Homoclinic Orbits in a First Order Superquadratic Hamiltonian System: Convergence of Subharmonic Orbits

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We consider the existence of homoclinic orbits for a first order Hamiltonian system

\[ i = JH_z(t, z). \]

We assume \( H(t, z) \) is of form \( H(t, z) = \frac{1}{2}(Az, z) + W(t, z) \), where \( A \) is a symmetric matrix with \( \sigma(JA) \cap i\mathbb{R} = \emptyset \) and \( W(t, z) \) is \( 2\pi \)-periodic in \( t \) and has superquadratic growth in \( z \). We prove the existence of a nontrivial homoclinic solution \( z_\infty(t) \) and subharmonic solutions \( (z_\infty(t))^n \), (i.e., \( 2\pi n \)-periodic solutions) of (HS) such that \( z_\infty(t) \rightarrow z_\infty(t) \) in \( C^1_{loc}(\mathbb{R}, \mathbb{R}^{2N}) \) as \( T \rightarrow \infty \).

0. Introduction

In this paper we consider the first order Hamiltonian system

\[ \dot{z}(t) = JH_z(t, z(t)), \quad (HS) \]

where \( \cdot = d/dt, z = (z_1, ..., z_{2N}) \in \mathbb{R}^{2N} \).

\[ J = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}, \]

and \( H(t, z) \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R}) \). We denote by \( (\cdot, \cdot) \) the standard inner product in \( \mathbb{R}^{2N} \) and throughout this paper we assume \( H(t, z) \) has the form

\[ H(t, z) = \frac{1}{2}(Az, z) + W(t, z), \quad (0.1) \]

where

(A) \( A \) is a \( 2N \times 2N \) symmetric matrix such that

\[ \sigma(JA) \cap i\mathbb{R} = \emptyset \]

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and $W(t, z)$ is a $2\pi$-periodic and \textit{globally superquadratic} function; more precisely $W(t, z)$ satisfies

(W1) $W(t, z) \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ is $2\pi$-periodic in $t$ and $W(t, 0) = 0$,

(W2) there is a $\mu > 2$ such that

$\mu W(t, z) \leq (W_z(t, z), z)$ \quad for all $(t, z) \in \mathbb{R} \times \mathbb{R}^{2N}$,

(W3) there are $\alpha > \mu$ and $k_1 > 0$ such that

$k_1 |z|^2 \leq W(t, z)$ \quad for all $(t, z) \in \mathbb{R} \times \mathbb{R}^{2N}$,

(W4) there are $k_2, k_3 > 1$ such that

$|W_z(t, z)| \leq k_2 |W_z(t, z), z| + k_3$ \quad for all $(t, z) \in \mathbb{R} \times \mathbb{R}^{2N}$,

(W5) $W_z(t, z) = o(|z|)$ at $z = 0$ uniformly in $t \in \mathbb{R}$.

Under the above conditions, we study the existence of (nontrivial) homoclinic orbits emanating from 0. In other words, we consider the existence of solutions of (HS) such that

$z(t) \to 0$ as $|t| \to \infty$. \hspace{1cm} (0.2)

We remark that 0 is an equilibrium point of (HS).

The existence of homoclinic orbits is studied by Coti-Zelati, Ekeland, and Séré [2] and Hofer and Wysocki [6]. More precisely, under the conditions of (A), (W1), (W2), (W3) with $\alpha = \mu$, and

(W4') there is a $k_2 > 0$ such that

$|W_z(t, z)| \leq k_2 |z|^n - 1$ \quad for all $(t, z) \in \mathbb{R} \times \mathbb{R}^{2N}$,

and strict convexity of $W(t, z)$ with respect to $z$, [2] used a \textit{dual variational formulation} and obtained the existence of homoclinic orbits. On the other hand, [6] studied (HS) under conditions (A), (W1), (W2), (W3) with $\alpha = \mu$, and (W4'). They used first order elliptic system and nonlinear Fredholm operator theory and obtained the existence of a homoclinic orbit. See also [1, 5, 9-12] for similar problems for second order Hamiltonian systems. We remark that (W4) is a weaker condition than (W4') under the conditions (W2) and (W3).

In this paper, we take another approach to this problem. We study the convergence of \textit{subharmonic} solutions to a nontrivial homoclinic solution; that is, we consider $2\pi T$-periodic solutions $z_T(t)$ ($T \in \mathbb{N}$) of (HS), which possess some minimax characterization, and try to pass to the limit as $T \to \infty$. 
In the case where $A$ satisfies

\[(A^c) \quad A \text{ is a } 2N \times 2N \text{ symmetric matrix such that} \]

$$\sigma(JA) \cap i\mathbb{R} \neq \emptyset,$$

the behavior of $(z_T(t))_{T \in \mathbb{N}}$ as $T \to \infty$ is studied by Rabinowitz [7] and Felmer [4]. They showed

$$\|z_T(t)\|_{L^\infty} \to 0 \quad \text{as} \quad T \to \infty \quad (0.3)$$

under suitable conditions on $W(t, z)$ and eigenvalues of $JA$.

Under the assumption (A), we remark that $0 \in \mathbb{R}^{2N}$ is a hyperbolic point of (HS) and (0.3) cannot take place in our setting of the problem (see Proposition 2.8 below). The main purpose of this paper is to prove the following theorem, which is in contrast to the result of [4, 7] and also ensures the existence of a homoclinic orbit of (HS).

**THEOREM 0.1.** Assume (A) and (W1)-(W5). Then there is a sequence $(z_T(t))_{T \in \mathbb{N}} \subset C^1(\mathbb{R}, \mathbb{R}^{2N})$ of solutions of (HS) such that

(i) $z_T(t)$ is a $2\pi T$-periodic solution of (HS);

(ii) there are constants $m, M > 0$ independent of $T \in \mathbb{N}$ such that

$$m \leq \int_0^{2\pi T} \left[ \frac{1}{2} (-J\dot{z}_T, z_T) - H(t, z_T) \right] dt \leq M; \quad (0.4)$$

(iii) moreover $(z_T(t))_{T \in \mathbb{N}}$ is compact in the following sense: for any sequence of integers $T_n \to \infty$, there is a subsequence $(T_{n_k})$ and a (nontrivial) homoclinic orbit $z_\infty(t)$ emanating from 0 such that

$$z_{T_{n_k}}(t) \to z_\infty(t) \quad \text{in} \quad C^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^{2N}).$$

**Remark.** In the case where $W(t, z)$ does not depend on $t \in \mathbb{R}$, the conclusion of the above theorem holds without assumption (W4). That is,

**THEOREM 0.2.** Assume (A), (W1)-(W3), (W5), and $W(z)$ is independent of $t \in \mathbb{R}$. Then the conclusion of Theorem 0.1 holds.

We also remark that the convergence of $2\pi T$-periodic solutions to a nontrivial homoclinic orbit is obtained for a second order Hamiltonian system by Rabinowitz [10] and our work is largely motivated by it.

The proofs of Theorems 0.1 and 0.2 are given in the following sections. In Section 1, we deal with $2\pi T$-periodic solutions of (HS); we introduce a variational formulation and minimax procedure and we prove the existence of $2\pi T$-periodic solutions $z_T(t)$ of (HS). At the same time, we obtain
uniform estimates (from above and from below) of corresponding critical values. In Section 2, we get uniform estimates of $z_T(t)$ and pass to the limit as $T \to \infty$ and complete the proofs of Theorems 0.1 and 0.2. Finally in Section 3, we give a proof to Proposition 1.1; we study properties of the operator $J(d/dt) + A$, especially the $L^p$-boundedness of some projection operators related to $J(d/dt) + A$. These properties are used in Sections 1 and 2 without proof.

1. 2πT-Periodic Solutions

1.1. Variational Formulation and Functional Framework

In this section we study the problem

\[
\begin{align*}
\dot{z} &= JH_z(t, z), \quad \text{in } \mathbb{R}, \quad \text{(HS:T)} \\
z(t + 2\pi T) &= z(t), \quad \text{in } \mathbb{R},
\end{align*}
\]

where $T \in \mathbb{N}$.

Since the growth rate of $W(t, z)$ as $|z| \to \infty$ is not restricted, we introduce modification of $W(t, z)$ as in [8, Chapter 6]. Let $K \geq 1$ and $\chi_K(s) \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi_K(s) = 1$ for $s \leq K$, $\chi_K(s) = 0$ for $s \geq K + 1$, and $\chi_K'(s) \in [-2, 0]$ for $s \in [K, K + 1]$. Set

\[
W_K(t, z) = \chi_K(|z|) W(t, z) + (1 - \chi_K(|z|)) r_K |z|^2,
\]

where

\[
r_K = \max_{K \leq |z| \leq K + 1} \frac{W(t, z)}{|z|^2} \quad (\geq k_1).
\]

Then $W_K(t, z)$ satisfies (W1)–(W5) with $k_3$ replaced by $\max \{k_3, \max_{|z| \leq 1, t \in \mathbb{R}} |W_z(t, z)|\}$. We denote the replaced constant by $k_3$ again.

First, we find $2\pi T$-periodic solutions of the problem

\[
\begin{align*}
\dot{z} &= J(Az + W_K(t, z)), \quad \text{in } \mathbb{R}, \quad \text{(HS:T,K)} \\
z(t + 2\pi T) &= z(t), \quad \text{in } \mathbb{R}.
\end{align*}
\]

There is one-to-one correspondence between solutions of (HS:T,K) and critical points of the functional

\[
I_{T,K}(z) = \frac{1}{2} \int_0^{2\pi T} (-\dot{z} - Az, z) \, dt - \int_0^{2\pi T} W_K(t, z(t)) \, dt. \quad (1.1)
\]
So we seek for a nontrivial critical point \( z_{T,K}(t) \) of \( I_{T,K}(z) \). Later in Section 2, we get estimates \( \| z_{T,K}(t) \|_{L^\infty} \leq K_0 \), where \( K_0 \) does not depend on \( T \in \mathbb{N} \) and \( K \geq 1 \); that is, we find \( z_{T,K}(t) \) is a solution of \( (HS:T) \) for \( K \geq K_0 \).

In what follows, for \( p \in [1, \infty) \) we denote by \( L^p_{2\pi T} \) the space of \( 2\pi T \)-periodic functions \( \mathbb{R} \rightarrow \mathbb{R}^{2N} \) whose \( p \)th powers are integrable on \((0, 2\pi T)\). We use the notations

\[
\| z \|_{L^p_{2\pi T}} = \left( \int_0^{2\pi T} |z(t)|^p \, dt \right)^{1/p}
\]

and

\[
(z, w)_{2\pi T} = \int_0^{2\pi T} (z(t), w(t)) \, dt
\]

for \( z \in L^p_{2\pi T} \) and \( w \in L^q_{2\pi T} \) with \( 1/p + 1/q = 1 \).

Let \( \Phi_{2\pi T} = - (J(d/dt) + A) : D(\Phi_{2\pi T}) \subset L^2_{2\pi T} \rightarrow L^2_{2\pi T} \) be a self-adjoint operator under periodic boundary conditions. In Section 3 we show

\[
(-a, a) \cap \sigma(\Phi_{2\pi T}) = \emptyset \quad \text{for some} \quad a > 0. \quad (1.4)
\]

We consider the absolute value \( |\Phi_{2\pi T}| \) of \( \Phi_{2\pi T} \) and let

\[ E_{2\pi T} = D(|\Phi_{2\pi T}|^{1/2}) \]

and

\[
\| z \|_{E_{2\pi T}} = \| |\Phi_{2\pi T}|^{1/2} z \|_{L^2_{2\pi T}} \quad \text{for} \quad z \in E_{2\pi T}.
\]

By (1.4), \( E_{2\pi T} \) has an orthogonal decomposition,

\[
E_{2\pi T} = E^+_{2\pi T} \oplus E^-_{2\pi T}, \quad (1.5)
\]

where the quadratic form \( z \mapsto (\Phi_{2\pi T} z, z)_{2\pi T} \) is positive (resp. negative) definite on \( E^+_{2\pi T} \) (resp. \( E^-_{2\pi T} \)). We denote by

\[
P^\pm_{2\pi T} : E_{2\pi T} \rightarrow E^\pm_{2\pi T}
\]

the corresponding orthogonal projections. Then we have

\[
(\Phi_{2\pi T} z, z)_{2\pi T} = \| P^+_{2\pi T} z \|_{E_{2\pi T}}^2 - \| P^-_{2\pi T} z \|_{E_{2\pi T}}^2
\]

for all \( z \in E_{2\pi T} \).

We can see

\[
I_{T,K}(z) = \frac{1}{2} (\Phi_{2\pi T} z, z)_{2\pi T} - \int_0^{2\pi T} W_K(t, z) \, dt
\]

\[
= \frac{1}{2} \| P^+_{2\pi T} z \|_{E_{2\pi T}}^2 - \frac{1}{2} \| P^-_{2\pi T} z \|_{E_{2\pi T}}^2 - \int_0^{2\pi T} W_K(t, z) \, dt. \quad (1.7)
\]

The following properties of \( E_{2\pi T} \) and \( P^\pm_{2\pi T} \) are proved in Section 3.
Proposition 1.1. (i) Let $H_{2\pi T}^{1/2}$ be a completion of $\text{span}\{ae^{ijjt}; j \in \mathbb{Z}, a \in \mathbb{C}^{2N}\}$ under the norm

$$
\|z\|_{H_{2\pi T}^{1/2}}^2 = 2\pi T \sum_{j \in \mathbb{Z}} \left(1 + \frac{|j|}{T}\right) |a_j|^2,
$$

where

$$
z(t) = \sum_{j \in \mathbb{Z}} a_j e^{ijjt} \quad (a_j \in \mathbb{C}^{2N}, a_{-j} = \bar{a}_j).
$$

Then $E_{2\pi T} = H_{2\pi T}^{1/2}$ and there are constants $c_0, c'_0 > 0$ independent of $T \in \mathbb{N}$ such that

$$
c_0 \|z\|_{H_{2\pi T}^{1/2}} \leq \|z\|_{E_{2\pi T}} \leq c'_0 \|z\|_{H_{2\pi T}^{1/2}} \quad \text{(1.8)}
$$

for all $z \in E_{2\pi T}$.

(ii) For any $p \in [2, \infty)$, there is a constant $c_p > 0$ independent of $T \in \mathbb{N}$ such that

$$
\|z\|_{L_p^{2\pi T}} \leq c_p \|z\|_{E_{2\pi T}} \quad \text{for all } z \in E_{2\pi T}. \quad \text{(1.9)}
$$

Moreover, the embedding $E_{2\pi T} \hookrightarrow L_p^{2\pi T}$ is compact for all $T \in \mathbb{N}$ and $p \in [2, \infty)$.

(iii) For any $p \in (1, \infty)$, there is a constant $\tilde{c}_p > 0$ independent of $T \in \mathbb{N}$ such that

$$
\|P_{2\pi T}^\pm z\|_{L_p^{2\pi T}} \leq \tilde{c}_p \|z\|_{L_p^{2\pi T}} \quad \text{for all } z \in E_{2\pi T}. \quad \text{(1.10)}
$$

By (1.7) and (ii) of Proposition 1.1, we have $I_{T,K}(z) \in C^1(E_{2\pi T}, \mathbb{R})$. Moreover we have the Palais–Smale compactness condition. This condition is required when we apply minimax methods to $I_{T,K}(z)$.

Proposition 1.2. Under the assumptions (A), (W1)–(W2), $I_{T,K}(z)$ satisfies the following Palais–Smale compactness condition:

(P.S.) Whenever a sequence $(z_j)_{j=1}^\infty$ in $E_{2\pi T}$ satisfies, for some $M > 0$,

$$
|I_{T,K}(z_j)| \leq M \quad \text{for all } j,
$$

$$
I'_{T,K}(z_j) \to 0 \quad \text{in } E_{2\pi T}^* \text{ as } j \to \infty,
$$

there is a subsequence of $(z_j)_{j=1}^\infty$ which converges in $E_{2\pi T}$.

Proof. As in [8, Chapter 6].
1.2. Minimax Procedure

To find a nontrivial critical point of $I_{r,K}(z)$, we use the following proposition which is a special case of a theorem of Rabinowitz [8, Theorem 5.29]. In what follows, $B_r(E)$ denotes the open ball of radius $r$ in a Hilbert space $E$ and $\partial B_r(E)$ denotes its boundary.

**Proposition 1.3.** Let $E$ be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$. Suppose $E$ admits an orthogonal decomposition $E = E^+ \oplus E^-$ and $I(u) \in C^1(E, \mathbb{R})$ satisfies the Palais–Smale compactness condition (P.S.) and the following conditions:

1° $I(u) = \frac{1}{2} \langle P^+ u - P^- u, u \rangle + b(u)$, where $P^\pm : E \rightarrow E^\pm$ are the orthogonal projections and $b'(u)$ is compact,

2° there are constants $m, \rho > 0$ such that $I|_{\partial B_r(E^+)} \geq m$, and

3° there is an $e \in \partial B_1(E^+)$ and $R > \rho$ such that

$$I|_{\partial N} \leq 0,$$

where $N = \{u + re; u \in B_r(E^-), 0 < r < R\}$.

Then $I(u)$ possesses a critical value $b \geq m$ which can be characterized as

$$b = \inf_{h \in \Gamma} \sup_{u \in N} I(h(1, u)) \geq m,$$

where

$$\Gamma = \{h \in C([0, 1] \times E, E); h \text{ satisfies } (\Gamma_1) - (\Gamma_3)\}.$$  \hfill (1.11)

Here

$(\Gamma_1)$ $h(0, u) = u$ for all $u \in N$,

$(\Gamma_2)$ $h(t, u) = u$ for $u \in \partial N$ and $t \in [0, 1]$, and

$(\Gamma_3)$ $h(t, u) = e^{\theta(t, u)(P^+ - P^-)}u + K(t, u)$, where $\theta \in C([0, 1] \times E, \mathbb{R})$ and $K$ is compact.

We apply the above proposition to $I = I_{r,K}$, $E^\pm = E_2^{\pm}$ and $e = e_T \equiv P_2^{+\tau}\varphi$, where $\varphi \in C_0^\infty((0, 2\pi), \mathbb{R}^{2N})$ is a function such that

$$\int_0^{2\pi} \left( -J \frac{d}{dt} A \right) \varphi, \varphi \right) dt > 0.$$

(We extend $\varphi$ to $(0, 2\pi T) \rightarrow \mathbb{R}^{2N}$ by setting $\varphi = 0$ on $[2\pi, 2\pi T)$ and we regard it as a $2\pi T$-periodic function on $\mathbb{R}$.)
LEMMA 1.4. (i) There are constants $a_1, a_2 > 0$ independent of $T \in \mathbb{N}$ such that
\[ a_1 \leq \| e_T \|_{E_{2nT}} \leq a_2 \quad \text{for all } T \in \mathbb{N}. \] (1.12)

(ii) For any $p \in (1, \infty)$, there are constants $a_{3,p}, a_{4,p} > 0$ independent of $T \in \mathbb{N}$ such that
\[ a_{3,p} \leq \| e_T \|_{L^p_{2nT}} \leq a_{4,p} \quad \text{for all } T \in \mathbb{N}. \] (1.13)

Proof. (i) For any $T \in \mathbb{N}$, we have
\[
\| e_T \|_{E_{2nT}}^2 = \| P_{2nT}^+ \varphi \|_{E_{2nT}}^2 \geq \| P_{2nT}^+ \varphi \|_{E_{2nT}}^2 - \| P_{2nT}^- \varphi \|_{E_{2nT}}^2
= (\Phi_{2nT} \varphi, \varphi)_{2nT}
= \int_0^{2\pi} \left( \left(-J \frac{d}{dt} - A \right) \varphi, \varphi \right) dt \equiv a_1^2 > 0.
\]
This follows the left hand side inequality of (1.12). Using (1.8), we have
\[
\| e_T \|_{E_{2nT}}^2 = \| P_{2nT}^+ \varphi \|_{E_{2nT}}^2 \leq \| \varphi \|_{E_{2nT}}^2
\leq c_0' \| \varphi \|_{H^1_{2nT}}^2 \leq c_0' \| \varphi \|_{H^1_{2nT}}^2
= c_0' \int_0^{2\pi} (|\varphi|^2 + |\dot{\varphi}|^2) dt \equiv a_3^2 < \infty.
\]
Thus we get the right hand side inequality of (1.12).

(ii) By (1.12) and (1.9), we have the right hand side inequality of (1.13). To get the left hand side inequality, we observe for $1/p + 1/q = 1$
\[
\| P_{2nT}^+ \varphi \|_{L^q_{2nT}} \geq \left( \int_0^{2\pi} \left( \left(-J \frac{d}{dt} + A \right) \varphi \right)^q dt \right)^{1/q}
\geq \left( P_{2nT}^+ \varphi, -\left( J \frac{d}{dt} + A \right) \varphi \right)_{2nT}
\geq \left( \varphi, -\left( J \frac{d}{dt} + A \right) \varphi \right)_{2nT}
= \int_0^{2\pi} \left( \left(-J \frac{d}{dt} - A \right) \varphi, \varphi \right) dt.
\]
Hence we have
\[
\| e_T \|_{L^q_{2nT}} \geq \left( \int_0^{2\pi} \left( \left(-J \frac{d}{dt} - A \right) \varphi, \varphi \right) dt \right)^{-1/q}
\times \left( \int_0^{2\pi} \left( J \frac{d}{dt} + A \right) \varphi \right)^q dt \equiv a_{3,p} > 0.
\]

We remark that $e_T \neq 0$ follows from Lemma 1.4.
By the compactness of the embedding $E_{2\pi T} \rightarrow L_{2\pi T}^2$, $I_{T,K}(z)$ satisfies the assumption 1° of Proposition 1.3. Next we verify the assumptions 2° and 3° of Proposition 1.3.

**Lemma 1.5.** For $K \geq 1$, there are constants $\rho_K, m_K > 0$ independent of $T \in \mathbb{N}$ such that

$$I_{T,K}(z) \geq m_K \quad \text{for all } z \in E_{2\pi T}^+ \text{ with } \|z\|_{E_{2\pi T}} = \rho_K. \quad (1.14)$$

**Proof.** From (W2), we can see

$$W_K(t, z) \leq |z|^{\mu} \max_{|z| - K + 1, t \in \mathbb{R}} \frac{W_K(t, z)}{(K + 1)^{\frac{1}{\mu}}} = r_K(K + 1)^{x - \frac{1}{\mu}} |z|^{\mu},$$

for $|z| \leq K + 1$.

By the definition of $W_K(t, z)$, $W_K(t, z) = r_K |z|^x$ for $|z| \geq K + 1$. Thus we have

$$W_K(t, z) \leq r_K(K + 1)^{x - \frac{1}{\mu}} |z|^{\mu} + r_K |z|^x \quad \text{for all } (t, z) \in \mathbb{R} \times \mathbb{R}^{2N}. \quad (1.15)$$

Hence we have for $z \in E_{2\pi T}^+$

$$I_{T,K}(z) = \frac{1}{2} \|z\|_{E_{2\pi T}}^2 - \int_0^{2\pi T} W_K(t, z) \, dt \geq \frac{1}{2} \|z\|_{E_{2\pi T}}^2 - r_K(K + 1)^{x - \frac{1}{\mu}} \|z\|_{E_{2\pi T}}^{\mu} - r_K \|z\|_{L_{2\pi T}}^2.$$  

By (1.9),

$$I_{T,K}(z) \geq \frac{1}{2} \|z\|_{E_{2\pi T}}^2 - c^\mu r_K(K + 1)^{x - \frac{1}{\mu}} \|z\|_{E_{2\pi T}}^{\mu} - c^2 r_K \|z\|_{E_{2\pi T}}^2.$$  

Therefore we get (1.14) for sufficiently small $\rho_K > 0$ and $m_K > 0$.  

**Lemma 1.6.** There is a constant $R > 0$ which is independent of $T \in \mathbb{N}$ and $K \geq 1$ such that

$$(I_{T,K})|_{N_{T,R}} \leq 0,$$

where

$$N_{T,R} = \{u + re_T \in E_{2\pi T}; u \in B_R(E_{2\pi T}^-), 0 < r < R\}.$$  

Moreover there is a constant $M > 0$ which is independent of $T \in \mathbb{N}$ and $K \geq 1$ such that

$$\sup_{z \in N_{T,R}} I_{T,K}(z) \leq M \quad \text{for all } T \in \mathbb{N} \text{ and } K \geq 1. \quad (1.16)$$
Proof. For \( z = u + re_T(u \in E_{2\pi T}, r > 0) \), we have from (W3) that
\[
I_{T,K}(u + re_T) = \frac{r^2}{2} \|e_T\|_{E_{2\pi T}}^2 - \frac{1}{2} \|u\|_{E_{2\pi T}}^2 - \int_0^{2\pi T} W_K(t, u + re_T) \, dt
\]
\[
\leq \frac{r^2}{2} \|e_T\|_{E_{2\pi T}}^2 - \frac{1}{2} \|u\|_{E_{2\pi T}}^2 - k_1 \|u + re_T\|_{L_{2\pi T}}^2.
\]
By (1.10), we have
\[
\|e_T\|_{L_{2\pi T}}^2 = \|P_{2\pi T}^+(u + re_T)\|_{L_{2\pi T}^2} \leq \bar{c}_\alpha \|u + re_T\|_{L_{2\pi T}^2}.
\]
Thus we have
\[
I_{T,K}(u + re_T) \leq \frac{r^2}{2} \|e_T\|_{E_{2\pi T}}^2 - \frac{1}{2} \|u\|_{E_{2\pi T}}^2 - k_1 (\bar{c}_\alpha)^{-2} r^2 \|e_T\|_{L_{2\pi T}^2}.
\]
We can easily deduce the desired results from Lemma 1.4.

Now we can apply Proposition 1.3 to \( I_{T,K}(z) \) and we get

**Proposition 1.7.** Assume (A) and (W1)-(W3). Then, for any \( T \in \mathbb{N} \) and \( K \geq 1 \), there is a nontrivial critical point \( z_{T,K}(t) \in E_{2\pi T} \) of \( I_{T,K}(z) \) (i.e., a \( 2\pi T \)-periodic solution of (HS : T, K)) and its critical value \( b_{T,K} = I_{T,K}(z_{T,K}) \) is characterized as
\[
b_{T,K} = \inf_{h \in \mathcal{I}_T} \sup_{z \in N_{T,K}} I_{T,K}(h(1, z)) \geq m_K > 0,
\]
where \( T \) is defined in a similar way to (1.11). Moreover there are constants \( m_K > 0 \) independent of \( T \in \mathbb{N} \), and \( M > 0 \) independent of \( T \in \mathbb{N} \) and \( K \geq 1 \) such that
\[
m_K \leq b_{T,K} = I_{T,K}(z_{T,K}) \leq M \quad \text{for all} \quad T \in \mathbb{N} \text{ and } K \geq 1.
\]

Proof. We need only to prove the right hand side inequality of (1.18). Since \( id \in \mathcal{I}_T \), we have from (1.17) that
\[
b_{T,K} \leq \sup_{z \in N_{T,K}} I_{T,K}(z).
\]
By (1.16), we obtain (1.18).

Remark 1.1. A regularity argument shows \( z_{T,K}(t) \in C^1(\mathbb{R}, \mathbb{R}^{2N}) \). See [8, Chapter 6].

2. Uniform Estimates and Limit Process for \( z_T(t) \)

In this section, first we get an \( L^\infty \) estimate of \( z_{T,K}(t) \), which is independent of \( T \in \mathbb{N} \) and \( K \geq 1 \). That is, we see for sufficiently large \( K_0 \geq 1 \) that \( z_{T,K_0}(t) \) is a \( 2\pi T \)-periodic solution of the original equation (HS : T) for each
Second, we establish further uniform estimates of \(z_{T,K}(t)\) and pass to the limit as \(T \to \infty\) and complete the proof of Theorems 0.1 and 0.2.

In what follows, we denote by \(C, C_0, C_1, \ldots\), various constants which are independent of \(T \in \mathbb{N}\) and \(K \geq 1\).

### 2.1. Uniform \(L^\infty\) Estimates of \(z_{T,K}(t)\)

Let \(z_{T,K}(t)\) be a \(2\pi T\)-periodic solution of (HS; \(T, K\)) obtained in Proposition 1.7; especially we have

\[
I'_{T,K}(z_{T,K}) = 0,
\]

\[
I_{T,K}(z_{T,K}) \in [m_K, M] \quad \text{for all} \quad T \in \mathbb{N} \quad \text{and} \quad K \geq 1.
\]

The purpose of this subsection is to get the following

**Proposition 2.1.** Assume (A), (W1)–(W4). Then there is a constant \(C_1 > 0\) independent of \(T \in \mathbb{N}\) and \(K \geq 1\) such that

\[
\|z_{T,K}\|_{L^\infty_{2\pi T}} \leq C_1
\]

for all \(T \in \mathbb{N}\) and \(K \geq 1\).

To prove the above Proposition, we need two lemmas. To state the first one, we introduce the following norm \(\| \cdot \|_{L^1_{2\pi T} + L^\infty_{2\pi T}}\) into the space \(L^1_{2\pi T}\):

\[
\|f(t)\|_{L^1_{2\pi T} + L^\infty_{2\pi T}} = \inf \{ \|f_1(t)\|_{L^1_{2\pi T}} + \|f_\infty(t)\|_{L^\infty_{2\pi T}} : f(t) = f_1(t) + f_\infty(t) \text{ with } f_1(t) \in L^1_{2\pi T}, f_\infty(t) \in L^\infty_{2\pi T} \).
\]

Clearly \(\| \cdot \|_{L^1_{2\pi T} + L^\infty_{2\pi T}}\) satisfies

\[
\|f\|_{L^1_{2\pi T} + L^\infty_{2\pi T}} \leq \|f\|_{L^1_{2\pi T}} \leq 2\pi T \|f\|_{L^1_{2\pi T} + L^\infty_{2\pi T}}
\]

for all \(f \in L^1_{2\pi T}\) and \(T \in \mathbb{N}\).

**Lemma 2.2.** There is a constant \(C_2 > 0\) independent of \(T \in \mathbb{N}\) and \(K \geq 1\) such that

\[
\|W_{K,z}(t, z_{T,K}(t))\|_{L^1_{2\pi T} + L^\infty_{2\pi T}} \leq C_2 \quad \text{for all} \quad T \in \mathbb{N} \quad \text{and} \quad K \geq 1.
\]

**Proof.** We have from (2.1), (2.2) and (W2) for \(W_{K,z}(t, z)\) that

\[
M \geq I_{T,K}(z_{T,K}) - \frac{1}{2} I'_{T,K}(z_{T,K}) z_{T,K}
\]

\[
= \int_0^{2\pi T} \left( \frac{1}{2} (W_{K,z}(t, z_{T,K}), z_{T,K}) - W_{K}(t, z_{T,K}) \right) dt
\]

\[
\geq \left( \frac{1}{2} - \frac{1}{\mu} \right) \int_0^{2\pi T} (W_{K,z}(t, z_{T,K}), z_{T,K}) dt,
\]

\[
\frac{1}{2} - \frac{1}{\mu} > 0.
\]
i.e.,
\[
\int_0^{2\pi T} (W_{Kz}(t, z_{T,K}), z_{T,K}) \, dt \leq \left( \frac{1}{2} - \frac{1}{\mu} \right)^{-1} M. \tag{2.4}
\]

Setting
\[
f_{\infty}(t) = \begin{cases} 
\min\{k_3, |W_{Kz}(t, z_{T,K}(t))|\} \frac{W_{Kz}(t, z_{T,K}(t))}{|W_{Kz}(t, z_{T,K}(t))|}, & \text{if } W_{Kz}(t, z_{T,K}(t)) \neq 0, \\
0, & \text{otherwise},
\end{cases}
\]

we have \(W_{Kz}(t, z_{T,K}(t)) = f_{\infty}(t) + f_1(t)\) and \(\|f_{\infty}(t)\|_{L^\infty_{2\pi T}} \leq k_3\).

By (W4) for \(W_K(t, z)\), we have
\[
|f_1(t)| = |W_{Kz}(t, z_{T,K}(t)) - f_{\infty}(t)| \\
= \max\{|W_{Kz}(t, z_{T,K}(t))| - k_3, 0\} \\
\leq k_2(W_{Kz}(t, z_{T,K}(t)), z_{T,K}(t)).
\]

Thus we get from (2.4) that
\[
\|f_1\|_{L^1_{2\pi T}} \leq k_2 \int_0^{2\pi T} (W_{Kz}(t, z_{T,K}), z_{T,K}) \, dt \\
\leq k_2 \left( \frac{1}{2} - \frac{1}{\mu} \right)^{-1} M.
\]

Therefore we have
\[
\|W_{Kz}(t, z_{T,K}(t))\|_{L^1_{2\pi T} + L^\infty_{2\pi T}} \leq \|f_1\|_{L^1_{2\pi T}} + \|f_{\infty}\|_{L^\infty_{2\pi T}} \\
\leq k_2 \left( \frac{1}{2} - \frac{1}{\mu} \right)^{-1} M + k_3.
\]

**Lemma 2.3.** Let \(z(t)\) be a \(2\pi T\)-periodic solution of
\[
\dot{z} = JAz + f(t), \tag{2.5}
\]
\(z(0) = z(2\pi T),\)

where \(f(t) \in L^1_{2\pi T}\). Then there is a constant \(C_3 > 0\) independent of \(T \in \mathbb{N}\) such that
\[
\|z(t)\|_{L^\infty_{2\pi T}} \leq C_3 \|f(t)\|_{L^1_{2\pi T} + L^\infty_{2\pi T}} \quad \text{for all } f(t). \tag{2.6}
\]
Before giving a proof of Lemma 2.3, we give a remark. Let $F_s$ (resp. $F_u$) be a stable (resp. unstable) subspace of $\mathbb{R}^{2N}$ of the flow defined by $\dot{x} = JAz$. By the assumption (A), we can see

$$\mathbb{R}^{2N} = F_s \oplus F_u,$$

and $F_s$ and $F_u$ are invariant under $JA$ and $e^{t(JA)}$. Moreover we have

$$|e^{t(JA)x}| \leq C_0 e^{-at} |x| \quad \text{for} \quad t \geq 0 \text{ and } x \in F_s,$$

$$|e^{-t(JA)y}| \leq C_0 e^{-at} |y| \quad \text{for} \quad t \geq 0 \text{ and } y \in F_u,$$

where $C_0 > 0$ and $a > 0$ are constants independent of $x$, $y$, and $t$. In what follows, we denote the projections by

$$\tilde{P}_s: \mathbb{R}^{2N} \to F_s \quad \text{and} \quad \tilde{P}_u: \mathbb{R}^{2N} \to F_u. \quad (2.8)$$

Proof of Lemma 2.3. Let $z(t)$ be a $2\pi T$-periodic solution of $(2.5)$. Then we have for $t \in [0, 2\pi T]$

$$z(t) = e^{t(JA)}z_0 + \int_0^t e^{(t - \tau)JA} \tilde{P}_s f(\tau) \, d\tau$$

$$+ e^{(t - 2\pi T)JA}z_0^u - \int_t^{2\pi T} e^{(t - \tau)JA} \tilde{P}_u f(\tau) \, d\tau \quad (2.9)$$

where

$$z_0 = (I - e^{2\pi T(JA)})^{-1} \int_0^{2\pi T} e^{(2\pi T - \tau)JA} \tilde{P}_s f(\tau) \, d\tau \in F_s,$$

$$z_0^u = - (I - e^{-2\pi T(JA)})^{-1} \int_0^{2\pi T} e^{-\tau JA} \tilde{P}_u f(\tau) \, d\tau \in F_u.$$

By $(2.7)$, there is a constant $C_0' > 0$ independent of $T \in \mathbb{N}$ such that

$$\left| \int_0^t e^{(t - \tau)JA} \tilde{P}_s f(\tau) \, d\tau \right| \leq C_0' \| f_1 \|_{L^1_{2\pi T}},$$

$$\left| \int_0^t e^{(t - \tau)JA} \tilde{P}_s f(\tau) \, d\tau \right| \leq C_0' \| f_{\infty} \|_{L^\infty_{2\pi T}},$$

for all $f_1 \in L^1_{2\pi T}$, $f_{\infty} \in L^\infty_{2\pi T}$ and $t \in [0, 2\pi T]$.

From the definition of the norm $\| \cdot \|_{L^1_{2\pi T} + L^\infty_{2\pi T}}$, we have

$$\left| \int_0^t e^{(t - \tau)JA} \tilde{P}_s f(\tau) \, d\tau \right| \leq C_0' \| f \|_{L^1_{2\pi T} + L^\infty_{2\pi T}}$$

for all $f \in L^1_{2\pi T}$ and $t \in [0, 2\pi T]$. 
Similarly there is a constant $C_0'' > 0$ independent of $T \in \mathbb{N}$ such that
\[
\left| \int_{t}^{2\pi T} e^{(t-\tau)J^\perp} \tilde{P}_u f(\tau) \, d\tau \right| \leq C_0'' \| f \|_{L^1_{2\pi T} + L^\infty_{2\pi T}}
\]
for all $f \in L^1_{2\pi T}$ and $t \in [0, 2\pi T]$.

We also have
\[
\| ((I - e^{2\pi T J^\perp})^{-1}) |_{F_u} \| \leq C_0'' ,
\]
\[
\| ((I - e^{-2\pi T J^\perp})^{-1}) |_{F_u} \| \leq C_0'' ,
\]
for a constant $C_0''' > 0$ independent of $T \in \mathbb{N}$.

Therefore we can deduce (2.6) from (2.9).

Proof of Proposition 2.1. Combining Lemmas 2.2 and 2.3, we get the desired result:
\[
\| z_{T,K}(t) \|_{L^\infty_{2\pi T}} \leq C_3 \| W_{K_2}(t, z_{T,K}(t)) \|_{L^1_{2\pi T} + L^\infty_{2\pi T}} \leq C_2 C_3 .
\]

In case $W(t, z)$ does not depend on $t \in \mathbb{R}$, we can get an $L^\infty$ estimate for $z_{T,K}(t)$ without assumption (W4) as follows:

**PROPOSITION 2.4.** Suppose that $W(z)$ is independent of $t \in \mathbb{R}$ and assume (A), (W1)-(W3). Then there is a constant $C_1' > 0$ independent of $T \in \mathbb{N}$ and $K \geq 1$ such that
\[
\| z_{T,K}(t) \|_{L^\infty_{2\pi T}} \leq C_1'
\]
for all $T \in \mathbb{N}$ and $K \geq 1$.

**Proof.** In case $W(z)$ is independent of $t$, so is $W_K(z)$ by its definition. Since $z_{T,K}(t)$ is a solution of (HS:T,K),
\[
H_{T,K} = H_{K}(z_{T,K}(t)) = \frac{1}{2} (A z_{T,K}(t), z_{T,K}(t)) + W_{K}(z_{T,K}(t))
\]
does not depend on $t$.

As in the proof of Lemma 2.2, we have
\[
M \geq I_{T,K}(z_{T,K}) - \frac{1}{2} I'_{T,K}(z_{T,K}) z_{T,K}
\]
\[
= \int_{0}^{2\pi T} \left( \frac{1}{2} (W_{K_2}(z_{T,K}), z_{T,K}) - W_{K}(z_{T,K}) \right) \, dt
\]
\[
\geq \left( \frac{\mu}{2} - 1 \right) \int_{0}^{2\pi T} W_{K}(z_{T,K}) \, dt .
\] (2.10)
By (W3), we have for \( z \in \mathbb{R}^{2N} \),
\[
|\langle Az, z \rangle| \leq C |z|^2 \leq C'(k_1 |z|^q)^{2/q} \leq C' W_K(z)^{2/q}.
\]
Hence we get
\[
\frac{1}{2} \langle Az, z \rangle + W_K(z) \leq \frac{1}{2} C' W_K(z)^{2/q} + W_K(z) \leq 2W_K(z) + C''
\]
for all \( z \in \mathbb{R}^{2N} \).

By (2.10), we get
\[
2\pi T \mathcal{H}_{T,K} \leq 2 \int_0^{2\pi T} W_K(z_{T,K}(t)) \, dt + 2\pi TC''
\]
\[
\leq 2 \left( \frac{\mu}{2} - 1 \right)^{-1} M + 2\pi TC''.
\]
Thus we get \( \mathcal{H}_{T,K} \leq C''' \). That is,
\[
\frac{1}{2} \langle Az_{T,K}(t), z_{T,K}(t) \rangle + W_K(z_{T,K}(t)) \leq C'''
\]
for all \( T \in \mathbb{N} \), \( K \geq 1 \), and \( t \in \mathbb{R} \). We can deduce the desired result from (2.11) and (W3) easily. 1

Since \( z_{T,K}(t) \) is a \( C^1 \) solution of (HS : T, K), we have the following as a corollary to Propositions 2.1 and 2.4.

**Corollary 2.5.** There is a constant \( C_4 > 0 \) independent of \( T \in \mathbb{N} \) and \( K \geq 1 \) such that
\[
\|z_{T,K}(t)\|_{C^1} \leq C_4
\]
for all \( T \in \mathbb{N} \) and \( K \geq 1 \).

By Propositions 2.1 and 2.4, we can see for sufficiently large \( K_0 \geq 1 \) that
\[
\|z_{T,K_0}(t)\|_{L^\infty_{2\pi T}} < K_0
\]
and \( z_{T,K_0}(t) \) is a \( 2\pi T \)-periodic solution of the original problem (HS : T). In what follows, we fix such a \( K_0 \geq 1 \) and denote \( z_{T,K_0}(t) \) by \( z_T(t) \), and consider the behavior of \( z_j(t) - z_{T,K_0}(t) \) as \( T \to \infty \). We remark here that (i), (ii) of Theorems 0.1 and 0.2 hold for \( m = m_{K_0} \). In the following subsections we prove (iii) of Theorems 0.1 and 0.2.

### 2.2. Uniform Estimates of \( \|z_T(t)\|_{E_{2\pi T}} \) and \( \|z_T(t)\|_{L^8_{2\pi T}} \)

In this subsection, we get some uniform estimates for \( z_T(t) = z_{T,K_0}(t) \). Using these estimates, we can pass to the limit as \( T \to \infty \).
Lemma 2.6. There is a constant $C_z > 0$ independent of $T \in \mathbb{N}$ such that

$$\|z_t\|_{L_{2nT}} \leq C_z \quad \text{for all} \quad T \in \mathbb{N}.$$ 

Proof. As in the proof of Lemma 2.2, we have

$$M \geq I_{T,K_0}(z_T) - \frac{1}{2} I_{T,K_0}(z_T)z_T$$

$$= \int_0^{2\pi T} \left( \frac{1}{2} (W_{K_0}(t, z_T), z_T) - W_{K_0}(t, z_T) \right) dt$$

$$\geq \left( \frac{\mu}{2} - 1 \right) \int_0^{2\pi T} W_{K_0}(t, z_T) dt.$$

Thus by (W3), we get

$$\|z_T\|_{L_{2nT}^2} \leq \left( \frac{\mu}{2} - 1 \right)^{-1/\alpha} k^{-1/\alpha} M^{1/\alpha}. \quad (2.12)$$

On the other hand, writing $z_T = z_T^+ + z_T^- \in E_{2nT}^+ \oplus E_{2nT}^-$, we have

$$0 = I_{T,K_0}(z_T)(z_T^+ - z_T^-) = \|z_T\|_{E_{2nT}^+}^2 - \int_0^{2\pi T} (W_{K_0}(t, z_T), z_T^+ - z_T^-) dt,$$

i.e.,

$$\|z_T\|_{E_{2nT}^+}^2 = \int_0^{2\pi T} (W_{K_0}(t, z_T), z_T^+ - z_T^-) dt. \quad (2.13)$$

By (W5), for any $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that

$$|W_{K_0}(t, z)| \leq \varepsilon |z| + C_\varepsilon |z|^{\alpha - 1} \quad \text{for all} \quad (t, z) \in \mathbb{R} \times \mathbb{R}^{2N}. \quad (2.14)$$

So we have

$$\int_0^{2\pi T} (W_{K_0}(t, z_T), z_T^+ - z_T^-) dt \leq \varepsilon \int_0^{2\pi T} |z_T||z_T^+ - z_T^-| dt$$

$$+ C_\varepsilon \int_0^{2\pi T} |z_T|^{\alpha - 1}|z_T^+ - z_T^-| dt$$

$$\leq \varepsilon \|z_T\|_{L_{2nT}}^2 + C_\varepsilon \|z_T\|_{L_{2nT}}^{\alpha - 1}\|z_T^+ - z_T^-\|_{L_{2nT}}.$$

By (1.9), we get

$$\int_0^{2\pi T} (W_{K_0}(t, z_T), z_T^+ - z_T^-) dt$$

$$\leq \varepsilon C_2 \|z_T\|_{L_{2nT}}^2 + C_\varepsilon C_\alpha \|z_T\|_{L_{2nT}}^{\alpha - 1}\|z_T\|_{E_{2nT}}.$$

(2.15)
Combining (2.12), (2.13), and (2.15), we get
\[
\|z_T\|_{E_{2\pi T}}^2 \leq c_2\|z_T\|_{E_{2\pi T}}^2 + C_\epsilon c_\epsilon \left(\frac{\mu}{2} - 1\right)^{-\frac{(\alpha - 1)}{\alpha}} k_1^{-\frac{(\alpha - 1)}{\alpha}} M^{\frac{(\alpha - 1)}{\alpha}} \|z_T\|_{E_{2\pi T}}.
\]
Choosing \(\epsilon > 0\) sufficiently small, we get
\[
\|z_T\|_{E_{2\pi T}} \leq \frac{C_\epsilon c_\epsilon}{1 - \epsilon c_2} \left(\frac{\mu}{2} - 1\right)^{-\frac{(\alpha - 1)}{\alpha}} k_1^{-\frac{(\alpha - 1)}{\alpha}} M^{\frac{(\alpha - 1)}{\alpha}}.
\]
Therefore we get the desired result.

**Corollary 2.7.** There is a constant \(C_6 > 0\) independent of \(p \in [2, \infty]\) and \(T \in \mathbb{N}\) such that
\[
\|z_T\|_{L^p_{2\pi T}} \leq C_6 \quad \text{for all } p \in [2, \infty] \text{ and } T \in \mathbb{N}. \quad (2.16)
\]

**Proof.** It follows from (1.9) and Lemma 2.6 that
\[
\|z_T\|_{L^1_{2\pi T}} \leq c_2 \|z_T\|_{E_{2\pi T}} \leq c_2 C_5.
\]
We get (2.16) from the above estimate and (2.3).

Next we obtain a uniform estimate of \(\|z_T\|_{L^\infty_{2\pi T}}\) from below.

**Proposition 2.8.** There is a constant \(\delta > 0\), which is independent of \(T \in \mathbb{N}\), such that
\[
\|z_T\|_{L^\infty_{2\pi T}} \geq \delta \quad \text{for all } T \in \mathbb{N}.
\]

**Proof.** By the assumption (W5), for any \(\epsilon > 0\) we can find a \(\delta_\epsilon > 0\) such that
\[
|W_{K_0}(t, z)| \leq \epsilon |z| \quad \text{for } |z| \leq \delta_\epsilon. \quad (2.17)
\]
Suppose that \(\|z_T(t)\|_{L^\infty_{2\pi T}} \leq \delta_\epsilon\). Then, using (2.17), we have as in the proof of Lemma 2.6
\[
\|z_T\|_{E_{2\pi T}}^2 = \int_0^{2\pi T} (W_{K_0}(t, z_T), z_T^+ - z_T^-) \, dt
\]
\[
\leq \epsilon \int_0^{2\pi T} |z_T| |z_T^+ - z_T^-| \, dt
\]
\[
\leq \epsilon \|z_T\|_{L^1_{2\pi T}}^2 \leq \epsilon c_2 \|z_T\|_{E_{2\pi T}}^2.
\]
Choosing \( \varepsilon \in (0, 1/c_2) \), we have \( z_T = 0 \). But this contradicts \( I_{T, K_0}(z_T) \geq m > 0 \). Therefore we have \( \|z_T\|_{L^2} \geq \delta_\varepsilon \).

2.3. Limit Process for \( z_T(t) \)—Proofs of Theorems 0.1 and 0.2

We can find a sequence \((l_T)_{T \in \mathbb{N}}\) of integers such that

\[
\max_{t \in [0, 2\pi]} |z_T(t + 2\pi l_T)| = \max_{t \in \mathbb{R}} |z_T(t)| \in [\delta_\varepsilon, C_1].
\]

(2.18)

We remark that \( \bar{z}_T(t) = z_T(t + 2\pi l_T) \) is a solution of (HS) satisfying (i), (ii) of Theorems 0.1 and 0.2 and \( I_T(\bar{z}_T) = I_{T, K_0}(z_T) \). In what follows, we show that \((\bar{z}_T(t))_{T \in \mathbb{N}}\) possesses the compactness property (iii) of Theorems 0.1 and 0.2.

By Corollary 2.5, we can extract a subsequence from any given sequence of integers \( T_n \to \infty \)—we still denote it by \( T_n \)—such that

\[
\bar{z}_{T_n} \equiv z_{T_n}(t + 2\pi l_{T_n}) \to z_\infty(t) \quad \text{in} \quad C^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^{2N}),
\]

(2.19)

where \( z_\infty(t) \in C^1(\mathbb{R}, \mathbb{R}^{2N}) \) is a solution of (HS). The following Lemma 2.9 completes the proofs of Theorems 0.1 and 0.2.

**Lemma 2.9.** \( z_\infty(t) \) satisfies the following

(i) \( z_\infty(t) \neq 0 \).

(ii) \( z_\infty(t) \in L^p(\mathbb{R}, \mathbb{R}^{2N}) \) for all \( p \in [2, \infty] \).

(iii) \( z_\infty(t) \to 0 \) as \( t \to \pm \infty \).

**Proof.** (i) By (2.18) and (2.19), we have

\[
\max_{t \in [0, 2\pi]} |z_\infty(t)| = \sup_{t \in \mathbb{R}} |z_\infty(t)| \in [\delta_\varepsilon, C_1].
\]

(2.20)

Therefore we have (i) via Proposition 2.8.

(ii) For \( p \in [2, \infty) \), we have from Corollary 2.7 for any \( R > 0 \)

\[
\int_{-R}^{R} |z_\infty(t)|^p dt = \lim_{n \to \infty} \int_{-R}^{R} |z_{T_n}(t + 2\pi l_{T_n})|^p dt \leq \limsup_{T \to \infty} \|z_T\|_{L^p_{2\pi}T}^p \leq C_6^p.
\]

Letting \( R \to \infty \), we get \( z_\infty(t) \in L^p(\mathbb{R}, \mathbb{R}^{2N}) \) for \( p \in [2, \infty) \). For \( p = \infty \), (ii) follows from (2.20).

(iii) Since \( z_\infty(t) \) is a solution of (HS), we can write

\[
z_\infty(t) = e^{t(JA)}z_0 + \int_{-\infty}^{t} e^{(t-\tau)JA} \bar{p}_\tau JW_{K_0z}(\tau, z_\infty(\tau)) d\tau - \int_{t}^{\infty} e^{(t-\tau)JA} \bar{p}_\tau JW_{K_0z}(\tau, z_\infty(\tau)) d\tau
\]

(2.21)
for some \( z_0 \in \mathbb{R}^{2N} \). By (ii) and (2.14), we have
\[
z_\infty(t) \in L^2(\mathbb{R}, \mathbb{R}^{2N}) \quad \text{and} \quad W_{K_0}(t, z_\infty(t)) \in L^2(\mathbb{R}, \mathbb{R}^{2N}).
\]
By (2.7), we see
\[
\int_{-\infty}^{\infty} e^{(t-\tau)J^A} \bar{P}_s JW_{K_0}(\tau, z_\infty(\tau)) d\tau \in L^2(\mathbb{R}, L^2(\mathbb{R}^{2N})).
\]
On the other hand, we have \( e^{(tJ^A)}z_0 \notin L^2(\mathbb{R}, \mathbb{R}^{2N}) \) for \( z_0 \neq 0 \). Thus \( z_0 = 0 \) follows from (2.21). Therefore we can easily deduce from (2.21) that \( z_\infty(t) \to 0 \) as \( |t| \to \infty \).

3. Proof of Proposition 1.1

This section is devoted to the proof of Proposition 1.1. Using Fourier series, we have the following representation of \( \Phi_{2\pi T} \)
\[
(\Phi_{2\pi T})(t) = \sum_{j \in \mathbb{Z}} \left( \frac{-ij}{T} J - A \right) a_j e^{ij/T} \quad (3.1)
\]
where
\[
z(t) = \sum_{j \in \mathbb{Z}} a_j e^{ij/T} \quad (a_j \in \mathbb{C}^{2N} \text{ with } a_{-j} = \bar{a}_j). \quad (3.2)
\]
We also have
\[
\|z(t)\|_{L^2_{2\pi T}}^2 = 2\pi T \sum_{j \in \mathbb{Z}} |a_j|^2. \quad (3.3)
\]
We remark that \( \text{span} \{ae^{ij/T} + \bar{a}e^{-ij/T} ; a \in \mathbb{C}^{2N} \} \subset L^2_{2\pi T} \) is invariant under the operator \( \Phi_{2\pi T} \) for all \( j \in \mathbb{N} \). \( E_{2\pi T} = D(\Phi_{2\pi T}^{1/2}) \) can be written
\[
E_{2\pi T} = \left\{ z = \sum_{j \in \mathbb{Z}} a_j e^{ij/T} ; \|z\|_{E_{2\pi T}}^2 = 2\pi T \sum_{j \in \mathbb{Z}} \left( \frac{ij}{T} J + A \right) a_j, a_j < \infty \right\}, \quad (3.4)
\]
where \( (x, y) = \sum_{k=1}^{2N} x_k \overline{y_k} \) for \( x = (x_1, ..., x_{2N}) \), \( y = (y_1, ..., y_{2N}) \in \mathbb{C}^{2N} \). Note that \( -i\theta J - A(\theta \in \mathbb{R}) \) is a \( 2N \times 2N \) Hermitian matrix and we can define \( |i\theta J + A| : \mathbb{C}^{2N} \to \mathbb{C}^{2N} \).
By the assumption \((A)\), we have
\[
0 \notin \sigma(-i\theta J - A) \quad \text{for all} \quad \theta \in \mathbb{R}. \tag{3.5}
\]
We can see that \(-i\theta J - A\) has \(N\) positive eigenvalues and \(N\) negative eigenvalues (counting multiplicities). In fact, the eigenvalues are solutions of
\[
\det(\lambda + (i\theta J + A)) = 0.
\]
By (3.5), we can see the number of positive (or negative) eigenvalues is independent of \(\theta \in \mathbb{R}\). Dividing by \(\theta > 0\), it equals to the number of positive (or negative) solutions of
\[
\det \left( \lambda + \left( iJ + \frac{1}{\theta} A \right) \right) = 0.
\]
Passing to the limit as \(\theta \to \infty\), we see it equals to \(N\). We denote the eigenvalues of \(-i\theta J - A\) by \(\lambda^-_N(\theta) \leq \cdots \leq \lambda^-_1(\theta) < 0 < \lambda^+_1(\theta) \leq \cdots \leq \lambda^+_N(\theta)\) and the corresponding eigenvectors by \(\xi^-_N(\theta), \ldots \xi^-_1(\theta), \xi^+_1(\theta), \ldots \xi^+_N(\theta)\). We remark
\[
\lambda^\pm_k(-\theta) = \lambda^\pm_k(\theta), \tag{3.6}
\]
and
\[
\xi^\pm_k(-\theta) = \overline{\xi^\pm_k(\theta)} \tag{3.7}
\]
for all \(\theta \in \mathbb{R}\) and \(k = 1, \ldots, N\).

**Lemma 3.1.** Under the assumption \((A)\), there are constants \(c, c' > 0\) independent of \(\theta \in \mathbb{R}\) such that
\[
c(1 + |\theta|) \leq |\lambda^\pm_k(\theta)| \leq c'(1 + |\theta|) \tag{3.8}
\]
for all \(\theta \in \mathbb{R}\) and \(k = 1, \ldots, N\).

**Remark 3.1.** (1.4) follows from (3.8).

**Proof.** Since \(\lambda^\pm_k(\theta)\) is a solution of
\[
\det \left( \frac{\lambda}{\theta} + \left( iJ + \frac{1}{\theta} A \right) \right) = 0,
\]
it is clear that
\[
\left| \frac{\lambda^\pm_k(\theta)}{\theta} \right| \to 1 \quad \text{as} \quad |\theta| \to \infty. \tag{3.9}
\]
On the other hand, by (3.5) we have
\[
0 < \inf \{|\lambda_+^k(\theta)|; |\theta| \leq L, 1 \leq k \leq N\}
\leq \sup \{|\lambda_-^k(\theta)|; |\theta| \leq L, 1 \leq k \leq N\} < \infty
\]  
(3.10)
for any \( L > 0 \). Combining (3.9) and (3.10), we get (3.8). □

Now we can prove (i), (ii) of Proposition 1.1.

**Proof of (i) of Proposition 1.1.** By (3.8), we have
\[
c \sum_{j \in \mathbb{Z}} \left(1 + \frac{|j|}{T}\right)|a_j|^2 \leq \sum_{j \in \mathbb{Z}} \left(\frac{|j|}{T} J + A \left| a_j, a_j \right| \right)
\]
\[
\leq c' \sum_{j \in \mathbb{Z}} \left(1 + \frac{|j|}{T}\right)|a_j|^2.
\]
Thus by the definition of \( \|z\|_{H^{1/2}_{2\pi T}} \) and (3.4), we get (1.8) and \( E_{2\pi T} = H^{1/2}_{2\pi T} \). □

**Proof of (ii) of Proposition 1.1.** It suffices to prove
\[
\|z\|_{L^p_{2\pi T}} \leq c_p \|z\|_{H^{1/2}_{2\pi T}} \quad \text{for } z \in H^{1/2}_{2\pi T} \text{ for } p \in [2, \infty). \hspace{1cm} (3.11)
\]
For \( z(t) \) of form (3.2), we have from Hausdorff-Young's inequality and Hölder's inequality,
\[
\|z\|_{L^p_{2\pi T}} \leq (2\pi T)^{1/p} \left(\sum_{j \in \mathbb{Z}} |a_j|^q\right)^{1/q}
\leq (2\pi T)^{1/p} \left(\sum_{j \in \mathbb{Z}} \left(1 + \frac{|j|}{T}\right)^{-q/(2-q)}\right)^{(2-q)/2q}
\times \left(\sum_{j \in \mathbb{Z}} \left(1 + \frac{|j|}{T}\right)|a_j|^2\right)^{1/2},
\]
where \( 1/p + 1/q = 1 \).

Since
\[
\sum_{j \in \mathbb{Z}} \left(1 + \frac{|j|}{T}\right)^{-q/(2-q)} \leq 1 + \int_{\mathbb{R}} \left(1 + \frac{|s|}{T}\right)^{-q/(2-q)} \, ds = 1 + c_q T,
\]
we get (3.11). □

Next we give a proof to (iii) of Proposition 1.1. We represent \( P_{2\pi T}^\pm \colon E_{2\pi T} \to E_{2\pi T}^\pm \) by means of Fourier series. Let \( Q_{\theta}^\pm \) be a matrix associated to the projection \( C^{2N} \to \text{span}\{\xi_k^\pm(\theta); 1 \leq k \leq N\} \). Then we can see from (3.1)
\[
(P_{2\pi T}^\pm z)(t) = \sum_{j \in \mathbb{Z}} (Q_{j/T}^\pm a_j) e^{j\mu/T}
\]  
(3.12)
for \( z(t) \) of form (3.2). By (3.6) and (3.7), we remark

\[ Q_{j,T}^+ a_j = \overline{Q_{j,T}^- a_j} \quad \text{for all } j \in \mathbb{Z}. \]

We can easily see from (3.12) that

\[ \| P_{2\pi T}^\pm z \|_{L^2_{2\pi T}} \leq \| z \|_{L^2_{2\pi T}}, \]

\[ \| P_{2\pi T}^\pm z \|_{L^2_{2\pi T}} \leq \| z \|_{L^2_{2\pi T}} \quad \text{(3.13)} \]

for all \( z \in E_{2\pi T} \).

To prove the continuity of \( P_{2\pi T}^\pm : L^p_{2\pi T} \to L^p_{2\pi T} \), we introduce the operator

\[ \widehat{P}_{2\pi T}^\pm : L^2_{2\pi} \to L^2_{2\pi} \]

defined by

\[ (\widehat{P}_{2\pi T}^\pm z)(t) = \sum_{j \in \mathbb{Z}} (Q_{j,T}^\pm a_j) e^{ijt} \quad \text{(3.14)} \]

for

\[ z(t) = \sum_{j \in \mathbb{Z}} a_j e^{ijt} \quad (a_j \in \mathbb{C}^{2^N} \text{ with } a_j = \overline{a_j}). \]

Since

\[ \sup \{ \| P_{2\pi T}^\pm z \|_{L^p_{2\pi T}} ; z \in L^p_{2\pi T}, \| z \|_{L^p_{2\pi T}} \leq 1 \} \]

\[ = \sup \{ \| \widehat{P}_{2\pi T}^\pm z \|_{L^p_{2\pi}} ; z \in L^p_{2\pi}, \| z \|_{L^p_{2\pi}} \leq 1 \}, \quad \text{(3.15)} \]

we estimate the right hand side instead of the left hand side. We rely on the following Stečkin’s theorem (Theorem 3.5 of [3]).

**Proposition 3.2.** Let \((\phi(j))_{j \in \mathbb{Z}}\) be a function of bounded variation on \( \mathbb{Z} \). Then for each \( p \in (1, \infty) \) the operator

\[ (T_{\phi} z)(t) = \sum_{j \in \mathbb{Z}} \phi(j) a_j e^{ijt} \quad \text{for } z(t) = \sum_{j \in \mathbb{Z}} a_j e^{ijt} \]

is continuous as \( L^p_{2\pi} \to L^p_{2\pi} \). Moreover there is a constant \( C_p > 0 \) independent of \( \phi \) such that

\[ \sup_{z \in L^p_{2\pi}} \| T_{\phi} z \|_{L^p_{2\pi}} \leq C_p \max \{ |\phi(0)|, \text{Var } \phi \}. \quad \text{(3.16)} \]
Proof of (iii) of Proposition 1.1. We apply Proposition 3.2 to (3.14). By (3.15) and (3.16), we need only to prove the existence of $C > 0$ such that

$$\text{Var}(Q_{j/T}^\pm) \leq C \quad \text{for all} \quad T \in \mathbb{N}. \quad (3.17)$$

We have

$$\text{Var}(Q_{j/T}^\pm) = \sum_{j \in \mathbb{Z}} |Q_{(j+1)/T}^\pm - Q_{j/T}^\pm|$$

$$\leq \int_{-\infty}^{\infty} \left| \frac{dQ_{\theta}^\pm}{d\theta} \right| d\theta. \quad (3.18)$$

In what follows, we see the right hand side is finite (clearly it is independent of $T \in \mathbb{N}$). We deal with only "+" case. The case "-" is treated similarly. First we prove $\int_1^{\infty} |dQ_{\theta}^+/d\theta| d\theta < \infty$.

Since $Q_{\theta}^+$ is a projection operator corresponding to $-i\theta J - A$, it is also corresponding to $-iJ - (1/\theta) A$. By Lemma 3.1, we can find constants $a, b > 0$ independent of $\theta \in [1, \infty)$ such that

$$a \leq \frac{\lambda_k^+(\theta)}{\theta} \leq b \quad \text{for all} \quad \theta \in [1, \infty) \text{ and } k = 1, \ldots, N.$$ 

Since $\lambda_k^+(\theta)/\theta$ are eigenvalues of $-iJ - (1/\theta) A$, we have

$$Q_{\theta}^+ = \frac{1}{2\pi i} \int_{\gamma} \left( \zeta + iJ + \frac{1}{\theta} A \right)^{-1} d\zeta.$$ 

Here, $\gamma$ is a cycle in the right half plane \{z $\in \mathbb{C}; \text{Re} \ z > 0\} which surrounds the interval $[a, b]$. Thus

$$\frac{dQ_{\theta}^+}{d\theta} = \frac{1}{2\pi i} \int_{\gamma} \theta^{-2} \left( \zeta + iJ + \frac{1}{\theta} A \right)^{1} A \left( \zeta + iJ + \frac{1}{\theta} A \right)^{-1} d\zeta.$$ 

Hence we have

$$\left| \frac{dQ_{\theta}^+}{d\theta} \right| \leq \frac{1}{2\pi} \int_{\gamma} \theta^{-2} \left| A \left( \zeta + iJ + \frac{1}{\theta} A \right)^{-1} \right|^2 |d\zeta| \leq C \theta^{-2},$$

where $C > 0$ is independent of $\theta > 1$. Therefore we have

$$\int_1^{\infty} \left| \frac{dQ_{\theta}^+}{d\theta} \right| d\theta < \infty. \quad (3.19)$$
Using the representation

$$Q^{+}_\theta = \frac{1}{2\pi i} \int_{\gamma'} (\zeta + i\theta J + A)^{-1} d\zeta,$$

where $\gamma'$ is a cycle in $\{ z \in \mathbb{C} ; \text{Re} \ z > 0 \}$ surrounding the set $\{ \lambda_k^+(\theta); k = 1, \ldots, N, |\theta| \leq 1 \}$, we obtain

$$\left| \frac{dQ^{+}_\theta}{d\theta} \right|_{-1}^{0} d\theta < \infty. \quad (3.20)$$

Similarly to (3.19), we obtain

$$\left| \frac{dQ^{+}_\theta}{d\theta} \right|_{-\infty}^{0} d\theta < \infty. \quad (3.21)$$

Combining (3.19)–(3.21), we obtain

$$\int_{-\infty}^{0} \left| \frac{dQ^{+}_\theta}{d\theta} \right| d\theta < \infty.$$ 

Thus we obtain (3.17). \qed

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