Connections of Berry and Hannay Type for Moving Lagrangian Submanifolds

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Berry's phase is the holonomy of the natural connection on the canonical circle bundle over projectivized quantum Hilbert space. Given a symplectic manifold $P$, a classical limit of this phase is constructed as the holonomy of a Berry connection over a finite-codimensional isodrastic foliation (defined by constancy of action integrals) on the space of lagrangian submanifolds in $P$ equipped with smooth densities of total measure 1. If the densities are determined by a Kähler metric compatible with the symplectic structure, the curvature of the Berry connection at the lagrangian submanifold $L$ is given by a simple formula involving curvature of $L$. In particular, the Berry connection defines a homotopy invariant for isodrastic loops of minimal lagrangian submanifolds in a simply connected Kähler manifold. A similar invariant constructed by the author for loops of symplectomorphisms is also a special case of the classical Berry phase. A classical analogue of Berry's phase was discovered by Hannay for moving families of completely integrable systems. Following Berry and Hannay, we interpret Hannay's angles as the holonomy of a Hannay connection on a bundle of tori over the isodrastic foliation on the space of lagrangian toral layers consisting of lagrangian tori with a flat affine structure and an extension of this structure to the first infinitesimal neighborhood. Finally, we show that the Hannay angles are the derivatives with respect to action variables of the classical Berry phase.

1. INTRODUCTION

Given a symplectic manifold $(P, \omega)$ which is the base of a $U(1)$ bundle $Q$ with connection $\alpha$ whose curvature is $\omega$, the holonomy associated to a loop in $P$ is an element of the structure group $U(1)$ which can be thought of as a “geometric phase” associated with the loop. This point of view comes from work of Berry [5], Simon [21], and Aharanov–Anandan [1] (among many others) on cyclic evolutions in quantum mechanics, where $Q$ is the unit sphere in a Hilbert space and $P$ is the corresponding projective space.

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A related construction was carried out by the author in [29], where a homotopy-invariant phase was constructed for systems in which all points in a compact phase space undergo a cyclic evolution, i.e., for loops in the group of automorphisms of a compact symplectic manifold. This phase can also be interpreted as the holonomy for a connection on a bundle, this time over the symplectomorphism group itself.

In both of the situations described above, the lifting from $P$ involves normalizing hamiltonians—either to be zero along a path in the first case or to have mean value zero in the second.

In this paper, we define a phase for the intermediate situation in which a lagrangian submanifold of $P$ undergoes a cyclic evolution. This construction, which may be thought of as a quasi-classical limit of Berry's phase, requires that we have a way of normalizing hamiltonians on our lagrangian submanifolds—this is done by providing the submanifolds with measures in some prescribed way. As suggested by the context of adiabatic motion in which Berry's phase was first studied, we must also restrict the motion of our lagrangian submanifolds to isodrasts defined by the constancy of action integrals. Finally, to have a suitable space in which to have the phases take their values, we must choose a prequantization of $P$, a bundle with a connection whose curvature is the symplectic structure.

Thus we introduce a space of "weighted lagrangian submanifolds" and find a natural "Berry connection" on a bundle over the isodrastic foliation. (From the semiclassical point of view, the weights represent WKB amplitudes, just as the lagrangian submanifolds themselves represent phase functions; see [20].)

If we have a prescription for assigning a weight to each lagrangian submanifold, we get a connection over the lagrangian submanifolds themselves, whose curvature depends on how we assigned the weights. Sometimes, this connection is flat, in which case we get a homotopy invariant for loops of lagrangian submanifolds. This occurs, for instance, if we look at graphs of symplectomorphisms with the symplectic volume as the weighting, as in [29]. On the other hand, if the weightings are determined by a Kähler metric on $P$ compatible with the symplectic structure, the Berry curvature vanishes at a lagrangian submanifold $L$ if and only if $L$ is minimal.

A classical analog (as opposed to limit) of Berry's phase was introduced by Hannay [11]. If a completely integrable hamiltonian system undergoes a cyclic evolution, then the integral manifold (a torus) with given values of the action variables returns to itself having undergone a translation by an element of the underlying toral group. This element, whose components are called Hannay's angles, may also be thought of as the holonomy for a connection, this time for a torus bundle over a space of lagrangian foliations [2, 13, 18]. In this paper, we prove in detail an assertion of Berry and
Hannay [7] that the "Hannay connection" can actually be defined over the space of lagrangian submanifolds provided with a free torus action on an infinitesimal neighborhood. In a preliminary version of the present paper, it was asserted that the connection could even be defined over the space of flat affine tori; this assertion, however, turns out to be incorrect, as we shall explain in Section 8.

Finally, since the affine structure of a torus also induces a weighting, the Berry and Hannay connections can be compared; we do this in the last section of the paper.

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2. ISODRASTIC DEFORMATIONS OF LAGRANGIAN SUBMANIFOLDS

Much of this section consists of a paraphrase of results in [26], to which we refer for further details.

For any symplectic manifold \((P, \omega)\), we define \(A(P)\) to be the space of closed, embedded lagrangian submanifolds of \(P\), considered informally as an infinite dimensional manifold. Using the cotangent coordinates of [25], we see that the tangent space to \(A(P)\) (as well as the local model space) at a point \(L\) is naturally isomorphic to the space \(Z'(L)\) of closed 1-forms on \(L\).

The isomorphism may also be described in the following way. Any 1-parameter family \(L_t\) of lagrangian submanifolds may be realized by deforming \(L = L_0\) along the trajectories of a time-dependent locally hamiltonian vector field \(\xi\), defined on a neighborhood of \(L\).\(^1\) The locally hamiltonian vector fields which leave \(L\) invariant are those which are tangent to \(L\). Thus a tangent vector to \(A(P)\) at \(L\) is represented by an equivalence class of locally hamiltonian vector fields defined near \(L\), modulo the fields which are tangent to \(L\). Under the correspondence between vector fields and forms, defined by the symplectic structure, the hamiltonian vector fields are mapped to closed 1-forms; since \(L\) is lagrangian, the vector fields tangent to \(L\) go to the forms which pull back to zero on \(L\). Thus a tangent vector to \(A(P)\) at \(L\) is represented by an equivalence class of closed 1-forms defined near \(L\) modulo the kernel of pullback to \(L\). Since every closed form on \(L\) extends to a closed form near \(L\) (by a tubular neighborhood projection, say), we obtain an isomorphism between the tangent space \(T_L(A(P))\) and \(Z'(L)\).

\(^1\) Think of closed curves in the plane to see why \(\xi\) might not be defined on all of \(P\).
The subspace $B'(L)$ in $Z'(L)$ consisting of exact 1-forms determines a distinguished subspace $T'(A(P))$ of $T(A(P))$ whose elements we shall call isodrastic since, as we shall see, deformations in these directions may be characterized by a condition of "constancy of action integrals." A path of lagrangian submanifolds whose tangent vector is everywhere isodrastic will be called an isodrastic deformation (or isotopy), and the endpoints of such a deformation will be called isodrastic submanifolds. Since isodrastic deformations can be obtained by flowing along globally hamiltonian vector fields, they are also called hamiltonian deformations. We remark here that one reason among many for the interest in isodrastic deformations is that they are realized by the trajectories of adiabatically varying completely integrable hamiltonian systems.

When $P$ is a cotangent bundle $T^*X$ with the canonical symplectic structure, the lagrangian submanifolds near the zero section $O$ are the graphs of the closed 1-forms on $X$. It is easily checked that two such graphs may be connected by an isodrastic deformation if and only if the corresponding 1-forms are cohomologous. Thus, near $O$ in $A(T^*X)$, the subspaces $T'_i(A(T^*X))$ are the tangent spaces to a foliation. Since any symplectic manifold is isomorphic to a cotangent bundle near each of its lagrangian submanifolds, we may conclude that the spaces $T'_i(A(P))$ are tangent to a foliation $\mathcal{F}$ for any symplectic $P$. We call each leaf of this foliation an isodrast.

To explain the role of action integrals, we begin with the case where $\omega = -d\beta$ for some 1-form $\beta$. In this case, the action integral $A(\gamma)$ is simply defined as the integral of $\beta$ around the loop $\gamma$. If $\gamma$ bounds a surface $S$ mapped into $P$, then by Stokes' theorem

$$A(\gamma) = \int_S -\omega$$

which is clearly independent of the choice of $\beta$. Equation (1) can also be used to define the action integral on contractible loops in any symplectic manifold, up to an element of the spherical period group consisting of the integrals of $\omega$ over all possible spherical surfaces in $P$. When all the elements of this period group are multiples of $2\pi$, the action integral is well defined as a phase. In any case, we can unambiguously define the difference between the action integrals around nearby loops in any symplectic manifold $P$ by integrating $-\omega$ over a narrow cylindrical surface joining the

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2 Rather than mix the Greek prefix iso with the Latin action, we have replaced the latter by its Greek equivalent to obtain this term. I thank George Bergman for his advice on this linguistic point.

3 The term exact deformation is also used by some authors.
two loops. (In the same way, we can unambiguously define the notion of an isodrastic deformation of a loop.) If the two loops are in nearby lagrangian submanifolds \( L \) and \( L' \), then the action difference remains constant as the loops are deformed within these submanifolds and so defines a real-valued invariant on the fundamental group \( \pi_1(L) \).

We may summarize this discussion as follows.

**Proposition 2.1.** Given a pair \( L \) and \( L' \) of nearby lagrangian submanifolds in the symplectic manifold \( P \), the difference in action integrals between nearby loops in \( L \) and \( L' \) gives a well defined invariant in \( H^1(L; \mathbb{R}) \). If \( L \) is fixed and \( L' \) undergoes a deformation, this action invariant remains constant if and only if the deformation is isodrastic.

**Remark 2.2.** For use in the next section, we note here that the construction of the action integral around a loop does not require the nondegeneracy of the \( \omega \)-form, but only the fact that it is closed.

### 3. Weighted Lagrangian Submanifolds

Our aim in this section is to assign a phase-like invariant to loops in the space \( A(P) \). Following the approach of Section 2 (see Remark 2.2), we should use a \( \omega \)-form on \( A(P) \) to do this. Such a form would assign a number to every pair of closed 1-forms on the submanifold, which cannot be done in any interesting invariant way of which we know.

To make the construction of the \( \omega \)-form possible, we shall alter the problem in a couple of ways. First of all, we shall restrict our attention to one isodrast at a time; second, we shall equip our lagrangian submanifolds with some extra structure.

Recall that the tangent space \( T^1_L(A(P)) \) to the isodrast through \( L \) is isomorphic to the space \( B^1(L) \) of exact 1-forms on \( L \). Assuming \( L \) to be connected (which we shall do), the latter space is isomorphic to the quotient \( C^\infty(L)/\mathbb{R} \), where \( \mathbb{R} \) consists of the constant functions. A 2-form \( \Omega \) on the isodrast, therefore, would assign to each pair of functions \( f \) and \( g \) on \( L \) a number \( \Omega(f, g) \) which is antisymmetric in \( f \) and \( g \) and which is unchanged when a constant is added to either function. Such an assignment can be made in a natural way if we attach to each lagrangian submanifold an extra piece of data which enables us to average functions on it. (Of course, we now have a larger tangent space to account for variations in this extra data.)

**Definition 3.1.** A *weighting* of the manifold \( L \) is a compactly supported smooth density \( \rho \) on \( L \) whose integral over \( L \) is equal to 1. The pair \( (L, \rho) \) is called a *weighted manifold*. We denote the space of weightings of \( L \) by \( WL \).
WL is an affine space modeled on the vector space of compactly supported smooth densities with integral 0 in $L$. The latter space is dual to the space $B^1(L) = C^\infty(L)/\mathbb{R}$. (More precisely, it is dense in the space of compactly supported distribution densities with integral zero, which is the full dual of $C^\infty(L)/\mathbb{R}$.) Thus, if $L$ is a lagrangian submanifold of the symplectic manifold $P$, WL is an affine space modeled on the space of (smooth) cotangent vectors to the isodrast through $L$. We shall denote the space of all weighted lagrangian submanifolds of $P$ by $W\Lambda(P)$; it is an affine bundle over $\Lambda(P)$ which is modeled on the vector bundle of smooth cotangent vectors along the isodrastic foliation $\mathcal{F}$. Using the projection from $W\Lambda(P)$ to $\Lambda(P)$ which forgets weightings, we obtain a pulled back foliation $\mathcal{F}_P$ of $W\Lambda(P)$.

**Theorem 3.2.** Let $P$ be a symplectic manifold. The form $\Omega$ given by (2) or (3) below is an invariantly defined, skew-symmetric bilinear form on the subbundle of $T(W\Lambda(P))$ tangent to $\mathcal{F}$. On each leaf of $W\mathcal{F}$, it defines a weakly nondegenerate closed 2-form.

**Proof.** We begin by giving an description of the tangent spaces of $W\Lambda(P)$ and the subspaces tangent to the isodrastic foliation $\mathcal{F}$. For convenience, we shall assume that our lagrangian submanifolds are oriented, so that we can use forms instead of densities as weightings.

**Lemma 3.3.** There is a natural 1–1 correspondence between tangent vectors to $W\Lambda(P)$ and equivalence classes of quadruples $(L, \rho, v, \phi)$, where $L$ is a lagrangian submanifold of $P$, $\rho$ is an $n$-form on a neighborhood of $L$ such that $\int_L \rho = 1$, $v$ is a locally hamiltonian vector field on a neighborhood of $L$, and $\phi$ is an $n$-form on a neighborhood of $L$ such that $\int_L (\mathcal{L}_v \rho + \phi) = 0$. Here, $(L, \rho, v, \phi)$ is equivalent to $(L, \rho', v', \phi')$ if and only if the following conditions hold: (1) $\rho$ and $\rho'$ have the same pullback to $L$; (2) $v' - v$ is tangent to $L$; (3) the pullbacks to $L$ of $\mathcal{L}_v \rho + \phi$ and $\mathcal{L}_{v'} \rho' + \phi'$ are equal. (Note that it is $v$ and not $v'$ which appears in the last expression.)

The tangent vector is isodrastic if and only if $v$ is globally hamiltonian near $L$. In this case, $v = H_f$ for a hamiltonian function $f$, and condition (2) means that $f' - f$ is constant on $L$.

**Proof.** A weighted lagrangian submanifold can be considered as an equivalence class of pairs $(L, \rho)$, where $L$ is lagrangian submanifold and $\rho$ is an $n$-form on a neighborhood of $L$ such that $\int_L \rho = 1$. Here, $(L, \rho)$ is equivalent to $(L, \rho')$ if and only if $\rho$ and $\rho'$ have the same pullback to $L$.

Thus, a tangent vector to $W\Lambda(P)$ at $(L, \rho)$ is an equivalence class of paths $(L_t, \rho_t)$ with $\int_L \rho_t = 1$, $L_0 = L$, and $\rho_0 = \rho$ on $L$. The derivative of $L_t$ with respect to $t$ is represented by a locally hamiltonian vector field $v$,
defined near \( L \) (modulo fields which are tangent to \( L_r \)); in particular, the derivative at \( t = 0 \) is represented by \( v = v_0 \).

Let \( \mu_1 \) be the 1-parameter family of symplectomorphisms generated by \( \nu_1 \). If the derivative of \( \rho \) with respect to \( t \) at \( t = 0 \) is \( \phi \), then the derivative with respect to \( t \) at \( t = 0 \) of \( \mu_1^*(\rho) \) equals \( \mathcal{L}_\rho \phi + \phi \). Differentiating the condition \( \int_L \rho_r = 1 \) gives \( \int_L \mathcal{L}_\rho \phi + \phi = 0 \). Furthermore, if the paths \((L, \rho)\) and \((L, \rho')\) are to represent the same tangent vector, we must have \( \mu_1^*(\rho) = \mu_1^*(\rho') \) to first order at \( t = 0 \), i.e., \( \mathcal{L}_\rho \phi + \phi = \mathcal{L}_{\rho'} \phi' + \phi' \). (It is easy to check that, if this condition holds for \( v \), it also holds for any equivalent \( v' \).)

For the final statement in the lemma, we simply recall that a hamiltonian vector field is tangent to the lagrangian submanifold \( L \) if and only if its hamiltonian is constant on \( L \). Q.E.D.

We return now to the proof of Theorem 3.2. Let \( \xi_1 \) and \( \xi_2 \) be isodrastic tangent vectors at \((L, \rho)\) represented by the quadruples \((L, \rho_1, f_1, \phi_1)\) and \((L, \rho_2, f_2, \phi_2)\), respectively. Then we define

\[
\Omega(\xi_1, \xi_2) = \int_L \left[ \{ f_1, f_2 \} \rho + f_1 (\mathcal{L}_{\rho_2} \rho + \phi_2) - f_2 (\mathcal{L}_{\rho_1} \rho + \phi_1) \right].
\] (2)

It is straightforward to check that this definition is independent of the choice of representatives of \( \xi_1 \) and \( \xi_2 \).

To show that the resulting 2-form is closed, we shall use cotangent coordinates. Thus, we assume that \( P \) is a cotangent bundle \( T^*X \) with its standard symplectic structure, and that the lagrangian submanifolds are graphs of 1-forms. We can then represent each isodrastic tangent vector \( \xi \) by a quadruple \((L, \rho, f, \phi)\) for which \( \rho, f, \phi \) are all pulled up from the base \( X \). The Poisson bracket and Lie derivative terms in (2) all vanish, leaving the simple formula

\[
\Omega(\xi_1, \xi_2) = \int_L (f_1 \phi_2 - f_2 \phi_1).
\] (3)

Since this is independent of \( L \) and \( \rho \), it defines a closed 2-form.

Using (3), we can also check the weak nondegeneracy of \( \Omega \). If \( \Omega(\xi_1, \xi_2) = 0 \) for all \( \xi_2 \), then we have \( \int_L (f_1 \phi_2 - f_2 \phi_1) = 0 \) for all \( f_2 \) and \( \phi_2 \). Since \( f_2 \) and \( \phi_2 \) are independent, with the only constraint on them being that \( \int_L \phi_2 = 0 \), we conclude that \( f_1 \) is constant and \( \phi_1 = 0 \), which means that \( \xi_1 = 0 \). Q.E.D.

Integrating \(-\Omega\) over a surface spanned by a contractible loop \( \lambda \) in a leaf of \( \mathcal{W} \), i.e., taking the action integral around \( \lambda \), we obtain a quantity which we call the Berry phase of \( \lambda \), defined modulo the period group of \( \Omega \). We will obtain some information about this group in the following section.
Given any weighting rule, i.e., a section $\sigma$ of $W\Lambda(P)$ over $\Lambda(P)$, we get a well-defined closed 2-form $\sigma^*\Omega$ on each isodrast in $\Lambda(P)$, which enables us to assign a Berry phase to each contractible loop in the isodrast. We will study the 2-forms arising from specific weighting rules in Section 6.

Remark 3.4. The form $\Omega$ may be considered as a Poisson structure [27] on the manifold $W\Lambda(P)$ with symplectic leaves given by the isodrastic foliation $\mathcal{F}$. When we apply a weighting rule to pull $\Omega$ down to $\Lambda(P)$ itself, we get a family of closed 2-forms along the leaves of $\mathcal{F}$. Since these forms may be degenerate, the result is usually a Dirac structure in the sense of [9] rather than a Poisson structure.

4. PREQUANTIZATION

We have noted in Section 3 that each leaf of $\mathcal{F}$ is an "affinized" version of the cotangent bundle of the corresponding leaf of $\mathcal{I}$. Such bundles are known to arise as reduced manifolds of principal circle bundles, with symplectic structure the curvature of a "universal connection." (See the Appendix.) In this section, we shall give this kind of interpretation for the form $\Omega$, which allows us to define the Berry phase around an arbitrary loop as the holonomy of the connection.

Denote by $\Gamma_\omega$ the period group of the form $\omega$ on $P$, i.e., the subset of $\mathbb{R}$ consisting of the integrals of $\omega$ over all integer 2-cycles in $P$. It is well known in the theory of geometric quantization (see, for instance, [22]) that, if $\Gamma_\omega$ is trivial or cyclic, so that $G_\omega = \mathbb{R}/\Gamma_\omega$ is $\mathbb{R}$ or a circle, then there is a principal $G_\omega$ bundle $\pi: Q \rightarrow P$ with a connection form $\alpha$ whose curvature form is the symplectic structure $\omega$. The action integral around a loop $\gamma$ in $P$ is then equal to the holonomy of the connection $\alpha$ around $\gamma$. In this way, we get a definition of the action integral for arbitrary loops in $P$, with values in $G_\omega$. Of course it depends on the choice of $(Q, \alpha)$, which is indeterminate up to a homomorphism from $\pi_1(P)$ to $G_\omega$. In any case, the holonomy around a loop is unchanged when the loop undergoes an isodrastic deformation.

It is still possible to interpret action integrals as holonomy when $\Gamma_\omega$ is a dense subgroup of $\mathbb{R}$. To do so, we must enlarge our working category from differentiable manifolds to diffeological spaces in the sense of [23]. The quotient group $G_\omega = \mathbb{R}/\Gamma_\omega$ is a diffeological group; if we consider principal $G_\omega$ bundles with connection in the diffeological category, their classification is the same as in the usual situation.

Now we shall look at lagrangian submanifolds. Given any element $L$ of $\Lambda(P)$, the connection $\alpha$ is flat over $L$, so we may find parallel sections (generally multiple valued) of the prequantization over $L$. We call the
images of these sections \textit{planckian} submanifolds, following \cite{22}, and we denote the set of all planckian submanifolds by $A(Q)$. The map $\pi$ induces a map $\pi$ from $A(Q)$ to $A(P)$.

The fibre of $\pi$ at each $L \in A(P)$ is naturally isomorphic to the quotient $G_L$ of $G$, by the image of the holonomy homomorphism $\pi_1(L) \to G$. (Note that, even if $G$ is an "honest" Lie group, $G_L$ may not be.) As $L$ varies within an isodraft $I$, the group $G_L$ remains fixed at a value which we will call $G$, and $A_I(Q) = \pi^{-1}(I) \subset A(Q)$ is a principal $G$ bundle over $I \subset A(P)$.

We consider now the tangent and cotangent bundles of $A_I(Q)$.

**Lemma 4.1.** The restriction of generating functions gives a natural isomorphism between the tangent space to $A_I(Q)$ at the planckian manifold $M$ and the space $C^\infty(L)$ of smooth functions on the underlying lagrangian submanifold $L = \pi(M)$.

\textbf{Proof.} Let $M$ be a planckian submanifold lying over $L \in I$. A tangent vector to $A_I(Q)$ at $M$ is given by a vector field $v$ along $M$ in $Q$ which is $\pi$-related to a vector field $w$ along $L$ in $P$. The condition that $v$ be tangent to the space of planckian submanifolds means that $0 = \mathcal{L}_v \alpha = v \cdot d\alpha + d(v \cdot \alpha)$. We may decompose $v$ into horizontal and vertical parts as $f \zeta + \tilde{w}$, where $\zeta$ is the generator of the $G$ action on $Q$ and $\tilde{w}$ is the horizontal lift of $w$. Since $\zeta \cdot \alpha = 1$, $\zeta \cdot \alpha = 0$, we have $0 = \tilde{w} \cdot d\alpha + df$. Since $d\alpha$ is nondegenerate on the horizontal space (being the lift of the symplectic form $\omega$), the vector $v$ is completely determined by the function $f$ on $M$.

It remains to show that $f$ is actually the pullback by $\pi$ of a function on $L$. Since $v$ is $\pi$-related to $w$ on $L$, we already know that $df$ is the pullback of a closed form $\eta$ on $L$. Since the tangent vector $w$ is isodrastic, $\eta$ must be exact. Thus, $df = \pi^* \delta g$ for $g \in C^\infty(L)$. Since $M$ is connected, $f - \pi^* g$ is a constant $c$, and so $f = \pi^*(g + c)$. \hfill Q.E.D.

**Corollary 4.2.** The smooth cotangent space to $A_I(Q)$ at the planckian manifold $M$ is naturally isomorphic to the space $D^\infty(L)$ of compactly supported smooth densities on the underlying lagrangian submanifold $L = \pi(M)$.

The action of the structure group $G$ on $A_I(Q)$ lifts to the (smooth) cotangent bundle $T^*A_I(Q)$. Identifying the dual of the Lie algebra of $G$ with $\mathbb{R}$, we calculate the momentum mapping $J$ of this lifted action as follows. At a point $(M, \rho)$, the value of $J$ is found by evaluating the cotangent vector $\rho$ on the generator of the $G$ action. Since this generator is represented by the constant function $1$ on $L = \pi(M)$, we have $J(M, \rho) = \int_{\pi(M)} \rho$.

We now apply the symplectic reduction procedure \cite{16} at the element $1$ of the dual of the Lie algebra. $J^{-1}(1)$ consists of pairs $(M, \rho)$ for which $M$ lies in $A_I(Q)$ and $\rho$ is a weighting of $\pi(M)$. Dividing by the action of the
group $G_1$, we obtain precisely the leaf $W_1 A(P)$ of $\mathcal{W}$ in the manifold $W A(P)$ of weighted lagrangian submanifolds.

By the general theory of reduction, the natural symplectic structure on $T^* A_1(Q)$ induces a symplectic structure on the reduced manifold $W_1 A(P)$. Using cotangent coordinates on $P$ and examining Eq. (3), we see that this reduced symplectic structure is equal to the 2-form $\Omega$ defined in Section 3.

We will now apply to the present situation the general construction which is described in the Appendix. Given a planckian submanifold $M$, a weighting $\rho$ of $L = \pi(M)$ determines by integration a projection from $C^\infty(L)$ to the space of constant functions. Identifying $C^\infty(L)$ with $T^\perp/h_0(Q)$ and the constant functions with the tangent space to the fibre of $\pi$, we may consider $\rho$ as defining a $G_1$-equivariant family of horizontal spaces along the fibre $\pi^{-1}(LK) \subset A_1(Q)$. In other words, we have:

**Proposition 4.3.** $W_1 A(P)$ is the bundle over $I$ whose sections are connections on the principal $G_1$ bundle $A_1(Q)$ over $I$. The symplectic structure on $W_1 A(P)$ is the curvature of the "universal" connection on the pullback of $A_1(Q)$ to $W_1 A(P)$.

Proposition 4.3 means that the Berry phase of a contractible loop in $W_1 A(P)$ equals the holonomy around this loop for the universal connection. As a consequence, the period group of $\Omega$ must be contained in the subgroup $\Pi(I)$ of $\mathbb{R}$ which is the kernel of the projection onto $G_1$. Furthermore, we can define the Berry phase of an arbitrary loop in $W_1 A(P)$ as the holonomy of this loop.

We will now give an explicit construction of this holonomy. First we must extend the definition of action integrals to paths, not necessarily loops, which begin and end on a given lagrangian submanifold. Thus, on the groupoid of paths in $P$ whose endpoints lie in a lagrangian submanifold $L$, there is a well-defined homomorphism $\mathcal{A}$ into $G_1$ given by assigning to each path $\gamma$ from $x$ to $y$ the holonomy around the loop obtained by adding to $\gamma$ any path in $L$ from $y$ to $x$. Note that this homomorphism is trivial on the subgroupoid of paths which lie entirely in $L$.

Suppose now that we are given an isodrastic loop of weighted lagrangian submanifolds $(L_t, \rho_t)$, where $L_t$ is obtained from $L = L_0$ by a family $s_t$ $(0 \leq t \leq T)$ of symplectomorphisms generated by a time dependent hamiltonian $f_t$. We do not assume that $s_T$ is the identity, but simply that it maps $L$ to itself.

The time dependent hamiltonian $f_t$ also generates a family $r_t$ of automorphisms of $(Q, \omega)$. (Specifically, the time dependent vector field which generates $r_t$ equals the sum of the horizontal lift of $H_{f_t}$ and $-f_t$ times the fundamental vertical vector field on $Q$.) If $M$ is any planckian manifold lying over $L$, then $r_t(M)$ also lies over $L$ and "differs" from $L$ by an
element $\theta$ of $G_L$. If the Hamiltonian satisfies the normalization condition
\[ \int_{L_t} f_t \rho_t = 0 \quad \text{for all } t, \]
then $\theta$ is just the Berry phase for the loop $L_t$.

To calculate $\theta$, it suffices to follow the trajectory of a single point $m$ in $M$ under the family $r_t$; $\theta$ is (the image in $G_L$ of) any element of $G_\omega$ for which $\theta \cdot M$ contains $r_T(m)$. Let $\tau$ be the path $t \mapsto s_\tau(\pi(m))$ in $P$ with endpoints in $L$. Following [11], we introduce for comparison the horizontal path $\kappa(t) = a(t) \cdot r_\tau(m)$ lying over $\tau$, where $a(t) = \int_0^t f_u(s_u(\pi(m)))) du$ reduced modulo $\tau_{\omega}$.

Let $\sigma$ be a loop in $P$ obtained by adding to $\tau$ a path in $L$ which returns from $s_\tau(\pi(m))$ to $\pi(m)$. If $\kappa$ is completed to a horizontal lift of $\sigma$, then the terminal point of $\kappa$ lies in $A(\tau) \cdot M$, by the definition of the action integral $A$ as a holonomy. Since $M$ is a horizontal manifold, it follows that $\kappa(T)$ lies in $A(\tau) \cdot M$ as well. Thus $r_\tau(m) = -a(T) \cdot \kappa(T)$ lies in $(A(T) - a(T)) \cdot M$, and so $\theta = A(\tau) - a(T)$.

To summarize, we have proved the following result.

**Proposition 4.4.** Let $(L_t, \rho_t)$ be an isodrastic loop of weighted lagrangian submanifolds, and let $f_t$ be a family of Hamiltonians whose flow $s_t$ maps $L = L_0$ to $L_t$, normalized so that $\int_{L_t} f_t \rho_t = 0$ for all $t$. Then the Berry phase of the loop equals the image in $G_L$ of
\[ A(\tau_t) - \int_0^T f_u(\tau_t(u)) du, \tag{4} \]
where $\tau_t: t \mapsto s_t(l)$ is the trajectory of any point $l$ in $L$.

5. POSITIVE WEIGHTINGS AND COADJOINT ORBITS

It is a well known result of Moser [19] that all the volume elements with fixed total volume on a compact manifold are equivalent under diffeomorphism. This result easily extends to densities (pass to the oriented double cover, if necessary), so that the formula (2) can be considerably simplified if we restrict attention to the open submanifold $W^pA(P)$ of positive weightings on (connected) lagrangian submanifolds. (Since weightings have compact support, the underlying submanifolds must be compact.)

Thanks to Moser's theorem, an isodrastic tangent vector to $W^pA(P)$ can always be represented (as in Lemma 3.3) by a quadruple $(L, \rho, f, \phi)$ for which $\mathcal{L}_{\mathcal{H}} \rho + \phi = 0$. Note that $\phi$ is now completely determined by the other entries. For two such quadruples, (2) simplifies to the form
\[ \Omega(\xi, \zeta) = \int_T \{ f_t, f_t \} \rho, \tag{5} \]
which allows us to give a Lie algebraic interpretation of the Poisson structure $\Omega$ on $W_{\rho}A(P)$.

The space of compactly supported distribution densities on $P$ is the dual $g^*$ of the Lie algebra $g = C^\infty(P)$, so it carries a natural Lie–Poisson structure, in which the symplectic leaves are the orbits of the natural action of the group $G$ of “exact” symplectic diffeomorphisms—i.e., those obtained from the identity by integrating time-dependent hamiltonian vector fields. (Actually, the group corresponding to $C^\infty(P)$ is a central extension of $G$, for instance the automorphism group of a prequantization, but the center of this larger group acts trivially on the Lie algebra and hence on its dual.)

A tangent vector to the “coadjoint orbit” through $\delta \in g^*$ is of the form $\mathcal{L}_{H_i}\delta$ for some $f \in C^\infty(P)$, and the value of the Lie–Kirillov–Souriau symplectic structure on a pair of such tangent vectors is

$$\Omega_{LKKS}(\mathcal{L}_{H_1}\delta, \mathcal{L}_{H_2}\delta) = \delta \{ f_1, f_2 \}. \quad (6)$$

There is a $G$-equivariant map $J: W_{\rho}A(P) \rightarrow g^*$ which assigns to each $(L, \rho)$ the linear functional $f \mapsto \int_L f\rho$. Comparing the general formula (6) with (5), we see that $J$ is a symplectic map from each isodrast onto a coadjoint orbit. In other words, we have:

**Theorem 5.1.** The map $J$ which assigns to each positively weighted lagrangian submanifold the corresponding delta-density on $P$ with total mass one is an embedding from $W_{\rho}A(P)$ with the Berry Poisson structure to a Poisson submanifold of the dual of $C^\infty(P)$. This embedding is essentially the momentum mapping of the pullback action of the automorphism group of $P$ on $W_{\rho}A(P)$.

### 6. Weighting Rules

According to Proposition 4.3, a weighting rule for the symplectic manifold $P$ (i.e., a section of $WA(P)$ over $A(P)$) can be considered as a family of connections on the bundles $A_i(Q)$ over the isodrasts in $A(P)$, so that the resulting 2-forms along the isodrasts are the curvatures of these connections. In this section, we will compute these 2-forms for certain types of weighting rules.

**Definition 6.1.** A lagrangian lagrangian on the symplectic manifold $P$ is a smooth mapping which assigns to each lagrangian subspace in (each fibre of) $TP$ a density on that subspace.

Obviously, a lagrangian lagrangian $\mu$ puts a smooth density, which we will denote by $\mu(L)$, on each lagrangian submanifold of $P$. We will call $L$
admissible for \( \mu \) if \( \mu(L) \) is compactly supported and if \( \int_L \mu(L) \neq 0 \). For instance, if \( \mu(l) \) is always positive, then every compact \( L \) is admissible.

**Definition 6.2.** Let \( \mu \) be a lagrangian lagrangian on \( P \), \( A^\mu(P) \) the corresponding set of admissible lagrangian submanifolds. The associated weighting rule \( \rho^\mu \) is defined on \( A^\mu(P) \) by

\[
\rho^\mu(L) = \mu(L) \int_L \mu(L).
\]

Considering \( \rho^\mu \) as a (possible partially defined) section of \( WA(P) \) over \( A(P) \) and thus as a connection on each \( A_1(Q) \), we will compute the 2-form \( \rho^\mu(\Omega) \) on each isodrast by using cotangent coordinates and Eq. (3).

Thus, we assume that \( P = T^*X \) and that the isodrastic in question contains the images of exact 1-forms on \( X \). An isodrastic tangent vector to \( A(P) = A(T^*X) \) at \( L = dg(X) \) is represented by a function \( f \) on \( X \) (pulled back to \( L \)), and the resulting tangent vector \( \xi \) to \( WA(P) \) is represented by \( (L, \rho^\mu(L), f, \phi) \), where \( \phi \) is the derivative of \( \rho^\mu \) at \( L \) in the direction of \( f \).

Differentiating (7) with respect to \( L \), we have

\[
\phi = T_L \rho^\mu[f] = \frac{T_L \mu[f] \int_L \mu(L) - \mu(L) \int_L T_L \mu[f]}{(\int_L \mu(L))^2}.
\]

Further analysis of this formula requires evaluating the "variational derivative" \( T_L \mu[f] \), which depends upon the particular form of the lagrangian.

Suppose that \( \mu \) is determined by a riemannian metric on \( P \); i.e., the density \( \mu(L) \) is just the riemannian density of the induced metric on \( L \). Assume in addition that the metric is compatible with the symplectic structure in the sense that there is an almost complex structure \( J \) for which \( \langle x, y \rangle = \omega(Jx, y) \). Since the normal spaces to \( L \) in such a metric are lagrangian as well, we may assume that our cotangent coordinates are arranged with \( L \) as the zero section and with the fibres perpendicular to \( L \).

It follows that, for any function \( f \) pulled back from \( L \) to \( T^*L \), the hamiltonian vector field \( H_f = -J\nabla f \) is perpendicular to \( L \).

Now, by a standard calculation in riemannian geometry \([24]\), the derivative \( T_L \mu[f] \) is equal to the density \( \mu(L) \) times the inner product of the mean curvature vector \( \mathcal{H}_L \) of \( L \) with \( H_f \). Thus we have

\[
T_L \mu[f] = \langle \mathcal{H}_L, -J\nabla f \rangle \mu(L) = \langle J\mathcal{H}_L, \nabla f \rangle \mu(L).
\]
Note that, since $L$ is lagrangian, the vector field $J\mathcal{H}_L$ is tangent to $L$, so the last inner product in (9) is $df(J\mathcal{H}_L)$. Thus we have

$$T_L\mu[f] = df(J\mathcal{H}_L)\mu(L). \quad (10)$$

Substituting (10) into (8) gives

$$\phi = T_L\rho^\mu[f] = \frac{df(J\mathcal{H}_L)\mu(L)\int_L\mu(L) - \mu(L)\int_L df(J\mathcal{H}_L)\mu(L)}{(\int_L\mu(L))^2}. \quad (11)$$

Now suppose that we have two isodrastic tangent vectors at $L$ given by the functions $f_1$ and $f_2$. Substituting (11) into (3) we obtain our main result.

**THEOREM 6.3.** Let $\rho^\mu$ be the weighting rule defined by an almost Kähler structure $(\langle , \rangle, J)$ on the symplectic manifold $(P, \omega)$. Then the induced closed "Berry" 2-form is defined, for a pair of isodrastic tangent vectors at $L \in \Lambda(P)$ represented by $f_1$ and $f_2$ in $C^\infty(L)$, by

$$\rho^\mu \Omega(f_1, f_2) = \int_L (f_1 df_2 - f_2 df_1)(J\mathcal{H}_L) \rho^\mu(L) - \int_L f_1 \rho^\mu(L) \int_L df_2(J\mathcal{H}_L) \rho^\mu(L)$$

$$+ \int_L f_2 \rho^\mu(L) \int_L df_1(J\mathcal{H}_L) \rho^\mu(L). \quad (12)$$

If $f_1$ and $f_2$ are normalized to have mean value zero on $L$, then the last two terms in (12) disappear, and we have simply

$$\rho^\mu \Omega(f_1, f_2) = \int_L (f_1 df_2 - f_2 df_1)(J\mathcal{H}_L) \rho^\mu(L). \quad (13)$$

**COROLLARY 6.4.** The Berry connection is flat over each isodrast in the space of minimal lagrangian submanifolds in an almost Kähler manifold.

The simplest case of Theorem 6.3 occurs when $P$ is 2-dimensional, so that we are dealing with closed curves on an oriented riemannian surface, with symplectic structure equal to the riemannian area element. In this case, (13) can be written as

$$\rho^\mu \Omega(f_1, f_2) = |L|^{-1} \int_L \kappa_g(f_1 df_2 - f_2 df_1), \quad (14)$$

where $\kappa_g$ is the geodesic curvature and $|L|$ is the arc length of the curve $L$. This Dirac structure on $\Lambda(P)$ (see Remark 3.4) is a Poisson structure when
restricted to the curves with nonvanishing geodesic curvature; at the other extreme, if we restrict it to the great circles on a round $S^2$, say, the resulting connection is flat over each isodrast.

We note the similarity of (14) to the symplectic structure 
\[ \int_{S^1} (f_1 \, df_2 - f_2 \, df_1) \] on $C^\infty(S^1)/\mathbb{R}$ which is commonly used in connection with the Korteweg–de Vries equation, and which has also arisen in the Hamiltonian formulation of the motion of vortex patches for planar incompressible fluids [17]. It would be interesting to find a physical interpretation of (14).

A completely different class of weighting rules is defined by differential forms on $P$. Supposing again for convenience that our lagrangian submanifolds are oriented, we fix a form $\psi$ on $P$ with degree equal to half that of $P$ and let the lagrangian lagrangian $\mu$ assign to each oriented lagrangian subspace $l$ the pullback of $\psi$ to $l$, considered as a density.

Rather than considering this case in its full generality, we shall restrict our attention to a class of situations in which the induced Berry 2-form is zero. Specifically, we suppose that the symplectic manifold $(P, \omega)$ is equipped with a submersion $F$ onto a manifold $P_0$ of half its dimension such that $P_0$ has a Poisson structure and $F$ is a Poisson map. We suppose further that the lagrangian submanifolds of interest are cross sections of $F$ and that the weighting rule is given by the pullback $\psi = F^*\psi_0$ of a fixed weighting $\psi_0$ which is invariant under the action of all Hamiltonian vector fields on $P_0$.

For example, $F$ could be the projection of a cotangent bundle onto its base, in which case $\psi_0$ could be any weighting of the base. Another important example is the case where $(P_0, \omega_0)$ is a symplectic manifold, $(P, \omega) = (P_0, -\omega_0) \times (P_0, \omega_0)$, $F$ is the projection onto the second factor, and $\psi_0$ is the normalized Liouville measure. In this case, the lagrangian submanifolds of interest are the graphs of symplectic automorphisms of $P_0$.

More generally, $F$ could be the projection of any symplectic groupoid (see [8, 12, 28]) with an invariant density on its base.

Denote by $\mathcal{A}^F(P)$ the open subset of $\mathcal{A}(P)$ consisting of lagrangian cross sections of $F$. Each isodrastic tangent vector at $L \in \mathcal{A}^F(P)$ can be represented by a function of the form $f = F^*f_0$ for $f_0 \in C^\infty(P_0)$. The image under the weighting is the tangent vector $\Omega\psi$ to $\mathcal{A}^F(P)$ represented, in the terms of Lemma 3.3, by the quadruple $(L, \psi, f, 0)$. To evaluate the Berry 2-form, we use the formula (2) to obtain, for a pair of tangent vectors,

\[
\Omega(\xi_1, \xi_2) = \int_{\xi_1} \left\{ f_1, f_2 \right\} \psi + f_1 \mathcal{L}_{H_{f_2}^\psi} \psi - f_2 \mathcal{L}_{H_{f_1}^\psi} \psi. \tag{15}
\]

The last two terms under the integral sign in (15) vanish because, for instance, $\mathcal{L}_{H_{f_2}^\psi}$ is the pullback of $\mathcal{L}_{H_{f_2,0}^\psi}$, which is zero by assumption.
For the first term, we have

\[ \int_L \{ f_1, f_2 \} \psi = \int_{P_0} \{ f_{1,0}, f_{2,0} \} \psi_0 \]

\[ = \int_{P_0} \mathcal{L}_{H_{1,0}} f_{1,0} \psi_0 - \int_{P_0} f_{1,0} \mathcal{L}_{H_{1,0}} \psi_0. \]

(16)

The first term on the right hand side of (16) is zero because it is the integral of a Lie derivative, while the second term is zero thanks to the invariance of \( \psi_0 \).

We have therefore proved the following result.

**Theorem 6.5.** For a weighting rule defined by an invariant density on the base of a Poisson submersion \( P \to P_0 \), the pulled back Berry 2-form is zero on the space \( \Lambda^f(P) \) of lagrangian cross sections. Consequently, if \( P \) is equipped with a prequantization \( Q \), the resulting connection is flat on each bundle \( \Lambda^f(Q) \) of planckian manifolds covering cross sections, and its holonomy defines a \( G_1 \)-valued homotopy invariant on isodrastic loops.

**Corollary 6.6.** The homotopy invariant defined for loops of symplectomorphisms in [29] is a special case of the classical Berry phase.

More generally, given an invariant weighting on a Poisson manifold, we can define a homotopy invariant for loops in the group of admissible lagrangian sections of the associated symplectic groupoid, when the groupoid is prequantized. It seems that this invariant should play a role in the study of quantization of Poisson manifolds.

7. THE BERRY PHASE AS AN AVERAGE HOLONOMY

In this section, we shall give an interpretation of the Berry phase for positively weighted lagrangian submanifolds in a way which generalizes the original definition in [29] for loops of symplectomorphisms. We return to the context of Proposition 4.4, in which the isodrastic loop \((L_t, \rho_t)\) is such that \( L_t \) is obtained from \( L = L_0 \) by a family \( s_t \) \((0 \leq t \leq T)\) of symplectomorphisms generated by a time dependent hamiltonian \( f_t \). If in addition we assume (as is always possible when the weightings \( \rho_t \) are positive), that \( f_t^* \rho_t = \rho_0 \) for all \( t \), then we obtain the following result, in which the hamiltonians have disappeared.

**Theorem 7.1.** Let \((L_t, \rho_t)\) be an isodrastic loop of weighted lagrangian submanifolds obtained from a fixed \((L, \rho)\) by applying a 1-parameter family
of symplectomorphisms $s_t$. Then the Berry phase of the loop is equal to the average (with respect to $\rho$) of the action integrals $A(\tau_t)$ of the trajectories $\tau_t: I \mapsto s_t(I)$.

**Proof.** According to Proposition 4.4, the expression in (4) is (as an element of $G_2$) independent of $I$. As a consequence, the function $l \mapsto A(\tau_l)$ is the reduction to $G_L$ of a real-valued function $B(l)$ on $L$. Hence we can average $B$ with respect to $\rho$ to obtain a well-defined element of $G_L$, which we call the average action integral. (We remark that an assumption of simple connectivity was unnecessarily invoked in [29] to carry out this averaging.)

If we now integrate (4) with respect to $I$, using the measure $\rho$, then the normalization condition on $f$ causes the second term to vanish, giving the desired result. Q.E.D.

If the trajectories in Theorem 7.1 are all closed and contractible loops, then the action integrals are, as usual, given by integrating the negative of the symplectic form over spanning discs.

**Example 7.2.** Consider the euclidean plane $\mathbb{R}^2$ with the usual area element $dq \wedge dp$ as its symplectic structure. Let $L$ be a circle of radius $r$ whose center is at a distance $R$ from the origin, equipped with the weighting given by the (normalized) arc length element. For the loop of weighted lagrangian submanifolds obtained by revolving $L$ once around the origin in the counterclockwise direction, the Berry phase is equal to minus the average area swept out by a point on $L$. This average area turns out to be simply the area swept out by the center of the circle, so the Berry phase is $-\pi R^2$, regardless of $r$. If we identify the space of circles of radius $r$ with the plane by mapping each circle to its center, the Berry curvature on this space becomes $-dq \wedge dp$ for any $r$.

**Example 7.3.** We replace the plane of the previous example by the sphere $P = S^2$ with the standard Kähler structure having area $4\pi$, for which $G_4 = \mathbb{R}/4\pi\mathbb{Z}$. Within $A(P)$, we shall consider for each $r$ between 0 and $\pi$ the space $C_r$ of geodesic circles of radius $r$. If the center of a loop of such circles itself sweeps out a circle of geodesic radius $R$, the average area swept out by a point on the moving circle is equal to $2\pi(1 - \cos r \cos R)$. Note that, unlike in the case of the plane, this area now depends on $r$; only in the limit as $r \to 0$ does it reach the value $2\pi(1 - \cos R)$ of the area swept out by the center of the circle. The Berry phase of our loop of circles is this $-2\pi(1 - \cos r \cos R)$.

We may identify $C_r$ with $S^2$ by mapping each circle to its center, as long as $r \neq \pi/2$. (The case of $C_{\pi/2}$ is discussed in the next paragraph.)
symmetry, the Berry curvature must be equal to the standard area element times a constant, which may be calculated as the quotient of the Berry phase around any loop by the area of the loop. Taking the loop to be a circle, we find this quotient to be \(-(1 - \cos r \cos R)/(1 - \cos R)\), which appears to depend on \(R\). This dependence is illusory, however. Since the Berry phase is defined only modulo the area \(2\pi (1 - \cos r)\) of the moving circles, we can get a form which is more useful when \(R\) is small by replacing \(-2\pi (1 - \cos r \cos R)\) by the equivalent form \(-2\pi (1 - \cos r \cos R) + 2\pi (1 - \cos r) = -2\pi \cos r (1 - \cos R)\). We see immediately that the Berry curvature of \(C\) is \(-C\), is \(-\cos r\) times the standard area element.

When \(r = \pi/2\), the Berry curvature becomes zero. This is consistent with Corollary 6.4, since the circles of radius \(\pi/2\) are geodesics. The space \(C_{\pi/2}\) is now a projective plane rather than a sphere; the holonomy of the flat Berry connection is determined by the Berry phase for the loop obtained from a great circle \(L\) through the north and south poles by rotating it through "half a day." This phase is half that of rotation through a full day, which is given by the calculations above with \(r = R = \pi/2\) as \(2\pi\). Thus the holonomy around the half-day loop is \(\pi\), which is an element of order 2 in the structure group \(G_t = \mathbb{R}/2\pi\mathbb{Z}\).

We may consider \(S^2\) as \(\mathbb{C}P^1\) and \(L\) as \(P^1\); it would interesting to generalize our calculation to loops of \(\mathbb{R}P^n\)'s in \(\mathbb{C}P^n\), and even to loops of "real flag manifolds" in arbitrary coadjoint orbits of compact semisimple Lie groups. (See [10] for other aspects of the symplectic geometry of these flag manifolds.)

8. The Vector (Hannay) Phase for Toral Layers

A completely integrable system is one for which the hamiltonian is constant on the leaves of a fibration of phase space by invariant lagrangian tori. (Singular tori are sometimes allowed.) We refer the reader to Chapter 10 of [3] for a discussion of the geometry of integrable systems as we will use it here. If the hamiltonian and the fibration vary, then we can follow a particular invariant torus by requiring that it undergo an isodrastic deformation. This means that, by fixing an isodrast, we obtain a torus bundle over the space of fibrations by lagrangian tori. In the recent papers [13, 18], a connection on this bundle has been constructed in a symplectically invariant way; parallel translation with respect to this connection defines the Hannay angles for a path in phase space which follows an isodrastic loop of tori. (Such paths occur, for instance, in the adiabatic approximation to motion under a slowly varying integrable hamiltonian which returns to its original value.)

Given a fibration by lagrangian tori, the associated action-angle
variables endow each fibre $L$ in an invariant way with the structure of an affine torus; i.e., there is a underlying torus group $\mathbb{T}_L$ which acts freely and transitively on $L$, so that each pair of points in $L$ has a well-defined "difference" in $\mathbb{T}_L$. It is nearly the case (and, in fact, the author originally believed it to be so) that the connection of [13] and [18] is the pullback of a connection which is already defined on the isodrasts of the natural torus bundle over the space of affine lagrangian tori in a symplectic manifold $P$. However, as Berry and Hannay [7] have already observed, the minimal space over which the connection can be defined consists of the lagrangian tori equipped with a torus action on an infinitesimal neighborhood. We shall define this connection below, compute its curvature, and relate it to that of the Berry connection which we considered in previous sections of this paper.

**Definition 8.1.** A toral structure on a compact connected manifold $L$ is a connection on $TL$ (also called an affine connection on $L$) which is globally flat and torsion free. A manifold with a toral structure is called an affine torus.

By globally flat, we mean that parallel translation is the same along any two paths with given endpoints. This condition implies that there are enough parallel vector fields to give a basis of the tangent space at each point of $L$. The torsion free condition means that these parallel vector fields commute, so that they generate an action of a commutative Lie group. Since $L$ is compact and connected, we can take this group to be a torus $\mathbb{T}_L$ which then acts freely and transitively on $L$.

We denote by $T_L P$ the pullback of $L$ of the tangent bundle of $P$, i.e., the set of all tangent vectors to $P$ with basepoints in $L$. Since $P$ is symplectic and $L$ is lagrangian, $T_L P$ is a symplectic vector bundle over $L$, and $TL$ is a lagrangian subbundle thereof. The normal bundle to $L$ in $P$, defined as the quotient $T_L P/TL$, is naturally isomorphic via the symplectic structure to the cotangent bundle $T^* L$ (see [25]). Thus we have the exact sequence

$$0 \to TL \to T_L P \to T^* L \to 0. \quad (17)$$

An affine lagrangian torus carries a flat connection on each of the outer terms in the exact sequence (17); however, this structure is just a bit weaker than what is needed to define the Hannay connection.

**Definition 8.2.** A lagrangian toral layer in the symplectic manifold $P$ is a lagrangian submanifold $L$ with a globally flat symplectic connection on $T_L P$ which induces a toral structure on $L$.

The last condition in Definition 8.2 means that the subbundle $TL$ is invariant under the parallel translation of the given connection on $T_L P$.
and that the induced connection on \( L \) is torsion free. The resulting action of \( \mathbb{T}_L \) on \( L \) lifts by parallel translation to give a free symplectic action of \( \mathbb{T}_L \) on \( T_L P \). Finally, the symplectic property implies that the induced connection and torus action on \( T_L P/TL \cong T^*L \) are dual to those on \( TL \). The extra structure on a lagrangian toral layer as compared with a lagrangian affine torus is that the latter carries a global means of propagating the choice of a normal subspace at a point of \( L \).

**Definition 8.3.** The space of all lagrangian toral layers in \( P \) will be denoted by \( TA(P) \). The tautological bundle \( BA(P) \) over \( TA(P) \) is the set of pairs \((L, x)\) with \( L \in U/A(P) \) and \( x \in L \).

The structure group of \( BA(P) \) is the symmetry group of an affine torus—the semidirect product of the additive group \( \mathbb{R}^n/\mathbb{Z}^n \) by its group \( GL(n, \mathbb{Z}) \) of automorphisms. The obstruction to reducing this structure group to \( \mathbb{R}^n/\mathbb{Z}^n \) is known as monodromy.

As was the case for weighted lagrangian submanifolds, the isodrastic foliation \( \mathcal{F} \) on \( A(P) \) pulls back to a foliation \( \mathcal{T} \mathcal{F} \) on \( TA(P) \). Our main goal in this section is to define the Hannay connection on \( BA(P) \) over the isodrasts in \( U/A(P) \).

Suppose that \( L_1 \) and \( L_2 \) are lagrangian toral layers in \( P \). Since any two affine tori of the same dimension are isomorphic, there is an affine diffeomorphism \( \lambda \) from \( L_1 \) to \( L_2 \). Using the parallel translations given by the toral layer structure, we can lift \( \lambda \) to a connection-preserving symplectic isomorphism from \( T_{L_1}P \) to \( T_{L_2}P \), which by the semiglobal Darboux theorem of [25] is tangent to a symplectic diffeomorphism from a neighborhood of \( L_1 \) to a neighborhood of \( L_2 \).

By a parametrized version of this argument, we can follow all curves in \( TA(P) \), and in fact all curves in \( BA(P) \), by integrating time-dependent, locally defined, locally hamiltonian vector fields. If the curves are isodrastic, the vector fields are globally hamiltonian (and globally defined). It follows that the isodrasts in \( TA(P) \) and \( BA(P) \) are homogeneous spaces for the group \( G \) of symplectomorphisms of \( P \) which are isodrastic to the identity.

To prepare for studying the Hannay connection on \( BA(P) \) as a torus bundle over \( TA(P) \), it will be useful to make a short digression to discuss connections on bundles of homogeneous spaces.

Let \( G \) be a Lie group, \( H \) a closed subgroup, and \( K \) a closed subgroup of \( H \), with the corresponding Lie algebra and its subalgebras denoted by \( g, \mathfrak{h}, \) and \( \mathfrak{k} \). The group \( G \) acts transitively on the total space and base of the bundle of homogeneous spaces \( H/K \to G/K \to G/H \); our aim here is to describe the \( G \)-invariant connections on this bundle and the computation of their curvature.

By a connection on a smooth fibre bundle, as opposed to a principal bundle, we shall mean a smooth family of complements to the tangent
spaces along the fibres or, equivalently, a projection from the tangent bundle of the total space to the tangent bundle along the fibres. In our homogeneous situation, such a connection is determined by its value at the image of the identity element of $G$; this value is a projection $\zeta$ from $g/\mathfrak{f}$ to $\mathfrak{h}/\mathfrak{t}$ which must be equivariant with respect to the adjoint action of $\mathfrak{f}$. In practice, we can construct $\zeta$ from a map $\eta : g \to \mathfrak{h}$ which satisfies the following conditions: (A) if $u \in \mathfrak{h}$, then $\eta(u) - u \in \mathfrak{t}$; (B) if $u \in g$ and $v \in \mathfrak{f}$, then $\eta([u, v]) - [\eta(u), v] \in \mathfrak{f}$. (Note that condition (A) implies that $\eta(1) \subseteq \mathfrak{t}$, so $\zeta$ is well defined by $\eta$.)

The curvature of a connection is found by taking the vertical component of the bracket of two horizontal vector fields. In our homogeneous situation, we begin with two elements $u$ and $v$ of $\eta^{-1}(\mathfrak{f})$, take the projection $\eta([u, v])$, and consider the result modulo $\mathfrak{f}$. It follows from condition (B) above that this construction gives a well-defined skew-symmetric form $\Phi$ on $\eta^{-1}(1)/\mathfrak{f}$ with values in $\mathfrak{h}/\mathfrak{t}$.

We now apply this construction to the case in which $G$ is the group of canonical transformations isodrastic to the identity on the symplectic manifold $(P, \omega)$, $H$ is the isotropy subgroup of a given lagrangian toral layer $L$, and $K$ is the isotropy subgroup of the pair $(L, x)$ in the tautological bundle. The resulting homogeneous spaces $G/H$ and $G/K$ are then isomorphic to isodrasts through $L$ and $(L, x)$ in $TA(P)$ and $BA(P)$, respectively.

The Lie algebra $\mathfrak{g}$ is identified as usual with $\mathcal{C}^{\infty}(P)/\mathbb{R}$. To describe the subalgebras $\mathfrak{h}$ and $\mathfrak{f}$, it is convenient to use cotangent coordinates on a neighborhood of $L$ in $P$, which in this case are just action-angle variables $(I, \theta)$, where $\theta$ is an $n$-tuple of $\mathbb{R}/2\pi\mathbb{Z}$-valued coordinates on $L$ and $I$ is an $n$-tuple of conjugate momenta. $L$ is the submanifold defined by $I = 0$, and the toral layer structure is induced by the natural flat connection given by these coordinates.

The group $H$ consists of transformations which leave $I = 0$ invariant and whose 1-jets at $I = 0$ are those of translations in the $\theta$ direction. Whereas the general element of the Lie algebra $\mathfrak{g}$ may be written in the form $a(\theta) + \sum b_{ij}(\theta) I_{i} + \sum c_{jk}(\theta) I_{i}I_{k} + O(I^{3})$, a function in $\mathfrak{h}$ must have constant $a$, $b_{i}$, and $c_{jk}$, and a function in $\mathfrak{f}$ must have in addition $b_{i} = 0$. Note that all these descriptions must be considered modulo constants, and that the term $O(I^{3})$ absorbs all the behavior outside the cotangent coordinate system.

It is easy now to define the map $\eta : \mathfrak{g} \to \mathfrak{h}$ which determines the connection, namely by averaging the functions $a$, $b_{i}$, and $c_{ij}$ with respect to $\theta$. More precisely, to take care of the fact that our functions are actually defined globally on $P$, we may introduce a cutoff function $\chi(I)$ which is identically 1 near $I = 0$ and which is supported inside the cotangent coordinate system. Denoting by $\langle \rangle$ the operation of averaging with respect to
\(\theta\), we set \(\eta(u) = \langle u \rangle\) (extended to \(P\) with the value 0 outside the cotangent coordinate system). We note that, although \(\eta\) depends upon the choice of \(\chi\) (and upon the choice of coordinates), the induced map on the quotients does not. This follows, for instance, from the invariance property (B) to be proved below.

To check condition (A), we observe that, for \(u \in \mathfrak{h}\), \(\eta(u) - u\) is \(O(I^3)\), which certainly belongs to \(\mathfrak{l}\). For condition (B), we take \(u \in \mathfrak{g}\) and \(v = a + \sum b_I I_I + \sum c_{jk} I_j I_k + O(I^3) \in \mathfrak{l}\) and must show that the difference

\[
\eta(\{u, v\}) - \{\eta(u), v\}
\]

belongs to \(\mathfrak{f}\). Since Poisson bracketing with a function of \(I\) kills any other function of \(I\) and preserves \(O(I^3)\), the second term in (18) is \(O(I^3)\) and hence belongs to \(\mathfrak{f}\). For the first term, we have \(\eta(\{u, v\}) = \langle \chi\{u, v\} \rangle = \chi\langle \{u, v\} \rangle = \chi\langle u, a + \sum b_I I_I + \sum c_{jk} I_j I_k + O(I^3) \rangle\). Now Poisson bracketing with a function of \(I\) produces functions whose average with respect to \(\theta\) is zero, so we are left with just \(\chi\langle \{u, O(I^3) \rangle\). This function is of the form \(O(I^2)\) and is independent of \(\theta\). It can therefore be written as \(\sum c_{jk} I_j I_k + O(I^3)\), so it belongs to \(\mathfrak{f}\).

We have therefore defined in a symplectically invariant way a connection, which we shall call the Hannay connection, on the tautological bundle over each isodrast in the space of lagrangian toral layers in a symplectic manifold \(P\). One may check that the connection defined in [18] is just the pullback of this one under a mapping from a parametrized family of integrable systems to the space of lagrangian toral layers.

We can now explain the sense in which the Hannay connection is not well-defined on the space of affine lagrangian tori, as opposed to toral layers. In action angle variables, the Lie algebra \(\mathfrak{h}\) would have to be replaced by the isotropy algebra of an affine lagrangian torus, consisting of functions of the form \(a + \sum b_I I_I + O(I^2)\); \(\mathfrak{f}\) would be replaced by the subalgebra of such functions having \(b_I = 0\). In this case, though, the map \(\eta\) defined by averaging over \(\theta\) would no longer have the necessary invariance property (B), so that it would not define a symplectically invariant connection.

According to the general theory of homogeneous connections, the curvature of the Hannay connection is given by the bilinear form \(\Phi(u, v) = \eta([u, v])\), considered modulo \(\mathfrak{f}\). In our situation, we obtain (ignoring the cutoff \(\chi\) since we are working only modulo \(O(I^3)\)) the simple formula

\[
\Phi(u, v) = \langle \{u, v\} \rangle.
\]

In principle, Eq. (19) applies only when \(u\) and \(v\) are in the kernel of \(\eta\), but in fact this expression is unchanged when any functions \(u\) and \(v\) are projec-
ted into this kernel by subtracting off their $\theta$-averages. This means that may just as well consider $u$ and $v$ as tangent vectors to the base of the tautological bundle. We summarize our result, which is consistent with Theorem 1 of [18], in the following theorem.

**Theorem 8.4.** The operation of averaging with respect to angle variables gives a symplectically invariant Hannay connection along the isodrastic leaves for the tautological torus bundle over the space of lagrangian toral layers. Given two tangent vectors at $L$ to $\mathbb{T}A(P)$ determined by hamiltonian functions $u$ and $v$, the curvature of the Hannay connection is given by (19), where the right hand side is considered as an element of the quotient $\mathfrak{h}/\mathfrak{l}$ of the isotropy Lie algebras. In terms of action-angle variables, this element is represented by a sum $\sum h_j l_j$; the coefficients $h_j$ are the (infinitesimal) Hannay angles.

9. **Relation between Berry and Hannay Connections**

Combining Theorem 8.4 with Eq. (5), we can obtain a simple relation between the Berry and Hannay curvatures which belongs entirely to symplectic geometry (as opposed to the semiclassical argument of [4]): the Hannay curvature is the derivative of the Berry curvature with respect to action variables.

To make the relation between Berry and Hannay connections more precise, we begin by noting that, given an element $c$ of the cohomology space $H^1(L; \mathbb{R})$ of an affine torus $L$, there is a unique representative $\mu_c$ of $c$ which is parallel with respect to the connection; this closed 1-form on $L$ can be considered as a tangent vector at $L$ to $A(P)$. If $L$ has the further structure of a lagrangian toral layer, then one can extend $\mu_c$ to a parallel section of the cotangent bundle $T^*_L(P)$ of $P$ along $L$, determined up to a parallel section which pulls back to zero on $L$. The symplectic structure on $L$ converts this to a “locally hamiltonian” section of $T_L(P)$, which can be considered as a tangent vector at the point $L$ in the space of affine lagrangian tori; this vector is independent of the choice of extension.

Since an affine torus carries a canonical weighting, there is a natural mapping $E: \mathbb{T}A(P) \to WA(P)$, and we can interpret the construction of the preceding paragraph as mapping each pair $(L, c)$ to a vector in $T_L WA(P)$. Fixing $L$ and letting $c$ vary, we get a mapping from $H^1(L; \mathbb{R})$ to $T_L WA(P)$ which when followed by the natural map from $T_L WA(P) \cong Z^1(L)$ to $H^1(L; \mathbb{R})$ gives the identity. Letting $L$ vary within a small open set $\mathbb{U}$ (to avoid problems of monodromy) in an isodrast in $\mathbb{T}A(P)$, and fixing $c$, we obtain a vector field $v_c$ along $E$ which is tangent to a family of mappings from $\mathbb{U}$ into $WA(P)$, each of which maps $\mathbb{U}$ into an isodrast. Since the
Berry curvature $\Omega$ is defined on each isodrast, it makes sense to form the Lie derivative $\mathcal{L}_v \Omega$, which is now a 2-form on $\mathcal{U}$.

The construction of the previous paragraph gives us a linear map $F$ from $H^1(L; \mathbb{R})$ to the space of 2-forms on $\mathcal{U}$, which can be considered on a 2-form on $\mathcal{U}$ with values in the homology space $H_1(L; \mathbb{R})$. Since $L$ has a toral structure, $H_1(L; \mathbb{R})$ can be identified with the tangent space of $L$ and hence with the Lie algebra of the underlying translation group $\mathbb{T}_L$, so that $F$ becomes a 2-form on $L$ with values in this Lie algebra. The main result of this section is that $-F$ is the Hannay curvature.

To verify this result, we use action angle variables. Given such variables, with $L$ taken as $I=0$, the typical element of $H^1(L; \mathbb{R})$ is the class $c$ of $\sum c_i d\theta_i$, for which the corresponding hamiltonian vector field $v_i$ is given by $-\sum c_i \partial_i \theta_j$. If $f$ and $g$ are two functions defined near $L$, considered as tangent vectors at $L$ to $\mathcal{T}A(P)$, the Hannay curvature evaluated on $(f, g)$ and then paired with $c$ is equal by Theorem 8.4 and Eq. (5) to the derivative with respect to $t$ of the Berry curvature $\Omega(f, g)$, where $f$ and $g$ are now considered as tangent vectors to $WA(P)$ at the manifolds $L_i$ defined by $I_i = -tc_i$.

To state our final result somewhat more formally, we introduce a final definition.

**Definition 9.1.** Let $AA(P)$ be the bundle over $\mathcal{T}A(P)$ whose fibre at each $L$ is the vector space $\mathcal{T}_L$ of translations of $L$. An open subset $\mathcal{U}$ of an isodrast in $\mathcal{T}A(P)$ is said to be monodromy free if the bundle $AA(P)$ is trivial over $\mathcal{U}$.

**Theorem 9.2.** The assignment of the canonical weighting to each affine lagrangian torus defines a mapping $E: \mathcal{T}A(P) \rightarrow WA(P)$. If $\mathcal{U}$ is any monodromy free subset of an isodrast in $\mathcal{T}A(P)$, then for each element $c$ of the typical fibre $H^1(LK; \mathbb{R})$ of the dual bundle $A^*A(P)$ over $\mathcal{U}$ there is an associated vector field $v_c$ along $E$ which is tangent to a family of mappings from $\mathcal{U}$ to isodrasts in $WA(P)$. The Lie derivative along $v_c$ of the Berry curvature $\Omega$ equals minus the result of pairing the Hannay curvature $\Omega$ with $c$.

**Remark 9.3.** Before going on to examples, we wish to note here an important distinction between action variables and action integrals. If we compare the tori $I_j = 0$ and $I_j = e_j$, the difference between the action integrals around the fundamental loop in the $\theta_j$ direction is $2\pi$ times $e_j$ and not just $e_j$. Equivalently, the homology basis element represented by this loop is $2\pi$ times the one which is associated with the vector field $\partial_j / \partial \theta_j$, via the identification of parallel 1 forms with cohomology classes.

**Example 9.4.** We will reconsider here Examples 7.2 and 7.3. Given a circle in the plane or the sphere, it may be embedded in the family of con-
centric circles which, as a lagrangian fibration, endows each circle with a toral layer structure. According to Remark 9.3, the action variable in each case may be taken to be \(-1/2\pi\) times the area enclosed by the circle, so that the Hannay curvature is \(2\pi\) times the derivative of the Berry curvature with respect to area.

For circles in the plane, we found in Example 7.2 that the Berry curvature on the space of circles of any radius is equal to minus the standard area element, independent of the radius of the circles. It follows that the Hannay curvature on each of these spaces of circles is zero.

For the sphere, on the other hand, the Berry curvature on the space \(C_r\) of circles of radius \(r\) was shown in Example 7.3 to be equal to \(-\cos r\) times the area element. Since the area \(A\) of the circle of radius \(r\) is \(2\pi(1 - \cos r)\), the Hannay curvature is equal to the standard area element, independent of \(r\). This result agrees with the calculation of Berry and Hannay [6].

The results of the two examples above can be confirmed by a topological argument. On any orientable riemannian surface, the tautological bundle over the space of geodesic circles of any sufficiently small radius is isomorphic to the unit tangent bundle, so the integral of the Hannay curvature must equal the integral of the gaussian curvature. Any surface of constant gaussian curvature is locally homogeneous: it follows that the Hannay curvature on such a surface is constant as well and must therefore equal the gaussian curvature. For a general surface, one can approximate the metric near each point by one of constant curvature; it should follow that the Hannay curvature on the space of circles of radius \(r\) approaches the gaussian curvature as \(r \to 0\).

APPENDIX: THE BUNDLE OF CONNECTIONS

The following results are now standard in the theory of reduction of symplectic manifolds. (See [14, 15].)

Let \(Q \to P\) be a principal \(G\) bundle, where \(G\) is the quotient of the real numbers by a cyclic subgroup. The action of \(G\) on \(Q\) lifts to the cotangent bundle \(T^*Q\) with a real-valued momentum map \(J\) which is linear on fibres. The reduced manifold \(C(Q) = J^{-1}(1)/G\) carries a symplectic structure \(\omega_r\). \(C(Q)\) is also a bundle over \(P\); any section \(\alpha\) of this bundle defines a diffeomorphism \(D_\alpha\) from \(C(Q)\) to \(T^*P\) taking the image of \(\alpha\) to the zero section and having the property that \(D_\alpha^*\omega_r\) is the sum of the cotangent bundle structure and a "magnetic term" which the pullback from \(P\) to \(T^*P\) of a closed 2-form \(\omega_\alpha\).

The purpose of this appendix is to add to these results the following statement, which was known to the author in the mid-1970s but does not seem to have appeared in print.
PROPOSITION A.1. With notation as above, $C(Q)$ is the bundle over $P$ whose sections are the connections on $Q \to P$. The pullback of $Q$ to $C(Q)$ carries a connection $\alpha$, with curvature $\omega_\alpha$. This connection is universal in the sense that, for any section $\alpha$ of $C(Q)$, the pullback by $\alpha$ of $\omega_\alpha$ is $\omega_\alpha$ itself. Furthermore, for any section $\alpha$, the curvature of this connection is $\omega_\alpha$.

Proof. The momentum map $J$ assigns to each element of $T^*Q$ the result of evaluating that cotangent vector on the vector field $X$ which generates the principal $G$ action on $Q$. Thus, $J^{-1}(1)$ consists of those cotangent vectors $\eta$ for which $\eta(X) = 1$, and an element of $C(Q) = J^{-1}(1)/G$ is a set of $G$-related cotangent vectors with this property based at all the points of a fibre of $Q \to P$. A section of $C(Q)$, therefore, is a $G$-invariant 1-form $\alpha$ on $Q$ for which $\alpha(X)$ is identically equal to 1—i.e., a connection.

Now $J^{-1}(1)$, being a principal $G$-bundle over $C(Q)$ with an equivariant map to $Q$, is itself the pullback of $Q$ to $C(Q)$. The negative of the canonical 1-form of $T^*Q$ pulls back to give the connection from $\alpha$, on $J^{-1}(1)$ whose curvature is the reduced canonical symplectic form $\omega_\alpha$.

We leave to the reader the straightforward task of deducing the universal property of $\alpha$, from that of the canonical 1-form on $T^*Q$. Q.E.D.

REFERENCES