We discuss inequalities between the rank counts $N(r, m, n)$ and between the crank counts $M(r, m, n)$, for $m = 2, 3, 4$, and state three conjectures.

1. INTRODUCTION

A partition $\pi = (\pi_0, \pi_1, \ldots , \pi_k)$ is a finite (weakly) descending sequence of positive integers (the parts of $\pi$). Thus $\pi_0$ is the largest part of $\pi$. $\# \pi = k$ is the length of $\pi$ and $w(\pi) = \pi_0 + \pi_1 + \cdots + \pi_k$ is the weight of $\pi$. If $w(\pi) = n$, $\pi$ is a partition of $n$. In 1944 Dyson [5] defined the rank of a partition, $\pi$, by

$$\text{rank}(\pi) := \pi_0 - \# \pi$$

and set

$$N(m, n) := \# \{ \pi : w(\pi) = n, \text{rank}(\pi) = m \}$$

$$N(r, m, n) := \# \{ \pi : w(\pi) = n, \text{rank}(\pi) \equiv r \mod m \}.$$ 

Noting that $\text{rank}(\pi) = -\text{rank}(\bar{\pi})$ (where $\bar{\pi}$ denotes the conjugate [1, p. 7] of $\pi$), it follows that

$$N(m, n) = N(-m, n) \quad \text{and} \quad N(r, m, n) = N(-r, m, n).$$
Dyson observed that several relations appeared to hold among the \( N(r, m, n) \) when \( m = 5 \) and \( 7 \), and his observations were shown to be universally valid by Atkin and Swinnerton-Dyer [4]. Some 35 years later, Garvan defined the crank for certain vector partitions and he and Andrews subsequently defined

\[
\text{crank}(\pi) := \begin{cases} 
\pi_0, & \text{if } \mu(\pi) = 0, \\
\nu(\pi) - \mu(\pi), & \text{if } \mu(\pi) > 0,
\end{cases}
\]

where \( \mu(\pi) \) denotes the number of ones in \( \pi \) and \( \nu(\pi) \) denotes the number of parts of \( \pi \) larger than \( \mu(\pi) \). Following Dyson’s suggestion [5], they set, for \( n > 1 \),

\[
M(m, n) = \# \{ \pi : w(\pi) = n, \text{crank}(\pi) = m \} \\
M(r, m, n) = \# \{ \pi : w(\pi) = n, \text{crank}(\pi) \equiv r \mod m \}.
\]

We suppose the rank and the crank of the empty partition of 0 are each 0 and that

\[
M(1, 1) = M(-1, 1) = 1, \quad M(0, 1) = -1, \quad \text{and} \quad M(m, 1) = 0, \quad (m \neq \pm 1, 0).
\]

So the numbers \( M(m, n) \) are the numbers \( \mathcal{N}_{v}(m, n) \) defined by Garvan [7–9]. We take \( z \) and \( q \) to be complex variables with \( z \neq 0 \) and \( |q| < 1 \) and we will use the familiar notation

\[
(z; q)_{n} := \prod_{k=0}^{n-1} (1 - zq^{k}) \\
(z; q)_{\infty} := \prod_{k=0}^{\infty} (1 - zq^{k}).
\]

For future reference, we note that

\[
\frac{1}{(-q; q)_{2n}} = \frac{(q; q^{2})_{n}}{(q^{2n+2}; q^{2})_{n}} \quad (1.1)
\]

and

\[
\frac{1}{(-q; q)_{2n+1}} = \frac{(q; q^{2})_{n}}{(q^{2n+2}; q^{3})_{n+1}} \quad (1.2)
\]
It is not difficult to see that the generating function of the numbers $N(m, n)$ is given by

\[
\sum_{n=0}^{\infty} \sum_{m=\infty}^{n} N(m, n) z^m q^n = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(zq; q)_k (z^{-1}q; q)_k}
\]

\[
= 1 + \sum_{k=1}^{\infty} \frac{z^{k-1}q^k}{(z^{-1}q; q)_k},
\]

(1.3)

and we also have

\[
\sum_{n=0}^{\infty} \sum_{m=\infty}^{n} M(m, n) z^m q^n = \sum_{n=0}^{\infty} \sum_{m=\infty}^{n} N_p(m, n) z^m q^n
\]

\[
= \frac{(q; q)_\infty}{(zq; q)_\infty (z^{-1}q; q)_\infty}.
\]

(1.4)

In (1.3), $k$ marks the size of the Durfee square [1, pp. 27, 28] of a partition and, in the alternative expression (1.4), $k$ is the size of the largest part. The generating function for the crank (1.5) was given by Garvan [79].

It is shown in [10] that

\[
N(0, 2n) < N(1, 2n) \quad \text{if} \quad n \geq 1 \quad \text{and}
\]

\[
N(1, 2n + 1) < N(0, 2n + 1) \quad \text{if} \quad n \geq 0.
\]

(1.6)

The proof given in [10] of (1.6) is combinatorial (bijective) in nature and consists of the construction of maps

\[
\{ \text{partitions of } 2n \text{ of even rank} \} \rightarrow \{ \text{partitions of } 2n \text{ of odd rank} \}
\]

\[
\{ \text{partitions of } 2n + 1 \text{ of odd rank} \} \rightarrow \{ \text{partitions of } 2n + 1 \text{ of even rank} \}
\]

that are injective, but not surjective.

Setting $z = -1$ in (1.3), we see that

\[
\sum_{n=0}^{\infty} (N(0, 2, n) - N(1, 2, n)) q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n^n} =: f(q),
\]

where $f(q)$ is one of the third-order mock theta functions [11]. Thus (1.6) is the statement that the signs of the coefficients of $f(q)$ are $+, +, -, +, -, -,$ (alternating thereafter) or, equivalently, that the signs of the coefficients in $f(-q)$ are $+, -, +, -, -,$ (and thereafter all negative).
In fact, (1.6) has a straightforward algebraic derivation which we include, since it foreshadows our later arguments. Setting \( z = -1 \) in (1.4), we have

\[
f(q) = 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{q^k}{(-q; q)_k}
\]

(1.7)

and so

\[
f(-q) = 1 - \sum_{k=1}^{\infty} \frac{q^k}{(q; -q)_k}
\]

which, by (1.1) and (1.2),

\[
= 1 - \left\{ \sum_{k=1}^{\infty} \frac{q^{2k}(-q; q^2)_k}{(q^{2k+1}; q^4)_k} + \sum_{k=1}^{\infty} \frac{q^{2k-1}(-q; q^2)_k}{(q^{2k-1}; q^4)_k} \right\}.
\]

The coefficients of the terms of each sum in the brackets are clearly positive and this settles 1.6.

A number of inequalities between the \( N(r, m, n) \) and the \( M(r, m, n) \) were found by Garvan [79] when \( m = 5, 7 \), and Ekin [6] gave some inequalities between the \( M(r, 11, n) \). Here we establish some inequalities between the \( M(r, m, n) \) and the \( N(r, m, n) \) when \( m = 2, 3, \) and 4. We also state a number of conjectures.

2. \( M = 2 \)

The numbers \( M(r, 2, n) \) satisfy inequalities that are the reverse of those for the rank (1.6). We prove

**Theorem 1.** For all \( n \geq 0 \),

\[
M(0, 2, 2n) > M(1, 2, 2n),
\]

\[
M(1, 2, 2n + 1) > M(0, 2, 2n + 1).
\]

**Proof.** By (1.5), we have

\[
\sum_{n=0}^{\infty} (M(0, 2, n) - M(1, 2, n)) q^n = \frac{(q; q)_{\infty}}{(q^2 q^2; q^2)_{\infty}} =: g(q),
\]

(2.1)
say, and we want to show that the coefficient of $q^n$ in $g(q)$ is positive/negative according to whether $n$ is even or odd. So we need to show that the coefficients of $g(-q)$ are all positive. But

$$g(-q) = \frac{(-q; -q)_\infty}{(q; -q)_\infty^2} = (-q; q^2)_\infty^3 (q^2; q^2)_\infty,$$

which, by Jacobi’s Triple Product Identity,

$$= (-q; q^2)_\infty \sum_{n=-\infty}^{\infty} q^{n^2}.$$

Since every positive integer is the sum of a perfect square and an odd number, the coefficients of $g(-q)$ are all positive.

3. $M = 3$

We have no solid facts about the case $m = 3$ and merely present two conjectures. We first note that setting $z = e^{2\pi i/3}$ in (1.3) gives

$$\sum_{n \neq 0} (N(0, 3, n) - N(1, 3, n)) q^n = \sum_{n \neq 0} \frac{q^{n^2}}{(1 + q + q^2) \cdots (1 + q^n + q^{2n})} = \sum_{n \neq 0} \frac{q^{n^2}(q; q)_n}{(q^3; q^3)_n} = \gamma(q),$$

where $\gamma(q)$ is one of the sixth-order mock theta functions [3]. Also, setting $z = e^{2\pi i/3}$ in (1.2) we have

$$\sum_{n \neq 0} (M(0, 3, n) - M(1, 3, n)) q^n = \frac{(q; q)_\infty^2}{(q^3; q^3)_\infty}.$$

Computer evidence suggests the following:

**Conjecture 1.** For all $n > 0$

$$N(0, 3, 3n) < N(1, 3, 3n), \quad (3.1)$$
$$N(0, 3, 3n + 1) > N(1, 3, 3n + 1), \quad (3.2)$$
$$N(0, 3, 3n + 2) < N(1, 3, 3n + 2). \quad (3.3)$$
Conjecture 2. For all \(n\),
\[
M(0, 3, 3n) > M(1, 3, 3n),
\]
\[
M(0, 3, 3n + 1) < M(1, 3, 3n + 1),
\]
\[
M(0, 3, 3n + 2) \leq M(1, 3, 3n + 2), \quad \text{if } n \neq 1,
\]
with strict inequality in (3.6) if \(n \neq 4, 5\).

These conjectures (Conjecture 2, in particular) seem to be related to the Borwein conjectures [2]. We have no proofs of any one of (3.1)–(3.6).

4. \(M = 4\)

Setting \(z = i\) in (1.3) gives
\[
\sum_{n=0}^{\infty} (N(0, 4, n) - N(2, 4, n)) q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2; q^2)_n} = \phi(q),
\]
which is one of the third-order mock theta functions [11]. We will prove

**Theorem 2.**
\[
N(0, 4, n) = N(2, 4, n), \quad \text{for } n = 2, 8, 10, \text{ and } 26,
\]
while, for other \(n\)
\[
N(0, 4, n) > N(2, 4, n), \quad \text{if } n \equiv 0, 1 \pmod{4},
\]
\[
N(0, 4, n) < N(2, 4, n), \quad \text{if } n \equiv 2, 3 \pmod{4}.
\]

**Proof.** Set \(\alpha(n) := N(0, 4, n) - N(2, 4, n)\). Then, with \(\phi(q) = \sum_{n=0}^{\infty} \alpha(n) q^n\), we will show that
\[
\alpha(n) = \begin{cases} 
0, & n = 2, 8, 10, 26, \\
> 0, & n \equiv 0, 1 \pmod{4}, \ n \neq 8, \\
< 0, & n \equiv 2, 3 \pmod{4}, \ n \neq 2, 10, 26.
\end{cases}
\]
We first note, by expanding the series for \(\phi(q)\), that \(\alpha(n) = 0\) for \(n = 2, 8, 10, 26\), thus verifying (4.2).

The \(q\)-binomial theorem [1, Theorem 3.3, p. 36] states that
\[
[z; q]_n = \sum_{i=0}^{n} \frac{(-1)^i z^i q^{i^2}}{[i]_{q^2}}
\]
and so we have
\[
\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n} q^{n+1}}{(q^{4n+4}; q^{4})_{n}} = \sum_{n=0}^{\infty} \frac{q^{4n+4} q^{4n+1}}{(q^{4n+4}; q^{4})_{n}}
\]
\[
= \sum_{n=0}^{\infty} \frac{q^{4n+4} q^{4n+1}}{(q^{4n+4}; q^{4})_{n}} \sum_{j=0}^{n} (-1)^{j} q^{2j} \left[ \begin{array}{c} n \\ j \end{array} \right] q^{j} + \sum_{n=0}^{\infty} \frac{q^{4n+4} q^{4n+1}}{(q^{4n+4}; q^{4})_{n+1}} \sum_{j=0}^{n+1} (-1)^{j} q^{2j} \left[ \begin{array}{c} n+1 \\ j \end{array} \right] q^{j}.
\]
(4.5)

But the coefficients of \( [n] \) are nonnegative (since \( [n] \) is the generating function for partitions into \( n-j \) or fewer parts, all no bigger than \( j \)) and (4.5) shows that \( \alpha(m) \geq 0 \) (\( \leq 0 \)), when \( m \equiv 0, 1 \text{ mod } 4 \) (\( \equiv 2, 3 \text{ mod } 4 \)).

Now the first terms of \( \phi(q) \) are
\[
1 + q(1-q^{2}) + q^{2}(1-q^{2}) + q^{3}(1-q^{2})(1-q^{4}) + q^{4}(1-q^{2})(1-q^{6})
\]
\[
= \frac{q^{25}(1-q^{2})(1-q^{4})(1-q^{10})}{(1-q^{8})(1-q^{12})(1-q^{16})} + \frac{q^{24}(1-q^{2})(1-q^{4})(1-q^{10})}{(1-q^{8})(1-q^{12})(1-q^{16})} + \frac{q^{24}(1-q^{2})(1-q^{4})(1-q^{10})}{(1-q^{8})(1-q^{12})(1-q^{16})} + \frac{q^{24}(1-q^{2})(1-q^{4})(1-q^{10})}{(1-q^{8})(1-q^{12})(1-q^{16})}
\]
We see that the term \( q(1-q^{2})(1-q^{4}) \) guarantees that \( \alpha(m) > 0 \), if \( m \equiv 1 \text{ mod } 4 \), and \( \alpha(m) < 0 \), if \( m \equiv 3 \text{ mod } 4 \). The term \( q^{4}(1-q^{2})(1-q^{4}) \) means that \( \alpha(m) > 0 \) if \( m \equiv 4 \text{ mod } 8 \) and \( q^{16}(1-q^{2})(1-q^{4})(1-q^{12})(1-q^{16}) \) means that \( \alpha(m) > 0 \) if \( m \equiv 0 \text{ mod } 8 \) and \( m \not\equiv 8 \). Hence \( \alpha(m) > 0 \) if \( m \equiv 0 \text{ mod } 4 \) and \( m \not\equiv 8 \). Finally, the term \( q^{4}(1-q^{2})(1-q^{4}) \) guarantees \( \alpha(m) < 0 \) if \( m \equiv 6 \text{ mod } 8 \), the term \( q^{16}(1-q^{2})(1-q^{4})(1-q^{12})(1-q^{16}) \) guarantees \( \alpha(m) < 0 \) if \( m \equiv 2 \text{ mod } 16 \) and \( m \geq 18 \), and the term \( q^{16}(1-q^{2})(1-q^{4})(1-q^{10})(1-q^{16})(1-q^{20})(1-q^{24}) \) guarantees \( \alpha(m) < 0 \) if \( m \equiv 10 \text{ mod } 16 \) and \( m \geq 42 \). So \( \alpha(m) < 0 \) if \( m \equiv 2 \text{ mod } 4 \) and \( m \not\equiv 2, 10, 26 \). This completes the proofs of (4.3) and (4.4).

Setting \( z = i \) in (1.5) we have
\[
\sum_{n=0}^{\infty} (M(0, 4, n) - M(2, 4, n)) q^n = \frac{(q; q)_{\infty}}{(-q^{2}; q^{2})_{\infty}} = \frac{(q; q)_{\infty} (q^{2}; q^{2})_{\infty}}{(q^{4}; q^{4})_{\infty}}.
\]
(4.6)

Again, there seem to be inequalities among the \( M(0, 4, n) \) and \( M(2, 4, n) \) that are periodic mod 4 and computer evidence suggests

Conjecture 3. For \( n \neq 5 \)
\[
M(0, 4, n) \geq M(2, 4, n), \quad \text{if } n \equiv 0, 3 \text{ mod } 4, \quad (4.7)
\]
\[
M(0, 4, n) \leq M(2, 4, n), \quad \text{if } n \equiv 1, 2 \text{ mod } 4, \quad (4.8)
\]
the inequalities being strict if \( n \neq 11, 15, 21 \). We have no proof of either (4.7) or (4.8). (In fact, \( M(0, 4, 5) - M(2, 4, 5) = 1 \), which suggests that this conjecture, if true, may be hard to prove.)

Now we have, by (2.1),

\[
\sum_{n=0}^{\infty} (M(0, 4, n) + M(2, 4, n) - 2M(1, 4, n)) q^n
= \sum_{n=0}^{\infty} (M(0, 4, n) - M(1, 2, n)) q^n = \frac{(q; q)_{\infty}}{(-q; q)_{\infty}}
\]

and, with (4.6), we have

\[
2 \sum_{n=0}^{\infty} (M(0, 4, n) - M(1, 4, n)) q^n
= \sum_{n=0}^{\infty} (M(0, 4, n) + M(2, 4, n) - 2M(1, 4, n))
+ (M(0, 4, n) - M(2, 4, n))
= (q; q)_{\infty} \left\{ \frac{1}{(q; q^2)_{\infty}^2} + \frac{1}{(-q^2; q^2)_{\infty}} \right\} =: \pi(q),
\]

say. Now

\[
\pi(-q) = (-q; -q)_{\infty} \left\{ \frac{1}{(q; -q^2)_{\infty}^2} + \frac{1}{(-q^2; q^2)_{\infty}} \right\}
= (-q; q^2)_{\infty} (q^2; q^2)_{\infty} \left\{ (-q; q^2)_{\infty}^2 + (q^2; q^4)_{\infty} \right\}
= (-q; q^2)_{\infty} (q^2; q^2)_{\infty} \left\{ (-q; q^2)_{\infty}^2 + (q^2; q^2)_{\infty} \right\}.
\]

But

\[
(-q; q^2)_{\infty}^2 (q^2; q^2)_{\infty} = \sum_{n=-\infty}^{\infty} q^n
\]

has non-negative coefficients and

\[
(-q; q^2)_{\infty} + (q; q^2)_{\infty} = 2 \sum_{n=0}^{\infty} a(n) q^n,
\]
where \( a(n) \) is the number of partitions of \( n \) into an even number of different odd numbers \((a(0) = 1)\). Thus

\[
a(-q) = \sum_{n=-\infty}^{\infty} q^{2n} \sum_{n=0}^{\infty} a(n) q^n
\]

\[
= (1 + 2q + 2q^4 + \cdots)(1 + q^4 + q^8 + 2q^8 + \cdots)
\]

has non-negative coefficients. It is easy to see that \( a(n) > 0 \) for even \( n > 2 \) and it follows that the coefficient of \( q^n \) in \( a(-q) \) are positive for \( n > 3 \).

In just the same way, we see that, if

\[
\beta(q) := \sum_{n=0}^{\infty} (M(2, 4, n) - M(1, 4, n)) q^n
\]

\[
= \frac{1}{2} (q; q) \frac{1}{(1 - q^2; q^2)} \frac{1}{1 - q^{2n}},
\]

then

\[
\beta(-q) = \frac{1}{2} (-q, q^2) \frac{1}{(1 - q^2; q^2)} \frac{1}{1 - q^{2n}},
\]

where \( b(n) \) is the number of partitions of \( n \) into an odd number of distinct odd parts \((b(0) = 0)\). We see that the coefficients of \( q^n \) in \( \beta(-q) \) are positive for \( n > 0 \) and we have proved

**Theorem 3.**
(i) \( M(0, 4, 2n) > M(1, 4, 2n) \), for \( n \neq 1 \),

(ii) \( M(0, 4, 2n - 1) < M(1, 4, 2n - 1) \), for \( n \neq 2 \),

(iii) \( M(2, 4, 2n) > M(1, 4, 2n) \), for \( n > 0 \),

(iv) \( M(2, 4, 2n - 1) < M(1, 4, 2n - 1) \), for \( n > 0 \).

If \( f(q) = \sum_{n=0}^{\infty} a_n q^n \) and \( g(q) = \sum_{n=0}^{\infty} b_n q^n \) are power series in \( q \), we write \( f(q) \leq g(q) \) to mean \( a_n \leq b_n \) for all \( n \). We now prove

**Theorem 4.**

\[
N(0, 4, 2n) < N(1, 4, 2n) \quad \text{for all } n \geq 1,
\]

\[
N(0, 4, 2n - 1) > N(1, 4, 2n - 1) \quad \text{for all } n \geq 1,
\]

\[
N(2, 4, 2n) < N(1, 4, 2n) \quad \text{for all } n \geq 1,
\]

\[
N(2, 4, 2n - 1) > N(1, 4, 2n - 1) \quad \text{for all } n \geq 2.
\]
Proof. We note first that
\[ 1 + \sum_{k=0}^{\infty} q^{2k+1} (-q; q^2)_k = (-q; q^2)_\infty = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q^2; q^2)_k}, \] (4.13)
since each of these expressions is the generating function of partitions into distinct odd parts.
We have, by (1.7) and (4.1),
\[ 2 \sum_{n=1}^{\infty} (N(0, 4, n) - N(1, 4, n)) q^n \]
\[ = \sum_{n=1}^{\infty} (N(0, 4, n) + N(2, 4, n) - 2N(1, 4, n)) q^n \\
+ \sum_{n=1}^{\infty} (N(0, 4, n) - N(2, 4, n)) q^n \\
= \sum_{n=1}^{\infty} (N(0, 2, n) - N(1, 2, n)) q^n + \sum_{n=1}^{\infty} (N(0, 4, n) - N(2, 4, n)) q^n \\
= \sum_{k=1}^{\infty} (-1)^k -1 \frac{q^k}{(-q; q)_k} + \sum_{k=1}^{\infty} \frac{q^{k^2}}{(-q^2; q^2)_k} \\
= f_1(q) + \phi_1(q), \]
say (where we have written \( f_1(q) \) and \( \phi_1(q) \) for \( f(q) - 1 \) and \( \phi(q) - 1 \), respectively). To prove (4.9) and (4.10) we must show that the coefficients of \( f_1(-q) + \phi_1(-q) \) are negative for \( n \geq 1 \).
Now
\[ \phi_1(-q) = \sum_{k=1}^{\infty} (-1)^k \frac{q^{k^2}}{(-q^2; q^2)_k} \]
\[ = \sum_{k=1}^{\infty} \frac{q^{k^2}}{(q^2; q^2)_k} \]
\[ = -1 + (-q; q^2)_\infty, \]
by (4.13), and
\[ f_1(-q) = - \sum_{k=1}^{\infty} \frac{q^k}{(q; -q)_k} \]
\[ = - \left( \sum_{k=1}^{\infty} \frac{q^{2k-1}}{(q; -q)_{2k-1}} + \sum_{k=1}^{\infty} \frac{q^{2k}}{(q; -q)_{2k}} \right). \]
which, by (1.1) and (1.2),

\[ \sum_{k=1}^{\infty} q^{2k-1} \frac{(-q; q^2)_k}{(q^{2k}; q^2)_k} + \sum_{k=1}^{\infty} q^{2k} \frac{(-q; q^3)_k}{(q^{2k+2}; q^2)_k} \]

\leq \left( \sum_{k=1}^{\infty} q^{2k-1} (-q; q^2)_{k-1} + \sum_{k=1}^{\infty} q^{2k} (1 + q) \right)

\leq 1 - (-q; q^3)_{\infty} - \sum_{k=1}^{\infty} q^{2k} (1 + q),

by (4.13). So \( f_1(-q) + \frac{1}{2} \phi_1(-q) \leq - \sum_{k \geq 2} q^k \), showing that (4.9) and (4.10) hold for \( n \geq 2 \). But \( N(0, 4, 1) = 1 \) and \( N(1, 4, 1) = 0 \), which completes the proofs of (4.9) and (4.10). Equations (4.11) and (4.12) may be proved in the same way, using

\[ 2 \sum_{n=1}^{\infty} (N(2, 4, n) - N(1, 4, n)) q^n = 2 \sum_{k=1}^{\infty} (-1)^{k-1} \frac{q^k}{(-q; q)_k} - \sum_{k=1}^{\infty} \frac{q^{2k}}{(-q^2; q^2)_k}. \]

REFERENCES


