac{t}{3}\right) - \log(1-t).$$

Following again the scheme of Proposition 3.1, we introduce the function

$$\xi_d(t) := \frac{\vartheta_d(t)}{\varphi'(t)} = \frac{(3-t)(1-t)^{d/4}}{(1+t)\sqrt{t}}. \tag{4.44}$$

Let us keep d fixed. It turns out that Eq. (4.38) are again satisfied with the present choice of φ and with $\xi = \xi_d$. Therefore, Proposition 3.1 gives the asymptotics, analogous to (4.39),

$$L_d(m) = \frac{\sqrt{3\pi}}{\sqrt{m}} + O\left(\frac{1}{m^{3/2}}\right) \quad \text{for } m \to +\infty; \tag{4.45}$$

inserting Eq. (4.45) into (4.43) we obtain the thesis (4.41). □

4.6. Lemma. *For fixed d and $n \to +\infty$, one has*

$$\sup_{u\in[0,+\infty)} \mathcal{C}_{nd}(u) = \sqrt{\pi}\,\frac{3^{d/2+1/2}}{2^{d/2}\sqrt{n}}\left(\frac{4}{3}\right)^n\left[1 + O\left(\frac{1}{n}\right)\right] \tag{4.46}$$

(*again, the right-hand side is as in Eq.* (4.31)).

Proof. We have

$$\mathcal{C}_{nd}\left(\frac{1}{2}\right) \leqslant \sup_{u\in[0,+\infty)} \mathcal{C}_{nd}(u) \leqslant A_{nd} + B_{nd} \tag{4.47}$$

(the upper bound follows from Eqs. (4.17), (4.19) and (4.30)). Both the above bounds on $\sup \mathcal{C}_{nd}$ have asymptotics as in the right-hand side of Eq. (4.31). For the lower bound, this is granted by Lemma 4.5. For the upper bound, this follows from Lemmas 4.4 for A_{nd} and 4.3 for B_{nd}: the latter is negligible with respect to the former, since

$$B_{nd} = A_{nd}\, O\left(\frac{(9/8)^n}{(4/3)^n}\right) = A_{nd}\, O\left(\left(\frac{27}{32}\right)^n\right) = A_{nd}\, O\left(\frac{1}{n^\sigma}\right) \quad \text{for any real } \sigma. \quad \square \tag{4.48}$$

Proof of Proposition 2.2, item (iii). Equation (4.16) and the definition of K_{nd}^+ in Eq. (2.2) give

$$\sqrt{\mathcal{S}_{nd}\left(\frac{1}{2}\right)} = \frac{1}{2^{d/2}\pi^{d/4+1/4}}\sqrt{\frac{\Gamma(2n-d/2)}{\Gamma(2n)}\frac{\Gamma(n+1/2)}{\Gamma(n)}}\sqrt{\mathcal{C}_{nd}\left(\frac{1}{2}\right)}, \tag{4.49}$$

$$K_{nd}^+ = \frac{1}{2^{d/2}\pi^{d/4+1/4}}\sqrt{\frac{\Gamma(2n-d/2)}{\Gamma(2n)}\frac{\Gamma(n+1/2)}{\Gamma(n)}}\sqrt{\sup_{u\in(0,+\infty)} \mathcal{C}_{nd}(u)}. \tag{4.50}$$

We know that $\mathcal{C}_{nd}(1/2)$ and $\sup_{[0,+\infty)} \mathcal{C}_{nd}$ have the same asymptotics up to $O(1/n)$, given by Lemmas 4.5 and 4.6; furthermore, Eq. (4.28) implies

$$\frac{\Gamma(2n-d/2)}{\Gamma(2n)} = \frac{1}{(2n)^{d/2}}\left[1+O\left(\frac{1}{n}\right)\right], \qquad \frac{\Gamma(n+1/2)}{\Gamma(n)} = \sqrt{n}\left[1+O\left(\frac{1}{n}\right)\right], \tag{4.51}$$

and inserting these results into Eqs. (4.49), (4.50) we obtain the thesis (2.7). □

Computing the upper bounds K^+_{nd}. (a) For $d/2 < n \leqslant d/2 + 1/2$, we have for K^+_{nd} the explicit expression (2.5); this was employed to compute the numerical values reported in Table 1 for these cases.

(b) In all the other cases, to compute K^+_{nd} one has to maximize the function $\mathcal{S}_{nd}$ given by Eq. (2.3), containing the hypergeometric function F_{nd} of Eqs. (1.15)–(1.16). For $n - d/2 - 1/2$ integer, $\mathcal{S}_{nd}$ has the elementary expression (1.17).

(c) Apart from simple exceptions, the maximization of $\mathcal{S}_{nd}$ must be performed numerically. In all the cases analyzed with $n > d/2 + 1/2$, we have found numerical evidence (and sometimes an analytical proof) that $\mathcal{S}_{nd}$ has a unique maximum point $u = u_{nd} > 1/2$ in the interval $(0, +\infty)$, so that

$$K^+_{nd} = \sqrt{\mathcal{S}_{nd}(u_{nd})}. \tag{4.52}$$

(d) Let us consider, for example, the case $d = 2$. For $n = 2$, Eqs. (2.3), (1.15) give

$$\mathcal{S}_{22}(u) = \frac{(1+4u)^2}{12\pi} F\left(3, 2, \frac{5}{2}; -u\right); \tag{4.53}$$

one finds numerically that $\mathcal{S}_{22}$ attains its maximum at $u_{22} \simeq 6.84$. For $n = 5/2$, using (2.3), (1.17), one finds

$$\mathcal{S}_{5/2,2}(u) = \frac{(1+4u)^{5/2}}{96\pi} \frac{6+u}{(1+u)^{7/2}}; \tag{4.54}$$

the point of absolute maximum of this function is $u_{5/2,2} = 16/5 = 3.2$, determined analytically by solving an algebraic equation of second degree. For larger, half-integer values of n, $\mathcal{S}_{n2}$ is again elementary, but the analytic determination of its maximum point involves algebraic equations of order increasing with n; thus, a numerical attack is necessary.

Table 1 also considers, for $d = 2$, the values $n = 4, 7, 16, 31, 61, 121$. In all these cases, one finds numerically a unique maximum point $u_{n2} \simeq 1.46, 0.915, 0.654, 0.576, 0.538, 0.519$. Note the approach of this point to the limit value $u = 1/2$ for large n, as expected from Eq. (2.7); due to this behavior, numerical maximization is simple even for very large values of n.

5. Proofs for the Bessel lower bounds on K_{nd}

Proof of Proposition 2.3, item (i). Eqs. (2.11)–(2.12) are obvious; we must justify the expressions (2.13)–(2.14) of $\|g_{\lambda nd}\|_n$, and (2.15)–(2.16) for $\|g^2_{\lambda nd}\|_n$.

Step 1. Computation of $\|g_{\lambda nd}\|_n$.

We have

$$\|g_{\lambda nd}\|_n^2 = \int_{\mathbf{R}^d} dk\big(1+|k|^2\big)^n |\mathcal{F}g_{\lambda nd}|^2 = \frac{1}{\lambda^{2d}} \int_{\mathbf{R}^d} dk \frac{(1+|k|^2)^n}{(1+|k|^2/\lambda^2)^{2n}}$$

$$= \frac{2\pi^{d/2}}{\Gamma(d/2)\lambda^{2d}} \int_0^{+\infty} d\rho\, \rho^{d-1} \frac{(1+\rho^2)^n}{(1+\rho^2/\lambda^2)^{2n}}$$

$$= \frac{\pi^{d/2}}{\Gamma(d/2)\lambda^{d}} \int_0^{+\infty} du\, u^{d/2-1} \frac{(1+\lambda^2 u)^n}{(1+u)^{2n}}. \tag{5.1}$$

In the last two passages we have used Eq. (3.1) for the integral of a radially symmetric function, depending only on $\rho := |k|$, and then we have changed the variable to $u = \rho^2/\lambda^2$.

For n arbitrary, the last integral in u is computed using the identity (3.11); this gives the thesis (2.13) (after using Eq. (1.10) with $w = n - d/2$).

For n integer, in the integral over u we expand $(1+\lambda^2 u)^n$ with the binomial formula, and integrate term by term; this gives Eq. (2.14) after treating each term by (1.14).

Step 2. Computation of $\|g^2_{\lambda nd}\|_n$. According to the definition (2.9), the function $g_{\lambda nd}$ is obtained from the g_{nd} of Eq. (1.20) rescaling by λ. From here, and from Eq. (4.13) for $\mathcal{F}g^2_{nd}$ we infer

$$\big(\mathcal{F}g^2_{\lambda nd}\big)(k) = \frac{1}{2^{d/2}\lambda^d} \frac{\Gamma(2n-d/2)}{\Gamma(2n)} F_{nd}\left(\frac{|k|^2}{4\lambda^2}\right), \tag{5.2}$$

with F_{nd} as in Eqs. (1.15) or (1.16); thus,

$$\|g^2_{\lambda nd}\|_n^2 = \int_{\mathbf{R}^d} dk\,\big(1+|k|^2\big)^n \big|\mathcal{F}g^2_{\lambda nd}(k)\big|^2$$

$$= \frac{\pi^{d/2}\Gamma^2(2n-d/2)}{2^{d-1}\Gamma(d/2)\Gamma^2(2n)\lambda^{2d}} \int_0^{+\infty} d\rho\, \rho^{d-1}\big(1+\rho^2\big)^n F^2_{nd}\left(\frac{\rho^2}{4\lambda^2}\right). \tag{5.3}$$

Now, introducing the scaled variable $u := \rho^2/(4\lambda^2)$ we readily obtain the expression (2.15) for $\|g^2_{\lambda nd}\|_n$.

Finally, let us consider the case $n - d/2 - 1/2$ integer and show that Eq. (2.15) becomes Eq. (2.16). In fact, in this case the function F_{nd} has the elementary expression (1.17); when this is substituted into the integral over u of Eq. (2.15), we get

$$\int_0^{+\infty} du\, u^{d/2-1}\big(1+4\lambda^2 u\big)^n F^2_{nd}(u)$$

$$= \sum_{\ell,m=0}^{n-d/2-1/2} \frac{(n)_\ell (d/2+1/2-n)_\ell}{(n+1/2)_\ell \ell!} \frac{(n)_m (d/2+1/2-n)_m}{(n+1/2)_m m!} \times \int_0^{+\infty} du\, u^{d/2+\ell+m-1} \frac{(1+4\lambda^2 u)^n}{(1+u)^{2n+\ell+m}}; \tag{5.4}$$

each of the above integrals can be computed via Eq. (3.11), and the conclusion is the thesis (2.16). □

To prove the second item in Proposition 2.3 we need an elementary bound for the hypergeometric-like function F_{nd}, to be substituted in Eq. (2.15) for $\|g^2_{\lambda n d}\|_n$; this will require some lemmas.

5.1. Lemma. *Assume*

$$f \in C\big([0,1],\mathbf{R}\big) \cap C^2\big([0,1),\mathbf{R}\big), \qquad R \in C\big([0,1],\mathbf{R}\big) \cap C^1\big([0,1),\mathbf{R}\big), \quad \epsilon > 0; \tag{5.5}$$

$$f'(w) = (1-w)^{\epsilon-1} R(w), \qquad R'(w) > 0 \quad \textit{for } w \in [0,1), \tag{5.6}$$

and consider the C^2 function

$$w \in [0,1) \mapsto \frac{f(1)-f(w)}{(1-w)^\epsilon}. \tag{5.7}$$

Then:

$$\frac{f(1)-f(w)}{(1-w)^\epsilon} \to \frac{R(1)}{\epsilon} \quad \textit{for } w \to 1^-, \tag{5.8}$$

$$\frac{d}{dw} \frac{f(1)-f(w)}{(1-w)^\epsilon} > 0 \quad \textit{for } w \in [0,1). \tag{5.9}$$

The previous facts imply

$$f(1)-f(0) < \frac{f(1)-f(w)}{(1-w)^\epsilon} < \frac{R(1)}{\epsilon} \quad \textit{for } w \in (0,1). \tag{5.10}$$

Proof. By the generalized Lagrange theorem,

$$\frac{F(1)-F(w)}{G(1)-G(w)} = \frac{F'(t_w)}{G'(t_w)} \quad \text{for some } t_w \in (w,1), \tag{5.11}$$

if $F, G \in C([0,1],\mathbf{R}) \cap C^1((0,1),\mathbf{R})$, G' never vanishes and $w \in [0,1)$. We apply this statement with

$$f := F, \qquad G(w) := -(1-w)^\epsilon, \tag{5.12}$$

taking into account Eq. (5.6); this gives

$$\frac{f(1)-f(w)}{(1-w)^{\epsilon}}=\frac{R(t_w)}{\epsilon} \quad \text{for } w\in[0,1), \text{ with } t_w\in(w,1), \tag{5.13}$$

and in the limit $w\to 1^-$ we obtain Eq. (5.8).

In order to prove (5.9), we observe that

$$\frac{d}{dw}\frac{f(1)-f(w)}{(1-w)^{\epsilon}}=\frac{\epsilon}{(1-w)^{\epsilon+1}}\big(f(1)-f(w)\big)-\frac{R(w)}{1-w} \quad \text{for } w\in[0,1). \tag{5.14}$$

On the other hand (intending $\int_w^1$ as an improper Riemann integral)

$$\begin{aligned}\epsilon\big(f(1)-f(w)\big)&=\epsilon\int_w^1 dt\,f'(t)=\epsilon\int_w^1 dt\,(1-t)^{\epsilon-1}R(t)\\&=(1-w)^{\epsilon}R(w)+\int_w^1 dt\,(1-t)^{\epsilon}R'(t),\end{aligned} \tag{5.15}$$

the last equality following from integration by parts. Inserting (5.15) into (5.14) we obtain

$$\frac{d}{dw}\frac{f(1)-f(w)}{(1-w)^{\epsilon}}=\frac{1}{(1-w)^{\epsilon+1}}\int_w^1 dt\,(1-t)^{\epsilon}R'(t), \tag{5.16}$$

and the positivity of R' gives the thesis (5.9).

Finally the function $w\in(0,1)\mapsto(f(1)-f(w))/(1-w)^{\epsilon}$ is increasing, so it is strictly bounded from below and above by its limits for $w\to 0^+$ and $w\to 1^-$; this yields Eq. (5.10). □

5.2. Lemma. *Let*

$$0<a,b<+\infty; \qquad a+b<c<a+b+1; \quad w\in(0,1). \tag{5.17}$$

Then

$$0<P(a,b,c)-1<\frac{P(a,b,c)-F(a,b,c;w)}{(1-w)^{c-a-b}}<Q(a,b,c) \tag{5.18}$$

where

$$P(a,b,c):=F(a,b,c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{5.19}$$

$$Q(a,b,c):=\frac{\Gamma(c)\Gamma(a+b+1-c)}{(c-a-b)\Gamma(a)\Gamma(b)}.$$

Proof. We apply the previous lemma with

$$f := F(a,b,c;\cdot), \qquad \epsilon := c-a-b. \tag{5.20}$$

In this case, the differentiation formula (3.12) and the subsequent application of the Kummer transformation (3.15) give

$$f'(w) = (1-w)^{\epsilon-1}R(w), \qquad R(w) := \frac{ab}{c}F(c-a,c-b,c+1;w). \tag{5.21}$$

On the other hand, the hypergeometric function $w \mapsto F(c-a,c-b,c+1;w)$ has positive derivative, due to (3.13) and to the assumptions (5.17) for a,b,c; the same assumptions ensure this function to be continuous also at $w=1$, where its value is determined by Eq. (3.7). Thus all conditions of the previous lemma are fulfilled by f, ϵ, R, and Eq. (5.10) gives

$$\begin{aligned} F(a,b,c;1) - F(a,b,c;0) &< \frac{F(a,b,c;1) - F(a,b,c;w)}{(1-w)^{c-a-b}} \\ &< \frac{ab}{c(c-a-b)}F(c-a,c-b,c+1;1). \end{aligned} \tag{5.22}$$

But

$$F(a,b,c;1) = P(a,b,c), \qquad F(a,b,c;1) - F(a,b,c;0) = P(a,b,c) - 1 > 0; \tag{5.23}$$

the last inequality holds because $F(a,b,c;\cdot)$ is increasing (see again Eq. (3.13)). Finally, the equality

$$\frac{ab}{c(c-a-b)}F(c-a,c-b,c+1;1) = Q(a,b,c) \tag{5.24}$$

is easily inferred from Eq. (3.7), using the identity (1.10) with $w=a$ and $w=c$. Eqs. (5.22)–(5.24) yield the thesis. □

Remark. The idea of employing (3.15) in the above proof has been suggested by [12], where the usefulness of this transformation has been pointed out in relation to similar inequalities for F.

5.3. Lemma. *Let a,b,c, $P(a,b,c)$, $Q(a,b,c)$ be as in Lemma* 5.2*, and*

$$q(a,b,c) := \begin{cases} Q(a,b,c) & \text{if } P(a,b,c) \geqslant Q(a,b,c), \\ P(a,b,c) - 1 & \text{if } P(a,b,c) < Q(a,b,c). \end{cases} \tag{5.25}$$

Then

$$\begin{aligned} F(a,b,c;w)^2 > {} & P(a,b,c)^2 - 2P(a,b,c)Q(a,b,c)(1-w)^{c-a-b} \\ & + q(a,b,c)^2(1-w)^{2(c-a-b)} \quad \text{for } w \in (0,1). \end{aligned} \tag{5.26}$$

Proof. *Step 1. The case* $P(a,b,c) \geqslant Q(a,b,c)$.

For any $w \in (0,1)$, the upper bound in Eq. (5.18) implies

$$F(a,b,c;w) > P(a,b,c) - Q(a,b,c)(1-w)^{c-a-b}. \tag{5.27}$$

The right-hand side in the above equation is positive, so we infer

$$F(a,b,c;w)^2 > \big(P(a,b,c) - Q(a,b,c)(1-w)^{c-a-b}\big)^2; \tag{5.28}$$

expanding the right-hand side we get the thesis (5.26), since in this case $Q(a,b,c) = q(a,b,c)$.

Step 2. The case $P(a,b,c) < Q(a,b,c)$.

We write

$$\begin{aligned} F(a,b,c;w)^2 &= \big[P(a,b,c) - \big(P(a,b,c) - F(a,b,c;w)\big)\big]^2 \\ &= P(a,b,c)^2 + \big(P(a,b,c) - F(a,b,c;w)\big)^2 \\ &\quad - 2P(a,b,c)\big(P(a,b,c) - F(a,b,c;w)\big). \end{aligned} \tag{5.29}$$

We insert here the bounds on $P(a,b,c) - F(a,b,c;w)$ coming from Eq. (5.18); this gives

$$\begin{aligned} F(a,b,c;w)^2 > P(a,b,c)^2 &+ \big(P(a,b,c) - 1\big)^2 (1-w)^{2(c-a-b)} \\ &- 2P(a,b,c)Q(a,b,c)(1-w)^{c-a-b}, \end{aligned} \tag{5.30}$$

and we have the thesis (5.26) since in this case $q(a,b,c) = P(a,b,c) - 1$. □

Proof of Proposition 2.3, item (ii). Throughout the proof, $d/2 < n \leqslant d/2 + 1/2$.

Step 1. For $w \in (0,1)$ *one has*

$$\begin{aligned} &F\left(n, \frac{d}{2} + \frac{1}{2} - n, n + \frac{1}{2}; w\right)^2 \\ &\quad \geqslant \frac{P_{nd}^2}{(n-d/2)^2} - \frac{2P_{nd}Q_{nd}}{(n-d/2)^2}(1-w)^{n-d/2} + \frac{q_{nd}^2}{(n-d/2)^2}(1-w)^{2n-d}, \end{aligned} \tag{5.31}$$

where P_{nd}, Q_{nd} *and* q_{nd} *are as in* (2.21).

For $n < d/2 + 1/2$, this follows from application of Lemma 5.3 with $a = n$, $b = d/2 + 1/2 - n$, $c = n + 1/2$; comparing the coefficients in this lemma with Eq. (2.21) we see that

$$P(a,b,c) = \frac{P_{nd}}{n-d/2}, \qquad Q(a,b,c) = \frac{Q_{nd}}{n-d/2}, \qquad q(a,b,c) = \frac{q_{nd}}{n-d/2}. \tag{5.32}$$

Let us pass to the limit case $n = d/2 + 1/2$; then, (5.31) holds as an equality because $P_{nd} = 1/2$, $Q_{nd} = 0$, $q_{nd} = 0$, $F(n, d/2 + 1/2 - n, n + 1/2; w) = F(d/2 + 1/2, 0, d/2 + 1; w) = 1$ (by (3.9), with $m = 0$).

Step 2. Proof of Eq. (2.17): $\|g^2_{\lambda nd}\|^2_n \geqslant \mathcal{G}_{nd}(\lambda)$, *with* $\mathcal{G}_{nd}(\lambda)$ *as in Eq.* (2.20).

We start from the expression (2.15) of $\|g^2_{\lambda nd}\|^2_n$; the function F_{nd} therein is expressed as in (1.16), and its square is bounded via the result of Step 1 (with $w = u/(1+u)$). This gives

$$\|g^2_{\lambda nd}\|^2_n \geqslant \frac{\pi^{d/2}\Gamma^2(2n-d/2)}{(n-d/2)^2\Gamma(d/2)\Gamma^2(2n)\lambda^d} \times \int_0^{+\infty} du\, u^{d/2-1}\frac{(1+4\lambda^2 u)^n}{(1+u)^{2n}}\left(P^2_{nd} - 2\frac{P_{nd}Q_{nd}}{(1+u)^{n-d/2}} + \frac{q^2_{nd}}{(1+u)^{2n-d}}\right). \tag{5.33}$$

The above integral can be written as the sum of three integrals of the form (3.11); after computing each of them by (3.11), we apply (1.10) with $w = n-d/2$, $2n-d$ and $3n-3d/2$, respectively. The final result is the minorant for $\|g^2_{\lambda nd}\|^2_n$ as in Eq. (2.20).

Step 3. The $n \to (d/2)^+$ *limit of* $\mathcal{K}^{BB}_{nd}(\lambda)$.

Let d and $\lambda \in (0,+\infty)$ be fixed. We start computing the limiting behavior of $\mathcal{G}_{nd}(\lambda)$. For $n \to (d/2)^+$, the coefficients P_{nd}, Q_{nd} and q_{nd} therein have the same behavior up to $O(n-d/2)$:

$$P_{nd}, Q_{nd}, q_{nd} = \frac{\Gamma(d/2+1/2)}{\sqrt{\pi}\,\Gamma(d/2)}\big[1 + O(n-d/2)\big]. \tag{5.34}$$

In the same limit, the three hypergeometric functions also have equal behavior:

$$F\left(-n, \frac{d}{2}, n; 1-4\lambda^2\right), F\left(-n, \frac{d}{2}, 2n-\frac{d}{2}; 1-4\lambda^2\right), F\left(-n, \frac{d}{2}, 3n-d; 1-4\lambda^2\right) = F\left(-\frac{d}{2}, \frac{d}{2}, \frac{d}{2}; 1-4\lambda^2\right) + O\left(n-\frac{d}{2}\right) = 2^d\lambda^d + O\left(n-\frac{d}{2}\right), \tag{5.35}$$

where the last equality follows from (3.8). Inserting Eqs. (5.34), (5.35) into (2.20), we find

$$\mathcal{G}_{nd}(\lambda) = \frac{2^d\pi^{d/2-1}\Gamma(d/2+1/2)^2}{3\Gamma^2(d)\Gamma(d/2)}\frac{1+O(n-d/2)}{(n-d/2)^3} = \frac{\pi^{d/2}}{32^{d-2}\Gamma(d/2)^3}\frac{1+O(n-d/2)}{(n-d/2)^3}; \tag{5.36}$$

the second equality in (5.36) follows from the first one applying the duplication formula (1.13) with $w = d/2$.

Let us pass to the $n \to (d/2)^+$ behavior of $\|g_{\lambda nd}\|_n$; from (2.13) and (3.8), we infer

$$\|g_{\lambda nd}\|^2_n = \frac{\pi^{d/2}}{\Gamma(d/2)\lambda^d}F\left(-\frac{d}{2}, \frac{d}{2}, \frac{d}{2}; 1-\lambda^2\right)\frac{1+O(n-d/2)}{n-d/2} = \frac{\pi^{d/2}}{\Gamma(d/2)}\frac{1+O(n-d/2)}{n-d/2}. \tag{5.37}$$

Since $\mathcal{K}^{BB}_{nd}(\lambda) = \sqrt{\mathcal{G}_{nd}(\lambda)}/\|g_{\lambda nd}\|^2_n$, from (5.36) and (5.37) we obtain

$$\mathcal{K}^{BB}_{nd}(\lambda) = \frac{1}{\sqrt{3}\, 2^{d/2-1}\pi^{d/4}\sqrt{\Gamma(d/2)}} \frac{1+O(n-d/2)}{\sqrt{n-d/2}}; \tag{5.38}$$

comparing this with the definition (2.6) of M_d, we get the thesis (2.22). □

Computing the Bessel lower bounds. (a) For all $n > d/2$, the lower bound $\mathcal{K}^B_{nd}(\lambda)$ is the ratio of $\|g^2_{\lambda nd}\|_n$ and $\|g_{\lambda nd}\|^2_n$. The norm of $g_{\lambda nd}$ has the analytic expression (2.13) in terms of a hypergeometric function, that becomes the elementary formula (2.14) for n integer.

The norm of $g^2_{\lambda nd}$ has the integral representation (2.15), involving the hypergeometric-like function F_{nd} of Eqs. (1.15), (1.16). For $n - d/2 - 1/2$ integer, this norm has the explicit expression (2.16) in terms of hypergeometric functions. For $n - d/2 - 1/2$ noninteger, the integral in (2.15) must be computed numerically. As anticipated, this is a difficult task for n very close to $d/2$, due to the slow convergence of the integral: the integrand behaves like $1/u^{1+(n-d/2)}$ for $u \to +\infty$ (as made evident by Eq. (1.16) for F_{nd}), and we are interested in situations where $n - d/2 = 10^{-4}$. In these cases it is convenient to compute, in place of $\|g^2_{\lambda nd}\|^2_n$, the minorant $\mathcal{G}_{nd}(\lambda)$ of Eq. (2.20), and from this the lower bound $\mathcal{K}^{BB}_{nd}(\lambda)$ of Eq. (2.19), both of them having analytic expressions in terms of hypergeometric functions.

(b) Assuming we are able to compute $\mathcal{K}^B_{nd}(\lambda)$ or $\mathcal{K}^{BB}_{nd}(\lambda)$, for each λ we have a lower bound for K_{nd}; the next step is maximization with respect to λ, to get K^B_{nd} or K^{BB}_{nd}. In general, this is done numerically (using some package for automatic maximization or for plotting these functions of λ, so as to read the maximum from the graph).

(c) Let us consider, for example, the case $d = 2$ and the values of n reported in Table 1. For $n = 3/2$, we have the elementary expression

$$\mathcal{K}^B_{3/2,2}(\lambda) = \frac{\lambda}{2\sqrt{2\pi}} \frac{\sqrt{F(1-4\lambda^2)}}{F(1-\lambda^2)},$$

$$F(w) := F(-3/2, 1, 3/2; w) = \frac{5-3w}{8} + \frac{3}{8}(1-w)^2\mathcal{F}(w), \tag{5.39}$$

$$\mathcal{F}(w) := \begin{cases} \operatorname{arctanh}(\sqrt{w})/\sqrt{w} & \text{if } 0 < w < 1, \\ 1 & \text{if } w = 0, \\ \arctan(\sqrt{-w})/\sqrt{-w} & \text{if } w < 0. \end{cases}$$

The function $\mathcal{K}^B_{3/2,2}$ attains its maximum at $\lambda \simeq 1.38$. $\mathcal{K}^B_{5/2,2}$ is also elementary, with its maximum at $\lambda \simeq 1.36$. For $n = 5/4, 2, 4, 7, 16, 31, 61$ the integral in $\|g^2_{\lambda n2}\|_n$ can be computed numerically; from the graph of $\mathcal{K}^B_{n2}$ we have found this function to get its maximum at $\lambda \simeq 1.40, 1.36, 1.39, 1.45, 1.53, 1.57, 1.58$, respectively.

For $n = 1 + 10^{-4}, 1 + 10^{-2}, 1 + 10^{-1}$ the numerical computation of $\mathcal{K}^B_{n2}$ and K^B_{n2} is difficult, so we have turned the attention to the simpler bound K^{BB}_{n2}; from the analytic expressions of $\mathcal{K}^{BB}_{n2}(\lambda)$ and numerical optimization, we have found the maximum of this function to be attained at $\lambda \simeq 1.42$ in each one of the three cases.

For all the cases in the table from $n = 5/4$ to $n = 61$, the previously mentioned Bessel bounds have been compared with the Fourier lower bounds K^F_{n2} or K^{FF}_{n2} of Proposition 2.4 (for the computation of these Fourier bounds, see the remarks at the end of the following section). In this way, we have found that the Fourier lower bounds are below the Bessel bounds up to $n = 5/2$, while the contrary happens for $n > 3/2$ (for example: $K^F_{5/4,2} < 0.610 K^B_{5/4,2}$ and $K^B_{61,2} < 0.411 K^{FF}_{61,2}$). Extrapolating, the Bessel bound K^B_{n2} is likely to be smaller than the Fourier bounds for the large value $n = 121$. Since the numerical computation of $\mathcal{K}^B_{121,2}$ and $K^B_{121,2}$ is difficult, in the construction of Table 1 we have chosen directly for $K^-_{121,2}$ a Fourier bound.

6. Proofs for the Fourier lower bounds on K_{nd}

We refer to the trial functions $f_{p\sigma d}$ of Eq. (2.23). Our aim is to prove all statements contained in Proposition 2.4; we will proceed in several steps.

6.1. Lemma. *For all $p, \sigma > 0$ and $n > d/2$, $\|f_{p\sigma d}\|_n$ is given by Eq.* (2.26).

Proof. The Fourier transform of $f_{p\sigma d}$ is elementary, and given by

$$(\mathcal{F} f_{p\sigma d})(k) = \frac{1}{\sigma^{d/2}} e^{-\frac{1}{2\sigma}|k-p\eta|^2}, \quad \eta := (1, 0, \ldots, 0); \tag{6.1}$$

thus

$$\begin{aligned}\|f_{p\sigma d}\|_n^2 &= \frac{1}{\sigma^d} \int_{\mathbf{R}^d} dk \big(1 + |k|^2\big)^n e^{-\frac{1}{\sigma}|k-p\eta|^2} \\ &= \frac{1}{\sigma^d} \int_{\mathbf{R}^d} dk \big(1 + |k|^2\big)^n e^{-\frac{|k|^2+p^2}{\sigma} + \frac{2p}{\sigma}\eta \bullet k}.\end{aligned} \tag{6.2}$$

To go on, let us first consider the case $d = 1$. Equation (6.2) gives

$$\begin{aligned}\|f_{p\sigma 1}\|_n^2 &= \frac{1}{\sigma} \int_{\mathbf{R}} dk \big(1 + k^2\big)^n e^{-\frac{k^2+p^2}{\sigma} + \frac{2p}{\sigma} k} \\ &= \frac{1}{\sigma} \int_0^{+\infty} d\rho \big(1 + \rho^2\big)^n e^{-\frac{\rho^2+p^2}{\sigma}} \big(e^{\frac{2p}{\sigma}\rho} + e^{-\frac{2p}{\sigma}\rho}\big)\end{aligned} \tag{6.3}$$

(in the last passage, we have used the variable $\rho = |k|$); this gives Eq. (2.26) for $d = 1$, since [14, p. 80]

$$e^s + e^{-s} = \sqrt{2\pi s}\, I_{-1/2}(s) \quad \forall s \in (0, +\infty). \tag{6.4}$$

Now, let us pass to the case $d \geqslant 2$. Equation (6.2) contains an integral of the form (3.2), where the integration variable is now k and $\chi(|k|, \eta \bullet k) = (1 + |k|^2)^n e^{-\frac{1}{\sigma}(|k|^2+p^2) + \frac{2p}{\sigma}\eta \bullet k}$; therefore,

$$\|f_{p\sigma d}\|_n^2 = \frac{2\pi^{d/2-1/2}}{\Gamma(\frac{d}{2} - \frac{1}{2})\sigma^d} \int_0^{+\infty} d\rho\, \rho^{d-1} \big(1 + \rho^2\big)^n e^{-\frac{\rho^2+p^2}{\sigma}} \int_0^{\pi} d\theta \sin\theta^{d-2} e^{\frac{2p}{\sigma}\rho\cos\theta}. \tag{6.5}$$

On the other hand [14, p. 79],

$$\int_0^{\pi} d\theta \sin\theta^{2\nu} e^{s\cos\theta} = \sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)\left(\frac{2}{s}\right)^{\nu} I_\nu(s); \tag{6.6}$$

inserting this result into the previous equation, we obtain the thesis (2.26). □

6.2. Lemma. *For all* $p, \sigma > 0$ *and integer* $n > d/2$, $\| f_{p\sigma d}\|_n$ *is given by Eq.* (2.27).

Proof. We return to the first equation (6.2), and expand $(1+|k|^2)^n$ by the binomial formula; this gives

$$\| f_{p\sigma d}\|_n^2 = \frac{1}{\sigma^d}\sum_{\ell=0}^{n}\binom{n}{\ell}\int_{\mathbf{R}^d} dk\,|k|^{2\ell} e^{-\frac{1}{\sigma}|k-p\eta|^2}. \tag{6.7}$$

Now, we write the integration variable as $k=(h,q)$, ($h \in \mathbf{R}$, $q \in \mathbf{R}^{d-1}$); so,

$$\begin{aligned}\| f_{p\sigma d}\|_n^2 &= \frac{1}{\sigma^d}\sum_{\ell=0}^{n}\binom{n}{\ell}\int_{\mathbf{R}\times\mathbf{R}^{d-1}} dh\,dq\,\left(h^2+|q|^2\right)^{\ell} e^{-\frac{(h-p)^2}{\sigma}} e^{-\frac{|q|^2}{\sigma}}\\ &= \frac{1}{\sigma^d}\sum_{\ell=0}^{n}\binom{n}{\ell}\sum_{j=0}^{\ell}\binom{\ell}{j}\int_{\mathbf{R}} dh\,h^{2j} e^{-\frac{(h-p)^2}{\sigma}}\int_{\mathbf{R}^{d-1}} dq\,|q|^{2\ell-2j} e^{-\frac{|q|^2}{\sigma}},\end{aligned} \tag{6.8}$$

where, in the last passage, we have used again the binomial formula to expand $(h^2+|q|^2)^{\ell}$. On the other hand,

$$\begin{aligned}\int_{\mathbf{R}} dh\,h^{2j} e^{-\frac{(h-p)^2}{\sigma}} &= \int_{\mathbf{R}} dh\,(h+p)^{2j} e^{-\frac{h^2}{\sigma}}\\ &= \sum_{m=0}^{2j}\binom{2j}{m} p^{2j-m}\int_{\mathbf{R}} dh\,h^m e^{-\frac{h^2}{\sigma}}\\ &= \sum_{g=0}^{j}\binom{2j}{2g}\frac{(2g-1)!!\sqrt{\pi}}{2^g} p^{2j-2g}\sigma^{1/2+g}.\end{aligned} \tag{6.9}$$

The last passage above depends on the evaluation of the integrals with h^m: these vanish for m odd, while in the even case $m=2g$ we have $\int_{\infty}^{+\infty} dh\,h^{2g} e^{-h^2/\sigma} = \sigma^{g+1/2}\Gamma(g+1/2) = \sigma^{g+1/2}2^{-g}(2g-1)!!\sqrt{\pi}$. Concerning the integrals over q, due to Eq. (3.1) we have

$$\int_{\mathbf{R}^{d-1}} dq\,|q|^{2\ell-2j} e^{-\frac{|q|^2}{\sigma}} = \frac{2\pi^{d/2-1/2}}{\Gamma(d/2-1/2)}\int_0^{+\infty} d\xi\,\xi^{d-2+2\ell-2j} e^{-\frac{\xi^2}{\sigma}}$$

$$= \pi^{d/2-1/2} \sigma^{d/2-1/2+\ell-j} \frac{\Gamma(d/2-1/2+\ell-j)}{\Gamma(d/2-1/2)}$$
$$= \pi^{d/2-1/2} \sigma^{d/2-1/2+\ell-j} (d/2-1/2)_{\ell-j}. \tag{6.10}$$

Inserting Eqs. (6.9), (6.10) into (6.8), we finally get the thesis (2.27). □

Proof of Proposition 2.4, item (i). This is given by the two previous lemmas. □

We pass to item (ii) of the same proposition, whose proof is more lengthy. The initial step concerns the expression of $\| f_{p\sigma d}\|_n$ when p is arbitrary and $\sigma = c/n$ $(c > 0)$; in this case, the already proved equation (2.26) becomes

$$\| f_{p,c/n,d}\|_n^2 = \frac{2\pi^{d/2} n^{d/2+1}}{c^{d/2+1} p^{d/2-1}} \int_0^{+\infty} d\rho\, \rho^{d/2} \left(1+\rho^2\right)^n e^{-n\frac{\rho^2+p^2}{c}} I_{d/2-1}\left(\frac{2np}{c}\rho\right). \tag{6.11}$$

We will analyze this formula in the limit $n \to +\infty$. In the first lemma, p and c will be arbitrary; in the subsequent ones, based on the theory of Laplace integrals, we will consider a specific choice, ultimately yielding Eq. (2.29).

6.3. Lemma. *Fix $p > 0$, $c > 0$ and d; for $n \to +\infty$,*

$$\| f_{p,c/n,d}\|_n^2 = \frac{\pi^{d/2-1/2} n^{d/2+1/2}}{c^{d/2+1/2} p^{d/2-1/2}} \left[X_{pc,d/2-1/2}(n) + O\left(\frac{X_{pc,d/2-3/2}(n)}{n}\right) + O\left(\left(1+p^2\right)^n\right)\right], \tag{6.12}$$

$$X_{pc\alpha}(n) := \int_p^{+\infty} d\rho\, \rho^{\alpha} \left(1+\rho^2\right)^n e^{-n\frac{(\rho-p)^2}{c}} \quad \textit{for all } \alpha \in \mathbf{R}. \tag{6.13}$$

Proof. We start from the relations

$$I_{d/2-1}(s) = \frac{e^s}{\sqrt{2\pi s}} h_d(s) = \frac{e^s}{\sqrt{2\pi s}}\left(1 + \frac{b_d(s)}{s}\right) \quad \text{for all } s \in (0, +\infty),$$
$$b_d, h_d \in L^{\infty}\left((0, +\infty), \mathbf{R}\right), \tag{6.14}$$

reflecting the asymptotic behavior of the Bessel functions $I_\nu(s)$ for $s \to 0^+$ and $s \to +\infty$ (see [14]).

To go on, in Eq. (6.11) we write $\int_0^{+\infty} = \int_p^{+\infty} + \int_0^p$; in these two integrals, we substitute the representations (6.14) of $I_{d/2-1}$ involving, respectively, b_d and h_d. This gives

$$\| f_{p,c/n,d}\|_n^2 = \frac{\pi^{d/2-1/2} n^{d/2+1/2}}{c^{d/2+1/2} p^{d/2-1/2}} \left[X_{pc,d/2-1/2}(n) + Y_{pcd}(n) + Z_{pcd}(n)\right], \tag{6.15}$$

where the X term is defined following Eq. (6.13), and

$$Y_{pcd}(n) := \frac{c}{2pn} \int_p^{+\infty} d\rho\, \rho^{d/2-3/2} \big(1+\rho^2\big)^n e^{-n\frac{(\rho-p)^2}{c}} b_d\left(\frac{2pn}{c}\rho\right), \tag{6.16}$$

$$Z_{pcd}(n) := \int_0^p d\rho\, \rho^{d/2-1/2} \big(1+\rho^2\big)^n e^{-n\frac{(\rho-p)^2}{c}} h_d\left(\frac{2np}{c}\rho\right).$$

We estimate these two integrals. Let $B_d := \sup_{(0,+\infty)} |b_d|$, $H_d := \sup_{(0,+\infty)} |h_d|$; then

$$\big|Y_{pcd}(n)\big| \leqslant \frac{B_d c}{2pn} X_{pc,d/2-3/2}(n), \tag{6.17}$$

$$\big|Z_{pcd}(n)\big| \leqslant H_d \big(1+p^2\big)^n \int_0^p d\rho\, \rho^{d/2-1/2} = H_d \big(1+p^2\big)^n \frac{p^{d/2+1/2}}{d/2+1/2},$$

whence

$$Y_{pcd}(n) = O\left(\frac{X_{pc,d/2-3/2}(n)}{n}\right), \qquad Z_{pcd}(n) = O\big(\big(1+p^2\big)^n\big) \quad \text{for } n \to +\infty. \tag{6.18}$$

Substituting Eq. (6.18) into (6.15) we obtain the thesis (6.12). □

To go on, we observe that Eq. (6.13) can be rephrased as

$$X_{pc\alpha}(n) = \int_p^{+\infty} d\rho\, \rho^\alpha e^{-n\Phi_{pc}(\rho)}, \qquad \Phi_{pc}(\rho) := \frac{(\rho-p)^2}{c} - \log\big(1+\rho^2\big). \tag{6.19}$$

In the sequel, we apply the Laplace analysis to the integral (6.19). We will consider the special choice

$$p := \frac{1}{2\sqrt{2}}, \quad c = \frac{3}{4} \tag{6.20}$$

and its double $(2p, 2c)$: this makes easy to compute the minimum point of Φ_{pc} and $\Phi_{2p,2c}$. We repeat here the remark made in Section 2, after stating Proposition 2.4: different choices of (p, c) complicate the computations, with no sensible increase in the dominant term of the Fourier bound $\mathcal{K}^F(p, c/n)$. (This conclusion is the result of a tedious analysis, that is not worthy to be reported here.)

6.4. Lemma. *Let p, c be as in* (6.20)*. For fixed $\alpha \in \mathbf{R}$ and $n \to +\infty$,*

$$X_{pc\alpha}(n) = \frac{3\sqrt{\pi/5}}{2^{\alpha/2+1/2}} \frac{(3/2)^n}{e^{n/6}\sqrt{n}} \left[1 + O\left(\frac{1}{n}\right)\right], \tag{6.21}$$

$$X_{2p,2c,\alpha}(n) = 3\sqrt{\pi/7}\,2^{\alpha/2}\frac{3^n}{e^{n/3}\sqrt{n}}\left[1 + O\left(\frac{1}{n}\right)\right]. \tag{6.22}$$

Proof. *Step 1. Proof of Eq.* (6.21).

We put for brevity

$$X_\alpha(n) := X_{pc\alpha}(n), \qquad \Phi := \Phi_{pc}. \tag{6.23}$$

Explicitly

$$\Phi(\rho) = \frac{4}{3}\left(\rho - \frac{1}{2\sqrt{2}}\right)^2 - \log\left(1 + \rho^2\right); \tag{6.24}$$

it is easily checked that

$$\Phi'(\rho) = \frac{2}{3}\left(\rho - \frac{1}{\sqrt{2}}\right)\frac{4\rho^2 + \sqrt{2}\rho + 2}{1 + \rho^2} \lesseqqgtr 0 \quad \text{for } \rho \lesseqqgtr \frac{1}{\sqrt{2}},$$

$$\Phi\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{6} - \log\left(\frac{3}{2}\right). \tag{6.25}$$

Now, following the scheme of (3.30) we re-express the integral under examination as

$$X_\alpha(n) = e^{-n\Phi(1/\sqrt{2})}\left[L_\alpha^-(n) + L_\alpha^+(n)\right] = \frac{(3/2)^n}{e^{n/6}}\left[L_\alpha^-(n) + L_\alpha^+(n)\right], \tag{6.26}$$

$$L_\alpha^-(n) := \int_0^{1/(2\sqrt{2})} dt\, \vartheta_\alpha^-(t) e^{-n\varphi^-(t)}, \qquad L_\alpha^+(n) := \int_0^{+\infty} dt\, \vartheta_\alpha^+(t) e^{-n\varphi^+(t)};$$

$$\vartheta_\alpha^\mp(t) := \left(\frac{1}{\sqrt{2}} \mp t\right)^\alpha, \tag{6.27}$$

$$\varphi^\mp(t) := \Phi\left(\frac{1}{\sqrt{2}} \mp t\right) - \Phi\left(\frac{1}{\sqrt{2}}\right) = \mp\frac{2\sqrt{2}}{3}t + \frac{4}{3}t^2 - \log\left(1 \mp \frac{2\sqrt{2}}{3}t + \frac{2}{3}t^2\right).$$

The above two integrals have the standard Laplace form discussed in Proposition 3.1. Following the usual scheme, we fix the attention on the functions

$$\xi_\alpha^\mp(t) := \frac{\vartheta_\alpha^\mp(t)}{\varphi^{\mp\prime}(t)} = \frac{3}{4t}\,\frac{3 \mp 2\sqrt{2}t + 2t^2}{5 \mp 5\sqrt{2}t + 4t^2}\left(\frac{1}{\sqrt{2}} \mp t\right)^\alpha. \tag{6.28}$$

For $t \to 0^+$, one easily checks that

$$\varphi^\mp(t) = \frac{10}{9}t^2 \mp \frac{20\sqrt{2}}{81}t^3 + O\left(t^4\right), \tag{6.29}$$

$$t = \frac{3}{\sqrt{10}}\sqrt{\varphi^{\mp}(t)} \pm \frac{\sqrt{2}}{10}\varphi^{\mp}(t) + O\big(\varphi^{\mp}(t)^{3/2}\big);$$

$$\begin{aligned}\xi_\alpha^{\mp}(t) &= \frac{1}{2^{\alpha/2}}\left[\frac{9}{20t} \mp \frac{1}{\sqrt{2}}\left(\frac{9\alpha}{10} - \frac{3}{10}\right)\right] + O(t)\\ &= \frac{1}{2^{\alpha/2+1/2}}\left[\frac{3}{2\sqrt{5}\sqrt{\varphi^{\mp}(t)}} \mp \left(\frac{9\alpha}{10} - \frac{1}{5}\right)\right] + O\Big(\sqrt{\varphi^{\mp}(t)}\Big).\end{aligned} \tag{6.30}$$

We can now apply Proposition 3.1 to both integrals $L_\alpha^{\mp}(n)$; this gives

$$L_\alpha^{\mp}(n) = \frac{1}{2^{\alpha/2+1/2}}\left[\frac{3\sqrt{\pi}}{2\sqrt{5}\sqrt{n}} \mp \left(\frac{9\alpha}{10} - \frac{1}{5}\right)\frac{1}{n}\right] + O\left(\frac{1}{n^{3/2}}\right) \quad \text{for } n \to +\infty, \tag{6.31}$$

and substituting these expansions into Eq. (6.26) we get the thesis (6.21). (Note the mutual cancellation of the terms $\mp(9\alpha/10 - 1/5)(1/n)$, in agreement with the remark concluding Section 3.)

Step 2. Proof of Eq. (6.22).

In this case, we put

$$X_\alpha(n) := X_{2p,2c,\alpha}(n), \qquad \Phi := \Phi_{2p,2c}. \tag{6.32}$$

One has

$$\Phi(\rho) = \frac{2}{3}\left(\rho - \frac{1}{\sqrt{2}}\right)^2 - \log\big(1 + \rho^2\big); \tag{6.33}$$

$$\Phi'(\rho) = \frac{2}{3}\big(\rho - \sqrt{2}\big)\frac{2\rho^2 + \sqrt{2}\rho + 1}{1 + \rho^2} \lesseqqgtr 0 \quad \text{for } \rho \lesseqqgtr \sqrt{2},$$

$$\Phi\big(\sqrt{2}\big) = \frac{1}{3} - \log 3.$$

We can write

$$X_\alpha(n) = e^{-n\Phi(\sqrt{2})}\big[L_\alpha^-(n) + L_\alpha^+(n)\big] = \frac{3^n}{e^{n/3}}\big[L_\alpha^-(n) + L_\alpha^+(n)\big], \tag{6.34}$$

$$L_\alpha^-(n) := \int_0^{1/\sqrt{2}} dt\, \vartheta_\alpha^-(t) e^{-n\varphi^-(t)}, \qquad L_\alpha^+(n) := \int_0^{+\infty} dt\, \vartheta_\alpha^+(t) e^{-n\varphi^+(t)};$$

$$\vartheta_\alpha^{\mp}(t) := \big(\sqrt{2} \mp t\big)^\alpha, \tag{6.35}$$

$$\varphi^{\mp}(t) := \Phi\big(\sqrt{2} \mp t\big) - \Phi\big(\sqrt{2}\big) = \mp\frac{2\sqrt{2}}{3}t + \frac{2}{3}t^2 - \log\left(1 \mp \frac{2\sqrt{2}}{3}t + \frac{1}{3}t^2\right).$$

We introduce the functions

$$\xi_\alpha^{\mp}(t) := \frac{\vartheta_\alpha^{\mp}(t)}{\varphi^{\mp\prime}(t)} = \frac{3}{2t}\frac{3 \mp 2\sqrt{2}t + t^2}{7 \mp 5\sqrt{2}t + 2t^2}\big(\sqrt{2} \mp t\big)^\alpha. \tag{6.36}$$

For $t \to 0^+$, comparing the expansions of $\varphi^{\mp}$, $\xi_\alpha^{\mp}$ in powers of t we get

$$\xi_\alpha^{\mp}(t) = 2^{\alpha/2}\left[\frac{3}{2\sqrt{7}\sqrt{\varphi^{\mp}(t)}} \mp \frac{1}{\sqrt{2}}\left(\frac{9\alpha}{14} - \frac{2}{49}\right)\right] + O\Big(\sqrt{\varphi^{\mp}(t)}\Big). \tag{6.37}$$

Applying Proposition 3.1 to $L_\alpha^{\mp}(n)$ we obtain

$$L_\alpha^{\mp}(n) = 2^{\alpha/2}\left[\frac{3\sqrt{\pi}}{2\sqrt{7}\sqrt{n}} \mp \frac{1}{\sqrt{2}}\left(\frac{9\alpha}{14} - \frac{2}{49}\right)\frac{1}{n}\right] + O\left(\frac{1}{n^{3/2}}\right) \quad \text{for } n \to +\infty, \tag{6.38}$$

and substituting these expansions into Eq. (6.34) we get the thesis (6.22). □

6.5. Lemma. *Let p, c be as in* (6.20). *For fixed d and $n \to +\infty$,*

$$\| f_{p,c/n,d} \|_n^2 = \frac{2^{3d/2}\pi^{d/2}}{3^{d/2-1/2}\sqrt{5}}\frac{(3/2)^n}{e^{n/6}} n^{d/2}\left[1 + O\left(\frac{1}{n}\right)\right], \tag{6.39}$$

$$\| f_{2p,2c/n,d} \|_n^2 = \frac{2^{d}\pi^{d/2}}{3^{d/2-1/2}\sqrt{7}}\frac{3^n}{e^{n/3}} n^{d/2}\left[1 + O\left(\frac{1}{n}\right)\right]. \tag{6.40}$$

Proof. To prove Eq. (6.39), we note that (6.21) implies

$$X_{pc,d/2-1/2}(n) = \frac{3\sqrt{\pi/5}}{2^{d/4+1/4}}\frac{(3/2)^n}{e^{n/6}\sqrt{n}}\left[1 + O\left(\frac{1}{n}\right)\right], \tag{6.41}$$

$$\frac{X_{pc,d/2-3/2}(n)}{n} = \frac{(3/2)^n}{e^{n/6}\sqrt{n}} O\left(\frac{1}{n}\right).$$

We insert these results into Eq. (6.12) for $\| f_{p,c/n,d} \|_n^2$, taking into account that the present choices of p, c imply

$$c^{d/2+1/2}p^{d/2-1/2} = \frac{3^{d/2+1/2}}{2^{7d/4+1/4}}; \qquad 1 + p^2 = \frac{9}{8} = \frac{3/2}{e^{1/6}}\theta, \quad 0.8 < \theta < 0.9;$$

$$\big(1 + p^2\big)^n = \frac{(3/2)^n}{e^{n/6}}\theta^n = \frac{(3/2)^n}{e^{n/6}\sqrt{n}} O\left(\frac{1}{n}\right). \tag{6.42}$$

The proof of Eq. (6.40) is very similar, depending on Eqs. (6.22), (6.12). □

Proof of Proposition 2.4, item (ii). This item concerns the $n \to +\infty$ limit for the special Fourier lower bound K^{FF}_{nd}; comparing the definition (2.28) of this bound with the notations of this section, we see that

$$K^{FF}_{nd} = \frac{\| f_{2p,2c/n,d} \|_n}{\| f_{p,c/n,d} \|^2_n}, \quad (p, c) \text{ as in (6.20).} \tag{6.43}$$

From Eqs. (6.39), (6.40) we infer, for $n \to +\infty$,

$$\begin{aligned} K^{FF}_{nd} &= \frac{\sqrt{5}}{7^{1/4}} \frac{3^{d/4-1/4}}{2^d \pi^{d/4}} \frac{(2/\sqrt{3})^n}{n^{d/4}} \frac{\sqrt{1+O(\frac{1}{n})}}{1+O(\frac{1}{n})} \\ &= \frac{(5/3)^{1/2}}{7^{1/4}} T_d \frac{(2/\sqrt{3})^n}{n^{d/4}} \left[1 + O\left(\frac{1}{n}\right)\right]. \end{aligned} \tag{6.44}$$

In the last passage we have used the definition (2.7) of T_d; our result is just the thesis (2.29). □

Computing the Fourier lower bounds. (a) For any n and d, the function $(p, \sigma) \to \mathcal{K}^F_{nd}(p, \sigma)$ in Eq. (2.24) is determined by the function $(p, \sigma) \to \| f_{p\sigma d} \|_n$. For n noninteger and given (p, σ), this can be computed via Eq. (2.26), evaluating numerically the integral therein; for n integer, we have the elementary expression (2.27).

The bound K^F_{nd} is obtained maximizing $\mathcal{K}^F_{nd}(p, \sigma)$ with respect to $(p, \sigma) \in (0, +\infty)^2$; in typical situations this must be done numerically, even for integer n (in any case, the maximization problem is not dramatic because $\mathcal{K}^F_{nd}(p, \sigma)$ is a lower bound for *all* choices of (p, σ), even not close to the maximizing pair).

For very large values of n, instead of maximizing $\mathcal{K}^F_{nd}(p, \sigma)$ one can evaluate it at $(p, \sigma) = (1/(2\sqrt{2}), 3/(4n))$; this yields the bound K^{FF}_{nd} of Eq. (2.28), that we know to be effective in this limit.

(b) Let us consider, for example, the case $d = 2$ and the values of n in Table 1. For the integer values $n = 2, 4, 7, 16, 31$ we have determined the analytic expression of $\mathcal{K}^F_{n2}$ using Eq. (2.27), and then maximized this function numerically; the maxima occur, respectively, at $(p, \sigma) \simeq (0.511, 1.05), (0.417, 0.309), (0.371, 0.148), (0.331, 0.0582), (0.316, 0.0290)$. For the large values $n = 61, 121$, we have used directly the lower bound $K^{FF}_{n2} = \mathcal{K}^F_{n2}(1/(2\sqrt{2}), 3/(4n))$; since n is integer, in principle this could be obtained again from Eq. (2.27), but in these two cases it is more convenient to compute it numerically, starting from the integral representation (2.26) of $\| f_{p\sigma 2} \|_n$ (note that this contains the non-elementary function I_0).

For $n = 5/4, 3/2, 5/2$, $\mathcal{K}^F_{n2}(p, \sigma)$ has been computed numerically for many sample values of (p, σ), starting again from (2.26); in this case, approximate maximization has been performed choosing the best value in the sample. The maxima are attained at $p \simeq 0.354$ in the three cases, and $\sigma \simeq 5.22, 2.41, 0.696$, respectively.

The numerical computation of $\mathcal{K}^F_{n2}$ and K^F_{nd} is difficult for the small values $n = 1 + 10^{-4}$, $1 + 10^{-3}$ and $1 + 10^{-1}$. On the other hand, for the reasons already explained at the end of the previous section the Fourier bounds should be below the Bessel bounds for these extreme values of n; therefore to construct Table 1 in these cases we have given up computing K^F_{n2}, and we have chosen directly for K^-_{n2} a Bessel lower bound.

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Appendix A. The integral $I_{\mu\nu}(h)$

This integral is defined by Eq. (3.16); we want to prove Eq. (3.17). We start from the identity [14, p. 440]

$$K_{\nu/2}^2(r) = 2\int_0^{+\infty} dt\, K_\nu(2r\cosh t), \tag{A.1}$$

and insert it into (3.16); this gives

$$I_{\mu\nu}(h) = 2\int_0^{+\infty} dt \int_0^{+\infty} dr\, r^{\mu+\nu+1} J_\mu(hr) K_\nu(2r\cosh t). \tag{A.2}$$

On the other hand [14, p. 410],

$$\begin{aligned}
&\int_0^{+\infty} dr\, r^{\mu+\nu+1} J_\mu(hr) K_\nu(2r\cosh t) \\
&\quad = \frac{\Gamma(\mu+\nu+1)h^\mu}{2^{\mu+2}\cosh^{2\mu+\nu+2} t} F\left(\mu+\nu+1, \mu+1, \mu+1; -\frac{h^2}{4\cosh^2 t}\right) \\
&\quad = \frac{\Gamma(\mu+\nu+1)h^\mu}{2^{\mu+2}\cosh^{2\mu+\nu+2} t}\left(1+\frac{h^2}{4\cosh^2 t}\right)^{-\mu-\nu-1},
\end{aligned} \tag{A.3}$$

where the last passage depends on (3.8). Returning to Eq. (A.2) we obtain

$$\begin{aligned}
I_{\mu\nu}(h) &= \frac{\Gamma(\mu+\nu+1)h^\mu}{2^{\mu+1}} \int_0^{+\infty} dt\, \frac{1}{\cosh^{2\mu+\nu+2} t}\left(1+\frac{h^2}{4\cosh^2 t}\right)^{-\mu-\nu-1} \\
&= \frac{\Gamma(\mu+\nu+1)h^\mu}{2^{\mu+2}} \int_0^1 ds\, s^{\mu+\nu/2}(1-s)^{-1/2}\left(1+\frac{h^2}{4}s\right)^{-\mu-\nu-1},
\end{aligned} \tag{A.4}$$

the last passage following with the change of variable $s = 1/\cosh^2 t$. Now, comparison with (3.10) gives the thesis (3.17).

Appendix B. Proof of Proposition 3.1 on Laplace integrals

We recall the notations and assumptions (3.18)–(3.21), and point out some consequences of our hypotheses.

First of all, by the monotonicity of φ, $\varphi(b) := \lim_{t\to b^-}\varphi(t)$ exists in $(0,+\infty]$, and φ is a C^1 diffeomorphism between $(0,b)$ and $(0,\varphi(b))$.

Moreover, by Eq. (3.21), there are a constant $\epsilon \in (0,b)$ and a bounded function $\beta \in C((0,\epsilon),\mathbf{R})$ such that

$$\xi(t) = \sum_{i=0}^{\ell-1} P_i\varphi(t)^{\alpha_i-1} + \beta(t)\varphi(t)^{\alpha_\ell-1} \quad \text{for all } t \in (0,\epsilon). \tag{B.1}$$

Putting the attention to Eq. (3.18) and dividing integration in two parts, we get

$$L(n) = M(n) + N(n), \tag{B.2}$$

$$M(n) := \int_0^\epsilon dt\,\vartheta(t)e^{-n\varphi(t)}, \qquad N(n) := \int_\epsilon^b dt\,\vartheta(t)e^{-n\varphi(t)}.$$

Let us estimate $M(n)$. Introducing the new variable $s=\varphi(t)$ and then using (B.1) we obtain

$$M(n) = \int_0^{\varphi(\epsilon)} ds\,\xi\big(\varphi^{-1}(s)\big)e^{-ns} = \sum_{i=0}^{\ell-1} P_i M_i(n) + \delta M_\ell(n), \tag{B.3}$$

$$M_i(n) := \int_0^{\varphi(\epsilon)} ds\,s^{\alpha_i-1}e^{-ns}, \qquad \delta M_\ell(n) := \int_0^{\varphi(\epsilon)} ds\,s^{\alpha_\ell-1}\beta\big(\varphi^{-1}(s)\big)e^{-ns}.$$

The above integrals are related to the incomplete Gamma function

$$\gamma(\alpha,u) := \int_0^u dv\,v^{\alpha-1}e^{-v} = \Gamma(\alpha) - \int_u^{+\infty} dv\,v^{\alpha-1}e^{-v}$$
$$= \Gamma(\alpha) + O\big(u^{\alpha-1}e^{-u}\big) \quad \text{for } u \to +\infty \quad (\alpha>0) \tag{B.4}$$

(concerning the asymptotics of γ for $u\to+\infty$, see [11]). As for M_i, with a variable change $s=v/n$ we get

$$M_i(n) = \frac{\gamma(\alpha_i, n\varphi(\epsilon))}{n^{\alpha_i}} = \frac{\Gamma(\alpha_i)}{n^{\alpha_i}} + O\left(\frac{e^{-n\varphi(\epsilon)}}{n}\right) \quad \text{for } n \to +\infty; \tag{B.5}$$

furthermore,

$$\big|\delta M_\ell(n)\big| \leqslant \Big(\sup_{(0,\epsilon)}|\beta|\Big)\int_0^{\varphi(\epsilon)} ds\, s^{\alpha_\ell-1}e^{-ns} = \Big(\sup_{(0,\epsilon)}|\beta|\Big)\frac{\gamma(\alpha_\ell, n\varphi(\epsilon))}{n^{\alpha_\ell}}$$
$$= \Big(\sup_{(0,\epsilon)}|\beta|\Big)\left[\frac{\Gamma(\alpha_\ell)}{n^{\alpha_\ell}} + O\left(\frac{e^{-n\varphi(\epsilon)}}{n}\right)\right] = O\left(\frac{1}{n^{\alpha_\ell}}\right) \quad \text{for } n \to +\infty. \tag{B.6}$$

To estimate $N(n)$, we fix $n_1 > n_0$ and write $N(n) = \int_\epsilon^b dt\, \vartheta(t)e^{-(n-n_1)\varphi(t)}e^{-n_1\varphi(t)}$; for all $n \in [n_1, +\infty)$, this implies

$$\big|N(n)\big| \leqslant e^{-(n-n_1)\varphi(\epsilon)}\int_\epsilon^b dt\big|\vartheta(t)\big|e^{-n_1\varphi(t)} = O\big(e^{-n\varphi(\epsilon)}\big) \quad \text{for } n \to +\infty \tag{B.7}$$

(recall that $0 < \varphi(\epsilon) \leqslant \varphi(t)$ for $t \in [\epsilon, b)$).

From Eqs. (B.2), (B.5), (B.6) and (B.7) we get the thesis (3.22). □

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