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The Brauer Group of Dimodule Algebras

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INTRODUCTION

In this paper we develope a Brauer group theory for algebras which are simultaneously H-modules and H-comodules, where H is a Hopf algebra. This work generalizes, and also arises naturally from, the work in [4] where we considered algebras acted on by a group and graded by the same group.

In Section 1 we consider the case where there is only an H-module action. This generalizes the equivariant Brauer group of Fröhlich and Wall [2] and at the end of Section 1 we show that some of Fröhlich and Wall's results extend to the Hopf algebra situation.

H-comodule action is studied in Section 2, and we show that this is a straightforward generalization of group grading. The two concepts of module and comodule action are applied simultaneously in Section 3. When they commute (in a sense made precise in Section 3) we call the resulting structure an H-dimodule. This is not the same as an H Hopf module (see [6]) as the commuting condition is different.

In Section 4 we say when an H-dimodule algebra is H-Azumaya. This generalizes the usual definition of Azumaya algebra to our situation and allows us to define "the Brauer group of dimodule algebras." We then obtain some of its properties.

We conclude the paper with an example. In Section 5 we consider the Brauer group of dimodule algebras over a field of characteristic p where the Hopf algebra is the group ring over that field of the group of p elements. This covers the outstanding case left from [4].

Throughout, R is a fixed commutative ring with 1, each $\otimes$, Hom, etc. is taken over R and each map is R-linear unless otherwise stated.

1. HOPF ALGEBRAS AND MODULE ALGEBRAS

DEFINITION 1.1. A *Hopf Algebra* H (over R) is an R-module H together with the following structure maps, multiplication $\cdot$: $H \otimes H \to H$ (written $h_1 \cdot h_2$)

unit 1_H: $R \to H$ (we also use 1_H to represent the unit element)

diagonalization Δ: $H \otimes H \to H$ (we write $\Delta(h) = \sum_{(h)} h^{(1)} \otimes h^{(2)}$)

counit ϵ: $H \to R$

antipode S: $H \to H$

so that H is an R-algebra with product $\cdot$ and unit $1_H(1)$ and an R-coalgebra under Δ and ϵ. This means that Δ, ϵ satisfy axioms dual to those satisfied by $\cdot$ and 1_H. Moreover, Δ and ϵ are algebra homomorphisms, or equivalently $\cdot$ and 1_H are coalgebra homomorphisms. S acts as a sort of inverse for both the algebra and coalgebra structures.

DEFINITION 1.2. A Hopf algebra is said to be *commutative* if the multiplication is commutative and *cocommutative* if the diagonalization is commutative, i.e., $\sum_{(h)} h^{(1)} \otimes h^{(2)} = \sum_{(h)} h^{(2)} \otimes h^{(1)}$.

For more details about Hopf algebras and a more rigorous definition, see [6]. We will sometimes use elementary results on Hopf algebras without stating them explicitly beforehand.

A classical example of a Hopf algebra is the group ring $R[\Gamma]$ where Γ is any group. This has the usual algebra structure and

$$\Delta(\gamma) = \gamma \otimes \gamma \quad \gamma \in \Gamma \text{ and all}$$
$$\epsilon(\gamma) = 1 \quad \text{extended by linearity}$$
$$S(\gamma) = \gamma^{-1} \quad \text{to the whole of } R[\Gamma].$$

In an arbitrary Hopf algebra H, an element g which satisfies $\Delta(g) = g \otimes g$ is said to be *group-like*.

DEFINITION 1.3. A *left H-module M* is an R-module together with a map $\rightharpoonup_M$: $H \otimes M \to M$ so that the following diagrams commute.

(i)

$$\begin{tikzcd}
H \otimes H \otimes M \arrow[r, "\cdot \otimes I"] \arrow[d, "I \otimes \rightharpoonup_M"'] & H \otimes M \arrow[d, "\rightharpoonup_M"] \\
H \otimes M \arrow[r, "\rightharpoonup_M"] & M
\end{tikzcd}$$

(ii)

$$\begin{tikzcd}
R \otimes M \arrow[r, "1_H \otimes I"] \arrow[dr] & H \otimes M \arrow[d, "\rightharpoonup_M"] \\
& M
\end{tikzcd}$$

PROPOSITION 1.4. *Let M, N be left H modules with actions $\rightharpoonup_M$, $\rightharpoonup_N$. Then the map*

(i)

$$\underset{M\otimes N}{\rightharpoonup}: H\otimes M\otimes N \xrightarrow{\Delta\otimes I\otimes I} H\otimes H\otimes M\otimes N \xrightarrow{I\otimes T\otimes I} H\otimes M\otimes H\otimes N \xrightarrow{\vec{M}\otimes\vec{N}} M\otimes N$$

gives a left H-modules structure to $M\otimes N$. $\mathrm{Hom}(M, N)$ *is also a left H-module by the following action:*

(ii) $(h\rightharpoonup f)(m) = \sum_{(h)} h^{(1)} \rightharpoonup_N [f(S(h)^{(2)}) \rightharpoonup_M m)]$ $h\in H$, $m\in M$, $f\in \mathrm{Hom}(M, N)$.

Proof. We will run through the proof of part (ii) in order to show the techniques involved, although these are quite straightforward. Part (i) is even more straightforward.

We verify diagram 1.3(i). Let $f\in \mathrm{Hom}(M, N)$, $m\in M$, $h_1, h_2\in H$.

$$\begin{aligned}
(h_1\cdot h_2\rightharpoonup f)(m) &= \sum_{(h_1\cdot h_2)} (h_1\cdot h_2)^{(1)} \underset{N}{\rightharpoonup} [f(S(h_1\cdot h_2)^{(2)}) \underset{M}{\rightharpoonup} m)] \quad \text{by definition of } \rightharpoonup \\
&= \sum_{(h_1)}\sum_{(h_2)} (h_1^{(1)}\cdot h_2^{(1)}) \underset{N}{\rightharpoonup} [f(S(h_1^{(2)}\cdot h_2^{(2)}) \underset{M}{\rightharpoonup} m)] \\
&= \sum_{(h_1)(h_2)} (h_1^{(1)}\cdot h_2^{(2)}) \underset{N}{\rightharpoonup} [f(S(h_2^{(2)})\cdot S(h_1^{(2)}) \underset{M}{\rightharpoonup} m)] \\
&= \sum_{(h_1)(h_2)} h_1^{(1)} \underset{N}{\rightharpoonup} \{h_2^{(1)} \underset{N}{\rightharpoonup} [f(S(h_2^{(2)}) \underset{M}{\rightharpoonup} (S(h_1^{(2)}) \underset{M}{\rightharpoonup} m))]\} \\
&\qquad \text{by 1.3(i) for } \underset{M}{\rightharpoonup} \text{ and } \underset{N}{\rightharpoonup} \\
&= \sum_{(h_1)} h_1^{(1)} \underset{N}{\rightharpoonup} [(h_2\rightharpoonup f)(S(h_1^{(2)}) \underset{M}{\rightharpoonup} m)]. \quad \text{by definition of } \rightharpoonup \\
&= (h_1\rightharpoonup (h_2\rightharpoonup f))(m) \quad \text{by definition of } \rightharpoonup
\end{aligned}$$

$$\therefore\ h_1\cdot h_2\rightharpoonup f = h_1\rightharpoonup (h_2\rightharpoonup f)$$

$$\Delta(1_H) = 1_H\otimes 1_H, \quad \text{and} \quad S(1_H) = 1_H,$$

so 1.3(ii) follows immediately. Q.E.D.

DEFINITION 1.5. If M, N are as above, and $f\in \mathrm{Hom}(M, N)$, then f is said to be an *H-module map* if the following diagram commutes.

$$\begin{array}{ccc} H \otimes M & \xrightarrow{I \otimes f} & H \otimes N \\ \Big\downarrow{\scriptstyle \underset{M}{\rightharpoonup}} & & \Big\downarrow{\scriptstyle \underset{N}{\rightharpoonup}} \\ M & \longrightarrow & N \end{array}$$

DEFINITION 1.6. A is a *left H-module algebra* if it is an R algebra and a left H-module so that the structure maps

$$1_A: R \to A \qquad \text{and} \qquad \cdot_A: A \otimes A \to A$$

are H-module maps. (R is a left H-module by $\rightharpoonup_R : H \otimes R \cong H \xrightarrow{\epsilon} R$, and $A \otimes A$ is a left H-module by 1.4).

These conditions say that $h \rightharpoonup 1_A = \epsilon(h) \cdot 1_A$ and

$$h \rightharpoonup (a_1 \cdot a_2) = \sum_{(h)} (h^{(1)} \rightharpoonup a_1) \cdot (h^{(2)} \rightharpoonup a_2).$$

If A is an algebra then let A^{op} denote the opposite algebra. This is isomorphic to A as R-module so, if A is an H-module then A^{op} is automatically an H-module as well.

PROPOSITION 1.7. *Let H be cocommutative. If A, B are left H-module algebras then so are $A \otimes B$ and A^{op}, and $A \otimes B$ and $B \otimes A$ are isomorphic as H-module algebras.*

Proof. Straightforward.

PROPOSITION 1.8. *Let H be cocommutative. If M is a left H-module then* $\mathrm{End}(M)$ *is a left H-module algebra. If M, N are left H-modules which are finitely generated projective over R, then the natural isomorphism*

$$\mathrm{End}(M) \otimes \mathrm{End}(N) \cong \mathrm{End}(M \otimes N)$$

is an H-module map.

Proof. We have that $\mathrm{End}(M) = \mathrm{Hom}(M, M)$ is a left H-module by 1.4(ii). We show that the H-module action respects the algebra structure in $\mathrm{End}(M)$.

We have $(h \rightharpoonup_E f)(m) = \sum_{(h)} h^{(1)} \rightharpoonup_M [f(S(h^{(2)}) \rightharpoonup_M m)]$. Let $h \in H$, $m \in M$, $f, g \in \mathrm{End}(M)$. Then

$$\begin{aligned} &\Big[\sum_{(h)} (h^{(1)} \rightharpoonup f) \cdot (h^{(2)} \rightharpoonup g)\Big](m) \\ &\quad = \sum_{(h)} (h^{(1)} \rightharpoonup f)[(h^{(2)} \rightharpoonup g)(m)] \\ &\quad = \sum_{(h)} (h^{(1)} \rightharpoonup f)[h^{(2)} \underset{M}{\rightharpoonup} (g(S(h^{(3)}) \underset{M}{\rightharpoonup} m))], \end{aligned}$$

(where $\sum_{(h)} h^{(1)} \otimes h^{(2)} \otimes h^{(3)}$ denotes $(I \otimes \Delta)\, \Delta(h)$)

$$= \sum_{(h)} h^{(1)} \underset{M}{\rightharpoonup} \{f[S(h^{(2)}) \underset{M}{\rightharpoonup} (h^{(3)} \underset{M}{\rightharpoonup} [g(S(h^{(4)}) \underset{M}{\rightharpoonup} m)])]\}$$

$$= \sum_{(h)} h^{(1)} \underset{M}{\rightharpoonup} \{f[(S(h^{(2)}) \cdot h^{(3)}) \underset{M}{\rightharpoonup} (g[S(h^{(4)}) \underset{M}{\rightharpoonup} m])]\}$$

$$= \sum_{(h)} h^{(1)} \underset{M}{\rightharpoonup} [f(\epsilon(h^{(2)}) \cdot 1_H \underset{M}{\rightharpoonup} [g(S(h^{(3)}) \underset{M}{\rightharpoonup} m)])]$$

$$= \sum_{(h)} h^{(1)}\epsilon(h^{(2)}) \underset{M}{\rightharpoonup} [f(g[S(h^{(3)}) \underset{M}{\rightharpoonup} m])]$$

($\epsilon(h^{(2)}) \in R$, so can be "pulled through")

$$= \sum_{(h)} h^{(1)} \underset{M}{\rightharpoonup} [f \cdot g(S(h^{(2)}) \underset{M}{\rightharpoonup} m)]$$

$$= (h \rightharpoonup f \cdot g)(m)$$

So,

$$h \rightharpoonup f \cdot g = \sum_{(h)} (h^{(1)} \rightharpoonup f) \cdot (h^{(2)} \rightharpoonup g).$$

$$\begin{aligned}(h \rightharpoonup I_M)(m) &= \sum_{(h)} h^{(1)} \underset{M}{\rightharpoonup} [I_M(S(h^{(2)}) \underset{M}{\rightharpoonup} m)] \\ &= \sum_{(h)} h^{(1)} \underset{M}{\rightharpoonup} (S(h^{(2)}) \underset{M}{\rightharpoonup} m) \\ &= \sum_{(h)} h^{(1)} \cdot S(h^{(2)}) \underset{M}{\rightharpoonup} m \\ &= \epsilon(h)m.\end{aligned}$$

That is, $h \rightharpoonup I_M = \epsilon(h) \cdot I_M$. Hence, End($M$) is an H-module algebra. The rest is easy. Q.E.D.

PROPOSITION 1.9. *Let H be cocommutative. If A is a left H-module algebra then the standard map $A \otimes A^{\text{op}} \to \text{End}(A)$ is an H-module algebra map.*

Proof. Straightforward.

We now have all the apparatus necessary to define the Brauer group for the category of left H-module algebras which are finitely generated projective faithful Azumaya algebras over R, (with H cocommutative).

We define $A \sim B$ (in this category) if $\exists$ finitely generated projective faithful R-modules M, N which are also left H-modules such that

$$A \otimes \text{End}(M) \cong B \otimes \text{End}(N) \qquad \text{as } H\text{-module algebras.}$$

It is easy to verify that $\sim$ is an equivalence relation and the quotient set is a group under the operations induced by $\otimes$.

DEFINITION 1.10. We call this group *the Brauer group of H-module algebras* and denote it by $BM(R, H)$.

This generalizes the equivariant Brauer group of Fröhlich and Wall (see [2]). We now prove that, for certain Hopf algebras H,

$$BM(K, H) \cong B(K) \times H^2(H, K),$$

where K is a field and $H^2(H, K)$ is the 2-cohomology group as defined in [7] (H acts trivially on K). This result is an extension of a result of Fröhlich and Wall.

By ignoring the H-structure, it is easy to see that one can get a group homomorphism from $BM(K, H)$ onto $B(K)$. Let the kernel of this homomorphism be denoted by $Bo(K, H)$. So we have a short exact sequence

$$0 \to Bo(K, H) \to BM(K, H) \to B(K) \to 0.$$

If we have any K-algebra A then we can give it the trivial H-module structure $h \rightharpoonup a = \epsilon(h)a$. This construction gives rise to an injective group homomorphism from $B(K)$ into $BM(K, H)$ which splits the above short exact sequence.

So we have already $BM(K, H) \cong B(K) \times Bo(K, H)$.

Now, $Bo(K, H)$ consists of classes of algebras which become (ordinary) Brauer trivial when we forget the H-structure. i.e., they are endomorphism rings over our base field K. So, when calculating $Bo(K, H)$ we had only consider endomorphism rings.

For the remainder of this section we suppose that H is a cocommutative Hopf algebra over K whose simple subcoalgebras are of the form Kg for some grouplike element g. Then [7, Theorem 9.5] applies and any H-action on a finite dimensional central simple K algebra A is A-inner. This means that there is a map $f: H \to A$ such that

$$h \rightharpoonup a = \sum_{(h)} f(h^{(1)})\, af^{-1}(h^{(2)}) \qquad \forall h \in H,\ a \in A.$$

Here, f^{-1} is the inverse of f under convolution in the algebra $\mathrm{Hom}(H, A)$ and satisfies

$$\sum_{(h)} f(h^{(1)}) \cdot f^{-1}(h^{(2)}) = \epsilon(h) \cdot 1_A .$$

Since $h_1 \rightharpoonup (h_2 \rightharpoonup a_2) = h_1 h_2 \rightharpoonup a$ we have

$$\sum_{(h_1)(h_2)} f(h_1^{(1)} h_2^{(1)})\, af^{-1}(h_1^{(2)} h_2^{(2)}) = \sum_{(h_1)(h_2)} f(h_1^{(1)}) f(h_2^{(1)})\, af^{-1}(h_2^{(2)}) f^{-1}(h_1^{(2)}),$$

i.e.,

$$\sum_{(h_1)(h_2)} af^{-1}(h_1^{(1)}h_2^{(1)})\, f(h_1^{(2)})\, f(h_2^{(2)}) = \sum_{(h_1)(h_2)} f^{-1}(h_1^{(1)}h_2^{(1)})\, f(h_1^{(2)})\, f(h_2^{(2)})a \quad \forall a \in A.$$

Since A is central we deduce that $\forall h_1, h_2 \in H$

$$\sum_{(h_1)(h_2)} f^{-1}(h_1^{(1)}h_2^{(1)})\, f(h_1^{(2)})\, f(h_2^{(2)}) = c(h_1, h_2)\, 1_A,$$

for some $c(h_1, h_2) \in K$. We can rewrite this as

$$f(h_1)\, f(h_2) = \sum_{(h_1)(h_2)} c(h_1^{(1)}, h_2^{(2)})\, f(h_1^{(2)}h_2^{(2)}).$$

Using associativity in A gives that c satisfies the 2-cocycle identity,

$$\sum_{(h_1)(h_2)} c(h_1^{(1)}, h_2^{(1)})\, c(h_1^{(2)}h_2^{(2)}, h_3) = \sum_{(h_2)(h_3)} c(h_1, h_2^{(1)}h_3^{(1)})\, c(h_2^{(2)}, h_3^{(2)}).$$

It is straightforward to check that if we had started with a different inner action given by $g\colon H \to A$ then we would have arrived at a 2-cocycle in the same cohomology class as c.

LEMMA 1.11. *The Azumaya H-module algebra $A = \mathrm{End}(V)$ is Brauer trivial (in $BM(K, H)$) if and only if its associated cocycle c is a coboundary.*

Proof. $\Rightarrow$: A is Brauer trivial so the H-action on A is derived from an H-action on V as in 1.4(ii). Hence, the inner action on A is given by

$$f(h) = f_h \qquad \text{where} \quad f_h(v) = h \rightharpoonup v.$$

Then $f(h_1)\, f(h_2) = f_{h_1} f_{h_2} = f_{h_1 h_2} = f(h_1 h_2)$. So, the associated cocycle is trivial, being $c(h_1, h_2) = \epsilon(h_1)\, \epsilon(h_2)$.

$\Leftarrow$: Suppose that $A = \mathrm{End}(V)$ gives rise to the cohomologically trivial cocycle c.

$$\therefore\ c(h_1, h_2) = \sum_{(h_1)(h_2)} g(h_1^{(1)})\, g(h_2^{(1)})\, g^{-1}(h_1^{(2)}h_2^{(2)})$$

for some invertible map $g\colon H \to K$.

$$f(h_1)\, f(h_2) = \sum_{(h_1)(h_2)} g(h_1^{(1)})\, g(h_2^{(1)})\, g^{-1}(h_1^{(2)}h_2^{(2)})\, f(h_1^{(3)}h_2^{(3)}), \tag{*}$$

where $f\colon H \to A$ gives the inner action $h \rightharpoonup a = \sum_{(h)} f(h^{(1)})\, af^{-1}(h^{(2)})$. Define an H-action on V by $h \rightharpoonup v = \sum_{(h)} g^{-1}(h^{(1)})\, f(h^{(2)})(v)$.

Then

$$\begin{aligned} h_1 \rightharpoonup (h_2 \rightharpoonup v) &= h_1 \rightharpoonup \Big[\sum_{(h_2)} g^{-1}(h_2^{(1)})\, f(h_2^{(2)})(v)\Big] \\ &= \sum_{(h_1)(h_2)} g^{-1}(h_2^{(1)})\, g^{-1}(h_1^{(1)})\, f(h_1^{(2)})\, f(h_2^{(2)})(v) \\ &= \sum_{(h_1)(h_2)} g^{-1}(h_2^{(1)})\, g^{-1}(h_1^{(1)})\, g(h_1^{(2)})\, g(h_2^{(2)})\, g^{-1}(h_1^{(3)}h_2^{(3)})\, f(h_1^{(4)}h_2^{(4)})(v) \\ &= \sum_{(h_1)(h_2)} g^{-1}(h_1^{(1)}h_2^{(1)})\, f(h_1^{(2)}h_2^{(2)})(v) \qquad \text{by } (*) \\ &= h_1h_2 \rightharpoonup v. \end{aligned}$$

Also, from (*) $f(1_H)\, f(h) = \sum_{(h)} g(1_H)\, g(h^{(1)})\, g^{-1}(h^{(2)})\, f(h^{(3)}) = g(1_H) f(h)$

$$\therefore g^{-1}(1_H)\, f(1_H) = 1_A .$$

Thus, $h \rightharpoonup v = \sum_{(h)} g^{-1}(h^{(1)})\, f(h^{(2)})(v)$ is an H-module action on V. Further, from (*),

$$\begin{aligned} \sum_{(h)} f(h^{(1)})\, f(S(h^{(2)})) &= \sum_{(h)} g(h^{(1)})\, g(S(h^{(2)}))\, g^{-1}(h^{(3)}S(h^{(4)}))\, f(h^{(5)}S(h^{(6)})) \\ &= \sum_{(h)} g(h^{(1)})\, g(S(h^{(2)}))\, g^{-1}(1_H)\, f(1_H) \\ &= \sum_{(h)} g(h^{(1)})\, g(S(h^{(2)}))\ 1_A \end{aligned}$$

$$\therefore \sum_{(h)} f(h^{(1)})[g^{-1}(h^{(2)})\, g^{-1}(S(h^{(3)}))\, f(S(h^{(4)}))] = 1_A ,$$

i.e.,

$$\sum_{(h)} g^{-1}(h^{(1)})\, g^{-1}(S(h^{(2)}))\, f(S(h^{(3)})) = f^{-1}(h). \qquad (^\dagger)$$

Now, our action on V gives an action on $\mathrm{End}(V)$ which we denote by $h \rightharpoondown a$. This is

$$\begin{aligned} (h \rightharpoondown a)(v) &= \sum_{(h)} h^{(1)} \rightharpoonup [a(S(h^{(2)}) \rightharpoonup v)] \\ &= \sum_{(h)} g^{-1}(h^{(1)})\, f(h^{(2)})[a(g^{-1}(S(h^{(3)}))\, f(S(h^{(4)}))(v))] \\ &= \sum_{(h)} f(h^{(1)})[a(f^{-1}(h^{(2)})(v))] \quad \text{by } (^\dagger) \\ &= (h \rightharpoonup a)(v); \end{aligned}$$

i.e., the action we have defined via V is the same as the original action. So, our original H-module algebra A is Brauer trivial. Q.E.D.

This lemma gives us a well defined injective map from $Bo(K, H)$ to $H^2(H, K)$. It is easy to check that it is, in fact, a group homomorphism.

It remains to prove that this homomorphism is onto and we do this by constructing, for any prescribed cocycle c, an H-module algebra A whose associated cocycle is c.

So, let c be a 2-cocycle from H to K. Let $A = \mathrm{End}(H)$ and define $f_{h_1} \in A$ by

$$f_{h_1}(h_2) = \sum_{(h_1)(h_2)} c(h_1^{(1)}, h_2^{(1)})\, h_1^{(2)} h_2^{(2)}.$$

Now define an inner action of H on A by

$$f\colon H \to A, \qquad f(h) = f_h .$$

Then $f^{-1}(h) = f_{S(h)}$ as is easily checked. Now,

$$\begin{aligned}
[f(h_1)\, f(h_2)](h) &= f_{h_1}(f_{h_2}(h)) \\
&= \sum_{(h_2)(h)} f_{h_1}[c(h_2^{(1)}, h^{(1)})\, h_2^{(2)} h^{(2)}] \\
&= \sum_{(h_1)(h_2)(h)} c(h_1^{(1)}, h_2^{(2)} h^{(2)})\, c(h_2^{(1)}, h^{(1)})\, h_1^{(2)} h_2^{(3)} h^{(3)} \\
&= \sum_{(h_1)(h_2)(h)} c(h_1^{(1)}, h_2^{(1)} h^{(1)})\, c(h_2^{(2)}, h^{(2)})\, h_1^{(2)} h_2^{(3)} h^{(3)} \\
&= \sum_{(h_1)(h_2)(h)} c(h_1^{(1)}, h_2^{(1)})\, c(h_1^{(2)} h_2^{(2)}, h^{(1)})\, h_1^{(3)} h_2^{(3)} h^{(2)} \\
&\qquad\qquad \text{(by the cocycle law)} \\
&= \sum_{(h_1)(h_2)} c(h_1^{(1)}, h_2^{(1)})\, f(h_1^{(2)} h_2^{(2)})(h)
\end{aligned}$$

$$\therefore f(h_1)\, f(h_2) = \sum_{(h_1)(h_2)} c(h_1^{(1)}, h_2^{(1)})\, f(h_1^{(2)} h_2^{(2)}),$$

and so A has associated with it the given cocycle c. This allows us to conclude with the following theorem.

THEOREM 1.12. *If H is a cocommutative Hopf algebra over a field K whose simple subcoalgebras are of the form Kg for some grouplike element g, then $Bo(K, H) \cong H^2(H, K)$ as groups. Furthermore*

$$BM(K, H) \cong B(K) \times H^2(H, K).$$

2. Comodule Algebras

Having dealt with H-modules, we now turn to the dual concept of H-comodules.

Definition 2.1. A right H-comodule M is an R-module together with a map $\chi_M: M \to M \otimes H$, so that the following diagrams commute.

(i)

$$\begin{tikzcd} M \arrow[r, "\chi_M"] \arrow[d, "\chi_M"'] & M \otimes H \arrow[d, "\chi_M \otimes I"] \\ M \otimes H \arrow[r, "I \otimes \Delta"] & M \otimes H \otimes H \end{tikzcd}$$

(ii)

$$\begin{tikzcd} M \arrow[r, "\chi_M"] \arrow[dr, "\sim"] & M \otimes H \arrow[d, "I \otimes \epsilon"] \\ & M \otimes R \end{tikzcd}$$

If M, N are right H-comodules with structure maps χ_M, χ_N respectively and $f \in \operatorname{Hom}(M, N)$, then f is said to be an *H-comodule map* if the following diagram commutes.

(iii)

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\chi_M"'] & N \arrow[d, "\chi_N"] \\ M \otimes H \arrow[r, "f \otimes I"] & N \otimes H \end{tikzcd}$$

We write $\chi_M(m) = \sum_{(m)} m^{(0)} \otimes m^{(1)}$, etc. $m^{(0)}$'s $\in M$, $m^{(1)}$'s $\in H$. The next proposition shows that H-comodules generalize Γ-graded modules.

Proposition 2.2. *Let Γ be a group. Then M is a (right) $R[\Gamma]$ comodule if and only if M is a Γ graded R-module.*

Proof. $\Leftarrow$: M graded by Γ, i.e., $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$. Define

$$\chi_M: M \to M \otimes R[\Gamma]$$

by $m \mapsto \sum_{\gamma \in \Gamma} m_\gamma \otimes \gamma$ where $m = \sum_{\gamma \in \Gamma} m_\gamma$, $m_\gamma \in M_\gamma$. This is obviously R-linear. Also: $\chi_M: m_\gamma \mapsto m_\gamma \otimes \gamma \ \forall m_\gamma \in M_\gamma$.

$$\therefore (\chi_M \otimes I) \circ \chi_M: m \mapsto \sum_{\gamma \in \Gamma} m_\gamma \otimes \gamma \otimes \gamma,$$

but $(I \otimes \Delta) \circ \chi_M: m \mapsto \sum_{\gamma \in \Gamma} m_\gamma \otimes \gamma \otimes \gamma$ so 2.1(i) is satisfied.

$$(I \otimes \epsilon) \circ \chi_M: m \mapsto \sum_{\gamma \in \Gamma} m_\gamma = m$$

(since $\epsilon(\gamma) = 1\ \forall \gamma \in \Gamma$). This is 2.1(ii). So (M, χ_M) is a right $R[\Gamma]$ comodule. $\Rightarrow$: $\chi_M: m \mapsto \sum_i m_i \otimes h_i\ \ h_i \in R[\Gamma]$. $\gamma \in \Gamma$ give an R basis for $R[\Gamma]$ so we can write

$$\chi_M: m \mapsto \sum_{\gamma \in \Gamma} m_\gamma \otimes \gamma.$$

By 2.1(ii) $m = \sum_{\gamma \in \Gamma} m_\gamma$. Now, $\forall \gamma \in \Gamma$ define $\gamma^* \in \mathrm{Hom}(R[\Gamma], R)$ by $\langle \gamma^*, \gamma \rangle = 1$, $\langle \gamma^*, \delta \rangle = 0\ \gamma \neq \delta$. Consider the map $\theta_\gamma: X \otimes R[\Gamma] \to X$ for any R-module X defined by $x \otimes h \to x\langle \gamma^*, h \rangle$. This is R-linear. Also, if $x_\gamma \in X$ s.t. $\sum_{\gamma \in \Gamma} x_\gamma \otimes \gamma = 0$, then applying θ_γ shows that $x_\gamma = 0\ \forall \gamma \in \Gamma$. From 2.1(i) we have $\sum_{\gamma \in \Gamma} \chi_M(m_\gamma) \otimes \gamma = \sum_{\gamma \in \Gamma} m_\gamma \otimes \gamma \otimes \gamma$

$$\therefore\ \chi_M(m_\gamma) = m_\gamma \otimes \gamma \qquad \forall \gamma \in \Gamma.$$

Also, $\sum_{\gamma \in \Gamma} m_\gamma' \otimes \gamma = \sum_{\gamma \in \Gamma} m_\gamma \otimes \gamma \Rightarrow m_\gamma' = m_\gamma\ \forall \gamma \in \Gamma$. So $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ where $M_\gamma = \{m \in M \text{ s.t. } \chi_M(m) = m \otimes \gamma\}$. This gives the grading. Q.E.D.

For an R-module X, we shall denote $\mathrm{Hom}(X, R)$ by X^*. Suppose, just for the moment, that H is finitely generated projective over R. Then we have an isomorphism,

$$\lambda: \mathrm{Hom}(M, M \otimes H) \to \mathrm{Hom}(H^* \otimes M, M)$$

given by $\lambda(f)(h^* \otimes m) = (I \otimes h^*) f(m)$. $f \in \mathrm{Hom}(M, M \otimes H)$, $h^* \in H^*$, $m \in M$.

PROPOSITION 2.3. (i) *If H is a Hopf algebra which is finitely generated projective as R-module then H^* is also a Hopf algebra.*

(ii) *$\chi: M \to M \otimes H$ defines a right H-comodule if and only if $\rightharpoonup = \lambda(\chi): H^* \otimes M \to M$ defines a left H^*-module.*

(iii) *If M, N are right H-comodules, and hence left H^*-modules, and $f \in \mathrm{Hom}(M, N)$, then f is an H-comodule map if and only if f is an H^*-module map.*

Proof. (i) The structure maps for H^* are given as follows

$$\cdot_{H^*}: H^* \otimes H^* \xrightarrow{\simeq} (H \otimes H)^* \xrightarrow{\Delta^*} H^*$$

$$1_{H^*}: R \xrightarrow{\simeq} R^* \xrightarrow{\epsilon^*} H^*$$

$$\Delta_{H^*}: H^* \xrightarrow{\cdot^*} (H \otimes H)^* \simeq H^* \otimes H^*$$

$$\epsilon_{H^*}: H^* \xrightarrow{1^*} R^* \longrightarrow R$$

$$S_{H^*}: H^* \xrightarrow{S^*} H^*$$

where, if $f: X \to Y$ is a morphism of R-modules, then $f^*: Y^* \to X^*$ is the map given by $\langle f^*(y^*), x\rangle = \langle y^*, f(x)\rangle$. It is straightforward (but tedious) to show that these new maps satisfy all the conditions for H^* to be a Hopf algebra.

For a proof of part (ii), see [5, Proposition 1]. The last part of the proposition is straightforward. Q.E.D.

We see from the previous propositions that if H is finitely generated projective we can go over to H^*-modules and study these instead of H-comodules. Note that, if Γ is a finite group, then $R[\Gamma]^*$ is always a Hopf algebra although it may not be a group ring. When H is not finitely generated projective we can proceed as follows.

PROPOSITION 2.4. *Let M, N be right H-comodules with actions χ_M, χ_N. Then the map*

(i)

$$\chi_{M\otimes N}: M \otimes N \xrightarrow{\chi_M \otimes \chi_N} M \otimes H \otimes N \otimes H \xrightarrow{I\otimes T\otimes I} M \otimes N \otimes H \otimes H$$
$$\xrightarrow{I\otimes I\otimes \cdot} M \otimes N \otimes H$$

gives $M \otimes N$ a right H-comodule structure.

Proof. Straightforward.

Suppose that M is finitely generated projective over R and $\chi_M: M \to M \otimes H$ gives M a right H-comodule structure.

Define a map $M^* \to \mathrm{Hom}(M, H)$ by $m^* \mapsto f_{m^*}$ where

$$f_{m^*}(m) = (m^* \otimes I)\,\chi(m) = \sum_{(m)} \langle m^*, m^{(0)}\rangle\, m^{(1)}.$$

Identifying $\mathrm{Hom}(M, H)$ with $H \otimes M^*$ gives a map.

$$\chi_{M^*}: M^* \to H \otimes M^*.$$

PROPOSITION 2.5. *χ_{M^*} gives a left H-comodule structure to M^*.*

Proof. Write $\chi_{M^*}(m^*) = \sum_{(m^*)} m^{*(-1)} \otimes m^{*(0)}$. Then

$$\sum_{(m^*)} m^{*(-1)}\langle m^{*(0)}, m\rangle = f_{m^*}(m) = \sum_{(m)} \langle m^*, m^{(0)}\rangle\, m^{(1)}. \quad \text{(i)}$$

Now,

$$\forall m \in M, \quad \sum_{(m^*)} \epsilon(m^{*(-1)})\langle m^{*(0)}, m\rangle = \epsilon\left(\sum_{(m^*)} m^{*(-1)}\langle m^{*(0)}, m\rangle\right)$$

$$= \epsilon\left(\sum_{(m)} \langle m^*, m^{(0)}\rangle\, m^{(1)}\right) \qquad \text{by (i) above}$$

$$= \sum_{(m)} \langle m^*, m^{(0)}\, \epsilon(m^{(1)})\rangle$$

$$= \langle m^*, m\rangle \qquad \text{by 2.1(ii)}$$

$\therefore\ m^* = \sum_{(m^*)} \epsilon(m^{*(-1)})\, m^{*(0)} = (\epsilon \otimes I) \circ \chi_{M^*}(m^*)$. This is the counit law for M^*.

It remains to prove that $(\Delta \otimes I) \circ \chi_{m^*} = (I \otimes \chi_{m^*}) \circ \chi_{m^*}$. These are maps $M^* \to H \otimes H \otimes M^*$. Now $H \otimes H \otimes M^* \simeq \mathrm{Hom}(M, H \otimes H)$ so we work in $\mathrm{Hom}(M, H \otimes H)$. We are required to show that

$$\sum_{(m^*)} \Delta(m^{*(-1)}) \otimes m^{*(0)} = \sum_{(m^*)} m^{*(-1)} \otimes \left[\sum_{(m^{*(0)})} m^{*(0)^{(-1)}} \otimes m^{*(0)^{(0)}}\right].$$

$$\forall m \in M, \quad \sum_{(m^*)} m^{*(-1)} \otimes \left[\sum_{(m^{*(0)})} m^{*(0)^{(-1)}}\langle m^{*(0)^{(0)}}, m\rangle\right]$$

$$= \sum_{(m^*)} m^{*(-1)} \otimes \left[\sum_{(m)} \langle m^{*(0)}, m^{(0)}\rangle\, m^{(1)}\right] \qquad \text{by (i)}$$

$$= \sum_{(m)} \sum_{(m^*)} m^{*(-1)}\langle m^{*(0)}, m^{(0)}\rangle \otimes m^{(1)}$$

$$= \sum_{(m)} \langle m^*, m^{(0)}\rangle\, m^{(1)} \otimes m^{(2)} \qquad \text{by (i) again and 2.1(i)}$$

$$\forall m \in M, \quad \sum_{(m^*)} \Delta(m^{*(-1)})\langle m^{*(0)}, m\rangle = \Delta\left(\sum_{(m^*)} m^{*(-1)}\langle m^{*(0)}, m\rangle\right)$$

$$= \Delta\left(\sum_{(m)} \langle m^*, m^{(0)}\rangle\, m^{(1)}\right) \qquad \text{(i) again}$$

$$= \sum_{(m)} \langle m^*, m^{(0)}\rangle\, m^{(1)} \otimes m^{(2)} \qquad \text{Q.E.D.}$$

Suppose M, N are right H comodules and M is finitely generated projective over R. We construct the following map.

$$\mathrm{Hom}(M, N) \xrightarrow[\simeq]{\omega} N \otimes M^* \xrightarrow{\chi_N \otimes \chi_{M^*}} N \otimes H \otimes H \otimes M^*$$

$$\xrightarrow{I \otimes I \otimes T} N \otimes H \otimes M^* \otimes H \xrightarrow{I \otimes T \otimes S} N \otimes M^* \otimes H \otimes H$$

$$\xrightarrow{\omega^{-1} \otimes \cdot} \mathrm{Hom}(M, N) \otimes H.$$

If $f \leftrightarrow \sum_i n_i \otimes m_i{}^*$ in ω, then

$$f \mapsto \Big[\sum_i \sum_{(n_i)} \sum_{(m_i{}^*)} n_i^{(0)} \otimes m_i^{*(0)} \otimes n_i^{(1)} \cdot S(m_i^{*(-1)})\Big] \qquad (\in N \otimes M^* \otimes H).$$

PROPOSITION 2.6. *This gives a right H-comodule structure to* $\mathrm{Hom}(M, N)$.

Proof. It is easy to check that $x \mapsto \sum_{(x)} x^{(-1)} \otimes x^{(0)}$ is a left-comodule structure iff $x \mapsto \sum_{(x)} x^{(0)} \otimes S(x^{(-1)})$ is a right comodule structure. Now use Propositions 2.4 and 2.5. (See also [3, p. 356].) Q.E.D.

LEMMA 2.7. *With the action described above*

$$\chi(f)(m) = \sum_{(m)(f(m^{(0)}))} f(m^{(0)})^{(0)} \otimes f(m^{(0)})^{(1)} \cdot S(m^{(1)}) \quad m \in M, f \in \mathrm{Hom}(M, N).$$

Proof. Let $f \leftrightarrow \sum_i n_i \otimes m_i{}^*$ in the isomorphism $\omega\colon \mathrm{Hom}(M, N) \cong N \otimes M^*$.

$$\therefore f(m) = \sum_i n_i \langle m_i{}^*, m\rangle.$$

$$\chi(f) = \sum_i \sum_{(n_i)(m_i{}^*)} n_i^{(0)} \otimes m_i^{*(0)} \otimes n_i^{(1)} \cdot S(m_i^{*(-1)}).$$

$$\therefore \chi(f)(m) = \sum_i \sum_{(n_i)(m_i{}^*)} n_i^{(0)} \langle m_i^{*(0)}, m\rangle \otimes n_i^{(1)} \cdot S(m_i^{*(-1)})$$

$$= \sum_i \sum_{(n_i)(m)} n_i^{(0)} \langle m_i{}^*, m^{(0)}\rangle \otimes n_i^{(1)} \cdot S(m^{(1)}) \qquad \text{by 2.5(i).}$$

$$= \sum_{(m)} \sum_{(f(m^{(0)}))} f(m^{(0)})^{(0)} \otimes f(m^{(0)})^{(1)} \cdot S(m^{(1)}). \qquad \text{Q.E.D.}$$

PROPOSITION 2.8. *If H is finitely generated projective over R, then the above structure is the same as that obtained by going over to H^*-modules.*

Proof. M is an H^*-module by $h^* \rightharpoonup m = \sum_{(m)} m^{(0)} \langle h^*, m^{(1)}\rangle$ and similarly for N (see Proposition 2.3).

So, regarding M and N as H^*-modules, we have that $\mathrm{Hom}(M, N)$ is a left H^*-module by

$$(h^* \rightharpoonup f)(m) = \sum_{(m^*)} h^{*(1)} \underset{N}{\rightharpoonup} [f(S(h^{*(2)}) \underset{M}{\rightharpoonup} m)] \qquad \text{(see Proposition 1.4(ii))}$$

$$= \sum_{(h^*)(m)} h^{*(1)} \underset{N}{\rightharpoonup} [f(m^{(0)} \langle S(h^{*(2)}), m^{(1)}\rangle)]$$

$$= \sum_{(h^*)(m)(f(m^{(0)}))} f(m^{(0)})^{(0)} \langle h^{*(1)}, f(m^{(0)})^{(1)}\rangle \langle h^{*(2)}, S(m^{(1)})\rangle$$

$$= \sum_{(m)(f(m^{(0)}))} f(m^{(0)})^{(0)} \langle h^*, f(m^{(0)})^{(1)} \cdot S(m^{(1)})\rangle.$$

Lemma 2.7 now gives the result. Q.E.D.

DEFINITION 2.9. A is a *right H-comodule algebra* if it is an R-algebra and a right H-comodule so that the structure maps

$$1_A\colon R \to A \qquad \text{and} \qquad \cdot_A\colon A \otimes A \to A$$

are H-comodule maps (R is a right H-comodule by $\chi\colon R \xrightarrow{1_H} H \simeq R \otimes H$, and $A \otimes A$ is a right H-comodule by 2.4).

In symbols these conditions become $\chi_A(1_A) = 1_A \otimes 1_H$

$$\chi_A(a_1 \cdot a_2) = \sum_{(a_1)(a_2)} a_1^{(0)} \cdot a_2^{(0)} \otimes a_1^{(1)} \cdot a_2^{(1)}.$$

If H is finitely generated projective then A is a right H-comodule algebra iff it is a left H^*-module algebra. Also, if $H = R[\Gamma]$ then A is a (right) H-comodule algebra iff it is a Γ-graded R-algebra.

PROPOSITION 2.10. *Let H be commutative. If, A, B are right H-comodule algebras then so are $A \otimes B$ and A^{op}, and $A \otimes B$ and $B \otimes A$ are isomorphic as H-comodule algebras.*

Proof. Straightforward.

PROPOSITION 2.11. *Let H be commutative. Let M, N be right H-comodules which are finitely generated projective over R. Then* $\operatorname{End}(M)$ *is a right H-comodule algebra and the natural isomorphism*

$$\operatorname{End}(M) \otimes \operatorname{End}(N) \simeq \operatorname{End}(M \otimes N)$$

is an H-comodule map.

Proof. We have that $\operatorname{End}(M) = \operatorname{Hom}(M, M)$ is a right H-comodule by 2.6.

We show that the H-comodule action respects the algebra structure in $\operatorname{End}(M)$

$$\begin{aligned}
\chi(I_M)(m) &= \sum_{(m)(I(m^{(0)}))} I(m^{(0)})^{(0)} \otimes I(m^{(0)})^{(1)} \cdot S(m^{(1)}) \qquad \text{by 2.7} \\
&= \sum_{(m)} m^{(0)} \otimes m^{(1)} \cdot S(m^{(2)}) \\
&= \sum_{(m)} m^{(0)} \epsilon(m^{(1)}) \otimes 1_H = m \otimes 1_H
\end{aligned}$$

$\therefore \chi(I_M) = I_M \otimes 1_H$. This is the first condition. We are required to prove that

$$\sum_{(f \cdot g)} (f \cdot g)^{(0)} \otimes (f \cdot g)^{(1)} = \sum_{(f)(g)} f^{(0)} \cdot g^{(0)} \otimes f^{(1)} \cdot g^{(1)}.$$

Now

$$\sum_{(f)(g)} f^{(0)} \cdot g^{(0)}(m) \otimes f^{(1)} \cdot g^{(1)}$$

$$= \sum_{(f)(m)(g(m^{(0)}))} f^{(0)}(g(m^{(0)})^{(0)}) \otimes f^{(1)} \cdot g(m^{(0)})^{(1)} \cdot S(m^{(1)}) \qquad \text{by 2.7 for } g$$

$$= \sum_{(m)(g(m^{(0)}))} f(g(m^{(0)})^{(0)})^{(0)} \otimes f(g(m^{(0)})^{(0)})^{(1)} \cdot S(g(m^{(0)})^{(1)})$$

$$\cdot\, g(m^{(0)})^{(2)} \cdot S(m^{(1)})^{(f(g(m^{(0)})^{(0)}))} \qquad \text{again by 2.7 for } f$$

$$= \sum_{(m)(f(g(m^{(0)})))} f(g(m^{(0)}))^{(0)} \otimes f(g(m^{(0)}))^{(1)} \cdot S(m^{(1)}) \qquad \text{using properties of } S \text{ and } \epsilon$$

$$= (f \cdot g)(m) \qquad \text{by 2.7 yet again.}$$

This completes the first part of the proposition. The second part is similar to the above, and uses that H is commutative. Q.E.D.

PROPOSITION 2.12. *Let H be commutative, A a right H-comodule algebra, finitely generated projective over R. Then the standard map $A \otimes A^{\mathrm{op}} \to \mathrm{End}(A)$ is an H-comodule map.*

Proof. The map is $F: A \otimes A^{\mathrm{op}} \to \mathrm{End}(A)$

$$F_{a \otimes b^{\mathrm{op}}}(c) = a \cdot c \cdot b$$

$$\therefore \chi(F_{a \otimes b^{\mathrm{op}}})(c) = \sum_{(c)(a \cdot c^{(0)} \cdot b)} (a \cdot c^{(0)} \cdot b)^{(0)} \otimes (a \cdot c^{(0)} \cdot b)^{(1)} \cdot S(c^{(1)}) \qquad \text{by 2.7}$$

$$= \sum_{(a)(b)(c)} a^{(0)} \cdot c^{(0)} \cdot b^{(0)} \otimes a^{(1)} \cdot c^{(1)} \cdot b^{(1)} \cdot S(c^{(2)}) \qquad \text{since } A \text{ is a comodule algebra.}$$

$$= \sum_{(a)(b)} a^{(0)} \cdot c \cdot b^{(0)} \otimes a^{(1)} \cdot b^{(1)} \qquad H \text{ commutative and use properties of } S, \epsilon$$

$$= \sum_{(a \otimes b^{\mathrm{op}})} F_{(a \otimes b^{\mathrm{op}})^{(0)}}(c) \otimes (a \otimes b^{\mathrm{op}})^{(1)}. \qquad \text{Q.E.D.}$$

Once again we can define the Brauer group, this time in the category of right H-comodule algebras which are finitely generated projective faithful Azumaya algebras over R.

DEFINITION 2.13. We call this group *the Brauer group of H-comodule algebras* and denote it by $BC(R, H)$.

If H is finitely generated projective then of course $BC(R, H)$ is isomorphic to $BM(R, H^*)$ as defined in 1.10.

3. Dimodules

Throughout this section H is commutative and cocommutative. In general, H-module actions will be written on the left, and H-comodule action on the right.

We use $\rightharpoonup_M: H \otimes M \to M$, and $\chi_M: M \to M \otimes H$ for these actions respectively.

Definition 3.1. (i) An *H-dimodule* is an R-module M which is also an H-module and an H-comodule so that the following diagram commutes.

$$\begin{tikzcd} H \otimes M \arrow[r, "\rightharpoonup_M"] \arrow[d, "I \otimes \chi_M"'] & M \arrow[d, "\chi_M"] \\ H \otimes M \otimes H \arrow[r, "\rightharpoonup_M \otimes I"] & M \otimes H \end{tikzcd}$$

(ii) If M, N are H-dimodules and $f \in \mathrm{Hom}(M, N)$, then f is said to be an *H-dimodule map* if it is simultaneously an H-module map and an H-comodule map.

(iii) An *H-dimodule algebra* is an R-algebra which is an H-dimodule so that it is an H-module algebra and an H-comodule algebra.

(iv) If A, B are H-dimodule algebras $f \in \mathrm{Hom}(A, B)$, then f is an *H-dimodule algebra map* if it is an H-dimodule map and an algebra map.

If H is finitely generated projective then we can take the H-comodule structure over to an H^*-module structure. It is then easy to verify that M is an H-dimodule iff it is an $H - H^*$-bimodule.

If M, N are H-dimodules then $M \otimes N$ is an H-module by 1.4(i) and an H-comodule by 2.4. Similarly, if A, B are H-dimodule algebras then $A \otimes B$ is an H-module algebra and an H-comodule algebra by 1.7 and 2.10, respectively.

However, we can put a new multiplication on $A \otimes B$ as follows.

$$A \otimes B \otimes A \otimes B \xrightarrow{I \otimes \chi_B \otimes I \otimes I} A \otimes B \otimes H \otimes A \otimes B$$

$$\xrightarrow{I \otimes I \otimes \rightharpoonup_A \otimes I} A \otimes B \otimes A \otimes B \xrightarrow{I \otimes T \otimes I} A \otimes A \otimes B \otimes B \xrightarrow{\cdot_A \otimes \cdot_B} A \otimes B.$$

Definition 3.2. $A \otimes B$ with this multiplication is denoted by $A \# B$ and is called the *smash product* of A and B. $A \# B$ has H-action inherited from the R-module $A \otimes B$.

In symbols

$$(a_1 \# b_1) \cdot (a_2 \# b_2) = \sum_{(b_1)} a_1 \cdot (b_1^{(1)} \underset{A}{\rightharpoonup} a_2) \# b_1^{(0)} \cdot b_2 .$$

Note that the multiplication in $A \# B$ depends only on the comodule structure on B and the module structure on A (and the algebra structure on A and B).

The smash product can be defined under more general circumstances.

THEOREM 3.3. *With the structure described above we have the following results. $M \otimes N$ is an H-dimodule. $A \otimes B$ and $A \# B$ are H-dimodule algebras and we have an isomorphism of H-dimodule algebras*

$$(A \# B) \# C \cong A \# B\,(B \# C),$$

given by

$$(a \# b) \# c \mapsto a \# (b \# c).$$

Proof. Straightforward (but tedious!).

DEFINITION 3.4. Let A be an H-dimodule algebra. Define $\bar{A}$ to be isomorphic to A as R-module with multiplication defined by

$$\bar{a} \cdot_{\bar{A}} \bar{b} = \overline{\sum_{(a)} (a^{(1)} \rightharpoonup b) \cdot a^{(0)}}$$

and H-actions inherited from A.

THEOREM 3.5. *$\bar{A}$ is a dimodule algebra. Furthermore, we have isomorphisms of H-dimodule algebras*

$$A \cong \bar{\bar{A}} \text{ given by } a \mapsto \overline{\overline{\sum_{(a)} a^{(1)} \rightharpoonup a^{(0)}}}$$

$$\bar{B} \# \bar{A} \cong \overline{A \# B} \text{ given by } \bar{b} \# \bar{a} \mapsto \overline{\sum_{(b)} (b^{(1)} \rightharpoonup a) \# b^{(0)}}.$$

Proof. That $\bar{A}$ is a dimodule algebra is again straightforward. We prove the isomorphism.

$a \mapsto \overline{\overline{\sum_{(a)} a^{(1)} \rightharpoonup a^{(0)}}}$ has inverse $\bar{\bar{a}} \mapsto \sum_{(a)} S(a^{(1)}) \rightharpoonup a^{(0)}$, so it is an isomorphism of R-modules.

It is easy to see that it respects the H-structures; we show that it is an algebra map.

$$\sum_{(a)(b)} \overline{\overline{(a^{(1)} \rightharpoonup a^{(0)})}} \cdot_{\bar{\bar{A}}} \overline{\overline{(b^{(1)} \rightharpoonup b^{(0)})}}$$

$$= \overline{\sum_{\substack{(a)(b)\\(a^{(1)} \rightharpoonup a^{(0)})}} \overline{[(a^{(1)} \rightharpoonup a^{(0)})^{(1)} \rightharpoonup (b^{(1)} \rightharpoonup b^{(0)})] \cdot_{\bar{A}} (a^{(1)} \rightharpoonup a^{(0)})^{(0)}}}$$

by definition of multiplication in $\bar{A}$

$$= \overline{\sum_{(a)(b)} \overline{[(a^{(1)} \rightharpoonup (b^{(1)} \rightharpoonup b^{(0)})] \cdot_{\bar{A}} (a^{(2)} \rightharpoonup a^{(0)})}} \qquad \text{since } A \text{ is a dimodule}$$

$$= \overline{\overline{\sum_{(a)(b)} (b^{(1)}a^{(2)} \rightharpoonup a^{(0)}) \cdot (a^{(1)}b^{(2)} \rightharpoonup b^{(0)})}} \qquad \text{by definition of } \bar{A} \text{ again}$$

$$= \overline{\overline{\sum_{(a)(b)} (a^{(1)}b^{(1)} \rightharpoonup a^{(0)}) \cdot (a^{(2)}b^{(2)} \rightharpoonup b^{(0)})}} \qquad \text{since } H \text{ is commutative and cocommutative}$$

$$= \overline{\overline{\sum_{(a)(b)} a^{(1)}b^{(1)} \rightharpoonup a^{(0)}b^{(0)}}} = \overline{\overline{\sum_{(ab)} ab^{(1)} \rightharpoonup ab^{(0)}}}.$$

The other isomorphism is dealt with similarly. Q.E.D.

By 1.7 and 2.10 A^{op} is also an H-dimodule algebra.

Let M, N be finitely generated projective over R. If M, N are H-dimodules then $\mathrm{Hom}(M, N)$ is an H-module by 1.4(ii) and an H-comodule by 2.6. As would be expected we now have the following theorem.

THEOREM 3.6. *If M, N are H-dimodules, then so is* $\mathrm{Hom}(M, N)$. *Furthermore,* $\mathrm{End}(M)$ *is an H-dimodule algebra.*

Proof. We already have that $\mathrm{Hom}(M, N)$ is an H-module and an H-comodule, so all that remains is to show that the diagram in 3.1(i) is satisfied, i.e., we have to show that $\chi(h \rightharpoonup f) = \sum_{(f)} h \rightharpoonup f^{(0)} \otimes f^{(1)}$ $f \in \mathrm{Hom}(M, N)$ $h \in H$. Now

$$\chi(h \rightharpoonup f)(m) = \sum_{(m)((h \rightharpoonup f)(m^{(0)}))} [(h \rightharpoonup f)(m^{(0)})]^{(0)} \otimes [(h \rightharpoonup f)(m^{(0)})]^{(1)} \cdot S(m^{(1)})$$

by Lemma 2.7

$$= \sum_{\substack{(m)(h)\\(h^{(1)} \rightharpoonup_N [f(S(h^{(2)}) \rightharpoonup_M m^{(0)}])}} (h^{(1)} \underset{N}{\rightharpoonup} [f(S(h^{(2)}) \underset{M}{\rightharpoonup} m^{(0)})])^{(0)}$$

$$\otimes (h^{(1)} \underset{N}{\rightharpoonup} [f(S(h^{(2)}) \underset{M}{\rightharpoonup} m^{(0)})])^{(1)} \cdot S(m^{(1)}) \qquad \text{by 1.4(ii)}$$

$$= \sum_{\substack{(m)(h)\\(f(S(h^{(2)}) \rightharpoonup_M m^{(0)}))}} h^{(1)} \underset{N}{\rightharpoonup} [(f(S(h^{(2)}) \underset{M}{\rightharpoonup} m^{(0)}))^{(0)}]$$

$$\otimes [f(S(h^{(2)}) \underset{M}{\rightharpoonup} m^{(0)})]^{(1)} \cdot S(m^{(1)}) \qquad \text{since } N \text{ is a dimodule}$$

$$= \sum_{\substack{(h)(S(h^{(2)}) \rightharpoonup_M m) \\ (f((S(h^{(2)}) \rightharpoonup_M m)^{(0)}))}} h^{(1)} \underset{N}{\rightharpoonup} [(f((S(h^{(2)}) \underset{M}{\rightharpoonup} m)^{(0)}))^{(0)}]$$

$$\otimes [f((S(h^{(2)}) \underset{M}{\rightharpoonup} m)^{(0)})]^{(1)} \cdot S((S(h^{(2)}) \underset{M}{\rightharpoonup} m)^{(1)})$$

since M is a dimodule

$$= \sum_{(h)(f)} h^{(1)} \underset{N}{\rightharpoonup} [f^{(0)}(S(h^{(2)}) \underset{M}{\rightharpoonup} m)] \otimes f^{(1)} \qquad \text{by 2.7 again}$$

$$= \sum_{(f)} (h \rightharpoonup f^{(0)})(m) \otimes f^{(1)}.$$

This is true $\forall m \in M$, hence $\chi(h \rightharpoonup f) = \sum_{(f)} h \rightharpoonup f^{(0)} \otimes f^{(1)}$ as required. That $\mathrm{End}(M)$ is a dimodule algebra now follows from 1.8 and 2.11. Q.E.D.

Let M be an H-dimodule; and for $m \in M$, $h \in H$ define f_h by $f_h(m) = h \rightharpoonup m$. Then $f_h \in \mathrm{End}(M)$. If $f \in \mathrm{End}(M)$ then 1.4(ii) gives us that $h \rightharpoonup f = \sum_{(h)} f_{h^{(1)}} \cdot f \cdot f_{S(h^{(2)})}$. Of course, $f_{h_1} f_{h_2} = f_{h_1 h_2}$ and $f_{1_H} = I_M$. By 2.7

$$\chi(f_h)(m) = \sum_{(m)(f_h(m^{(0)}))} f_h(m^{(0)})^{(0)} \otimes f_h(m^{(0)})^{(1)} \cdot S(m^{(1)})$$

$$= \sum_{(m)(h \rightharpoonup m^{(0)})} (h \rightharpoonup m^{(0)})^{(0)} \otimes (h \rightharpoonup m^{(0)} \cdot)^{(1)} \cdot S(m^{(1)}) \text{ by definition } f_h$$

$$= \sum_{(m)} h \rightharpoonup m^{(0)} \otimes m^{(1)} \cdot S(m^{(2)}) \qquad \text{as } M \text{ is a dimodule}$$

$$= h \rightharpoonup m \otimes 1_H \qquad \text{by properties of } S, \text{ etc.}$$

$$= f_h(m) \otimes 1_H$$

$$\therefore \chi(f_h) = f_h \otimes 1_H.$$

The above notation is useful in the next proposition.

PROPOSITION 3.7. *Let M be an H-dimodule which is finitely generated projective over R, and let B be an H-dimodule algebra. The map*

$$\phi \colon \mathrm{End}(M) \# B \to \mathrm{End}(M) \otimes B$$

given by $\phi(f \# b) = \sum_{(b)} f \cdot f_{b^{(1)}} \otimes b^{(0)}$ is an isomorphism of H-dimodule algebras.

Proof. (i) we show that ϕ is an algebra map.

$$\phi((f_1 \# b_1) \cdot (f_2 \# b_2)) = \phi\Big(\sum_{(b_1)} f_1 \cdot (b_1^{(1)} \rightharpoonup f_2) \# b_1^{(0)} \cdot b_2\Big)$$

$$= \sum_{(b_1)(b_2)} f_1 \cdot f_{b_1^{(1)}} \cdot f_2 \cdot f_{S(b_1^{(2)})} f_{b_1^{(3)} \cdot b_2^{(1)}} \otimes b_1^{(0)} \cdot b_2^{(0)}$$

$$= \sum_{(b_1)(b_2)} f_1 \cdot f_{b_1^{(1)}} \cdot f_2 \cdot f_{b_2^{(1)}} \otimes b_1^{(0)} b_2^{(0)}$$

by properties of S, etc.

$$= \Big(\sum_{(b_1)} f_1 \cdot f_{b_1^{(1)}} \otimes b_1^{(0)}\Big) \cdot \Big(\sum_{(b_2)} f_2 \cdot f_{b_2^{(1)}} \otimes b_2^{(0)}\Big)$$

$$= \phi(f_1 \# b_1) \cdot \phi(f_2 \# b_2)$$

$\phi(I \# 1_B) = I \otimes 1_B$ trivially.

(ii) ϕ is an H-module map.

$$\phi(h \rightharpoonup (f \# b)) = \phi\Big(\sum_{(h)} (h^{(1)} \rightharpoonup f) \# (h^{(2)} \rightharpoonup b)\Big)$$

$$= \sum_{(h)(b)} f_{h^{(1)}} f f_{S(h^{(2)})} \cdot f_{b^{(1)}} \otimes h^{(3)} \rightharpoonup b^{(0)}$$

$$= \sum_{(h)(b)} f_{h^{(1)}} f f_{b^{(1)}} f_{S(h^{(2)})} \otimes h^{(3)} \rightharpoonup b^{(0)} \quad \text{since } H \text{ is commutative}$$

$$= h \rightharpoonup \phi(f \# b).$$

(iii) ϕ is an H-comodule map

$$\chi(\phi(f \# b)) = \chi\Big(\sum_{(h)} f \cdot f_{b^{(1)}} \otimes b^{(0)}\Big)$$

$$= \sum_{(b)(f)} f^{(0)} \cdot f_{b^{(2)}} \otimes b^{(0)} \otimes f^{(1)} \cdot b^{(1)} \quad \text{remember } \chi(f_h) = f_h \otimes 1_H$$

$$= \sum_{(f)(b)} f^{(0)} \cdot f_{b^{(1)}} \otimes b^{(0)} \otimes f^{(1)} \cdot b^{(2)} \quad \text{since } H \text{ is commutative}$$

$$= (\phi \otimes I)(\chi(f \# b))$$

$f \otimes b \mapsto \sum_{(b)} f \cdot f_{S(b^{(1)})} \# b^{(0)}$ is easily seen to be an inverse to ϕ, hence ϕ is an isomorphism of dimodule algebras. Q.E.D.

COROLLARY 3.8. *If M, N are H-dimodules which are finitely generated projective over R, then we have*

$$\mathrm{End}(M) \# \mathrm{End}(N) \simeq \mathrm{End}(M \otimes N) \qquad \textit{as dimodule algebras.}$$

Proof. $\mathrm{End}(M) \# \mathrm{End}(N) \cong \mathrm{End}(M) \otimes \mathrm{End}(N)$ by Proposition 3.7.

$\cong \mathrm{End}(M \otimes N)$ by 1.8 and 2.11. Q.E.D.

In the above, the function $f \mapsto f \cdot f_h$ can be regarded as an action of H on $\mathrm{End}(M)$. In order to prove that $B \# \mathrm{End}(M) \cong B \otimes \mathrm{End}(M)$ we must define the dual action and derive some of its properties.

Let M be an H-dimodule which is finitely generated projective over R, let $f \in \mathrm{End}(M)$ and suppose $f \leftrightarrow \sum_i m_i \otimes m_i{}^*$ in the isomorphism $\mathrm{End}(M) \cong M \otimes M^*$. Then we define a map ξ: $\mathrm{End}(M) \to H \otimes \mathrm{End}(M)$ by

$$\xi(f) = \sum_i \sum_{(m_i)} S(m_i^{(1)}) \otimes (m_i^{(0)} \otimes m_i{}^*).$$

We write $\xi(f) = \sum_{[f]} f^{[-1]} \otimes f^{[0]}$.

LEMMA 3.9. (i) $\xi(f)(m) = \sum_{(f(m))} S(f(m)^{(1)}) \otimes f(m)^{(0)}$, $m \in M$.

(ii) $\xi(h \rightharpoonup f) = \sum_{[f]} f^{[-1]} \otimes h \rightharpoonup f^{[0]} \quad h \in H$.

Proof. (i) We have $f(m) = \sum_i m_i \langle m_i{}^*, m\rangle$ where $f \leftrightarrow \sum_i m_i \otimes m_i{}^*$ as above

$$\xi(f)(m) = \sum_i \sum_{(m_i)} S(m_i^{(1)}) \otimes m_i^{(0)} \langle m_i{}^*, m\rangle$$

$$= \sum_{(f(m))} S(f(m)^{(1)}) \otimes f(m)^{(0)}.$$

(ii) now follows fairly easily, using (i). Q.E.D.

PROPOSITION 3.10. *Let M be an H-dimodule which is finitely generated projective over R, and let B be an H-dimodule algebra. Then the map ρ: $B \# \mathrm{End}(M) \to B \otimes \mathrm{End}(M)$ given by $\rho(b \# f) = \sum_{[f]} (f^{[-1]} \rightharpoonup b) \otimes f^{[0]}$ is an isomorphism of H-dimodule algebras.*

Proof. (i) We show that ρ is an algebra map.

$$\rho[(b_1 \# f_1) \cdot (b_2 \# f_2)] = \rho\left[\sum_{(f_1)} b_1(f_1^{(1)} \rightharpoonup b_2) \# f_1^{(0)} \cdot f_2\right]$$

$$= \sum_{\substack{(f_1)\\ [f_1^{(0)} f_2]}} (f_1^{(0)} f_2)^{[-1]} \rightharpoonup [b_1 \cdot (f_1^{(1)} \rightharpoonup b_2)] \otimes (f_1^{(0)} f_2)^{[0]}.$$

Acting on $m \in M$ this gives

$$\sum_{(f_1)(f_1^{(0)}f_2(m))} S((f_1^{(0)}f_2(m))^{(1)}) \rightharpoonup [b_1 \cdot (f_1^{(1)} \rightharpoonup b_2)] \otimes (f_1^{(0)}f_2(m))^{(0)} \qquad \text{by 3.9(i)}$$

$$= \sum_{\substack{(f_2(m))\\(f_1(f_2(m)^{(0)}))}} S(f_1(f_2(m)^{(0)})^{(1)}) \rightharpoonup [b_1 \cdot_B (f_1(f_2(m)^{(0)})^{(2)} \cdot_H S(f_2(m)^{(1)}) \rightharpoonup b_2)]$$

$$\otimes f_1(f_2(m)^{(0)})^{(0)} \qquad \text{by 2.7}$$

$$= \sum_{(f_2(m))(f_1(f_2(m)^{(0)}))} (S(f_1(f_2(m)^{(0)})^{(1)}) \rightharpoonup b_1) \cdot_B (S(f_2(m)^{(1)}) \rightharpoonup b_2)$$

$$\otimes f_1(f_2(m)^{(0)})^{(0)}$$

$$= \sum_{[f_1](f_2(m))} (f_1^{[-1]} \rightharpoonup b_1) \cdot (S(f_2(m)^{(1)}) \rightharpoonup b_2) \otimes f_1^{[0]}(f_2(m)^{(0)}) \qquad \text{by 3.9(i)}$$

$$= \sum_{[f_1][f_2]} (f_1^{[-1]} \rightharpoonup b_1) \cdot (f_2^{[-1]} \rightharpoonup b_2) \otimes f_1^{[0]}(f_2^{[0]}(m)) \qquad \text{by 3.9(i) again.}$$

This is $\rho(b_1 \# f_1) \cdot \rho(b_2 \# f_2)$ acting on m. Hence $\rho(b_1 \# f_1) \cdot \rho(b_2 \# f_2) = \rho[(b_1 \# f_1) \cdot (b_2 \# f_2)]$. $\rho(1_B \# I_M)$ acting on m is $\sum_{(m)} (S(m^{(1)}) \rightharpoonup 1_B) \otimes m^{(0)} = 1_B \otimes m \;\therefore\; \rho(1_B \# I_M) = 1_B \otimes I_M$.

(ii) ρ is a comodule map

$$\chi(\rho(b \# f)) = \chi\left[\sum_i \sum_{(m_i)} (S(m_i^{(1)}) \rightharpoonup b) \otimes (m_i^{(0)} \otimes m_i{}^*)\right], \qquad \text{where}$$

$$f \leftrightarrow \sum_i m_i \otimes m_i{}^*$$

$$= \sum_{(b)(m_i)(m_i{}^*)} (S(m_i^{(2)}) \rightharpoonup b^{(0)}) \otimes (m_i^{(0)} \otimes m_i^{*(0)}) \otimes b^{(1)} m_i^{(1)} S(m_i^{*(-1)})$$

$$= (\rho \otimes I)\, \chi(b \# f).$$

(iii) ρ is an H-module map

$$\rho(h \rightharpoonup (b \# f)) = \rho\left(\sum_{(h)} (h^{(1)} \rightharpoonup b) \# (h^{(2)} \rightharpoonup f)\right)$$

$$= \sum_{(h)[f]} h^{(1)} \rightharpoonup [f^{[-1]} \rightharpoonup b] \otimes h^{(2)} \rightharpoonup f^{[0]} \qquad \text{by 3.9(ii)}$$

$$= h \rightharpoonup \rho(b \# f).$$

The map $b \otimes f \mapsto \sum_{[f]} [S(f^{[-1]}) \rightharpoonup b] \# f^{[0]}$ is inverse to ρ, so it is an isomorphism of dimodule algebras. Q.E.D.

COROLLARY 3.11. *M and B as above, then $B \# \mathrm{End}(M) \simeq \mathrm{End}(M) \# B$ as dimodule algebras.*

Proof. $$\begin{aligned} B \# \operatorname{End}(M) &\cong B \otimes \operatorname{End}(M) && \text{by 3.10} \\ &\cong \operatorname{End}(M) \otimes B && \text{by 1.7 and 2.10} \\ &\cong \operatorname{End}(M) \# B && \text{by 3.7.} \end{aligned}$$ Q.E.D.

PROPOSITION 3.12. *Let M be an H-dimodule which is finitely generated projective over R. Then the map*

$$\tau: \overline{\operatorname{End}(M)} \to \operatorname{End}(M)^{\mathrm{op}}$$

given by

$$\tau(\bar{f})(m) = \sum_{(m)} (m^{(1)} \rightharpoonup f)(m^{(0)})$$

is an isomorphism of dimodule algebras.

Proof. Note τ can be defined by going via the isomorphism $\operatorname{End}(M) \cong M \otimes M^*$ to show that it is well defined.

We show that τ is an algebra map.

$$\bar{f} \cdot \bar{g} = \overline{\sum_{(f)} (f^{(1)} \rightharpoonup g) f^{(0)}} \qquad \text{by definition of } {}^-$$

$$\begin{aligned} \therefore\ \tau(\bar{f} \cdot \bar{g})(m) &= \sum_{(m)(f)} (m^{(1)} \rightharpoonup [(f^{(1)} \rightharpoonup g) \cdot f^{(0)}])(m^{(0)}) \\ &= \sum_{(m)(f)} (m^{(1)} f^{(1)} \rightharpoonup g) \cdot (m^{(2)} \rightharpoonup f^{(0)})(m^{(0)}) \\ &= \sum_{(m)(f)} (m^{(1)} f^{(1)} \rightharpoonup g)(m^{(2)} \underset{M}{\rightharpoonup} [f^{(0)}(S(m^{(3)}) \underset{M}{\rightharpoonup} m^{(0)})]) \\ &= \sum_{\substack{(m) \\ (f(S(m^{(4)}) \rightharpoonup m^{(0)}))}} \{[m^{(1)}(f(S(m^{(4)}) \underset{M}{\rightharpoonup} m^{(0)}))^{(1)} S(m^{(2)})] \rightharpoonup g\} \\ &\qquad (m^{(3)} \underset{M}{\rightharpoonup} [f(S(m^{(4)}) \underset{M}{\rightharpoonup} m^{(0)})]^{(0)}) \qquad \text{by 2.7} \\ &= \sum_{\substack{(m) \\ (f(S(m^{(2)}) \rightharpoonup m^{(0)}))}} [f(S(m^{(2)}) \underset{M}{\rightharpoonup} m^{(0)})^{(1)} \rightharpoonup g] \\ &\qquad (m^{(1)} \underset{M}{\rightharpoonup} [f(S(m^{(2)}) \underset{M}{\rightharpoonup} m^{(0)})]^{(0)}) \\ &= \tau(\bar{g}) \Big(\sum_{(m)} (m^{(1)} \rightharpoonup f)(m^{(0)}) \Big) \\ &= \tau(\bar{g})\, \tau(\bar{f})(m) \end{aligned}$$

$$\therefore\ \tau(\bar{f} \cdot \bar{g}) = \tau(\bar{g}) \cdot \tau(\bar{f}) \qquad \text{in } \operatorname{End}(M)$$

$$\therefore\ \tau(\bar{f} \cdot \bar{g}) = \tau(\bar{f}) \cdot \tau(\bar{g}) \qquad \text{in } \operatorname{End}(M)^{\mathrm{op}}.$$

Obviously $\tau(I_M) = I_M$. That τ respect the H actions is fairly straightforward and is left to the reader. The inverse of τ is given by

$$\tau^{-1}(f^{\mathrm{op}})(m) = \sum_{(m)} (S(m^{(1)}) \rightharpoonup f)(m^{(0)})$$

as is easily checked. Hence, τ is an isomorphism of H-dimodule algebras. Q.E.D.

PROPOSITION 3.13. *Let M be an H-dimodule which is finitely generated projective over R. Then*

$$\mathrm{End}(M)^{\mathrm{op}} \cong \mathrm{End}(M^*) \qquad \textit{as dimodule algebras.}$$

Proof. The map $\mathrm{End}(M)^{\mathrm{op}} \to \mathrm{End}(M^*)$ is given by $f^{\mathrm{op}} \mapsto f^*$.

It is well known that this gives an isomorphism of algebras and it is straightforward (but tedious) to check that this map preserves the H-structures. Q.E.D.

4. THE BRAUER GROUP $BD(R, H)$

We are now in a position to be able to define the concept of H-Azumaya.

From now on, all R-modules (except H) will be finitely generated projective and faithful over R.

Let A be an H-dimodule algebra. We define two maps

$$F\colon A \# \bar{A} \to \mathrm{End}(A)$$

$$G\colon \bar{A} \# A \to \mathrm{End}(A)^{\mathrm{op}}$$

by

$$F(a \# \bar{b}) = F_{a\#\bar{b}} \qquad \text{where} \quad F_{a\#\bar{b}}(c) = \sum_{(b)} a \cdot (b^{(1)} \rightharpoonup c) \cdot b^{(0)}$$

$$G(\bar{a} \# b) = G^{\mathrm{op}}_{\bar{a}\#b} \qquad \text{where} \quad G_{\bar{a}\#b}(c) = \sum_{(c)} (c^{(1)} \rightharpoonup a) \cdot c^{(0)} \cdot b.$$

PROPOSITION 4.1. *F, G are H-dimodule algebra maps.*

Proof. We show this for F, and leave G to the reader.

$$(a_1 \# \bar{b}_1) \cdot (a_2 \# \bar{b}_2) = \sum_{(b_1)} a_1 \cdot (b_1^{(1)} \rightharpoonup a_2) \# \overline{b_1^{(0)}} \cdot \bar{b}_2$$

$$= \sum_{b_1} a_1 \cdot (b_1^{(2)} \rightharpoonup a_2) \# \overline{(b_1^{(1)} \rightharpoonup b_2) \cdot b_1^{(0)}}$$

$$\therefore F_{[(a_1 \# \bar{b}_1)\cdot(a_2 \# \bar{b}_2)]}(c)$$

$$= \sum_{(b_1)(b_2)} a_1(b_1^{(3)} \rightharpoonup a_2)(b_2^{(1)}b_1^{(1)} \rightharpoonup c)(b_2^{(1)} \rightharpoonup b_2^{(0)})\, b_1^{(0)}$$

$$= \sum_{(b_1)(b_2)} a_1(b_1^{(1)} \rightharpoonup a_2)[(b_1^{(2)} \rightharpoonup (b_2^{(1)} \rightharpoonup c)](b_1^{(3)} \rightharpoonup b_2^{(0)})\, b_1^{(0)}$$

since H is commutative and cocommutative

$$= \sum_{(b_1)(b_2)} a_1[b_1^{(1)} \rightharpoonup (a_2(b_2^{(1)} \rightharpoonup c)\, b_2^{(0)})]\, b_1^{(0)}$$

$$= F_{(a_1 \# \bar{b}_1)}F((_{(a_2 \# \bar{b}_2)}(c)).$$

$F_{1_A \# \bar{1}_A} = I_A$ trivially, so F is an algebra map.

$$F_{[h \rightharpoonup (a \# \bar{b})]}(c) = \sum_{(h)(b)} (h^{(1)} \rightharpoonup a)(b^{(1)} \rightharpoonup c)(h^{(2)} \rightharpoonup b^{(0)})$$

$$= \sum_{(h)(b)} (h^{(1)} \rightharpoonup a)(\epsilon(h^{(2)})\, b^{(1)} \rightharpoonup c)(h^{(3)} \rightharpoonup b^{(0)})$$

$$= \sum_{(h)(b)} (h^{(1)} \rightharpoonup a)(h^{(2)}b^{(1)}S(h^{(3)}) \rightharpoonup c)(h^{(4)} \rightharpoonup b^{(0)})$$

$$= \sum_{(h)(b)} h^{(1)} \rightharpoonup [a(b^{(1)} \rightharpoonup (S(h^{(2)}) \rightharpoonup c))\, b^{(0)}]$$

$$= \sum_{(h)} h^{(1)} \rightharpoonup [F_{(a \# \bar{b})}(S(h^{(2)} \rightharpoonup c)]$$

$$= (h \rightharpoonup F_{a \# \bar{b}})(c),$$

so F is an H-module map.

Finally

$$\chi(F_{a \# \bar{b}})(c) = \sum_{\substack{(c) \\ (F_{a \# \bar{b}}(c^{(0)}))}} [F_{a \# \bar{b}}(c^{(0)})]^{(0)} \otimes [F_{a \# \bar{b}}(c^{(0)})]^{(1)}\, S(c^{(1)}) \qquad \text{by 2.7}$$

$$= \sum_{(a)(b)(c)} a^{(0)} \cdot (b^{(1)} \rightharpoonup c^{(0)}) \cdot b^{(0)} \otimes a^{(1)} \cdot c^{(1)} \cdot b^{(2)} \cdot S(c^{(2)})$$

$$= \sum_{(a)(b)} a^{(0)} \cdot (b^{(2)} \rightharpoonup c) \cdot b^{(0)} \otimes a^{(1)} \cdot b^{(1)}$$

$$= (F \otimes I)\, \chi(a \# \bar{b})(c),$$

so F is a comodule map. Q.E.D.

DEFINITION 4.2. If A is an H-dimodule algebra (which is finitely generated projective and faithful over R) such that F and G are isomorphisms, then A is said to be *H-Azumaya*.

THEOREM 4.3. (i) *If M is an H-dimodule, then* $\mathrm{End}(M)$ *is H-Azumaya.*

(ii) *A, B H-Azumaya $\Rightarrow$ $A \# B$ is H-Azumaya.*

(iii) *A H-Azumaya $\Rightarrow$ $\bar{A}$ is H-Azumaya.*

Proof. (i) $\mathrm{End}(M) \# \overline{\mathrm{End}(M)} \cong \mathrm{End}(M) \otimes \mathrm{End}(M)^{\mathrm{op}}$ by 3.7 and 3.12
$\cong \mathrm{End}(\mathrm{End}(M))$ as algebra, since $\mathrm{End}(M)$ is Azumaya (ordinary).

Now suppose that R is a field. Then $\mathrm{End}(M) \# \overline{\mathrm{End}(M)}$ is simple as algebra, and hence $F\colon \mathrm{End}(M) \# \overline{\mathrm{End}(M)} \to \mathrm{End}(\mathrm{End}(M))$ must have zero kernel. Comparing dimensions gives that it is an isomorphism. The general case is now done by localization. Similar arguments apply to G.

(ii)
$$\begin{aligned}
(A \# B) \# \overline{(A \# B)} &\cong (A \# B) \# (\bar{B} \# \bar{A}) && \text{by 3.5.}\\
&\cong A \# (B \# \bar{B}) \# \bar{A} && \text{by 3.3}\\
&\cong A \# \mathrm{End}(B) \# \bar{A} && \text{as } B \text{ is } H\text{-Azumaya}\\
&\cong A \# \bar{A} \# \mathrm{End}(B) && \text{by 3.11}\\
&\cong \mathrm{End}(A) \# \mathrm{End}(B) && \text{as } A \text{ is } H\text{-Azumaya}\\
&\cong \mathrm{End}(A \otimes B) && \text{by 3.8}\\
&\cong \mathrm{End}(A \# B) && \text{since } A \# B = A \otimes B \text{ as } H\text{-dimodule.}
\end{aligned}$$

We now prove that F and G are isomorphisms as in (i).

$$\overline{(A \# B)} \# (A \# B) \cong \mathrm{End}(A \# B)^{\mathrm{op}} \qquad \text{is similar.}$$

(iii) This can be seen from the symmetry in the definition of H-Azumaya. Q.E.D.

DEFINITION 4.4. Let A, B be H-Azumaya. We say A and B are *Brauer equivalent as H-dimodule algebras* (denoted $A \sim B$) if $\exists$ H-dimodules M, N such that

$$A \# \mathrm{End}(M) \cong B \# \mathrm{End}(N) \quad \text{as } H\text{-dimodule algebras.}$$

THEOREM 4.5. *$\sim$ is an equivalence relation which respects the operation $\#$. The quotient set is a group under the multiplication induced by $\#$, with inverse induced by* $^{-}$.

Proof. Straightforward, using 3.3, 3.8, 3.11, 3.12, and 3.13.

DEFINITION 4.6. We denote this group by $BD(R, H)$ and call it the *Brauer group of dimodule algebras.*

Let T be a commutative ring with 1_T and suppose that we have a ring homomorphism $f\colon R \to T$. Then we can regard T as an R-module via f by $r \cdot t = f(r)t$, $r \in R$, $t \in T$. If M is an R-module define $M_T = M \otimes_R T$. This gives us a functor from R-modules to T-modules. Note that $M_T \otimes_T N_T \cong (M \otimes_R N)_T$ the maps being

$$(m \otimes t_1) \otimes (n \otimes t_2) \mapsto (m \otimes n) \otimes t_1 t_2\colon M_T \otimes_T N_T \to (M \otimes_R N)_T$$

$$(m \otimes n) \otimes t \mapsto (m \otimes t) \otimes (n \otimes 1_T)\colon (M \otimes_R N)_T \to M_T \otimes_T N_T .$$

THEOREM 4.7. *Let H be a Hopf Algebra over R. Let M, N be H-dimodules, A, B H-dimodule algebra. Then*

(i) *H_T is a Hopf Algebra over T.*

(ii) *M_T is an H_T-dimodule.*

(iii) *A_T is an H_T-dimodule algebra.*

(iv) *The isomorphism $M_T \otimes_T N_T \cong (M \otimes_R N)_T$ is an isomorphism of H_T-dimodules.*

(v) *$(A_T \#_T B_T) \cong (A \# B)_T$ is an isomorphism of H_T-dimodule algebras, as is the natural map $(\bar{A})_T \cong (\overline{A_T})$.*

(vi) *The natural isomorphism $(\mathrm{End}_R(M))_T \cong \mathrm{End}_T(M_T)$ is a map of H_T-dimodule algebras.*

Proof. (i) The structure maps for H_T are given by

$$\cdot : H_T \otimes_T H_T \cong H \otimes_R H \otimes_R T \xrightarrow{\cdot_H \otimes I} H \otimes_R T = H_T ,$$

$$1\colon T \cong R \otimes_R T \xrightarrow{1_H \otimes I} H \otimes_R T = H_T ,$$

$$\Delta\colon H \otimes_R T \xrightarrow{1_H \otimes I} H \otimes_R H \otimes_R T \cong H_T \otimes_T H_T ,$$

$$\epsilon\colon H \otimes_R T \xrightarrow{\epsilon_H \otimes I} R \otimes_R T = T,$$

$$S\colon H \otimes_R T \xrightarrow{S_H \otimes I} H \otimes_R T.$$

That these do define a Hopf algebra structure on H_T is straightforward.

(ii) M_T is an H_T-module by

$$H_T \otimes_T M_T \cong H \otimes_R M \otimes_R T \xrightarrow[M]{\rightharpoonup \otimes I} M_R \otimes_R T = M_T$$

and an H_T-comodule by

$$M_T = M \otimes_R T \xrightarrow{\chi_M \otimes I} M \otimes_R H \otimes_R T \cong M_T \otimes_T H_T .$$

Again we leave the details to the reader.

(iii) and (iv) Straightforward.

(v) The isomorphism $A_T \#_T B_T \cong (A \# B)_T$ is given by

$$(a \otimes s) \# (b \otimes t) \mapsto (a \# b) \otimes st, \qquad s, t \in T,\ a \in A,\ b \in B.$$

We check that this is an algebra map.

$$\begin{aligned}
&[(a_1 \otimes s_1) \# (b_1 \otimes t_1)][(a_2 \otimes s_2) \# (b_2 \otimes t_2)] \\
&\quad = \sum_{(b_1 \otimes t_1)} (a_1 \otimes s_1)[(b_1 \otimes t_1)^{(1)} \rightharpoonup (a_2 \otimes s_2)] \# (b_1 \otimes t_1)^{(0)}(b_2 \otimes t_2) \\
&\quad = \sum_{(b_1)} (a_1 \otimes s_1)([b_1^{(1)} \rightharpoonup a_2) \otimes s_2] \# b_1^{(0)} b_2 \otimes t_1 t_2 \\
&\quad = \sum_{(b_1)} [a_1(b_1^{(1)} \rightharpoonup a_2) \otimes s_1 s_2] \# b_1^{(0)} b_2 \otimes t_1 t_2 \\
&\quad = \sum_{(b_1)} [a_1(b_1^{(1)} \rightharpoonup a_2) \# b_1^{(0)} b_2] \otimes s_1 s_2 t_1 t_2 \\
&\quad = [(a_1 \# b_1) \otimes s_1 t_1][(a_2 \# b_2) \otimes s_2 t_2].
\end{aligned}$$

The rest is equally straightforward.

(vi) The map $\operatorname{End}(M) \otimes T \to \operatorname{End}_T(M_T)$ is given by

$$f \otimes t \mapsto f_t \qquad \text{where} \quad f_t(m \otimes t') = f(m) \otimes tt'.$$

Again, it is easy to check the details. Q.E.D.

COROLLARY 4.8. (i) *A is H-Azumaya $\Rightarrow A_T$ is H_T-Azumaya.*

(ii) *A, B Brauer equivalent as H-dimodule algebras $\Rightarrow A_T$, B_T are Brauer equivalent as H_T-dimodule algebras.*

Proof. (i) F_T: $A_T \#_T (\bar{A}_T) \to \operatorname{End}_T(A_T)$ is given by

$$A_T \#_T (\bar{A}_T) \cong A_T \#_T (\bar{A})_T \cong (A \# \bar{A})_T \xrightarrow{F \otimes I} (\operatorname{End}_R(A))_T \cong \operatorname{End}_T(A_T)$$

as is easily checked. So, if F is an isomorphism then so is F_T. Similarly for G.

(ii) This is similar. Q.E.D.

COROLLARY 4.9. *The functor $M \mapsto M_T$ induces a group homomorphism $BD(R, H) \to BD(T, H_T)$ by mapping the class of A to the class of A_T.*

We now exhibit some subgroups of $BD(R, H)$.

If M is an R-module then we can give M trivial H-module and comodule structures by

$$h \underset{M}{\rightharpoonup} m = \epsilon(h)m \qquad \forall h \in H,\ m \in M;$$

$$\chi_M(m) = m \otimes 1_H \qquad \forall m \in M.$$

If A is an H-module algebra or an H-comodule algebra, then giving A the appropriate trivial H structure makes A into an H-dimodule algebra.

We then have $\bar{A} = A^{\mathrm{op}}$, $A \# B = A \otimes B$, etc. So, suppose A is an H-module algebra which is Azumaya. Then giving A the trivial comodule structure makes it an H-dimodule algebra and

$$A \# \bar{A} \cong A \otimes A^{\mathrm{op}} \cong \mathrm{End}(A),$$
$$\bar{A} \# A \cong A^{\mathrm{op}} \otimes A \cong \mathrm{End}(A)^{\mathrm{op}},$$

$\therefore$ A is H-Azumaya. It is therefore easy to see that we have a map:

$$\Theta: BM(R, H) \to BD(R, H) \quad \text{of groups.}$$

THEOREM 4.10. *Θ is an injection.*

Proof. Suppose that A represents a class in $BM(R, H)$ which is in the kernel of Θ, i.e., A (with the trivial comodule structure) is Brauer trivial in $BD(R, H)$.

$\therefore \exists$ H-dimodules M, N such that $A \# \mathrm{End}(M) \cong \mathrm{End}(N)$ as H-dimodule algebras.

Let M' be isomorphic to M as H-module, but with trivial comodule structure, and let N' be similarly related to N. Then, since A has trivial comodule structure, we have

$$A \# \mathrm{End}(M') \cong \mathrm{End}(N')$$

$\therefore$ A is Brauer trivial in $BM(R, H)$. Q.E.D.

This theorem enables us to regard $BM(R, H)$ as a subgroup of $BD(R, H)$. Similarly, $BC(R, H)$ can be embedded in $BD(R, H)$. Furthermore, by giving trivial H-actions to R-algebras without any H-structure we can embed the ordinary Brauer group of R, $B(R)$, is both $BM(R, H)$ and $BC(R, H)$.

Hence, we have the following arrangement of groups.

$$\begin{tikzcd}[column sep=small]
 & BM(R, H) \arrow[dr, hook] & \\
B(R) \arrow[ur, hook] \arrow[dr, hook] & & BD(R, H) \\
 & BC(R, H) \arrow[ur, hook] &
\end{tikzcd}$$

5. An Example of $BD(R, H)$

In this section we consider the case when R is a field of characteristic p (we shall denote this by K instead of R), and H is the group algebra $K[C_p]$, where C_p is the cyclic group of order p. Throughout this section γ will be a fixed generator of C_p.

The case $H = K[C_n]$, K of characteristic *not* dividing n has been dealt with in [4]. There we were concerned only with the group C_n and did not mention the Hopf algebra $K[C_n]$. However, we implicitly used the fact that the Hopf algebra dual to $K[C_n]$ was again $K[C_n]$ (under suitable conditions on K). (See Proposition 2.3(i) of this paper for a definition of the dual Hopf algebra.)

It is necessary to investigate what happens to the dual Hopf algebra in our present situation.

Proposition 5.1. *Let K be of characteristic p, $H = K[C_p]$. Then H^* has basis $1^*, d, d^2, \ldots, d^{p-1}$ where d satisfies $d^p = d$, $\Delta(d) = 1^* \otimes d + d \otimes 1^*$.*

Proof. $K[C_p]$ has basis $1, y, \ldots, y^{p-1}$ where $y = \gamma - 1$ and so $y^p = 0$. (Remember, γ is our fixed generator of C_p).

$$\epsilon(y) = 0, \qquad \therefore \langle 1^*, y^s \rangle = 0 \quad \forall s > 0.$$

(1^* is the identity in H^*, see Proposition 2.3(i).)

Now,

$$\begin{aligned}
\Delta(y) &= \Delta(\gamma - 1) = \gamma \otimes \gamma - 1 \otimes 1 \\
&= y \otimes y + y \otimes 1 + 1 \otimes y \\
\therefore \Delta(y^s) &= \Delta(y)^s = (y \otimes y + y \otimes 1 + 1 \otimes y)^s \\
&= \sum_{t=0}^{s} \binom{s}{t} (y \otimes y)^{s-t} (y \otimes 1 + 1 \otimes y)^t
\end{aligned}$$

$$= \sum_{t=0}^{s} \sum_{u=0}^{t} \binom{s}{t}\binom{t}{u} (y \otimes y)^{s-t}(y \otimes 1)^{t-u}(1 \otimes y)^{u}$$

$$= \sum_{t=0}^{s} \sum_{u=0}^{t} \binom{s}{t}\binom{t}{u} (y^{s-u} \otimes y^{s-t+u}) \qquad (s \geqslant 1). \tag{1}$$

Note that the summation variables must satisfy

$$0 \leqslant u \leqslant t \leqslant s. \tag{2}$$

Define $d \in H^*$ by $\langle d, y^s \rangle = \delta_{s,1}$ (Kronecker delta). Notice $\langle d, 1 \rangle = 0$

$$\therefore \langle d^2, 1 \rangle = \langle d \otimes d, \Delta(1) \rangle = \langle d, 1 \rangle \langle d, 1 \rangle = 0,$$

and by induction, $\langle d^r, 1 \rangle = 0 \; \forall r > 0$. Now,

$$\langle d^{r+1}, y^s \rangle = \langle d \cdot d^r, y^s \rangle = \langle d \otimes d^r, \Delta(y^s) \rangle$$

$$= \sum_{t=0}^{s} \sum_{u=0}^{t} \binom{s}{t}\binom{t}{u} \langle d, y^{s-u} \rangle \langle d^r, y^{s-t+u} \rangle \qquad \text{by (1).}$$

But, $\langle d, y^{s-u} \rangle = 0$ unless $s - u = 1$, i.e., $u = s - 1$. Then, (2) requires $t = u = s - 1$ or $t = s$

$$\therefore \langle d^{r+1}, y^s \rangle = \binom{s}{s-1}\binom{s-1}{s-1} 1 \langle d^r, y^s \rangle + \binom{s}{s}\binom{s}{s-1} 1 \langle d^r, y^{s-1} \rangle$$

$$= s(\langle d^r, y^s \rangle + \langle d^r, y^{s-1} \rangle) \qquad (s \geqslant 1). \tag{3}$$

Induction now gives that

$$\langle d^r, y \rangle = 1 \qquad \forall r > 0,$$
$$\langle d^r, y^s \rangle = 0 \qquad \text{for} \quad p > s > r > 0,$$
$$\langle d^r, y^r \rangle = r! \neq 0 \qquad \text{for } p > r > 0.$$

So, the d^r, $r = 0, 1, \ldots, p - 1$ are linearly independent and must form a basis of H^* over K.

Now, $\langle \Delta(d), y^s \otimes y^t \rangle = \langle d, y^s \cdot y^t \rangle = \langle d, y^{s+t} \rangle = 0$ unless $y^{s+t} = y$.

$\therefore \Delta(d) = 1^* \otimes d + d \otimes 1^*$ as required.

$$\begin{aligned} \langle d^{p+1}, y^s \rangle &= s\langle d^p, y^s + y^{s-1} \rangle \qquad \text{for} \quad s \geqslant 1 \text{ by (3)} \\ &= s\langle \Delta(d)^p, y^{s-1} \otimes (y + 1) \rangle \\ &= s\langle 1^* \otimes d^p + d^p \otimes 1^*, y^{s-1} \otimes (y + 1) \rangle \qquad \text{as we are in char}^{\text{c}}.\ p \\ &= s\langle d^p, y^{s-1} \rangle \qquad \text{for} \quad s > 1. \end{aligned}$$

However, from our recursion formula (3) again.

$$\langle d^{p+1}, y^s\rangle = s\langle d^p, y^s\rangle + s\langle d^p, y^{s-1}\rangle$$

$$\therefore s\langle d^p, y^s\rangle = 0 \qquad \text{for} \quad s > 1$$

$$\therefore \langle d^p, y^s\rangle = 0 \qquad \text{for} \quad p > s > 1.$$

We have already proved that $\langle d^p, y\rangle = 1$ and $\langle d^p, 1\rangle = 0$ so finally $d^p = d$.
Q.E.D.

Notice that $\Delta(d) = 1^* \otimes d + d \otimes 1^*$ gives that

$$d(ab) = ad(b) + d(a)b.$$

So, in characteristic p, C_p-graded algebras correspond to algebras on which acts a derivation d satisfying $d^p = d$.

PROPOSITION 5.2. *Let K be algebraically closed of characteristic p, $H = K[C_p]$. Then any H-Azumaya algebra over K is Azumaya.*

Proof. Let A be an H-Azumaya algebra over K.

Then H is a group ring and we can apply [4, Theorem 1.9(i)]. So, A is semisimple, i.e., a direct sum of matrix rings with orthogonal central idempotents $e_1, \dots, e_r$. We have the derivation d on A corresponding to the grading. Let $d(e_i) = \sum_{j=1}^r a_{ij}e_j$. Then

$$\begin{aligned}\sum_{j=1}^{r} a_{ij}e_j = d(e_i) = d(e_ie_i) &= e_id(e_i) + d(e_i)\, e_i \\ &= e_i \left(\sum_{j=1}^{r} a_{ij}e_j\right) + \left(\sum_{j=1}^{r} a_{ij}e_j\right) e_i \\ &= a_{ii}e_i + a_{ii}e_i\end{aligned}$$

$\therefore a_{ij} = 0$, $i \neq j$ and $a_{ii} = a_{ii} + a_{ii}$

$\therefore a_{ii} = 0$ also. So, $d(e_i) = 0$ $\forall i = 1, \dots, r$. i.e., all the idempotents have grade 1.

$$\therefore F_{(1_A \# \bar{e}_i)}(c) = ce_i = e_ic = F_{(e_i \# \bar{1}_A)}(c) \qquad \forall c \in A.$$

$\therefore 1_A \# \bar{e}_i = e_i \# \bar{1}_A$ since $F\colon A \# \bar{A} \to \mathrm{End}(A)$ is an isomorphism

$\therefore e_i = 1_A$.

Hence, there is only one matrix ring and A is Azumaya. Q.E.D.

PROPOSITION 5.3. *Let K be algebraically closed of* char^c. *p. Then if A is an Azumaya H-dimodule algebra (over K) $\exists u$, $x \in A$ such that $u^p = 1_A$, $x^p = x$, ${}^{\gamma}a = uau^{-1}\ \forall a \in A$ and a is of grade $\gamma^t \Leftrightarrow xa - ax = ta$. Further, $xu - ux = su$ for some $s \in \mathbb{F}_p \subseteq K$. ($\mathbb{F}_p$ is the field of p elements.)*

Proof. γ is an automorphism of the Azumaya algebra A, and hence is inner

$$\therefore\ \exists u \in A \text{ such that } {}^{\gamma}a = uau^{-1} \quad \forall a \in A.$$

Then u^p is central in A and since K is perfect we can choose u such that $u^p = 1_A$.

Now A is an H-dimodule algebra and hence an H-comodule algebra. So, by Proposition 2.3(ii) A is an H^*-module algebra. So, the derivation d of Proposition 5.1 acts on A. Because A is Azumaya, any derivation is inner. (See [1, Chapter III, Proposition 1.6].)

$$\therefore\ \exists x' \in A \text{ such that } d(a) = x'a - ax'\ \forall a \in A.$$

Notice that x' and $x' + k1_A$, $k \in K$, give the same derivation. A straightforward induction gives that

$$d^s(a) = \sum_{r=0}^{s} (-1)^r \binom{s}{r} x'^{s-r}ax'^r.$$

So $d^p(a) = x'^pa - ax'^p$ as we are in characteristic p. But,

$$\begin{aligned} d^p = d, \qquad &\therefore\ x'^pa - ax'^p = x'a - ax' \quad \forall a \in A \\ &\therefore\ (x'^p - x')a = a(x'^p - x') \quad \forall a \in A \\ &\therefore\ x'^p = x' + k'1_A, \quad k' \in K, \text{ since } A \text{ is central.} \end{aligned}$$

Hence, $(x' + k1_A)^p = x'^p + k^p1_A = x' + (k^p + k')\,1_A$. It is always possible to solve $z^p - z + k' = 0$ in K since K is separably closed. Then, if k is a solution of this equation, we have

$$k^p + k' = k$$

so

$$(x' + k1_A)^p = x' + k1_A.$$

Putting $x = x' + k1_A$ we have $x^p = x$ as required. Now $\gamma = y + 1$ (same notation as in Proposition 5.1).

$$\therefore\ \gamma^s = (y + 1)^s = 1 + sy + \cdots + y^s.$$

$$\therefore\ a \in A \text{ has grade } \gamma^s \Leftrightarrow d(a) = sa.$$

Since A is a dimodule algebra, the actions of γ and d commute

$$\therefore\ xuau^{-1} - uau^{-1}x = u(xa - ax)\, u^{-1} \quad \forall a \in A$$

$$\therefore\ xuau^{-1} - uxau^{-1} = uau^{-1}x - uaxu^{-1} \quad \forall a \in A$$

$$\therefore\ (u^{-1}xu - x)a = a(u^{-1}xu - x) \quad \forall a \in A$$

$$\therefore\ u^{-1}xu - x = s1_A, \qquad s \in K, \text{ since } A \text{ is central}$$

$$\therefore\ xu - ux = su$$

$$\therefore\ x^p u - ux^p = s^p u = su \quad \text{since } x^p = x$$

$$\therefore\ s^p = s$$

$$\therefore\ s \in \mathbb{F}_p. \qquad \text{Q.E.D.}$$

As K is algebraically closed and Azumaya algebra is a matrix ring over K. So we can take $A = \operatorname{End}(V)$, and then we can give V an H-module structure by ${}^{\gamma}v = u(v)$ and an H-comodule structure (i.e., a C_p-grading) by demanding that v is homogeneous of grade $\gamma^r \Leftrightarrow x(v) = rv$. These structures on V induce the original structure on $\operatorname{End}(V)$. Note that V will be an H-dimodule if and only if u and x commute, i.e., if and only if $s = 0$. Proposition 5.3 allows us to associate to every Azumaya H-dimodule algebra an element $s \in \mathbb{F}_p$ and we have just seen that the algebra is equal to the endomorphism ring of an H-dimodule (i.e., it is in the trivial class in $BD(K, H)$) if and only if $s = 0$. Suppose now that our A is H-Azumaya. Then the map $F\colon A \# \bar{A} \to \operatorname{End}(A)$ is an isomorphism.

We show that this implies $s \neq 1$.

Assume, in order to get a contradiction, that $s = 1$. Then u has grade γ (notation as in Proposition 5.3). So

$$\begin{aligned} F_{(1_A \# \bar{u})}(a) &= ({}^{\gamma}a) \cdot u \\ &= uau^{-1} \cdot u \\ &= ua \\ &= F_{(u \# \bar{1}_A)}(a) \qquad \forall a \in A. \end{aligned}$$

But F is an isomorphism,

$$\therefore\ 1_A \# \bar{u} = u \# \bar{1}_A.$$

So, we must have, $u = k1_A$ for some $k \in K$. This is impossible since u has grade $\gamma \neq 1$. We conclude that $1 - s$ is not zero in $\mathbb{F}_p$, as required.

Lemma 5.4. *If A is as above and B is any H-dimodule algebra, then $A \# B \cong A \otimes B$ as H-module algebra.*

Note. This isomorphism does not preserve the comodule structure.

Proof. The isomorphism is given by the map

$$\theta\colon a \# b \mapsto a \cdot u^r \otimes b \quad \text{where } b \text{ is homogeneous of grade } \gamma^r,$$
$$\text{and } u \text{ is the element described in Proposition 5.3.}$$

θ is obviously an isomorphism of K-spaces, with inverse $a \otimes b \mapsto a \cdot u^{-r} \# b$, where b is homogeneous of grade γ^r. We show that θ is multiplicative.

$$\begin{aligned}
\theta((a_1 \# b_1) \cdot (a_2 \# b_2)) &= \theta(a_1({}^{\gamma^{r_1}}a_2) \# b_1b_2) \qquad b_i \text{ of grade } \gamma^{r_i} \\
&= \theta(a_1u^{r_1}a_2u^{-r_1} \# b_1b_2) \\
&= a_1u^{r_1}a_2u^{-r_1}u^{(r_1+r_2)} \otimes b_1b_2 \\
&= a_1u^{r_1}a_2u^{r_2} \otimes b_1b_2 \\
&= (a_1u^{r_1} \otimes b_1)(a_2u^{r_2} \otimes b_2) \\
&= \theta(a_1 \# b_1)\,\theta(a_2 \# b_2).
\end{aligned}$$

Obviously $\theta(1_A \# 1_B) = 1_A \otimes 1_B$. Also

$$\begin{aligned}
\theta({}^{\gamma}(a \# b)) &= \theta({}^{\gamma}a \# {}^{\gamma}b) = \theta(uau^{-1} \# {}^{\gamma}b) \\
&= uau^{-1}u^r \otimes {}^{\gamma}b \quad (\text{remember, } {}^{\gamma}b \text{ has the same grade as } b) \\
&= u(au^r)u^{-1} \otimes {}^{\gamma}b \\
&= {}^{\gamma}(\theta(a \# b)).
\end{aligned}$$

So θ preserves the H-action also. Q.E.D.

Now let A and B be Azumaya H-dimodule algebras. Let A have u with grade γ^s as above and let B have v with grade γ^r.

$$\begin{aligned}
{}^{\gamma}(a \otimes b) &= {}^{\gamma}a \otimes {}^{\gamma}b = uau^{-1} \otimes vbv^{-1} \\
&= (u \otimes v)(a \otimes b)(u \otimes v)^{-1}.
\end{aligned}$$

So the element corresponding to u in $A \otimes B$ is $u \otimes v$. But $A \otimes B \cong A \# B$ as H-module algebras so the corresponding element in $A \# B$ is

$$\begin{aligned}
\theta^{-1}(u \otimes v) &= u \cdot u^{-r} \# v \qquad (\text{see Lemma 5.4}) \\
&= u^{1-r} \# v.
\end{aligned}$$

This has grade $\gamma^{s(1-r)+r} = \gamma^{s+r-sr}$. If A has u of grade γ^s then we say A is of type $1 - s$. We have seen that "type" is a map from the set of H-Azumaya algebras which are also Azumaya into the nonzero elements of $\mathbb{F}_p$. Let A be of type s, B of type r, then by our argument above $A \# B$ is of type

$$1 - ((1-s) + (1-r) - (1-s)(1-r)) = sr.$$

Furthermore, A is Brauer trivial in $BD(R, H)$ if and only if A is of type 1. Hence, "type" lifts to a well defined injective group homomorphism from $BD(K, H)$ into $U(\mathbb{F}_p)$ (the nonzero elements of the field $\mathbb{F}_p$). We show that "type" is, in fact, onto by constructing Azumaya algebras of the appropriate type.

CONSTRUCTION 5.5. Let $f(Z) = Z^p - Z$ and let $V = K[Z]/(f(Z))$. Then V is a p-dimensional K-algebra. Denote the image of Z in V by z, then $z^p = z$. Let x be the endomorphism of V given by $x\colon v \mapsto zv$, so $x^p = x$. Define an algebra endomorphism of $K[Z]$ by $u\colon Z \mapsto Z - s$, $s \in \mathbb{F}_p$. Then

$$\begin{aligned} f(u(Z)) &= f(Z-s) = (Z-s)^p - Z + s \\ &= Z^p - Z \qquad \text{since} \quad s \in \mathbb{F}_p \\ &= f(Z). \end{aligned}$$

So, u induces an algebra endomorphism, also denoted by u, on V.

$$\begin{aligned} u(z) &= z - s \\ (xu - ux)(v) &= z \cdot u(v) - u(zv) \\ &= z \cdot u(v) - u(z) \cdot u(v) \qquad \text{since } u \text{ is an algebra map} \\ &= su(v). \end{aligned}$$

$$\therefore\ xu - ux = su.$$

Hence, $A = \operatorname{End}(V)$ if of type $1 - s$. It remains to prove that A is H-Azumaya, and for this we need the following lemma.

LEMMA 5.6. *Let* $\operatorname{End}(V)$ *be an H-dimodule algebra of type* $1 - s$. *Then there is an algebra map* $\tau\colon \overline{\operatorname{End}(V)} \to \operatorname{End}(V)^{\mathrm{op}}$ *given by* $\tau(\bar{f})(v) = ({}^{\gamma^r}f)(v)$ *where v is homogeneous of grade γ^r. If* $1 - s$ *is nonzero in $\mathbb{F}_p$ then τ is an isomorphism.*

Proof. Obviously $\tau(I) = I$ (I is the identity map on V).

$$\begin{aligned} \bar{f} \cdot \bar{g} &= \overline{({}^{\gamma^t}g) \cdot f} \qquad \text{where } f \text{ is of grade } \gamma^t, \\ \therefore\ \tau(\bar{f} \cdot \bar{g})(v) &= {}^{\gamma^r}(({}^{\gamma^r}g)f)(v) \qquad (v \text{ of grade } \gamma^r) \\ &= ({}^{\gamma^{r+t}}g)({}^{\gamma^r}f)(v) \\ &= \tau(\bar{g}) \cdot \tau(\bar{f})(v) \qquad \text{since } ({}^{\gamma^r}f)(v) \text{ is of grade } \gamma^{r+t}. \end{aligned}$$

So τ is an algebra map: $\overline{\operatorname{End}(V)} \to \operatorname{End}(V)^{\mathrm{op}}$. Now, v is of grade $\gamma^r \Leftrightarrow x(v) = rv$. (See Proposition 5.3.)

$$\therefore\; xu(v) = ux(v) + su(v) = (r+s)\,u(v)$$

$$\therefore\; {}^{\gamma}v = u(v) \quad \text{is of grade } \gamma^{r+s}.$$

Suppose $1 - s$ is nonzero in $\mathbb{F}_p$ with inverse t. Then σ, defined by $\sigma(f^{\mathrm{op}})(v) = ({}^{\gamma^{-rt}}f)(v)$ where v is of grade γ^r, is the inverse of τ.

$$\begin{aligned}
\tau\sigma(f^{\mathrm{op}})(v) &= ({}^{\gamma^r}(\sigma f^{\mathrm{op}}))(v)\\
&= {}^{\gamma^r}[(\sigma f^{\mathrm{op}})({}^{\gamma^{-r}}v)] && \text{by definition of } H\text{-action on } \operatorname{End}(V)\\
&= {}^{\gamma^r}[({}^{\gamma^{-(r-sr)t}}f)({}^{\gamma^{-r}}v)] && \text{since } {}^{\gamma^{-r}}v \text{ is of grade } \gamma^{r-st}\\
&= {}^{\gamma^r}[({}^{\gamma^{-r}}f)({}^{\gamma^{-r}}v)]\\
&= f(v).
\end{aligned}$$

Similarly $\sigma\tau(f)(v) = f(v)$. Q.E.D.

Proposition 5.7. *If* $\operatorname{End}(V)$ *is an* H*-dimodule algebra of type* $t \neq 0$, *then* $\operatorname{End}(V)$ *is* H*-Azumaya.*

Proof. We have to show that

$$F\colon \operatorname{End}(V) \# \overline{\operatorname{End}(V)} \to \operatorname{End}(\operatorname{End}(V))$$

and

$$G\colon \overline{\operatorname{End}(V)} \# \operatorname{End}(V) \to \operatorname{End}(\operatorname{End}(V))$$

are isomorphisms. Now,

$$\begin{aligned}
\operatorname{End}(V) \# \overline{\operatorname{End}(V)} &\cong \operatorname{End}(V) \# \operatorname{End}(V)^{\mathrm{op}} && \text{by Lemma 5.6}\\
&\cong \operatorname{End}(V) \otimes \operatorname{End}(V)^{\mathrm{op}} && \text{by Lemma 5.4}\\
&\cong \operatorname{End}(\operatorname{End}(V)) && \text{well known.}
\end{aligned}$$

So, as algebra, $\operatorname{End}(V) \# \overline{\operatorname{End}(V)}$ is isomorphic to $\operatorname{End}(\operatorname{End}(V))$. Hence, it is simple, so F must be injective ($F(I \# \bar{I}) = I$, so $F \neq 0$). Comparing dimensions gives that F is an isomorphism. Similarly one proves that G is an isomorphism. Q.E.D.

So, we have constructed H-Azumaya algebras of the appropriate type and we can conclude with the following theorem.

THEOREM 5.8. *Let K be algebraically closed of characteristic p, and let $H = K[C_p]$. Then $BD(K, H) \cong C_{p-1}$.*

Proof. "Type" is a group homomorphism from $BD(K, H)$ onto the group of nonzero elements of $\mathbb{F}_p$, which is isomorphic to the cyclic group of order $p - 1$. Q.E.D.

We now describe $BD(K, H)$ when K is not algebraically closed. Proposition 5.1 still holds, but Proposition 5.3 needs some modification. Let $\bar{K}$ be the algebraic closure of K, then by Corollary 4.9 we have a group homomorphism

$$\theta\colon BD(K, H) \to BD(\bar{K}, H_{\bar{K}}).$$

Of course, $H_{\bar{K}} = \bar{K}[C_p]$ so we know that $BD(\bar{K}, H_{\bar{K}}) \cong C_{p-1}$. Also, the algebras constructed in 5.5 did not require their base field to be algebraically closed. Hence they can be constructed over K and so θ above is onto. Let its kernel be N. Then we have a short exact sequence

$$1 \to N \to BD(K, H) \to BD(\bar{K}, H_{\bar{K}}) \to 1.$$

Now the ordinary Brauer group of K, $B(K)$, can be represented as a subgroup of $BD(K, H)$ (see the end of Section 4). Any class of algebras in $BD(K, H)$ which is in fact in $B(K)$ can be represented by an algebra A with trivial H-module and H-comodule structures. This implies that for any H-dimodule algebra B

$$A \# B \cong A \otimes B \cong B \otimes A \cong B \# A \qquad \text{as } H\text{-dimodule algebras.}$$

So, $B(K)$ is a normal subgroup of $BD(K, H)$. $B(\bar{K})$ is trivial, and we get the following diagram of groups.

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
 & B(K) \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & X \arrow[d] & \\
1 \arrow[r] & B(K) \arrow[r] \arrow[d] & BD(K, H) \arrow[r, "\phi"] \arrow[d, "\theta"] & BD(K, H)/B(K) \arrow[r] \arrow[d] & 1 \\
 & 1 \arrow[r] \arrow[d] & BD(\bar{K}, H_{\bar{K}}) \arrow[r] \arrow[d] & BD(\bar{K}, H_{\bar{K}}) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

Here ϕ is the natural surjection $BD(K, H) \twoheadrightarrow BD(K, H)/B(K)$, and X is the kernel of the induced map

$$BD(K, H)/B(K) \to BD(\bar{K}, H_{\bar{K}}).$$

It is easy to see from the above diagram that we have the exact sequence

$$1 \to B(K) \to N \to X \to 1.$$

We will find the group X. Let A be an H-dimodule algebra (over K) which is H-Azumaya and such that $\phi((A)) \in X$. (A) denotes the class of A in $BD(K, H)$. Let B be isomorphic to A^{op} as K-algebra and have trivial H-module and comodule structure. Then

$$A \# B \cong A \otimes B \cong \operatorname{End}(A) \qquad \text{as } K\text{-algebras.}$$

$A \# B$ is an H-dimodule algebra so we can use this isomorphism to give an H-dimodule structure to $\operatorname{End}(A)$. (This will not, in general, be the one induced by A.)

Now A is H-Azumaya so $A_{\bar{K}}$ is $H_{\bar{K}}$-Azumaya (Corollary 4.8(i)) and hence $A_{\bar{K}}$ is Azumaya as $\bar{K}$-algebra (Proposition 5.2). Thus A is Azumaya as K-algebra (see [1, Chapter III, Corollary 2.9]). Furthermore, $(B) \in B(K)$ since B must be Azumaya and has trivial H-structure.

$$\therefore (A) \cdot (B) = (\operatorname{End}(A)),$$

$$\therefore \phi((A)) = \phi((\operatorname{End}(A))) \qquad \text{since} \quad \phi((B)) = 1.$$

Further, if $\phi((A)) \in X$ then $\theta((A)) = 1$ also, by our definition of X. So, when studying X we can restrict our attention to H-dimodule algebras which are endomorphism rings over K and which become Brauer trivial in $BD(\bar{K}, H_{\bar{K}})$.

We now try to emulate Proposition 5.3. Let $A = \operatorname{End}(V)$ be an H-dimodule algebra which is H-Azumaya (and hence Azumaya) and which becomes Brauer trivial in $BD(\bar{K}, H_{\bar{K}})$. Since A is Azumaya and γ is an automorphism of A,

$$\exists u \in A \text{ s.t.} \qquad {}^{\gamma}a = uau^{-1} \quad \forall a \in A.$$

However, we can no longer ensure that $u^p = 1_A$. Instead we must be satisfied with $u^p = k \cdot 1_A$ where $k \in K^*$. Of course, u and k_1u ($k_1 \in K^*$) give the same inner automorphism and $(k_1u)^p = k_1{}^pk1_A$. Hence, the automorphism γ can be associated with a class in $K^*/(K^*)^p$. Now, as in Proposition 5.3, the derivation d acts on A and must be inner. So $\exists x \in A$ s.t.

$$d(a) = xa - ax \quad \forall a \in A.$$

However, again we may no longer be able to solve the necessary equation to ensure that $x^p = x$. Instead we have only that $x^p = x + l1_A$, $l \in K$. Now x and $x + l_1 1_A$ give rise to the same inner derivation and $(x + l_1 1_A)^p = (x + l_1 1_A) + (l_1^p - l_1 + l)\, 1_A$. So the derivation d can be associated with a class in the group $K/\wp(K)$, where $\wp(l) = l^p - l$.

On tensoring up with $\bar{K}$, u and x are preserved except that u may become ku, $k \in \bar{K}^*$ and x may become $x + l1_A$, $l \in \bar{K}$. Now A becomes Brauer trivial in $BD(\bar{K}, H_{\bar{K}})$ and so u and x must commute over $\bar{K}$. This implies that u and x must commute over K, i.e., u has grade 1 and γ acts trivially on x. (Recall that a has grade $\gamma^r \Leftrightarrow xa - ax = ra$.) We have associated with an algebra whose class has its image under ϕ in X an element of the group $K^*/(K^*)^p \times K/\wp(K)$. We show in fact that

$$X \cong K^*/(K^*)^p \times K/\wp(K) \qquad \text{as groups.}$$

If $u^p = 1_A$ and $x^p = x$ then we can define an H-module structure on V by ${}^\gamma v = u(v)$ and an H-comodule structure by demanding that v has grade γ^r if and only if $x(v) = rv$. These structures on V induce the original structures on $\operatorname{End}(V)$ as is easily checked and make V into an H-dimodule. Conversely, if V is an H-dimodule then we have u and x defined in this way and they satisfy $u^p = 1_A$, $x^p = x$, respectively. Thus, $k = 1$ and $l = 0$ if and only if A is Brauer trivial in $BD(K, H)$.

Now let A and B be H-Azumaya algebras such that $\phi((A)) \in X, \phi((B)) \in X$. Let A have u_A, x_A as above satisfying $u_A{}^p = k_A 1_A$, $x_A{}^p = x_A + l_A 1_A$ and corresponding to this let B have u_B, x_B satisfying $u_B{}^p = k_B 1_B$, $x_B{}^p = x_B + l_B 1_B$. Let $a \in A$, $b \in B$, then

$${}^\gamma(a \# b) = {}^\gamma a \# {}^\gamma b = u_A a u_A^{-1} \# u_B b u_B^{-1} = (u_A \# u_B)(a \# b)(u_A \# u_B)^{-1}$$

since u_B has grade 1 and obviously ${}^\gamma u_A = u_A$. So

$$u_{A\#B} = u_A \# u_B, \qquad \text{and} \qquad u_{A\#B}^p = u_A{}^p \# u_B{}^p = k_A \cdot k_B 1_A \# 1_B$$

$$\therefore\ k_{A\#B} = k_A \cdot k_B.$$

We next prove that $x_{A\#B} = 1_A \# x_B + x_A \# 1_B$. Remember, x_A is defined by the property that a has grade $\gamma^r \Leftrightarrow x_A a - a x_A = ra$. Suppose $a \in A$ has grade γ^r, $b \in B$ has grade γ^s. Then

$$\begin{aligned}
&(1_A \# x_B + x_A \# 1_B)(a \# b) - (a \# b)(1_A \# x_B + x_A \# 1_B) \\
&\quad = a \# x_B b + x_A a \# b - a \# b x_B - a x_A \# b \\
&\qquad \text{since } x_B \text{ has grade 1 and } {}^\gamma x_A = x_A \\
&\quad = a \# (x_B b - b x_B) + (x_A a - a x_A) \# b = a \# sb + ra \# b \\
&\quad = (s + r)(a \# b)
\end{aligned}$$

and, of course, $a \# b$ has grade γ^{s+r}. So $x_{A\# B} = 1_A \# x_B + x_A \# 1_B$ as required

$$\begin{aligned}\therefore x^p_{A\# B} &= 1_A \# x_B{}^p + x_A{}^p \# 1_B = 1_A \# (x_B + l_B 1_B) + (x_A + l_A 1_A) \# 1_B \\ &= 1_A \# x_B + x_A \# 1_B + (l_B + l_A)\, 1_A \# 1_B \\ &= x_{A\# B} + (l_B + l_A)\, 1_{A\# B}\,. \\ \therefore l_{A\# B} &= l_A + l_B\,.\end{aligned}$$

Thus by associating with the algebra A the elements k_A and l_A we get a well defined injective group homomorphism

$$X \to K^*/(K^*)^p \times K/\wp(K).$$

($K^*/(K^*)^p$ is a multiplicative group, $K/\wp(K)$ is additive.) It remains to construct algebras satisfying the required properties to prove that this map is onto.

Let V be a p^2 dimensional K-space with basic $v_{i,j}$ $i, j = 1,..., p$. Define u and x in $A = \mathrm{End}(V)$ by

$$u(v_{r,s}) = \begin{cases} v_{r+1,s} & r = 1,..., p-1, \\ kv_{1,s} & r = p, \end{cases}$$

$$x(v_{r,s}) = \begin{cases} v_{r,2} + lv_{r,p} & s = 1, \\ v_{r,s+1} & s = 2,..., p-2, \\ v_{r,1} & s = p-1, \\ v_{r,2} & s = p. \end{cases}$$

It is straightforward to check that u and x satisfy

$$u^p = k1_A\,, \qquad x^p = x + l1_A \qquad xu = ux.$$

For $p = 2$, we define x by

$$x(v_{r,1}) = v_{r,1} + lv_{r,2}$$

$$x(v_{r,2}) = v_{r,1}\,,$$

instead of the formulae given above.

This gives us an H-dimodule algebra which is H-Azumaya because it is certainly H-Azumaya when tensored up with $\bar{K}$ (in fact it becomes Brauer trivial in $BD(\bar{K}, H_{\bar{K}})$) and if either of the maps F, G had a kernel over K then this would be preserved over $\bar{K}$. So we have $X \cong K^*/(K^*)^p \times K/\wp(K)$. We sum up with the following theorem.

THEOREM 5.9. *Let K be a field of characteristic p, $\bar{K}$ its algebraic closure and let $H = K[C_p]$. Then we have the following exact sequences*

$$1 \to N \to BD(K, H) \to BD(\bar{K}, H_{\bar{K}}) \to 1$$

$$1 \to B(K) \to N \to K^*/(K^*)^p \times K/\wp(K) \to 1.$$

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REFERENCES

1. H. BASS, "Lectures on Topics in Algebraic K-Theory," Tata Institute of Fundamental Research, Bombay, 1967.
2. A. FRÖHLICH AND C. T. C. WALL, Generalisations of the Brauer group I, to appear.
3. R. G. LARSON, Characters of Hopf algebras, *J. Algebra* **17** (1971), 352–368.
4. F. W. LONG, A generalization of the Brauer group of graded algebras, *Proc. London Math. Soc.*, to appear.
5. B. PAREIGIS, When Hopf algebras are Frobenius algebras, *J. Algebra* **18** (1971), 588–596.
6. M. E. SWEEDLER, "Hopf Algebras," Benjamin, New York, 1969.
7. M. E. SWEEDLER, Cohomology of algebras over Hopf algebras, *Trans. Amer. Math. Soc.* 133 (1968), 205–239.