Note
Partition of a directed bipartite graph into two directed cycles

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Abstract
Let \( D = (V_1, V_2; A) \) be a directed bipartite graph with \(|V_1| = |V_2| = n \geq 2\). Suppose that \( d_D(x) + d_D(y) \geq 3n + 1 \) for all \( x \in V_1 \) and \( y \in V_2 \). Then \( D \) contains two vertex-disjoint directed cycles of lengths \( 2n_1 \) and \( 2n_2 \), respectively, for any positive integer partition \( n = n_1 + n_2 \). Moreover, the condition is sharp for even \( n \) and nearly sharp for odd \( n \).

1. Introduction

We discuss only finite simple graphs and strict directed graphs. The terminology and notation concerning graphs is that of [4], except as indicated. A directed graph \( D \) is called a directed bipartite graph if there exists a partition \( \{V_1, V_2\} \) of \( V(D) \) such that the two induced directed subgraphs \( D[V_1] \) and \( D[V_2] \) of \( D \) contain no arcs of \( D \). We denote by \( (V_1, V_2; A) \) a directed bipartite graph with \( \{V_1, V_2\} \) as its bipartition and \( A \) as its arc set. Similarly, \( (V_1, V_2; E) \) represents a bipartite graph with \( \{V_1, V_2\} \) as its bipartition and \( E \) as its edge set.

In 1963, Corrádi and Hajnal [5] investigated the maximum number of vertex-disjoint cycles in a graph. They proved that if \( G \) is a graph of order at least \( 3k \) with minimum degree at least \( 2k \), then \( G \) contains \( k \) vertex-disjoint cycles. In particular, when the order of \( G \) is exactly \( 3k \), then \( G \) contains \( k \) vertex-disjoint triangles. In 1984, El-Zahar [6] proved that if \( G \) is a graph of order \( n = n_1 + n_2 \) with \( n_1 \geq 3 \), \( n_2 \geq 3 \) and minimum degree at least \( \lceil n_1/2 \rceil + \lceil n_2/2 \rceil \), then \( G \) contains two vertex-disjoint cycles of lengths \( n_1 \) and \( n_2 \), respectively. In 1991, Amar and Raspaud [2] investigated vertex-disjoint directed cycles in a strongly connected directed graph of order \( n \) with

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Little and Wang [7] proved that if $D$ is a directed graph of order $n \geq 4$ with minimum degree at least $(3n - 3)/2$, then $D$ contains two vertex-disjoint directed cycles of lengths $n_1$ and $n_2$, respectively, for any integer partition $n = n_1 + n_2$ with $n_1 \geq 2$ and $n_2 \geq 2$. In this paper, we prove the following result.

**Theorem.** Let $D = (V_1, V_2; A)$ be a directed bipartite graph with $|V_1| = |V_2| = n \geq 2$. Suppose that $d_D(x) + d_D(y) \geq 3n + 1$ for all $x \in V_1$ and $y \in V_2$. Then $D$ contains two vertex-disjoint directed cycles of lengths $2n_1$ and $2n_2$, respectively, for any positive integer partition $n = n_1 + n_2$.

We recall some terminology and notation. Let $G$ be a graph and $D$ a directed graph. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of $G$, respectively. We denote $|E(G)|$ by $e(G)$. We use $V(D)$ and $A(D)$ to denote the vertex set and the arc set of $D$, respectively. For a vertex $x \in V(D)$, $N_D^-(x)$ is the set of all vertices $y$ of $D$ with $(x, y) \in A(D)$. We similarly define $N_D^+(x)$ and let $N_D(x) = N_D^-(x) \cup N_D^+(x)$. We also define $d_D^-(x) = |N_D^-(x)|$, $d_D^+(x) = |N_D^+(x)|$ and $d_D(x) = d_D^-(x) + d_D^+(x)$. For two vertices $x$ and $y$ of $D$, we say that $x$ is joined to $y$ in $D$ if either $(x, y)$ or $(y, x)$ is an arc of $D$. For a vertex $u \in V(G)$ and a subgraph $H$ of $G$, we define $d_G(u, H) = |N_G(u) \cap V(H)|$. Hence, $d_G(u, G) = d_G(u)$, the degree of $u$ in $G$. For a subset $U \subseteq V(G)$, $G[U]$ is the subgraph of $G$ induced by $U$. For a subset $X \subseteq V(D)$, $D[X]$ is the directed subgraph of $D$ induced by $X$. A graph or directed graph is said to be traceable if it contains a hamilton path or directed hamilton path, respectively. A graph or directed graph is called hamiltonian if it contains a hamilton cycle or directed hamilton cycle, respectively. For any two vertices $x$ and $y$ of $G$, we define $e(xy) = 1$ if $xy$ is an edge of $G$ and $e(xy) = 0$ otherwise.

2. Lemmas

In the following, $G = (V_1, V_2; E)$ is a bipartite graph with $|V_1| = |V_2| = n \geq 2$.

**Lemma 2.1.** Let $P = x_1y_1 \cdots x_ky_k$ be a path of $G$. Let $x \in V_1$ and $y \in V_2$ be vertices not on $P$. Then the following two statements hold:

(a) If $d_G(x, P) + d_G(y, P) \geq k + 2 - e(xy)$, then $G$ contains a path $P'$ from $x_1$ to $y_k$ such that $V(P') = V(P) \cup \{x, y\}$.

(b) If $d_G(x, P) + d_G(y, P) \geq k + 1 - e(xy)$, then $G$ contains a path $P'$ such that $V(P') = V(P) \cup \{x, y\}$.

**Proof.** We may assume that $x_1 \in V_1$. To prove (a), we see that if $e(xy) = 1$, then there exists $i \in \{1, 2, \ldots, k\}$ such that $\{x_1y_i, x_iy_k\} \subseteq E(G)$. Then the path $x_1y_1 \cdots y_{i-1}x_iy_1x_{i+1} \cdots x_ky_k$ is the required path. If $e(xy) = 0$, then there exist $i, j \in \{1, 2, \ldots, k\}$ with $i < j$ such that $\{x_1y_i, x_jy_j, x_iy_j, x_jy_1\} \subseteq E(G)$. Then the path $x_1y_1 \cdots y_{i-1}x_iy_jy_{j-1} \cdots x_{i+1}y_1x_jy_{j+1} \cdots x_ky_k$ is the required path.
To prove (b), we first assume that $\varepsilon(xy) = 1$. Then (b) is true if $xy_k$ or $x_1y$ is an edge of $G$. Hence we may assume that both $xy_k \notin E$ and $x_1y \notin E$. Then $d_G(x, P - x_1 - y_k) + d_G(y, P - x_1 - y_k) \geq k$. By (a), $G$ has a path $P'$ from $y_1$ to $x_k$ such that $V(P') = V(P - x_1 - y_k) \cup \{x, y\}$. Then the path $x_1P'x_k$ satisfies the requirement.

Next, we assume $\varepsilon(xy) = 0$. If both $xy_k \in E$ and $x_1y \in E$, then (b) is true. Thus, we may assume that $xy_k \notin E$. If $xy_k \notin E$, then by (a), $G$ has a path $L$ from $y_1$ to $x_k$ such that $V(L) = V(P - x_1 - y_k) \cup \{x, y\}$. Therefore, the path $x_1Lx_k$ satisfies the requirement. Therefore, we may further assume that $xy_k \notin E$. By the proof of (a), we may assume that there exists a unique $i \in \{1, 2, \ldots, k\}$ such that $\{xy_i, x_iy\} \subseteq E$. Therefore, for every $j \in \{1, 2, \ldots, k\}$ with $j \neq i$, we have that $xy_j \in E$ if and only if $x_jy \notin E$. Hence, $xy_1 \in E$. Let $j$ be the smallest integer in $\{1, 2, \ldots, k\}$ such that $x_jy \notin E$. Thus, $1 < j \leq i$ and $xy_{j-1} \in E$. Then the path $x_1y_1 \cdots x_{j-1}y_{j-1}x_k \cdots y_jx_jy$ satisfies the requirement. This proves the lemma. □

**Lemma 2.2** (Bondy and Chvátal [3]). The following two statements hold:

(a) Let $P = x_1y_1 \cdots x_ky_k$ be a path of $G$ with $k \geq 2$. If $d_G(x_1, P) + d_G(y_k, P) \geq k + 1$, then $G$ has a cycle $C$ such that $V(C) = V(P)$.

(b) If $d_G(x) + d_G(y) \geq n + 1$, for any two non-adjacent vertices $x$ and $y$ with $x \in V_1$ and $y \in V_2$, then $G$ is hamiltonian.

The following lemma is Lemma 6 of [1].

**Lemma 2.3.** Let $C = x_1y_1 \cdots x_ky_kx_1$ be a cycle of $G$. Let $i, j \in \{1, 2, \ldots, k\}$. Suppose that $d_G(x_i, C) + d_G(y_j, C) \geq k + 2$. Then $G$ has a path $P$ from $y_j$ to $x_{j+1}$ such that $V(P) = V(C)$, where subscripts are reduced modulo $k$.

**Proof.** Construct a new graph $G'$ from $G$ by adding two new vertices $u$ and $v$ of degree 2 to $G$ such that $y_iux_{j+1}$ is a path of $G'$. Observe that $P' = (C - x_iy_i - x_{j+1}y_j) \cup \{y_iu, uv, ex_{j+1}\}$ is a path of $G'$ from $x_i$ to $y_j$. By Lemma 2.2(a), $G'$ has a cycle $C'$ with $V(C') = V(P')$. Clearly, $C'$ contains the path $y_iux_{j+1}$. Hence, the path $C' - u - v$ satisfies the requirement. □

**Lemma 2.4.** Suppose that $G$ has a hamilton path and for any two endvertices $u$ and $v$ of a hamilton path of $G$, $d_G(u) + d_G(v) \geq k$ holds, where $k$ is an integer greater than $n$. Then for every $x \in V_1$ and every $y \in V_2$, $d_G(x) + d_G(y) \geq k$.

**Proof.** By Lemma 2.2(a), $G$ is hamiltonian. Let $C = x_1y_1 \cdots x_ny_nx_1$ be a hamilton cycle of $G$. Suppose, for a contradiction, that $d_G(x_i) + d_G(y_i) \leq k - 1$ for some $i, j \in \{1, 2, \ldots, n\}$. Then $G$ has no hamilton path from $x_i$ to $y_j$. By the hypothesis, we have that $d_G(y_{i-1}) + d_G(x_i) \geq k$ and $d_G(x_{j+1}) + d_G(y_j) \geq k$, where subscripts are reduced modulo $n$. Hence, $d_G(y_{i-1}) + d_G(x_i) \geq k + 1 \geq n + 2$. By lemma 2.3, $G$ has a hamilton path from $x_i$ to $y_j$. This contradiction proves the lemma. □
3. Proof of the theorem

Let \( D = (V_1, V_2; A) \) be a directed bipartite graph with \( |V_1| = |V_2| = n \geq 2 \) such that \( d_\rho(x) + d_\rho(y) \geq 3n + 1 \) for all \( x \in V_1 \) and \( y \in V_2 \). Suppose, for a contradiction, that \( D \) does not contain two vertex-disjoint directed cycles of lengths \( 2n_1 \) and \( 2n_2 \), respectively, for some positive integer partition \( n = n_1 + n_2 \).

We construct a bipartite graph \( G = (V_1, V_2; E) \) from \( D \) such that \( xy \in E \) if and only if both \( (x, y) \) and \( (y, x) \) belong to \( A \). Then \( G \) does not contain two vertex-disjoint cycles of lengths \( 2n_1 \) and \( 2n_2 \), respectively.

**Claim 1.** For all \( x \in V_1 \) and \( y \in V_2 \), \( d_G(x) + d_G(y) \geq n + 1 \).

**Proof.** We have

\[
d_G(x) + d_G(y) = d_\rho(x) + d_\rho(y) - (|N_\rho(x)| + |N_\rho(y)|)
\geq 3n + 1 - 2n = n + 1.
\]

By Claim 1 and Lemma 2.2(b), \( G \) is Hamiltonian. Hence, we can choose two vertex-disjoint induced subgraphs of \( G \), say \( G_1 = (A_1, B_1; E_1) \) and \( G_2 = (A_2, B_2; E_2) \), of order \( 2n_1 \) and \( 2n_2 \) respectively, such that both \( G_1 \) and \( G_2 \) are traceable.

Subject to (3), we may further choose \( G_1 \) and \( G_2 \) such that

\[
e(G_1) + e(G_2) \text{ is maximum.}
\]

**Claim 2.** Let \( u \) and \( v \) be two endvertices of a Hamilton path of \( G_1 \) and let \( x \) and \( y \) be two endvertices of a Hamilton path of \( G_2 \). Suppose that \( uy \in E \) and \( vx \in E \). Then

\[
d_G(u, G_1) + d_G(v, G_1) + d_G(x, G_2) + d_G(y, G_2)
\geq d_G(u, G_2) + d_G(v, G_2) + d_G(x, G_1) + d_G(y, G_1).
\]

**Proof.** Suppose that (5) does not hold. Then either \( d_G(u, G_2) + d_G(x, G_1) > d_G(u, G_1) + d_G(x, G_2) \), or \( d_G(v, G_2) + d_G(y, G_1) > d_G(v, G_1) + d_G(y, G_2) \). We may assume w.l.o.g. that the former holds. Then \( e(G_1 - u + x) + e(G_2 - x + u) > e(G_1) + e(G_2) \). As \( uy \in E \) and \( vx \in E \), both \( G_1 - u + x \) and \( G_2 - x + u \) are traceable. We obtain a contradiction with (4). \( \square \)

**Claim 3.** Let \( u \) and \( v \) be two endvertices of a Hamilton path of \( G_1 \) and let \( x \) and \( y \) be two endvertices of a Hamilton path of \( G_2 \) such that \( u \in V_1 \) and \( x \in V_1 \). Let \( G'_1 = G_1 - u - v + x + y \) and \( G'_2 = G_2 - x - y + u + v \). If both \( G'_1 \) and \( G'_2 \) are
traceable, then
\[ d_G(u, G_1) + d_G(v, G_1) + d_G(x, G_2) + d_G(y, G_2) \]
\[ \geq d_G(u, G_2) + d_G(v, G_2) + d_G(x, G_1) + d_G(y, G_1) \]
\[ - 2(e(uv) + e(vx)) + 2(e(uv) + e(xy)). \] (6)

In particular, if \( d_G(u, G_2) + d_G(v, G_2) \geq n_2 + 2 \) and \( d_G(x, G_1) + d_G(y, G_1) \geq n_1 + 2 \), then (6) holds.

**Proof.** If \( d_G(u, G_2) + d_G(v, G_2) \geq n_2 + 2 \) and \( d_G(x, G_1) + d_G(y, G_1) \geq n_1 + 2 \), then, by Lemma 2.1(b), both \( G'_1 \) and \( G'_2 \) are traceable. As both \( G'_1 \) and \( G'_2 \) are traceable, we have, by (4), that \( e(G'_1) + e(G'_2) \leq e(G_1) + e(G_2) \), which implies (6). \( \square \)

Let \( P_1 = u_1u_2 \ldots u_{2n_1} \) and \( P_2 = x_1x_2 \ldots x_{2n_2} \) be two hamilton paths of \( G_1 \) and \( G_2 \), respectively. We may assume that \( u_1 \in A_1, x_1 \in A_2, A_1 \cup A_2 = V_1 \) and of course \( B_1 \cup B_2 = V_2 \). As either \( G_1 \) or \( G_2 \) is not hamiltonian or isomorphic to \( K_2 \), we may assume that \( G_2 \) is not hamiltonian or isomorphic to \( K_2 \). Then \( n_2 \geq 2 \), and by Lemma 2.2(a), we have
\[ d_G(x_1, G_2) + d_G(x_{2n_2}, G_2) \leq n_2. \] (7)

**Claim 4.** \( G_1 \) is hamiltonian or isomorphic to \( K_2 \). Furthermore, \( d_G(u, G_1) + d_G(v, G_1) \geq n_1 + 1 \) for any two vertices \( u \) and \( v \) of \( G_1 \) with \( u \in V_1 \) and \( v \in V_2 \).

**Proof.** Suppose, for a contradiction, that the claim is not true. Then \( n_1 \geq 2 \), and by Lemma 2.2(b) and 2.4, there exist two endvertices \( u \) and \( v \) of a hamiltonian path of \( G_1 \) such that \( d_G(u, G_1) + d_G(v, G_1) \leq n_1 \). Without loss of generality, we may assume that \( d_G(u_1, G_1) + d_G(u_{2n_1}, G_1) \leq n_1 \). By Claim 1, it follows that \( d_G(u_1, G_2) + d_G(u_{2n_1}, G_2) \geq n_3 + 1 \) and \( d_G(x_1, G_1) + d_G(x_{2n_2}, G_2) \geq n_1 + 1 \). By Claim 2, we see that \( e(u_1x_{2n_1}) + e(x_1u_{2n_1}) \leq 1 \). Let \( G'_1 = G_1 - u_1 - u_{2n_1} + x_1 + x_{2n_2} \) and \( G'_2 = G_2 - x_1 - x_{2n_1} + u_1 + u_{2n_1} \). By Lemma 2.1(b), both \( G'_1 \) and \( G'_2 \) are traceable. By Claim 3, we have that \( n_1 + n_2 \geq (n_2 + 1) + (n_1 + 1) - 2(e(u_1x_{2n_1}) + e(x_1u_{2n_1})) \geq n_1 + n_2 \). This implies that \( d_G(u_1) + d_G(u_{2n_1}) = n + 1 = d_G(x_1) + d_G(x_{2n_2}) \). By (1) and (2), we see that \( N_P(u_1) = V_2 = N_P(x_1) \) and therefore both \( D[V(G_1)] \) and \( D[V(G_2)] \) are hamiltonian, a contradiction. \( \square \)

As \( D[V(G_2)] \) is not hamiltonian, \( \{(x_1, x_{j+1}), (x_{2n_2}, x_j)\} \not\in E \) and \( \{(x_{j+1}, x_1), (x_{2n_2}, x_1)\} \not\in E \) for all \( j \in \{1, 3, \ldots, 2n_2 - 1\} \). This implies that \( d_P(x_1, G_2) + d_P(x_{2n_2}, G_2) \leq n_2 \) and \( d_P(x_1, G_1) + d_P(x_{2n_2}, G_1) \leq n_2 \). Therefore, we have
\[ 4n_1 \geq d_P(x_1, G_1) + d_P(x_{2n_2}, G_1) \geq 3n + 1 - 2n_2 = 3n_1 + n_2 + 1. \] (8)

This implies
\[ n_1 \geq n_2 + 1 \quad \text{and} \quad d_G(x_1, G_1) + d_G(x_{2n_2}, G_1) \geq n_1 + n_2 + 1 \geq n_1 + 3. \] (9)
As \( G_1 \) is hamiltonian, we may assume that \( u_i u_{2n_i} \in E_1 \). In the following, the subscripts of the \( u_i \)'s will be reduced modulo \( 2n_1 \).

**Claim 5.** For each \( i \in \{1, 2, \ldots , 2n_1 \} \), \( d_G(u_i, G_1) + d_G(u_{i+1}, G_1) \geq n_1 + 2 \).

**Proof.** Suppose that the claim fails. Then, without loss of generality, we may assume that \( d_G(u_1, G_1) + d_G(u_{2n_1}, G_1) \leq n_1 + 1 \). By Claim 1, \( d_G(u_1, G_2) + d_G(u_{2n_1}, G_2) \geq n \). By (7), (9) and Claim 2, we see that either \( u_1 x_{2n_2} \notin E \) or \( x_{2n_2} u_1 \notin E \). By Lemma 2.1(b), both \( G_1' = G_1 - u_1 - u_{2n_1} + x_1 + x_{2n_2} \) and \( G_2' = G_2 - x_1 - x_{2n_2} + u_1 + u_{2n_1} \) are traceable. By Claim 3, we have

\[
(n_1 + 1) + n_2 \geq n_2 + (n_1 + 3) - 2(e(u_1 x_{2n_2}) + e(x_1 u_{2n_1})) + 2 \geq n_1 + n_2 + 3,
\]

a contradiction. \( \square \)

**Claim 6.** For every \( i \in \{1, 2, \ldots , 2n_1 \} \), \( D[V(G_i - u_i - u_{i+1} + x_1 + x_{2n_2})] \) is hamiltonian.

**Proof.** Without loss of generality, we show that \( D[V(G_i - u_1 - u_{2n_1} + x_1 + x_{2n_2})] \) is hamiltonian. By (9), \( d_G(x_i, G_i - u_1 - u_{2n_1}) + d_G(x_{2n_2}, G_i - u_1 - u_{2n_1}) \geq n_1 + 3 - 2 = n_1 + 1 \). By Lemma 2.1(a), we see that if either \( G_i - u_1 - u_{2n_1} \) is hamiltonian or \( u_2 \) is joined to \( u_{2n_1} \) in \( D \), then \( D[V(G_i - u_1 - u_{2n_1} + x_1 + x_{2n_2})] \) is hamiltonian. Therefore, we may assume that \( u_2 \) is not joined to \( u_{2n_1} \) in \( D \) and \( G_i - u_1 - u_{2n_1} \) is not hamiltonian. Then \( d_G(u_2, G_i - u_1 - u_{2n_1}) + d_G(u_{2n_1}, G_i - u_1 - u_{2n_1}) \leq n_1 - 1 \) by Lemma 2.2(a). Thus, we have that \( d_G(u_2, G_i) + d_G(u_{2n_1}, G_i) \leq n_1 + 1 \). By Claim 5, \( d_G(u_1, G_i) + d_G(u_{2n_1}, G_i) \geq n_1 + 2 \). This implies, by applying Lemma 2.1(a) to the path \( u_2 u_3 \ldots u_{2n_1} \), that \( G_i \) has a hamilton path from \( u_2 \) to \( u_{2n_1} \). By (1), we see that \( d_G(u_2) + d_G(u_{2n_1}) \geq n + 3 \) as \( |N_D(u_2)| \leq n - 1 \) and \( |N_D(u_{2n_1})| \leq n - 1 \). Hence, \( d_G(u_2, G_2) + d_G(u_{2n_1}, G_2) \geq n_2 + 2 \). By Claim 3, we have that \( (n_1 + 1) + n_2 \geq (n_2 + 2) + (n_1 + 3) - 2(e(x_1 u_2) + e(x_{2n_2} u_{2n_1} - 1)) \geq n_1 + n_2 + 1 \). This implies that \( \{x_i u_2, x_{2n_2} u_{2n_1} - 1\} \subseteq E \). By Claim 2, we should have \( e(x_i u_2) + e(x_{2n_2} u_{2n_1} - 1) \leq 1 \), a contradiction. This proves the claim. \( \square \)

We are now in a position to prove the theorem. If \( F \) is a subgraph of \( G \) or directed subgraph of \( D \), we define \( d^*_F(u, F) \) and \( d_F(u, F) \) to be \( |N^+_F(u) \cap V(F)| \) and \( |N_F(u) \cap V(F)| \), respectively, for any \( u \in V(D) \). Let \( H = G_2 - x_1 - x_{2n_2} \). By Claim 6 and the assumption that the theorem fails for \( D \), we have that \( D[V(H + u_i + u_{i+1})] \) is not hamiltonian for any \( i \in \{1, 2, \ldots , 2n_1\} \). Let \( x \) and \( y \) be any two endvertices of a hamilton path of \( H \) with \( x \in A_2 \). Then for any \( i \in \{1, 3, \ldots , 2n_1 - 1\} \), we have that \( \{x, u_{i+1}\}, \{u_i, y\} \notin A \) and \( \{u_{i+1}, x\}, \{y, u_i\} \notin A \). This implies:

\[
d_G(x, G_1) + d_G(y, G_1) \leq n_1, \quad \tag{12}
d^*_F(x, G_1) + d^*_F(y, G_1) \leq n_1, \quad \tag{13}
d_F(x, G_1) + d_F(y, G_1) \leq n_1. \quad \tag{14}
\]
By (13) and (14), we have that
\[
4n_2 \geq d_D(x, G_2) + d_D(y, G_2)
\]
\[
\geq d_D(x) + d_D(y) - 2n_1 \geq 3n + 1 - 2n_1 = 3n_2 + n_1 + 1.
\]
This implies that \(n_2 \geq n_1 + 1\), contradicting (9). This proves the theorem. \(\square\)

To see the sharpness of the condition of the theorem, we construct a direct bipartite graph \(B_n\) of order \(2n\) for every integer \(n \geq 2\). We use \(K^*_{a,b}\) to denote the complete directed bipartite graph \((V_1, V_2; A)\) with \(|V_1| = a\) and \(|V_2| = b\) such that both \((x, y)\) and \((y, x)\) belong to \(A\) for all \(x \in V_1\) and \(y \in V_2\). Let \(D_1 = (X_1, Y_1; A_1)\) and \(D_2 = (X_2, Y_2; A_2)\) be two vertex-disjoint directed bipartite graphs such that \(D_1\) is isomorphic to \(K^*_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}\) and \(D_2\) is isomorphic to \(K^*_{\lceil n/2 \rceil, \lceil n/2 \rceil}\). Then \(B_n\) consists of \(D_1\) and \(D_2\) and all arcs \((u, v)\) and \((x, y)\) for \(u \in X_1\), \(v \in Y_2\), \(x \in Y_1\) and \(y \in X_2\). It is easy to see that \(d_{B_n}(a) + d_{B_n}(b) \geq 3n - \varepsilon_n\) for all \(a \in X_1 \cup X_2\) and \(b \in Y_1 \cup Y_2\) with equality if \(a \in X_1\) and \(b \in Y_1\), where \(\varepsilon_n = 0\) if \(n\) is even and \(\varepsilon_n = 1\) otherwise. But \(B_n\) does not contain two vertex-disjoint directed cycles of lengths \(2n_1\) and \(2n_2\), respectively, for any positive integer partition \(n = n_1 + n_2\) with \(\{n_1, n_2\} \neq \lfloor n/2 \rfloor, \lfloor n/2 \rfloor\).

We conjecture the following.

**Conjecture.** Let \(D = (V_1, V_2; A)\) be a directed bipartite graph with \(|V_1| = |V_2| = n \geq 2\). Suppose that \(n\) is odd and \(d_D(x) + d_D(y) \geq 3n\) for all \(x \in V_1\) and \(y \in V_2\). Then \(D\) contains two vertex-disjoint directed cycles of lengths \(2n_1\) and \(2n_2\), respectively, for any positive integer partition \(n = n_1 + n_2\).

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**References**