On maximal sublattices of finite lattices

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Abstract

We discuss the possible structures for and mutual relationships between a finite distributive lattice \( L \), a maximal sublattice \( M \) of \( L \) and the corresponding 'remainder' \( \mathcal{R} = L \setminus M \) with the aid of Birkhoff duality, and contrast the results with the analogous situations for a general finite lattice \( L \). © 1999 Elsevier Science B.V. All rights reserved

1. Introduction

Maximal sublattices of finite lattices have been considered extensively in the literature, especially for the distributive case. The first papers devoted to this topic are [6, 9] where a maximal sublattice \( M \) of a finite distributive lattice \( L \) is characterized in terms of the structure of its ‘remainder’ \( \mathcal{R} = L \setminus M \) and, moreover, it is shown [9] that \( |\mathcal{R}| \leq |L|/3 \). In the 1970s numerous papers coauthored between Chen, Koh, Poh and Tan were focused on Frattini sublattices, that is, the intersection of all maximal sublattices of some given lattice. More recently, questions regarding numbers and sizes of maximal sublattices of finite lattices received some attention, see [2, 3]. The references given in these two papers may also serve as port of entry to the literature on the subject.

In the present paper, we treat \( L \), \( M \) and \( \mathcal{R} \) as given above as equal partners and call \( L \) a ‘minimal extension’ of \( M \) whenever \( M \) is a maximal sublattice of \( L \); \( \mathcal{R} \) is a sublattice of \( L \) whenever \( L \) is distributive. We show that a given finite distributive lattice \( M \) has only finitely many minimal extensions \( L \) preserving 0 and 1 and describe an algorithm to effectively list them (Theorem 3.3). In contrast, given a finite distributive lattice \( \mathcal{R} \), there are infinitely many pairs \((M, L)\) such that \( \mathcal{R} \cong L \setminus M \) (Theorem 3.4 and Corollary 3.5), again, a list of all these ‘hosts’ \( L \) may be generated effectively. We use Birkhoff duality to work these questions as problems about finite ordered sets; as a side

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benefit, we obtain that the ratio \( |R| = |L|/3 \) occurs exactly if \( L \cong K \times 3 \) and \( M \cong K \times 2 \) for some finite distributive lattice \( K \) (Corollary 3.6).

In Section 4, we contrast these results with those of [3] which show that for general finite lattices \( L \) the situation is radically different, and formulate a number of open questions.

### 2. Preliminaries and notation

Our notation is fairly standard. All lattices and ordered sets considered in this note are finite unless specified otherwise. If \((P, \leq)\) is an ordered set and \(a, b \in P\), then \([a, b] = \{x \in P; a \leq x \leq b\}\) stands for the closed interval between \(a\) and \(b\). We call \(a\) a lower cover of \(b\) and correspondingly \(b\) an upper cover of \(a\) iff \(a \neq b\) and \([a, b] = \{a, b\}\). \(\text{Cov}(P)\) stands for the set of all such covering pairs. For a subset \(A \subseteq P\), we write \(\downarrow A\) for \(\{x \in P; x \leq a\ \text{for some} \ a \in A\}\); \(A\) is a down-set iff \(A = \downarrow A\). If \(A = \{y\}\) is a singleton, the notation is simplified to \(\downarrow y\). The dual concept is that of an up-set, denoted analogously by \(\uparrow\). The \(n\)-element chain is written \(n\). Finally, we use \(\oplus\) to denote disjoint union of ordered sets.

We write 0 and 1 for the least resp. largest element of a finite lattice \(L\). A sublattice \(K\) of \(L\) is a \((0, 1)\)-sublattice iff it includes the 0 and 1 of \(L\). An element \(x \in L\) is join-irreducible iff \(x \neq 0\) and \(x\) has exactly one lower cover, we write \(J(L)\) for the set of all join irreducibles of \(L\). Meet-irreducible is defined dually, with \(M(L)\) standing for the set of all such elements in \(L\). Both \(J(L)\) and \(M(L)\) will be considered as ordered sets under the order induced from \(L\).

Suppose \(M\) is a maximal sublattice of \(L\). We look at this situation from below and say that \(L\) is a minimal extension of \(M\). We also consider the difference set \(L \setminus M\) with the order induced from \(L\) and call it the remainder of \(M\).

We shall be concerned primarily with distributive lattices; \(\varnothing\) stand for the class of all such lattices, finite or not. The interplay between finite distributives lattices, their maximal sublattices and the remainders thereof will be analyzed by means of (finite) Birkhoff duality; we give a bare-bones sketch of the parts we need (a good reference for the duality theory of arbitrary distributive lattices is [7]). Let \(D\) be the category of all finite distributive lattices with \((0, 1)\)-preserving lattice homomorphisms, and \(P\) the category of all finite ordered sets with order-preserving maps. To each object \(L \in D\) we assign the ordered set \(P(L) := J(L) \in P\); conversely, to each object \(P \in P\) we assign the lattice \(D(P)\) with carrier \(\{U \subseteq P; U\ \text{is a down-set in} \ P\}\) and set intersection resp. union as lattice operations. It turns out that \(L\) and \(DP(L)\) resp. \(P\) and \(PD(P)\) are canonically isomorphic; this sets up the object part of Birkhoff duality. The morphisms part is not quite so straightforward; fortunately, we only need to know that for a \((0, 1)\)-preserving lattice homomorphism \(f : L_1 \to L_2\) \((L_1, L_2 \in D)\) there is a canonical dual map \(P(f) : P(L_2) \to P(L_1)\); \(P(f)\) is order-preserving and related to \(f\) by \(f = P(f)^{-1}\) (see [7, pp. 171ff] for details). \(f\) is one-to-one iff \(f\) is monic iff \(P(f)\) is onto, and \(f\) is epic iff \(P(f)\) is one-to-one iff \(P(f)\) is monic. This describes one half of the morphism part.
of Birkhoff duality which is a full categorical coequivalence between the categories \( \mathcal{D} \) and \( \mathcal{P} \).

3. The distributive case

Our starting point is the following characterization of proper maximal sublattices of finite distributive lattices, due to Chen et al. [6] and to Rival [9]:

**Theorem 3.1.** If \( L \) is a finite distributive lattice and \( M \) is a proper maximal sublattice of \( L \), then the remainder \( R \) of \( M \) is either a singleton, consisting of 0,1 or a doubly irreducible element of \( L \), or else a proper interval \([a,b]\) with \( a \in J(L) \setminus M(L), \ b \in M(L) \setminus J(L) \) and \( x \notin J(L) \cup M(L) \) for all \( a < x < b \).

In other words, remainders are very special intervals of \( L \). Now, every finite distributive lattice \( M \) has a (proper) minimal extension in \( \mathcal{P} \): Just add a new top or bottom element. We may even preserve 0 and 1: Represent \( M \) as a lattice \( \mathcal{L} \) of subsets of some finite set \( X \) such that \( \emptyset \) and \( X \) belong to \( \mathcal{L} \) but \( \mathcal{L} \) is strictly contained in the power set lattice \( \mathcal{P}(X) \), then choose a sublattice \( \mathcal{L}' \) of \( \mathcal{P}(X) \) minimal with \( \mathcal{L} \subseteq \mathcal{L}' \) but \( \mathcal{L}' \neq \mathcal{P}(X) \).

**Fact 3.2.** Any \( M \subset \mathcal{P} \) has a proper minimal \((0,1)\)-extension in \( \mathcal{P} \).

Given the rather special nature of remainders, it is not so obvious whether any given finite distributive lattice will occur as a remainder within some other finite distributive lattice. We will show that this is indeed the case. To this end, we translate the question into the dual category; as a side benefit, this will lead to a new proof of Theorem 3.1 as well as of Rival’s result (Theorem 2 in [9]) that the cardinality of the remainder is at most \( \frac{1}{3} \) of the cardinality of its parent lattice.

Let \((P, \leq)\) be any ordered set. An ordered pair \((x, y)\) of elements of \( P \) is called **critical** iff \( x < y \) and for all \( u, v \in P \), \( u < x \) implies \( u < y \) and \( v > y \) implies \( v > x \). Equivalently, \((x, y)\) is critical iff \( \leq \cup \{(x, y)\} \) is an order relation on \( P \). This is a fundamental concept of order theory, see, e.g., Trotter [11]. In a slightly weakened form, it provides the key for dealing with maximal sublattices (of finite distributive lattices) within the category \( \mathcal{P} \). This observation goes back to Hashimoto [8] and, in a more precise form, to Adams [1]:

Call an ordered pair \((x, y)\) of elements of \((P, \leq)\) **critical** iff \( y \not< x \) and for all \( u, v \in P \), \( u < x \) implies \( u < y \) and \( v > y \) implies \( v > x \). Every critical pair is *critical*, but not vice versa: A *critical* pair may be comparable, and this happens exactly iff \((y, x) \in \text{Cov}(P)\) and, moreover, \( y \) is the unique lower cover of \( x \) and \( x \) is the unique upper cover of \( y \). In [8, 1] it is shown that there is a bijective correspondence between the proper maximal \((0,1)\)-sublattices of \( L \in \mathcal{D} \) on one side and the *critical* pairs of \( \mathcal{P}(L) \in \mathcal{P} \) on the other. See also [2] for a detailed analysis of *critical* pairs.
Birkhoff duality works at its best for the category $D$. This does not impose any serious restrictions within our context as shown by Fact 3.2, so we will stay within $D$ henceforth unless explicitly mentioned otherwise. We summarize the salient facts concerning duals of maximal $(0, 1)$-sublattices as follows: Let $L \in D$ and $P = P(L)$. Consider a proper maximal $(0, 1)$-sublattice $M$ of $L$. Then there exists a (unique) *critical pair $(x, y)$ in $P$ such that (the canonical copy of) $M$ may be recovered within $D(P)$ as

$$M \cong \{ U \in D(P); \ y \in U \Rightarrow x \in U \}. \quad (1)$$

Conversely, every *critical pair $(x, y)$ in $P$ determines a maximal proper $(0, 1)$-sublattice of $D(P)$ in this way. It follows that the remainder $R = L \setminus M$ may be found within $D(P)$ as

$$R \cong \{ U \in D(P); \ y \in U \text{ but } x \notin U \}. \quad (2)$$

Next, we want to describe the duals $P_{xy} := P(M)$ of $M$ and $Q_{xy} := P(R)$ of $R$. For $M$, two cases arise: If $x \parallel y$ (that is, if $(x, y)$ is even critical), then

$$P_{xy} \text{ is } P \text{ with the pair } (x, y) \text{ added to the order of } P. \quad (3)$$

Moreover, $f_{xy} = id_P : P \rightarrow P_{xy}$ is onto and order-preserving. Writing $i : M \rightarrow L$ for the natural embedding, we have $f_{xy} = P(i) : P(L) \rightarrow P(M)$; the two morphisms are related by $f_{xy}^{-1} = i$.

If $(x, y)$ is *critical but not critical (that is, $y < x$), then

$$P_{xy} \text{ is obtained from } P \text{ by identifying } x \text{ and } y. \quad (4)$$

Here, $f_{xy} : P \rightarrow P_{xy}$ given by $f_{xy}(y) = x$ and $f_{xy}(z) = z$ for all $z \in P$, $z \neq y$, is onto and order-preserving. Again, $f_{xy} = P(i)$ and $f_{xy}^{-1} = i$.

Turning to the remainder $R = L \setminus M$, observe that if $z \in P$ and $z \not\in \downarrow y \cup \uparrow x$, then $z \parallel x$ and $z \parallel y$ as $(x, y)$ is *critical. It follows that $U \mapsto U \setminus (\downarrow y \cup \uparrow x)$ maps the down-sets in $P$ containing $y$ but excluding $x$ bijectively onto the down sets in $P \setminus (\downarrow y \cup \uparrow x)$. Hence, $R$ as given by (2) may also be constructed as

$$R \cong D(P \setminus (\downarrow y \cup \uparrow x)). \quad (5)$$

We conclude that

$$Q_{xy} = P \setminus (\downarrow y \cup \uparrow x). \quad (6)$$

We will now use the dual of $M$ to determine the minimal $(0, 1)$-extensions of $M$. Two such extensions may be isomorphic as lattices but are considered different as extensions if the copies of $M$ they contain occur in different positions.

**Theorem 3.3.** Let $M \in D$, $P = P(M)$, $n = |P|$ and $m = |\text{Cov}(P)|$. Then there exist at most $n + m$ different (proper) minimal $(0, 1)$-extensions of $M$ within $D$.
Proof. Let \( P = P(M) \) and choose \( x \in P \). ‘Split’ \( x \) into two different points \( x^+ \) and \( x^- \). Order the resulting set \( P' \) by keeping the order of \( P \) on \( P' \setminus \{x^-, x^+\} \) and by adding the relations \( x^- < x^+ \), \( u < x^- \) whenever \( u < x \) in \( P \), and \( v > x^+ \) whenever \( v > x \) in \( P \). It is immediate that \( (x^+, x^-) \) is *critical in \( P' \), and that contracting \( (x^+, x^-) \) as specified by (4) yields \( P \). Hence, \( L = D(P') \) contains \( M = D(P) \) as a proper maximal \((0, 1)\)-sublattice. Obviously, this construction provides \( n \) minimal extensions of \( M \).

If \( (x, y) \in \text{Cov}(P) \), deleting \( (x, y) \) in the order of \( P \) may produce a new ordered set \( P' \) in which \( (x, y) \) is critical. As \( P'_x = P \) by (3), \( L = D(P') \) contains \( M = D(P) \) as a proper maximal \((0, 1)\)-sublattice. This procedure provides at most \( m \) minimal extensions of \( M \) and so their total number is at most \( n + m \). \( \Box \)

As mentioned above, lattices obtained in this way by ‘reversing’ (3) resp. (4) may turn out isomorphic for different *critical pairs. However, the bound given by Theorem 3.3 is tight: Let \( M = 3^k \) \((k \in \mathbb{N})\), hence \( P = P(M) = 2 \oplus 2 \oplus \cdots \oplus 2 \) \((k \text{ summands} 2)\). Splitting a point yields a copy of \( L_1 = 4 \times 3^{k-1} \) as the corresponding minimal extension, while deleting any pair \( (x, y) \in \text{Cov}(P) \) from the order relation of \( P \) produces the critical pair \( (x, y) \) in \( P' \), with corresponding minimal extension a copy of \( L_2 = 2 \times 2 \times 3^{k-1} \). It is easy to check that there are indeed \( n = 2k \) embeddings of \( M \) into \( L_1 \) and \( m = k \) such embeddings into \( L_2 \).

The proof of Theorem 3.3 together with (5) may be read as a rough algorithm to effectively determine, given \( M \), all minimal \((0, 1)\)-extensions \( L \) of \( M \) together with their remainders \( R = L \setminus M \): Determine the dual \( P(M) = (P, \leq) \) and split each point \( x \in P \), obtaining a family \( \mathcal{P}_1 = \{P_x, x \in P\} \). Then test any covering pair \( (x, y) \) in \( P \) for criticality in \( P_{xy} = (P, \leq \setminus \{(x, y)\}) \) and put \( P_{xy} \) into a second family \( \mathcal{P}_2 \) in the affirmative case. For any \( P_{uv} \in \mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \) determine the \( P \)-epimorphisms \( f_{uv} \) as specified after (3) resp. (4). Determine all ordered sets \( Q_{uv} = P_{uv} \setminus (\downarrow u \cup \uparrow u) \) for \( P_{uv} \in \mathcal{P} \). Finally, replace ordered sets by their respective collections of down-sets, and order-preserving maps by their inverses.

As announced above, a proof of Theorem 3.1 may be read off (2) and (5): \( R \) is a singleton iff \( Q_{xy} = \emptyset \), in which case \( \downarrow y \) is the corresponding doubly irreducible element of \( D(P) \) (it is join-irreducible as a principal down-set and meet-irreducible since \( \downarrow \{(x, y)\} \) is its unique upper cover). So suppose that \( R \) is not a singleton, that is, \( Q_{xy} \neq \emptyset \). The smallest set \( U \) qualifying in (2) is \( \downarrow y \) which still is join-irreducible but no longer meet-irreducible in \( D(P) \): Picking a minimal element \( z \in Q_{xy} \), observe that \( \downarrow \{z, y\} \) and \( \downarrow \{x, y\} \) are distinct upper covers. On the other hand, the largest set \( U \) qualifying in (5) is \( \downarrow (Q_{xy} \cup \{y\}) \) which is clearly join-reducible in \( D(P) \) but has unique upper cover \( \downarrow (Q_{xy} \cup \{x, y\}) \), making it meet-irreducible. Any further qualifying set \( U \) has the form \( \downarrow (V \cup \{y\}) \) with \( \emptyset \subset V \subset Q_{xy} \), and the same arguments show that it is doubly reducible in \( D(P) \). It is clear that the collection of all these sets forms an interval in \( D(P) \).

We are now prepared to show that any finite distributive lattice may serve as a remainder:
Theorem 3.4. For any given lattice $R \in D$ there exist a lattice $M \in D$ and a minimal \((0,1)\)-extension $L$ of $M$ such that $R$ is (isomorphic to) the remainder $L \setminus M$.

Proof. The idea is to use (6) and to manufacture $P \in \mathbb{P}$ and a *critical pair $(x, y)$ in $P$ such that the dual $Q = P(R)$ coincides with $Q_{xy}$. Let $P := Q \cup \{ x, y \}$ with $x, y \notin Q$ and $x \neq y$ and add $(y, x)$ to the order relation of $Q$. Obviously $(x, y)$ is *critical in the resulting ordered set $P$, and $Q_{xy} = P \setminus (\downarrow y \cup \uparrow x) = Q$ as desired. Finally, let $L := D(P)$ and $M := D(P_{xy})$ (where $P_{xy} \cong Q \cong 1$).

Observe that for $P$ and $Q$ as constructed in the preceding proof the down-sets of $P$ are of the form $U$, $U \cup \{ y \}$ or $U \cup \{ x, y \}$ where $U$ is a down-set of $Q$. Hence, the construction realizes Rival's [9] upper bound of $|L|/3$ for the size $|R|$ of the remainder. Corollary 3.6 below will show that this is essentially the unique way to realize this bound.

In fact, there are plenty of possibilities to realize $R$ as a remainder; the proof of the following corollary will provide a complete list:

Corollary 3.5. In the notation of Proposition 3.4, there are infinitely many pairs $(M, L)$ such that $R \in L \setminus M$.

Proof. Let $X$ be any ordered set with least element $x$, and similarly $Y$ any ordered set with greatest element $y$. Form $P := Q \oplus Y \oplus X$ and denote the order relation of $P$ by $\leq$. Optionally, add $(y, x)$ to $\leq$. In both cases, $(x, y)$ will be *critical in $P$ and $Q_{xy} = P \setminus (\downarrow y \cup \uparrow x) = Q$. The same conclusion is valid if either $\leq$ or $\leq \cup (y, x)$ is extended to some order relation $\leq'$ on $P$ such that $x$ and $y$ are incomparable to any $z \in Q$ with respect to $\leq'$. In fact, this procedure exhausts all realizations of (6) and thus describes all possible pairs $(M, L)$ (as $M := D((P, \leq')_{xy})$ and $L := D((P, \leq'))$).

Again, the preceding proof may be converted into an 'algorithm' to systematically produce the desired pairs $(M, L)$ (using a 'list' of orders with least element to enumerate the candidates for $X$ and $Y$).

The following corollary is an extension of Theorem 2 of [9] and provides an alternative proof thereof:

Corollary 3.6. In the notation of Proposition 3.4, we have $|R|/|L| \leq \frac{1}{3}$ (and thus $|M|/|L| \geq \frac{2}{3}$). These bounds are tight, and equality occurs exactly if $L \cong R \times 3$ and $M \cong R \times 2$.

Proof. Let $L, M$ and $R$ be given as specified with duals $P, P_{xy}$ and $Q_{xy}$ (with respect to the appropriate *critical pair $(x, y)$ in $P$). Define $P'$ as $Q_{xy} \cup \{ x, y \}$ (with the induced order) and let $L := D(P')$. The pair $(x, y)$ is still *critical in $P'$, and $Q_{xy}' = Q_{xy}$. Let $M' := D(P_{xy}')$ and $R' := D(Q_{xy}) \cong R$. The mapping $\alpha : U \mapsto \alpha(U) := U \cap P'$ maps the
down-sets of \( P \) onto the down-sets of \( P' \) — that is, \( \alpha \) maps \( L \) onto \( L' \), thus \( |L'| \leq |L| \). As \((x, y)\) is *critical in \( P' \), we have either \( x \parallel y \) or \((y, x) \in \text{Cov}(P')\). Hence, either \( P' \cong Q_{xy} \oplus 1 + 1 \) or \( P' \cong Q_{xy} \oplus 2 \). In the first case \( P'_{xy} \) is obtained from \( P' \) by adding \((x, y) \) to the order relation of \( P' \) and so \( P'_{xy} \cong Q_{xy} \oplus 2 \), in the second case \( P'_{xy} \) is obtained from \( P' \) by collapsing \( x \) and \( y \) and so \( P'_{xy} \cong Q_{xy} \oplus 1 \). Dualizing, we obtain \( L' \cong R \times 2 \times 2 \) and \( M' \cong R \times 3 \) in the first case resp. \( L' \cong R \times 3 \) and \( M' \cong R \times 2 \) in the second. The remainder size is larger in the second case and we see that indeed \( |R|/|L| \leq |R|/|L'| = |R|/|R \times 3| = \frac{1}{3} \), thus \( |M|/|L| \geq \frac{2}{3} \). To establish the uniqueness claim, observe that \( \alpha \) is not injective whenever \( P \geq P' \), so \( |L| \geq |L'| \) in this case and hence \( |R|/|L| < |R|/|L'| \). To give this ratio its maximal value we must thus have \( P = P' \), that is, one of the two cases examined above. The ratio being \( \frac{1}{4} \) in the first, we are stuck with the second to obtain \( \frac{1}{3} \).

Corollary 3.6 suggests that maximal sublattices of minimal possible size occur only in a rather special situation. Indeed, there are quite different types of (distributive) lattices where all proper maximal \((0,1)\)-sublattices are of the maximal possible size, obtained by deleting a doubly irreducible element: For one, this is clearly the case in chains \( n \) with \( n - 2 \) maximal sublattices each of size \( n - 1 \). More interesting is the case of \( F_n \), the free distributive \((0,1)\)-lattice on \( n \) generators: Its dual is the ordered set \( P_n \) obtained from the Boolean lattice \( 2^n \) by deleting the top and the bottom element. There are \( n \) *critical pairs — all of them critical — of the form \((a, c)\) where \( a \) is an atom of \( 2^n \) and \( c \) the coatom which is incomparable with \( a \). As is easily seen, \( Q_{ac} = \emptyset \) in this case, and \( P_{ac} \) is the dual of the maximal sublattice of \( F_n \) obtained by deleting one of the doubly irreducible generators of \( F_n \).

The appearance of \( n \) and \( F_n \) in the preceding discussion is no coincidence: Recently, Berman and Bordalo have investigated finite distributives generated by their doubly irreducible elements (see [4]) and determined that a lattice falls into this class iff all of its maximal \((0,1)\)-sublattices are obtained by deleting a doubly irreducible [5].

We recall from [2,3] that a distributive lattice of size \( n \) has at most \( n \) general (that is, not necessarily containing 0 and 1) maximal sublattices, with \( n \) being attained exactly for the chain \( n \). So we may say that, roughly, finite distributive lattices tend to contain few but rather large maximal sublattices. The following section is devoted to show that for arbitrary finite lattices the situation is quite different.

4. Beyond distributivity

Not unexpectedly, the situations considered in Section 3 regarding extensions, remainders, numbers and sizes are far less transparent if one looks at general finite lattices. Some of these questions have been addressed in [3]; we discuss some results and raise a few questions.

Given a finite lattice \( M \), it is trivial to find a proper minimal \((0,1)\)-extension \( L \): Let \( L := M \cup \{u\} \) with \( u \not\in M \) and with \( 0 < u < 1 \) as the only new comparabilities.
It is equally easy to add two new points to $M$ provided $M$ is not a chain: Pick
$a \in M$ maximal with at least two lower covers $u$ and $v$, and let $u'$ be a lower cover of
$u$. Add two new elements $x$ and $y$ to $M$ with new comparabilities $u' < x < u$, $v < y < a$
and $x < y$.

To start with, the analog of Theorem 3.3 fails in a rather spectacular way. Indeed,
there exists a lattice $M$ with 14 elements which is $(0,1)$-embedded, as a maximal
sublattice, in finite lattices of unbounded sizes and even in a countably infinite lattice,
see [3]. Hence

**Question 1.** Are there finite lattices $M$ which admit only a finite number of different
(proper) minimal extensions (resp. $(0,1)$-extensions)? If yes, how can these lattices be
characterized?

The situation is clear as far the number of maximal sublattices of a finite lattice
is concerned: As shown also in [3], there is, in general, no polynomial bound on the
number of sublattices a finite lattice may have (in the number of elements of the
lattice).

A natural question at this point is whether there exists a reasonable class of finite
lattices with the same behavior regarding sizes and numbers of maximal sublattices as
the class of finite distributive lattices? More precisely,

**Question 2.** Are there varieties $\mathcal{V} \neq \emptyset$ of lattices which admit numbers
$k_\mathcal{V} \in \omega$
and $1 > \rho_\mathcal{V} > 0$ such that for any finite $L \in \mathcal{V}$ (i) the number of maximal sublattices
of $L$ does not exceed $|L|^{k_\mathcal{V}}$ and (ii) for any maximal sublattice $M$ of $L$ we have
$|M|/|L| \geq \rho_\mathcal{V}$?

Of course, $k_\mathcal{V} = 1$ and $\rho_\mathcal{V} = \frac{2}{3}$.

It is quite easy to construct examples showing that for $M$ a maximal sublattice of a
finite lattice $L$ the remainder $L \setminus R$ need not be a sublattice of $L$. This brings us to

**Question 3.** Suppose $P$ is finite ordered set occurring as a suborder of a finite lattice.
Is there a finite lattice $L$ and a maximal sublattice $M$ of $L$ such that $P \cong L \setminus M$? How
can such ordered sets be characterized?

As Gabriela Bordalo has pointed out [5], such $P$ is quite restricted: If $|P| \geq 2$, then
$P$ cannot contain an isolated element, in particular, $P$ cannot be an antichain. Indeed,
let $|P| \geq 2$ and consider $a \in P = L \setminus M$. $M \cup \{a\}$ cannot be sublattice of $L$ since $M$
is maximal. So we find $x \in M$ such that $x \lor a \in P \setminus \{a\}$ or $y \in M$ such that $y \land a \in P \setminus \{a\}$. Evidently, both $x \lor a$ and $y \land a$ are different from but comparable with $a$.

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