Heteroclinic Cycles in Dynamical Systems with Broken Spherical Symmetry

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0. INTRODUCTION

In mathematics and physics the phrase "symmetry breaking" has two distinct meanings. The first refers to the frequently observed phenomenon that a configuration of a physical system satisfying a law (a set of equations) which is invariant under a group of transformations, may itself only be invariant under a subgroup of this group. This is referred to as spontaneous symmetry breaking. The second meaning refers to the process of explicitly adding symmetry breaking terms to the equations which describe the system. This we call induced or forced symmetry breaking.

Over the past decade spontaneous symmetry breaking has attracted a considerable amount of attention within the context of bifurcation and dynamical systems theory (see, for example, [13]). Previous work on forced symmetry breaking has mainly concentrated on the persistence of equilibrium solutions under small symmetry breaking perturbations of symmetric systems [3, 4, 2, 7, 8, 12, 22, 23]. The main aim of this paper is to show that such perturbations may also give rise to more complex dynamical behavior, in particular to heteroclinic cycles.

We first present a simple example of our constructions (for the particular choices of groups $G = SO(3)$, $H = SO(2)$, and $K = \mathbb{T}$, described below) in order to illustrate these general ideas, without too many technicalities; we return to a rigorous discussion of these types of problems in Section 2.
Take the ambient manifold $X = \mathbb{R}^3$ and consider the rotation group $G = SO(3)$ acting on $\mathbb{R}^3$ as usual. The orbit $M = SO(3)x_0$ of (say) $x_0 = (0, 0, 1)$ is the unit sphere $M = S^2 \subset \mathbb{R}^3$. Then the rotation subgroup $H = SO(2)$ leaves $x_0$ fixed, and we can identify the coset space $G/H = SO(3)/SO(2)$ with the sphere $M = S^2$.

Consider the spherically symmetric flow $\Phi$ in $\mathbb{R}^3$, invariant under the action of $SO(3)$, with trajectories along the rays through the origin and flowing towards $S^2$ both from the inside and the outside. Of course, $x_0$, and each point of $S^2$, is an equilibrium point of $\Phi$.

Now take some perturbation $\Phi_\varepsilon$ of $\Phi$ in $\mathbb{R}^3$ and then the invariant sphere $M - S^2$ will be perturbed (by normal hyperbolicity) to a $\Phi_\varepsilon$-invariant surface $M_\varepsilon$, which is known to be diffeomorphic to $M$. If $\Phi_\varepsilon$ were still required to be spherically symmetric, then $M_\varepsilon$ would also be a rigid sphere filled by $\Phi_\varepsilon$-equilibrium points. But suppose instead that we “break the symmetry” and require that $\Phi_\varepsilon$ and $M_\varepsilon$ be invariant merely under a finite group $K$, say $K = \mathbb{T} \subset SO(3)$, the group of rotational symmetries of the tetrahedron.

In this case there must be at least one equilibrium point, say $x_\varepsilon$, for $\Phi_\varepsilon$ on the invariant surface $M_\varepsilon$. If $x_\varepsilon$ is a “vertex of the tetrahedron,” then it is fixed under a 3-cyclic group of rotations. Such an $x_\varepsilon$ lies on a $\mathbb{T}$-orbit of four equilibrium points, and we can investigate whether these four points are joined by heteroclinic orbits of $\Phi_\varepsilon$ on $M_\varepsilon$.

But this process is simplified by Proposition 1.2, where the flow $\Phi_\varepsilon$ on $M_\varepsilon$ is shown to be $\mathbb{T}$-equivariantly diffeomorphic to some other perturbation of the flow $\Phi$ restricted to $M = S^2$. Thus we are led to the study of flows on the sphere $S^2$ (with small vector fields—a restriction removed by time re-scaling), with invariance under the tetrahedral group $\mathbb{T}$.

More generally, let $X$ be a smooth manifold with a smooth action of a compact Lie group $G$ and let $\Phi$ be a smooth $G$-equivariant flow on $X$. Suppose that $x_0$ is an equilibrium point of $\Phi$ which is fixed only by the subgroup $H$ of $G$ (thus symmetry is broken spontaneously from $G$ to $H$). Then the orbit $M = Gx_0$ is a manifold of equilibrium solutions diffeomorphic to the coset space $G/H$. If $M$ is normally hyperbolic with respect to $\Phi$ then it persists to give a diffeomorphic invariant manifold $M_\varepsilon$ for any small perturbation $\Phi_\varepsilon$ of $\Phi$. The flow on $M_\varepsilon$ will typically be nontrivial. If $\Phi_\varepsilon$ is $K$-equivariant for some subgroup $K$ of $G$ then $M_\varepsilon$ is $K$-invariant and $K$-equivariantly diffeomorphic to $M$, and hence $G/H$ (Proposition 1.2).

Moreover the flow of $\Phi_\varepsilon$ on $M_\varepsilon$ is $K$-equivariant and so can be identified with a $K$-equivariant flow on $G/H$. Conversely, up to a rescaling of time, any $K$-equivariant flow on $G/H$ can be obtained in this way from some $K$-equivariant perturbation of $\Phi$ (Proposition 1.3 and the following remark).

Essentially the same results apply, more generally, to relative equilibria of the flow $\Phi$. These are orbits of the group action which are invariant
under the flow. The only difference is that the flows on $G/H$ which can be realised by perturbations of $\Phi$ must themselves be perturbations of the flow on $G/H$ induced by $\Phi$. This flow is necessarily quasiperiodic [10, 11, 16].

The results on the persistence of relative equilibria are special cases of an equivariant version of the theorem of [14] on the persistence of normally hyperbolic invariant submanifolds of a flow (Proposition 1.1). The equivariant result is an easy consequence of the nonequivariant. Following Hirsch, Pugh and Shub our results are restricted to flows on finite dimensional manifolds, though many of the more interesting potential applications require infinite dimensional settings. We give such an extension, and consider some applications, in [18].

Having reduced the study of symmetry breaking perturbations of relative equilibria to that of $K$-equivariant flows on $G/H$, we can study the latter by methods which are primarily group theoretical. The key tool is the partially ordered set of isotropy subgroups of the action of $K$ on $G/H$. The set of points with non principal isotropy subgroups forms a flow invariant subcomplex of $G/H$. In the examples considered in Section 2 this is one dimensional and so consists of equilibrium points with connecting orbits. Heteroclinic cycles thus occur very naturally, and structurally stably within the context of $K$-equivariant flows.

The study of the dynamic stability of these invariant complexes is still at an early stage. In the first example we are able to apply a result of dos Reis [9] to obtain simple criteria, inequalities between eigenvalues of the linearised flow at equilibrium points, for asymptotic stability. In the second example we use an extension of the dos Reis result, pointed out by Melbourne [20], to similar effect. More interestingly, in the third example we apply the main result of Melbourne [20] to show that some heteroclinic cycles in an invariant complex, although not asymptotically stable, can be "essentially asymptotically stable," in the sense that they attract almost all nearby points.

Our examples are for dynamical systems with broken spherical symmetry. The initial motivation for the work came from potential applications to convection in slowly rotating spherical fluid shells such as the Earth's mantle (see, for example, [5]) and to the buckling of (almost) spherical shells subjected to (slightly) anisotropic forces. Of course these applications require the infinite dimensional extensions referred to above. An extra simplification for this paper is our restriction to the group $SO(3)$ of orientation preserving orthogonal transformations, rather than the full orthogonal group $O(3)$ which is relevant to the applications. These restrictions are relaxed in [18].
1. Perturbations of Relative Equilibria

In this section we study the persistence of flow invariant group orbits under perturbation. Then we look at some general features of flows on group orbits.

1.1. Perturbing Symmetric Invariant Manifolds

Let \( X \) be a smooth finite dimensional manifold with a smooth action of a compact Lie group \( G \):

\[
G \times X \rightarrow X
\]

\[
(g, x) \mapsto gx.
\]

Let \( \Phi: X \times \mathbb{R} \rightarrow X \) be a flow on \( X \) which commutes with the action of \( G \), i.e.,

\[
\Phi(gx, t) = g\Phi(x, t) \quad \text{for all } t \in \mathbb{R}.
\]

Let \( M \) be a \( G \)-invariant submanifold of \( X \) which is also invariant under the flow. We suppose that \( M \) is normally hyperbolic. We adopt the convention of [14] that "normally hyperbolic" stands for "immediately, relatively 1-normally hyperbolic." Let us recall this terminology. Let \( TX \) be the tangent bundle of \( X \), \( TX|_M \) be its restriction to \( M \) and \( TM \) be the tangent bundle of \( M \). Assume we have a continuous splitting

\[
TX|_M = TM \oplus W^u \oplus W^s \tag{1.1.1}
\]

which is invariant under \( D\Phi(\cdot, t) \) for all \( t \in \mathbb{R} \). We write \( \Phi|_M \) for the restriction of \( \Phi \) to \( M \) and \( (D\Phi)(x, t) \) for the derivative of the diffeomorphism \( \Phi(\cdot, t) \) at \( x \). In order to define normal hyperbolicity we need the minimal norm of a linear operator \( A \) on some Banach space \( E \), defined by

\[
m(A) = \inf\{ \|Ax\| : \|x\| = 1 \}.
\]

For \( x \in M \) let \( (T\Phi)(x, t) \) denote the restriction of \( (D\Phi)(x, t) \) to \( TM \) and similarly \( (U\Phi)(x, t) \) be the restriction of \( (D\Phi)(x, t) \) to \( W^u(W^s) \). The manifold \( M \) is called immediately, relatively \( r \)-normally hyperbolic if there is a Riemannian structure on \( TX \) such that for all \( x \in M \) and for all \( 0 \leq k \leq r \) the following inequalities hold:

\[
m((U\Phi)(x, t)) > \|T\Phi(x, t)\|^k
\]

\[
\|S\Phi(x, t)\|^k < m((S\Phi)(x, t)). \tag{1.1.2}
\]

**Theorem [14, Theorem 4.1].** Assume \( X \) is a compact finite dimensional manifold and \( v: X \rightarrow TX \) is a \( C^r \)-vector field on \( X \), such that the \( C^r \)-manifold \( M \subset X \) is invariant under the flow \( \Phi \) corresponding to \( v \) and normally hyperbolic. Then there exists some \( \varepsilon > 0 \) such that for any vector field \( v' \) with \( \|v - v'\|_{C^r} < \varepsilon \) there exists a unique invariant manifold \( M' \) near \( M \) and \( C^r \)-diffeomorphic to \( M \).
Remarks. The theorem is stated for $X$ a compact finite dimensional manifold. However, it follows that it is true for any finite dimensional manifold which may be smoothly compactified, for example $\mathbb{R}^n$. Using a smooth function which vanishes in a neighbourhood of $\infty$ the vector field can be altered so that

(a) it is unchanged near $M$

(b) it is smooth on the whole compactified manifold.

Hirsch, Pugh and Shub [14] do not explicitly mention the fact that the two manifolds $M$ and $M'$ are $C'$-diffeomorphic. However, in their proof they construct the invariant manifold using a $C'$-section of the normal bundle of $M$. This defines a $C'$-diffeomorphism.

**PROPOSITION 1.1.** Assume $v: X \to TX$ is a $G$-equivariant $C'$-vector field on $X$, $r \geq 1$. Let $M \subset X$ be a compact submanifold which is invariant under the flow $\Phi$ (corresponding to $v$) and the action of $G$. Assume that $M$ is normally hyperbolic. Let $K \subset G$ be a subgroup and $w: X \to TX$ be a $K$-equivariant $C'$-vector field with $\|w - v\|_{C'} < \varepsilon$. Then, if $\varepsilon$ is sufficiently small, there exists a unique $C'$-manifold $M_w$ near $M$ which is invariant under the flow $\Phi_w$ corresponding to $w$. Moreover there exists a $C'$-diffeomorphism $M \to M_w$ which is $K$-equivariant.

**Proof.** This follows almost immediately from the main theorem of [14]. As a consequence of the uniqueness statement for a hyperbolic splitting as in (1.1.1) in [14, Proposition 1.2] the $G$-equivariance of $v$ implies that the subbundles appearing in (1.1.1) are $G$-subbundles of $TX|_M$. Moreover the uniqueness of the section constructed by [14] implies that it is $K$-equivariant. 

**Remark.** Let us point out, that in the special case where the flow on $M$ is trivial, Proposition 1.1 becomes much simpler. In this case (1.1.2) is satisfied for any $r$ and any hyperbolic splitting. Thus it suffices to assume that $M$ is invariant under the flow and under the group action and has a continuous hyperbolic splitting. Moreover it is easily seen that such a splitting exists if

for any $m \in M$ the linearization $Dv: T_mM \to T_mM$ has $0$ as an eigenvalue of multiplicity $\dim M$ and no other eigenvalues on the imaginary axis. 

(1.1.3)

This condition is generically satisfied if we consider the manifolds of solutions of equivariant bifurcation problems near bifurcation points [10, Theorem A.20]. Let us summarize.
PROPOSITION 1.2. If \( M \subset X \) is a smooth, \( G \)-invariant manifold of equilibria for the smooth, \( G \)-equivariant vector field \( v \), such that for all \( m \in M \) condition (1.1.3) is satisfied. Then any smooth \( K \)-equivariant vector field \( w \) which is sufficiently close to \( v \) in the \( C^1 \) topology has a smooth, \( K \)-invariant and flow invariant manifold \( M_w \) close to \( M \). It is \( K \)-equivariantly and \( C^\infty \)-diffeomorphic to \( M \).

Remark. A similar statement applies to relative equilibria. In [11, Theorem 6.2] a spectral characterisation of normal hyperbolicity for relative equilibria is given, see also [16].

Remark. Proposition 1.1 is the only place where we require that \( X \) is finite dimensional. It would be nice to be able to extend the results to Hilbert spaces, because there are interesting applications in, for example, \( L^2(\Omega) \), where \( \Omega \) is a ball in \( \mathbb{R}^3 \). To obtain such an extension one could use inertial manifold theory. Unfortunately the best result in this respect [19] applies only to the standard cube in \( \mathbb{R}^3 \) because it is not known whether the principle of spatial averaging holds on balls or other domains. Instead, in [18] we present a more direct extension of the result of [14] to some infinite dimensional problems.

In the next proposition we show that all \( K \)-equivariant flows on \( M \) can be realised as restrictions of \( K \)-equivariant flows on \( X \). Moreover the extensions have certain continuity properties.

PROPOSITION 1.3. Let \( \Phi \) be a \( K \)-equivariant flow on \( X \), \( \Phi_M \) be its restriction to \( M \). For each compact neighbourhood \( W \) of \( M \) and for all \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for any flow \( \Psi_M \) on \( M \) with

\[
\| \Psi_M - \Phi_M \|_{C^1} < \delta
\]

there exists a \( K \)-equivariant flow \( \Psi \) on \( X \) such that \( \Psi|_M = \Psi_M \) and

\[
\| \Phi - \Psi \|_{C^1(W)} < \varepsilon.
\]

Proof. It is sufficient to extend the flow \( \Psi_M \) to a tubular neighbourhood of \( M \) controlling the \( C^1 \) distance to \( \Phi \). It can then be extended to the whole of \( X \) using a bump function. Let \( v: X \to TX \) be the vector field corresponding to \( \Phi \) and let \( w_M: M \to TM \) be the vector field of \( \Psi_M \). Here we regard \( TM \) as a subbundle of \( TX \) in the natural way. Consider the vector field \( w_M - v|_M \) on \( M \). For any point \( m \in M \) there exists some neighbourhood \( U \subset M \) such that the tubular neighbourhood of \( M \) is locally isomorphic to the direct product of \( U \times D \), where \( D \) is some disc in some Euclidean space. Using the product structure one can trivially extend \( w - v|_M \) to \( U \times D \). Using a partition of unity these trivial extensions can be patched together to form a vector field \( u \) defined on some tubular
neighbourhood of $M$. Averaging over $K$ makes it $K$-equivariant. Observe that the trivial extension does not alter the $C^1$ norm and averaging is continuous and therefore we can control the $C^1$ norm. Define $w = v + u$. Then $w|_M = v|_M + u|_M = w_M$. This completes the proof. }

**Remark.** In the examples below the restriction of $\Phi$ to $M$ is trivial. Suppose any flow $w_M$ on $M$ is given. Applying the proposition to $\varepsilon w$ for some small $\varepsilon$ shows that up to a rescaling of time any smooth $K$-equivariant flow on $M$ may be realised as the restriction of an appropriate $K$-equivariant flow near $\Phi$.

1.2. **Group Actions on Homogeneous Spaces**

Let $\Phi_0$ be a $G$-equivariant flow on $X$ and suppose $H \subset G$ is an isotropy subgroup of some point $x_0 \in X$. We assume that the compact manifold $M_{x_0} = \{gx_0 \mid g \in G\}$ is invariant under $\Phi_0$.

Our first observation is that $M_{x_0}$ is diffeomorphic to $G/H$, where $G/H$ denotes the homogeneous space of (left) cosets of $H$, i.e., $G/H = \{gH \mid g \in G\}$. We will write $[g]$ for $gH$. If we consider cosets with respect to another subgroup $K$ say, we write $[g]_K$. Diffeomorphisms $G/H \to M_{x_0}$ are easily constructed: let $\mu_{x_0} : G \to X$ be defined by $\mu_{x_0}(g) = gx_0$ and $\tilde{\mu}_{x_0} : G/H \to X$ by $\tilde{\mu}_{x_0}([g]) = gx_0$. Then $\tilde{\mu}_{x_0}$ is well defined and both mappings are smooth. Since $H$ is the isotropy of $x_0$ we have the obvious relation

$$\mu_{x_0}(g) = \tilde{\mu}_{x_0}([g]). \quad (1.2.1)$$

The maps $\mu_{x_0}$ and $\tilde{\mu}_{x_0}$ are $G$-equivariant,

$$\mu_{x_0}(g'g) = g'\mu_{x_0}(g)$$

and

$$\tilde{\mu}_{x_0}([g'g]) = [g'g]x_0 = g'gHx_0 = g'gx_0 = g'\tilde{\mu}_{x_0}([g]),$$

and $\tilde{\mu}_{x_0}$ is a diffeomorphism. Up to the choice of the basepoint $x_0$ it is the unique $G$-equivariant diffeomorphism $G/H \to M_{x_0}$.

Suppose $M_{x_0}$ is a normally hyperbolic invariant manifold with respect to the flow $\Phi_0$. Let $\Phi_\varepsilon$ be a $K$-equivariant perturbation of $\Phi_0$ for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ and let $M_\varepsilon$ be the manifold given by Proposition 1.1. We have

**Proposition 1.4.** The actions of $K$ on $M_\varepsilon$ and $G/H$ are equivariantly diffeomorphic: there exists a diffeomorphism

$$v_\varepsilon : G/H \to M_\varepsilon$$
such that
\[ v_\epsilon([kg]) = k v_\epsilon([g]). \]

Here the action of \( K \) on \( G/H \) comes from left multiplication and the action on \( M_\epsilon \) is the action induced by the action on \( X \).

**Proof.** Let \( \mu_\epsilon \) denote the \( K \)-equivariant diffeomorphism from \( M \) to \( M_\epsilon \) given by Proposition 1.2 and take the composition \( v_\epsilon = \mu_\epsilon \circ \bar{\mu}_{x_0} : G/H \to X \).

This proposition allows us to study the action of \( K \) on \( G/H \) instead of the action on an a priori unknown manifold \( M_\epsilon \). Along an orbit of the group action the isotropy changes. One expects that the results do not depend on the choice of the point \( x_0 \) and therefore do not depend on \( H \). This is the content of the next proposition. If two subgroups \( L \) and \( L' \) are conjugate we write \( L \simeq L' \).

**Proposition 1.5.** Let \( H, H', K, K' \) be two conjugate pairs of subgroups, i.e., \( H \simeq H' \) and \( K \simeq K' \). Choose \( h \in G \) such that \( h^{-1}Kh = K' \). There exists a diffeomorphism
\[ \mu' : G/H \to G/H' \]
such that
\[ \mu'([kg]) = \kappa_h(k) \mu'([g]). \tag{1.2.2} \]
where \( \kappa_h(k) = h^{-1}kh \).

In other words the actions of \( K \) on \( G/H \) and \( K' \) on \( G/H' \) are diffeomorphic.

**Proof.** We give the proof in two steps. In the first step we will show that we can fix \( H \) in its conjugacy class if we are ready to replace \( K \) by one of its conjugates. In the second step we consider the actions of two conjugate groups \( K \) and \( K' \) on a fixed homogeneous space and show that they are diffeomorphic.

Suppose now \( H \simeq H' \) and \( K \) acts on \( G/H \). Let \( H' = h^{-1}Hh \) for some \( h \in G \). Let
\[ \mu'' : G/H \to G/H' : [g] \mapsto [h^{-1}gh]_{H'}. \]
This mapping is well defined, since \( g^{-1}g' \in H \) implies that \( h^{-1}g^{-1}g'h \in H' \). Moreover
\[ \mu''([kg]) = h^{-1}kgH' = h^{-1}kh^{-1}ghH' = \kappa(k) \mu''([g]). \]

Here, as said before \( \kappa_h : K \to K' = h^{-1}Kh : k \mapsto h^{-1}kh \).
To prove the second step, suppose $H$ is fixed and $K \simeq K'$ are subgroups acting on $G/H$ by left multiplication. We need a diffeomorphism $\mu: G/H \to G/H$ such that

$$\mu([k'g]) = \kappa_k(k') \mu([g]),$$

where $\kappa_k$ is the conjugation $K$ to $K'$. Define

$$\mu([g]) = [k^{-1}g].$$

The equivariance is readily seen,

$$\mu([g'g]) = [k^{-1}g'g] = [k^{-1}g'k^{-1}k_g] = \kappa_k(g') \mu([g]). \square$$

In view of this proposition we may study $K'$-equivariant flows on $G/H'$ for any choice of subgroups $K'$, $H'$ in the conjugacy classes of $K$ and $H$, respectively. The next proposition is basic for the study of equivariant flows. If $X$ is a smooth $G$-manifold and $L \subset G$ a subgroup, then the fixed point subset of $L$ in $X$ is a smooth submanifold denoted by $\text{Fix}(L)$ and defined by $\text{Fix}(L) = \{x \in X | lx = x, \forall l \in L\}$. Its importance comes from the next simple proposition.

**Proposition 1.6.** Let $X$ be a smooth $G$-manifold. Suppose $\Phi$ is a $G$-equivariant flow on $X$. Then $\text{Fix}(L)$ is flow invariant for any subgroup $L \subset G$.

**Proof.** Let $x \in \text{Fix}(L)$ and $l \in L$. The $G$-equivariance gives

$$l\Phi(x, t) = \Phi(lx, t) = \Phi(x, t), \quad \forall t \in \mathbb{R}.$$ 

Therefore $\Phi(x, t) \in \text{Fix}(L)$ for all $t \in \mathbb{R}.$ $\square$

Due to this observation many features of $G$-equivariant flows can be determined by looking at the fixed point subspaces of subgroups of $G$. In the context of flows on the homogeneous space $G/H$ these fixed point sets are related to the set $N(K', H)$ which were introduced by Ihrig and Golubitsky [15]. These are defined by

$$N(K', H) = \{g \in G | K' \subset gHg^{-1}\}.$$ 

If $g \in N(K', H)$ it is clear that $g^{-1}K'g \subset H$. If such a relation is true we say $K'$ is subconjugate to $H$.

**Proposition 1.7.** Let $K$ act on $G/H$ by left multiplication and let $K'$ be a subgroup of $K$. We have

(a) $\text{Fix}(K') \neq \emptyset \iff K'$ is subconjugate to $H$;

(b) $\text{Fix}(K') = N(K', H)/H$. 

Proof. Let \( g \in G \) be given with \([g] \in \text{Fix}(K')\) for some subgroup \( K' \subset K \). For \( k \in K' \) we have the equality \([kg] = [g]\) or \( g^{-1}kg \in H\). Clearly it follows that \( g^{-1}K'g \subset H \). On the other hand if \( g \in G \) is such that \( g^{-1}K'g \subset H \) then for each \( k \in K' \) we obtain \( g^{-1}kg \in H \) or \([kg] = [g]\).

For \( g \in G \) let \( \text{Stab}([g]) \) denote the isotropy subgroup of \([g]\) (under left multiplication), i.e., \( \text{Stab}([g]) = \{k \in K | k[g] = [g]\}\). The set of isotropy subgroups forms a partial ordered set (poset) with respect to inclusion.

**Proposition 1.8.** \( \text{Stab}([g]) = K \cap gHg^{-1} \).

**Proof.** Obvious! \( \Box \)

This proposition characterizes isotropy subgroups as those subgroups which are the intersection of \( K \) with some conjugate of \( H \). If

\[
K' = K \cap gHg^{-1}
\]

then obviously

\[
H' = g^{-1}Kg \cap H
\]

is a subgroup of \( H \) which fixes the coset \([g^{-1}]_{K} \in G/K\) under the action of \( H \) on \( G/K \). Therefore we have an isomorphism between the posets of isotropy subgroups of the action of \( K \) on \( G/H \) and the action of \( H \) on \( G/K \). There is also a corresponding duality between the fixed point sets. If (1.2.3) and (1.2.4) give the relation between isotropy subgroups of \( H \) and \( K \) respectively the induced relation between the fixed sets is given by

\[
[g]_{H} \in \text{Fix}(K') \Leftrightarrow [g^{-1}]_{K} \in \text{Fix}(H').
\]

It should be clear that the topologies of the fixed sets are different in general. We shall see examples later on.

The strategy for determining the poset of isotropy subgroups is clear. The poset of closed subgroups is known (in many cases at least). The only remaining problem is to find out which of these subgroups are intersections of \( K \) with a conjugate of \( H \). We know already that the actions are diffeomorphic if we replace one or both groups by conjugate groups. This can simplify the procedure significantly.

**Remark.** Not every closed subgroup of \( K \) which is subconjugate to \( H \) is an isotropy subgroup. It is also not true that the duality gives a bijection of conjugacy classes of isotropy subgroups. This can be seen from the following examples. Let \( G = SO(3), H = SO(2) \), and \( K = D_6 \) then \( Z_3 \subset D_6 \) is subconjugate to \( SO(2) \) but any element which conjugates \( Z_3 \) into \( SO(2) \) also conjugates \( Z_6 \) into \( SO(2) \). For the second statement choose
G = SO(3), H = \mathbb{Z}_2, and K = \mathbb{O}, the octahedral group. Within K there are subgroups G-conjugate to H which are not conjugate to each other as subgroups of K.

1.3. Quotients

Let M be a smooth compact manifold with a smooth left action of a compact Lie group K. Then the quotient K\M has an induced topology, but is not in general a smooth manifold. However, K\M does have a natural smooth stratification—i.e., it can be decomposed as the union of a finite number of disjoint subsets, each of which is a smooth manifold. More precisely, if L is a subgroup of K let M^{(L)} denote the set of points in M with isotropy subgroup conjugate to L. Then M^{(L)} is a smooth K-invariant submanifold of M and the quotient space K\M^{(L)} is naturally a smooth manifold which can be identified with a subset of K\M we denote by (K\M)^{(L)}. Clearly K\M is the union of these orbit type strata.

Although K\M is not in general a smooth manifold it can be given a smooth structure by defining the smooth functions on K\M to be those real valued functions on K\M which pull back to smooth K-invariant functions on M—i.e., C^\infty(K\M) := C^\infty(M)^K (see [21]).

A smooth vector field on K\M is defined to be an \mathbb{R}-linear derivation of C^\infty(K\M), or equivalently of C^\infty(M)^K. Let V^\infty(K\M) denote the space of all smooth vector fields on K\M. A smooth vector field is said to be tangent to the orbit type stratum (K\M)^{(L)} if it maps the space of functions vanishing on (K\M)^{(L)} into itself. A smooth vector field is said to be stratum preserving if it is tangent to all the orbit type strata of K\M. We denote the space of all stratum preserving smooth vector fields on K\M by \mathcal{X}^\infty(K\M). It is not difficult to see that any smooth K-equivariant vector field on M (or, equivalently, any K-equivariant \mathbb{R}-linear derivation of C^\infty(M)) induces a smooth stratum preserving vector field on K\M. Thus there is a well defined map: \mathcal{X}^\infty(M)^K \rightarrow \mathcal{X}^\infty(K\M), where \mathcal{X}^\infty(M)^K is the space of K-equivariant vector fields on M. It is the main result of [21] that this map is surjective; i.e., every stratum preserving smooth vector field on K\M lifts to a smooth K-equivariant vector field on M. If K is finite then the map is an isomorphism, but if \dim K \geq 1 then it will not be, since passing to the quotient “forgets” the components of the vector field along group orbits.

1.4. Duality

We consider the relationship between K-equivariant vector fields on G/H and H-equivariant vector fields on G/K. In accordance with our previous notation let G/K be the quotient with respect to the right action of K on G and K\G the quotient with respect to the left action. Note that the action of H on G/K induced by the left action on G is isomorphic to the
action of $H$ on $K \backslash G$ induced by the right action of $H$ on $G$. An isomorphism is induced by the map $G \to G: g \mapsto g^{-1}$. Let $K \backslash G/H$ denote the quotient of $G/H$ by the left action of $K$ or equivalently $K \backslash G$ by the right action of $H$. Then we have the following commutative diagram of quotient maps:

\[
\begin{array}{ccc}
G & \to & K \backslash G \\
\downarrow & & \downarrow \\
K \backslash G/H & \to & G/H \\
\end{array}
\]

By our remarks above, a $K$-equivariant smooth vector field on $G/H$ induces a smooth stratum preserving vector field on $K \backslash G/H$, which in turn can be lifted by Schwarz's theorem to a smooth $H$-equivariant vector field on $K \backslash G$ (or $G/K$). And of course, the same holds if $K$ and $H$ are interchanged. If $H$ and $K$ are both finite this defines an isomorphism between the space of $K$-equivariant vector fields on $G/H$ and that of $H$-equivariant vector fields on $K \backslash G$. This is not true in general. However, the dynamics of corresponding vector fields will be similar, since they induce the same flow on $K \backslash G/H$. In particular relative equilibria will match up bijectively. An example will be given in Section 2 below.

2. HETEROCLINIC CYCLES

We now consider some examples of the dynamics that can occur in perturbations of dynamical systems with $SO(3)$ symmetry, near $SO(3)$ orbits of equilibrium points of the unperturbed system. No attempt is made to give a systematic survey of all the possibilities. Instead three closely related examples are presented to illustrate the general theory described in the first section, and to show how one particular type of dynamics, heteroclinic cycles, arises naturally in this context.

2.1. Subgroups of $SO(3)$

We begin by recalling some basic facts about the subgroups of $SO(3)$ from [6, 15, 13, 17]. The closed subgroups of $SO(3)$ are well known. We distinguish two types, planar subgroups and exceptional subgroups. The planar subgroups are all conjugate to one of $O(2)$, $SO(2)$, $D_n$, and $Z_n$. The group $SO(2)$ consists of all rotations about a given axis, while $O(2)$ contains $SO(2)$ together with all rotations by $\pi$ about axes perpendicular
to the given one. The group $\mathbb{Z}_n$ is the $n$-element subgroup of $SO(2)$ and $D_n$ is the $2n$-element subgroup of $O(2)$ containing $\mathbb{Z}_n$ together with rotations by $\pi$ about $n$ axes perpendicular to the given one with angles $2\pi/n$ between them.

The exceptional subgroups are $T$, $O$, and $I$, the groups of rigid motions of a tetrahedron, an octahedron and an icosahedron, respectively. These three groups have 12, 24, and 60 elements, respectively.

If $K$ is a subgroup of $H$ we write $K \subset H$. If $K$ is subconjugate to $H$, i.e., $K$ is conjugate in $SO(3)$ to a subgroup of $H$, we write $K \lhd H$. The subconjugacy relations between planar subgroups are given by

(a) $\mathbb{Z}_n < D_n < O(2)$,
(b) $\mathbb{Z}_n < \mathbb{Z}_m$ and $D_n < D_m$ if $n|m$,
(c) $\mathbb{Z}_2 < D_n$,
(d) $\mathbb{Z}_n < SO(2) < O(2)$.

The subconjugacy relations involving exceptional groups are summarised in Fig. 1.

In Table I we list $N(K, H)$, $N(K, H)/H$, and $N_H(K)$ for every pair of subgroups $K$, $H$ with $K \subset H$. Recall, from Proposition 1.5, that up to diffeomorphism $N(K, H)$ and $N(K, H)/H$ depend only on the conjugacy classes of $K$ and $H$ and, from Proposition 1.7, that $N(K, H) = \emptyset$ if $K$ is not subconjugate to $H$, so these sets are effectively given for all pairs of subgroups of $SO(3)$. Note that in Table I an integer in the $N(K, H)/H$ column indicates a set of points with that number of elements. The union

![Fig. 1. Subconjugacy relations involving the exceptional subgroups of $SO(3)$.](image-url)
symbols in the same column (though not in others) stand for disjoint unions.

The symbol $\bigcup N(D_2) \emptyset$ stands for the union of the sets $N(D_2) \emptyset$ taken over all four subgroups in $\emptyset$ which are conjugate to $D_2$.

2.2. **Example 1** ($H = O(2); K = \mathbb{T}$)

In this subsection we prove the following result.

**TABLE I**

<table>
<thead>
<tr>
<th>$K$</th>
<th>$H$</th>
<th>Condition</th>
<th>$N(K, H)$</th>
<th>$N(K, H)/H$</th>
<th>$N_H(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}_m$</td>
<td>$\mathbb{Z}_n$</td>
<td>$m \mid n$</td>
<td>$O(2)$</td>
<td>$S^1 \cup S^1$</td>
<td>$\mathbb{Z}_n$</td>
</tr>
<tr>
<td>$\mathbb{Z}_m$</td>
<td>$D_n$</td>
<td>$m \mid n \neq 2$</td>
<td>$O(2)$</td>
<td>$S^1$</td>
<td>$D_n$</td>
</tr>
<tr>
<td>$\mathbb{Z}_2$</td>
<td>$D_n$</td>
<td>$2 \mid n$</td>
<td>$O(2) D_{2\ell}$</td>
<td>$S^1$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>$\mathbb{Z}_2$</td>
<td>$O(2)$</td>
<td>$m &gt; 2$</td>
<td>$O(2)$</td>
<td>$S^1 \cup S^1 \cup S^1$</td>
<td>$D_4$ or $D_2$</td>
</tr>
<tr>
<td>$\mathbb{Z}_m$</td>
<td>$SO(2)$</td>
<td>$O(2)$</td>
<td>$2$</td>
<td>$SO(2)$</td>
<td></td>
</tr>
<tr>
<td>$\mathbb{Z}_2$</td>
<td>$O(2)$</td>
<td>$m &gt; 2$</td>
<td>$O(2)$</td>
<td>$1$</td>
<td>$O(2)$</td>
</tr>
<tr>
<td>$\mathbb{Z}_2$</td>
<td>$O(2)$</td>
<td>$2 \mid n$</td>
<td>$O(2) O D_{2\ell}$</td>
<td>$S^1 \cup S^1 \cup S^1$</td>
<td>$D_4$ or $O(2)$</td>
</tr>
<tr>
<td>$\mathbb{Z}_m$</td>
<td>$O(2)$</td>
<td>$m &gt; 2$</td>
<td>$O(2)$</td>
<td>$S^1 \cup S^1$</td>
<td>$\mathbb{Z}_3$</td>
</tr>
<tr>
<td>$\mathbb{Z}_m$</td>
<td>$D_n$</td>
<td>$2 \mid n, n &gt; 2$</td>
<td>$O(2)$</td>
<td>$3$</td>
<td>$D_3$</td>
</tr>
<tr>
<td>$\mathbb{Z}_m$</td>
<td>$O(2)$</td>
<td>$m &gt; 2$</td>
<td>$O(2)$</td>
<td>$1$</td>
<td>$D_{2\ell}$</td>
</tr>
<tr>
<td>$\mathbb{Z}_m$</td>
<td>$O(2)$</td>
<td>$2 \mid n$</td>
<td>$O(2)$</td>
<td>$3$</td>
<td>$D_4$</td>
</tr>
<tr>
<td>$\mathbb{Z}_m$</td>
<td>$\mathbb{T}$</td>
<td>$O(2)$</td>
<td>$S^1 \cup S^1$</td>
<td>$\mathbb{T}$</td>
<td></td>
</tr>
<tr>
<td>$\mathbb{Z}_m$</td>
<td>$\emptyset$</td>
<td>$O(2)$</td>
<td>$2$</td>
<td>$\mathbb{Z}_3$</td>
<td></td>
</tr>
<tr>
<td>$\mathbb{Z}_m$</td>
<td>$\emptyset$</td>
<td>$O(2)$</td>
<td>$1$</td>
<td>$\mathbb{O}$</td>
<td></td>
</tr>
<tr>
<td>$\mathbb{Z}_m$</td>
<td>$O(2)$</td>
<td>$2 \mid n$</td>
<td>$O(2)$</td>
<td>$S^1$</td>
<td>$D_2$</td>
</tr>
<tr>
<td>$\mathbb{Z}_m$</td>
<td>$O(2)$</td>
<td>$m &gt; 2$</td>
<td>$O(2)$</td>
<td>$S^1$</td>
<td>$D_3$</td>
</tr>
<tr>
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<td>$2 \mid n$</td>
<td>$O(2)$</td>
<td>$S^1$</td>
<td>$D_5$</td>
</tr>
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<td>$O(2)$</td>
<td>$m &gt; 2$</td>
<td>$O(2)$</td>
<td>$S^1$</td>
<td>$D_4$</td>
</tr>
<tr>
<td>$\mathbb{Z}_m$</td>
<td>$O(2)$</td>
<td>$2 \mid n$</td>
<td>$O(2)$</td>
<td>$S^1$</td>
<td>$D_5$</td>
</tr>
<tr>
<td>$\mathbb{Z}_m$</td>
<td>$O(2)$</td>
<td>$m &gt; 2$</td>
<td>$O(2)$</td>
<td>$1$</td>
<td>$O(2)$</td>
</tr>
<tr>
<td>$\mathbb{Z}_m$</td>
<td>$SO(3)$</td>
<td>$SO(3)$</td>
<td>$1$</td>
<td>$O(2)$</td>
<td></td>
</tr>
<tr>
<td>$\mathbb{Z}_m$</td>
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<td>$SO(3)$</td>
<td>$1$</td>
<td>$O(2)$</td>
<td></td>
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<td>$SO(3)$</td>
<td>$1$</td>
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<td></td>
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<tr>
<td>$\mathbb{Z}_m$</td>
<td>$SO(3)$</td>
<td>$SO(3)$</td>
<td>$1$</td>
<td>$O(2)$</td>
<td></td>
</tr>
<tr>
<td>$\mathbb{Z}_m$</td>
<td>$SO(3)$</td>
<td>$SO(3)$</td>
<td>$1$</td>
<td>$O(2)$</td>
<td></td>
</tr>
</tbody>
</table>
Proposition 2.1. Let \( v \) be an \( SO(3) \) equivariant vector field on \( X \) and \( M \) an \( SO(3) \) orbit of equilibrium points with isotropy subgroups conjugate to \( O(2) \) and satisfying condition (1.1.3).

1. If \( w \) is any \( T \)-equivariant vector field on \( X \) which is sufficiently close to \( v \) in the \( C^1 \) topology and \( M_w \) is the invariant submanifold for \( w \) near \( M \) (as in Proposition 1.2), then the flow on \( M_w \) has equilibrium points with isotropy subgroups conjugate to \( Z_3 \) and \( D_2 \), and either equilibrium points with isotropy subgroups conjugate to \( Z_2 \), or a system of heteroclinic cycles connecting the equilibrium points with \( D_2 \) symmetry.

2. There exist \( T \)-equivariant vector fields, \( w \), on \( X \) which are arbitrarily close to \( v \) in the \( C^1 \) topology such that the flows on the perturbed submanifolds \( M_w \) have any one of the following possibilities:
   
   (a) stable equilibrium points with \( Z_3 \) symmetry,
   
   (b) stable equilibrium points with \( D_2 \) symmetry and equilibrium points with \( Z_2 \) symmetry,
   
   (a) stable heteroclinic cycles.

Proof. Let \( M \) be an \( SO(3) \) orbit with isotropy subgroups conjugate to \( H = O(2) \). Then \( M \) is isomorphic to \( SO(3)/O(2) \), which, in turn, is isomorphic to \( \mathbb{P}^2 \), the two dimensional real projective space. This is easily seen by considering first the standard action of \( SO(3) \) on the unit sphere \( S^2 \) in \( \mathbb{R}^3 \) and then the induced action on \( \mathbb{P}^2 \), obtained by identifying antipodal points in \( S^2 \). The action on \( \mathbb{P}^2 \) is transitive, with isotropy subgroups conjugate to \( O(2) \). Thus \( \mathbb{P}^2 \) is isomorphic to \( SO(3)/O(2) \). Moreover the action of \( SO(3) \) on \( M \) and the action on \( SO(3)/O(2) \) induced by the left action on \( SO(3) \) are isomorphic to the action on \( \mathbb{P}^2 \) induced from the standard action on \( S^2 \).

Suppose \( M \) is an orbit of equilibrium points for an \( SO(3) \)-equivariant vector field which is perturbed to a \( T \)-equivariant vector field, \( w \), which is \( C^1 \) near to \( v \). If \( M \) satisfies (1.1.3) then, by Proposition 1.1, there exists a flow invariant, \( T \)-invariant manifold, \( M_w \), for \( w \) near \( M \). This manifold is \( T \)-equivariantly diffeomorphic to \( M \), and hence to \( \mathbb{P}^2 \) with the restriction to \( T \) of the standard \( SO(3) \) action on \( \mathbb{P}^2 \). The poset of conjugacy classes of subgroups of \( T \) is given in Fig. 2.

The diffeomorphism type of the fixed point set in \( \mathbb{P}^2 \) for each subgroup of \( T \) can be read off from Table I, or can be seen more directly from the geometric description of the \( T \) action:

\[
\begin{align*}
\text{Fix}(T) & \cong N(T, O(2))/O(2) = \emptyset \\
\text{Fix}(Z_3) & \cong N(Z_3, O(2))/O(2) = 1 \text{ point} \\
\text{Fix}(D_2) & \cong N(D_2, O(2))/O(2) = 3 \text{ points} \\
\text{Fix}(Z_2) & \cong N(Z_2, O(2))/O(2) = 1 \text{ circle } \cup 1 \text{ point}.
\end{align*}
\]
There are four subgroups of $\mathbb{T}$ conjugate to $\mathbb{Z}_3$, so the points in $\mathbb{P}^2$ with isotropy subgroups conjugate to $\mathbb{Z}_3$ form a single $\mathbb{T}$ orbit of four points. The subgroup $D_2$ is unique in its conjugacy class (i.e., it is normal in $\mathbb{T}$) so its fixed point set forms a single $\mathbb{T}$ orbit of three points. There are three subgroups of $\mathbb{T}$ conjugate to $\mathbb{Z}_2$ and the union of their fixed point sets is a set of three circles. For each pair of $\mathbb{Z}_2$s there is a pair of circles which intersect in a single point which is also fixed by the third $\mathbb{Z}_2$ and so has isotropy subgroup $D_2$. These fixed point sets are illustrated in Fig. 3. Figure 4 shows the quotient space $\mathbb{P}^2/\mathbb{T} \cong \mathbb{T}/SO(3)/O(2)$ with its orbit type stratification.

It follows that all $\mathbb{T}$-equivariant flows on $SO(3)/O(2)$ must have four equilibrium points with $\mathbb{Z}_3$ symmetry and three equilibrium points with $D_2$ symmetry. There must also be three invariant circles with $\mathbb{Z}_2$ symmetry, two passing through each $D_2$ equilibrium point. The action of $\mathbb{T}$ on these
circles maps any segment of such a circle between two $D_2$ equilibrium points to any other. Thus the flow on the circles is completely determined by the flow on any one segment. It is easily seen that if these segments consist of single trajectories; i.e., there are no equilibrium points on the circles other than those with $D_2$ symmetry, then they form heteroclinic cycles connecting the $D_2$ equilibria.

A $\mathbb{T}$-equivariant flow on $\mathbb{P}^2$ with asymptotically stable heteroclinic cycles and unstable $Z_3$ equilibria is shown in Fig. 3. The time reversal of this flow has stable $Z_3$ equilibria and unstable heteroclinic cycles. Flows with stable $D_2$ equilibria and additional equilibria with $Z_2$ isotropy are equally easy to construct. These can be "lifted" back to $X$ using Proposition 1.3, to complete the proof of the proposition.

When they exist, the stability of the heteroclinic cycles is determined (generically) by the eigenvalues of the linearizations of the flow at the $D_2$ equilibria. If the heteroclinic cycles exist then these equilibria are saddle points, and since the equilibria form a single $\mathbb{T}$ orbit the linearizations must all be the same. If the eigenvalues are $-s$ and $u$ with $s, u > 0$ then by a result of dos Reis [9] the cycles are asymptotically stable if $s > u$ and unstable if $s < u$.

The flow shown in Fig. 3, together with its time reversal, are in some sense the simplest $\mathbb{T}$-equivariant flows on $\mathbb{P}^2$. By the converse of the "principle of the fragility of all good things" [1] one might expect these to occur for "most" perturbations of the original flow.

2.3. Example 2 ($H = \mathbb{T}; K = O(2)$)

For our second example we consider the "dual" of the first example. That is we consider $O(2)$-equivariant flows on the perturbation of an
SO(3) orbit with isotropy subgroups conjugate to T. By the remarks following Proposition 1.8 we know that there is a bijection between the isotropy subgroups of the T action on SO(3)/O(2) and the O(2) action on SO(3)/T. Moreover, by Section 1.4 the quotient space T \SO(3)/O(2) and O(2) \SO(3)/T are isomorphic and since there is a close correspondence between the K-equivariant flows on SO(3)/H and the stratum preserving flows on K \SO(3)/H, it follows that there is a close correspondence between O(2)-equivariant flows on SO(3)/T and T-equivariant flows on SO(3)/O(2).

**Proposition 2.2.** Let v be an SO(3) equivariant vector field on X with an SO(3) orbit of equilibrium points, M, with isotropy subgroups conjugate to U and satisfying condition (1.1.3).

1. If w is any O(2)-equivariant vector field on X which is sufficiently close to v in the C¹ topology and Mₜ is the invariant submanifold for w near M (as in Proposition 1.2), then the flow on Mₜ has a circle of equilibrium points with isotropy subgroups conjugate to D₂, two invariant circles of points (generically periodic orbits) with isotropy groups conjugate to Z₃, and either equilibrium points with isotropy subgroups conjugate to Z₂, or a circle of heteroclinic cycles connecting the equilibrium points with D₂ symmetry.

2. There exist O(2)-equivariant vector fields, w, on X which are arbitrarily close to v in the C¹ topology such that the flows on the perturbed submanifolds Mₜ have any one of the following possibilities:

   (a) stable periodic orbits with Z₃ symmetry,
   (b) stable equilibrium points with D₂ symmetry and equilibrium points with Z₂ symmetry,
   (c) a stable union of heteroclinic cycles.

Note that the Z₃ periodic orbits of this flow are the relative equilibria "dual" to the Z₃ equilibrium points of example 1. The D₂ equilibria in the two examples also correspond to each other.

**Proof.** By the duality with Example 1 we know that the isotropy subgroups of the O(2) action on SO(3)/T are isomorphic to Z₃, D₂, Z₂, and 1. For Z₃, D₂, and 1 the conjugacy classes in O(2) are uniquely determined. However, there are two conjugacy classes of subgroups isomorphic to Z₂. One consists of the two element subgroup of SO(2), a normal subgroup of O(2) we denote by Z₂. The other consists of an infinite number of subgroups which do not lie in SO(2). We shall continue to use the label Z₂ for this class. From Table 1 we obtain
\[
\text{Fix}(Z_3) \cong N(Z_3, \mathbb{T})/\mathbb{T} = \text{disjoint union of 2 circles}
\]
\[
\text{Fix}(D_2) \cong N(D_2, \mathbb{T})/\mathbb{T} = 2 \text{ points}
\]
\[
\text{Fix}(Z_2^\eta) \cong N(Z_2^\eta, \mathbb{T})/\mathbb{T} = 1 \text{ circle}
\]
\[
\text{Fix}(Z_2) \cong N(Z_2, \mathbb{T})/\mathbb{T} = 1 \text{ circle.}
\]

The subgroup \(Z_3\) is normal in \(O(2)\) and so its fixed point set is invariant under the \(O(2)\) action. The action of \(SO(2)\) is by translation along the circles, while the other elements of \(O(2)\) map the circles to each other. It follows that in any \(O(2)\) invariant flow on \(M\) these circles will form either periodic orbits or, exceptionally, circles of equilibrium points. The normalizer of \(D_2\) in \(O(2)\) is \(D_4\) and so for any \(D_2\) the two points in \(\text{Fix}(D_2)\) lie in the same \(O(2)\) orbit. Since this orbit is isomorphic to \(O(2)/D_2\) it follows that the set of points in \(M\) with isotropy subgroups conjugate to \(D_2\) forms a single circle. In an \(O(2)\)-equivariant flow on \(M\) this will always be a circle of equilibrium points. The subgroup \(Z_2^\eta\) is contained in all the subgroups isomorphic to \(D_2\) and so \(\text{Fix}(Z_2^\eta)\) will contain this circle. Since \(\text{Fix}(Z_2^\eta)\) is itself a circle the two sets must be identical and hence \(Z_2^\eta\) is not an isotropy subgroup.

In each \(D_2\) there are two subgroups conjugate in \(O(2)\) to \(Z_2\). Their fixed point sets are two circles which intersect precisely in the two points of \(\text{Fix}(D_2)\). The subgroup \(D_4\) which normalizes \(D_2\) in \(O(2)\) acts by interchanging the two points in \(\text{Fix}(D_2)\) and the two circles. It follows that if the flow on one circle is away from one of the equilibrium points, then that on the other circle is away from the other equilibrium point. In the absence of other equilibria on these circles we obtain two trajectories connecting the two equilibrium points in each direction, as in Fig. 5. This is repeated

\[\text{circle of equilibria with isotropy conjugate to } D_2\]

\[Z_2\]

\[D_2\]

**Fig. 5.** The heteroclinic trajectories associated to a subgroup \(D_2\) in \(O(2)\), assuming that the only equilibria on the \(Z_2\) circles are those in \(\text{Fix}(D_2)\).
for each subgroup of $O(2)$ conjugate to $D_2$, yielding a circle of heteroclinic cycles.

The flow illustrated in Fig. 4 may be regarded as a flow on the quotient space $O(2) \backslash SO(3)/\mathbb{T}$, and as such can be lifted, by the results described in Section 1.4, to a flow on $SO(3)/\mathbb{T}$ for which the union of the heteroclinic cycles is asymptotically stable. The time reversed flow can be lifted to yield stable $Z_3$ periodic orbits. Flows with stable $D_2$ equilibria and additional $Z_2$ equilibria can be constructed similarly, and all these extended to $X$ by Proposition 1.3.

2.4. Example 3 ($H = \mathbb{T}; K = D_n$)

For our final example we consider perturbations of $SO(3)$ orbits of equilibria with isotropy subgroups conjugate to $\mathbb{T}$, to invariant submanifolds of systems with $D_n$ symmetry. The quotient space $D_n \backslash SO(3)/\mathbb{T}$ is three dimensional so the perturbed flows can be genuinely three dimensional.

The isotropy subgroups of the action of $D_n$ on $SO(3)/\mathbb{T}$ are subgroups of $D_n$ which are subconjugate to $\mathbb{T}$. These are listed in Table II along with information on their fixed point sets and the number of groups in each $D_n$ conjugacy class. Note that there are two conjugacy classes, denoted $D_2^{(1)}$ and $D_2^{(2)}$, of groups isomorphic to $D_2$ when $n \equiv 0 \mod 4$ (each containing $n/4$ groups), but only one (containing $n/2$ groups) when $n \equiv 2 \mod 4$. Similarly there is just one conjugacy class of subgroups conjugate to $Z_2$ when $n$ is odd, and three when $n$ is even. In this latter case we denote the normal subgroup by $Z_2^n$ and the groups in the other two conjugacy classes by $Z_2^{(1)}$ and $Z_2^{(2)}$, respectively. The superscripts are assigned so that $Z_2^{(1)} < D_2^{(j)}$ for $j = 1, 2$.

<table>
<thead>
<tr>
<th>Isomorphism Class of Isotropy Subgroup</th>
<th>Conditions</th>
<th>Number of Groups in $D_n$ Conjugacy Class</th>
<th>Fixed Point Set</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_3$</td>
<td>$3 \mid n$</td>
<td>1</td>
<td>Disjoint union of 2 circles</td>
</tr>
<tr>
<td>$D_2$</td>
<td>$4 \mid n$</td>
<td>$n/4; n/4$</td>
<td>2 Points</td>
</tr>
<tr>
<td></td>
<td>$2 \mid n; 4 \mid n$</td>
<td>$n/2$</td>
<td></td>
</tr>
<tr>
<td>$Z_2$</td>
<td>$2 \mid n$</td>
<td>$n$</td>
<td>1 Circle</td>
</tr>
<tr>
<td></td>
<td>$2 \mid n$</td>
<td>$1; n/2; n/2$</td>
<td></td>
</tr>
<tr>
<td>$1$</td>
<td></td>
<td>1</td>
<td>$SO(3)/\mathbb{T}$</td>
</tr>
</tbody>
</table>
The results of this section are summarised in the following proposition.

**Proposition 2.3.** Let \( v \) be an \( SO(3) \) equivariant vector field on \( X \) and \( M \) an \( SO(3) \) orbit of equilibrium points \( M \) with isotropy subgroups conjugate to \( T \) and satisfying condition (1.1.3). If \( w \) is any \( D_\pi \)-equivariant vector field on \( X \) which is sufficiently close to \( v \) in the \( C^1 \) topology and \( M_w \) is the invariant submanifold for \( w \) near \( M \), as in Proposition 1.2, then the flow on \( M_w \) has:

1. equilibrium points or periodic orbits with \( Z_3 \) symmetry, if \( 3 \mid n \),
2. equilibrium points or periodic orbits with \( Z_2 \) symmetry, if \( n \) is odd,
3. equilibrium points with \( D_2^{(1)} \) and \( D_2^{(2)} \) symmetry, and heteroclinic cycles connecting the \( D_2^{(1)} \) (resp. \( D_2^{(2)} \)) points or equilibrium points with \( Z_2^{(1)} \) (resp. \( Z_2^{(2)} \)) symmetry, if \( n \equiv 0 \mod 4 \),
4. equilibrium points with \( D_2 \) symmetry and heteroclinic cycles and/or equilibrium points with \( Z_2, Z_2^{(1)} \) and \( Z_2^{(2)} \) symmetry (in combinations described in more detail below), if \( n \equiv 2 \mod 4 \).

The proof, together with further comments, takes up the rest of the section.

If \( 3 \mid n \) then there are always two circles with isotropy groups \( Z_3 \) which are invariant under any \( D_\pi \) invariant flow on \( M \). The subgroups \( Z_n \) of \( D_\pi \) preserve each circle, each orbit containing \( n/3 \) points, while the other elements of \( D_\pi \) map the circles to each other. It follows that the flows near the circles must be identical to each other, and must commute with the action of \( Z_n \). Thus, if there are no equilibrium points on the circles they will be periodic orbits which generically are asymptotically stable in either forward or backward time—they are not of saddle type.

We treat the dynamics of the \( Z_2 \) and \( D_2 \) points in three separate cases: \( n \) odd, \( n \equiv 0 \mod 4 \), and \( n \equiv 2 \mod 4 \).

**n odd.** When \( n \) is odd \( D_2 \) is not a subgroup of \( D_\pi \), and there is only a single conjugacy class of isotropy subgroups conjugate to \( Z_2 \). Their fixed point sets give a set of \( n \) disjoint invariant circles in \( M \). An elements of \( D_\pi \) can be found mapping any one of these circles to any other, so the flows near the circles must all be identical. In particular their stability properties are all the same.

**n \equiv 0 \mod 4.** In this case there are two conjugacy classes of subgroups isomorphic to \( D_2 \), denoted \( D_2^{(1)} \) and \( D_2^{(2)} \), and three conjugacy classes isomorphic to \( Z_2 \), denoted \( Z_2^{(1)}, Z_2^{(2)}, \) and \( Z_2^{(3)} \). The subconjugacy relations are shown in Fig. 6.

The fixed point sets, for \( n = 4 \), of these subgroups are illustrated schematically in Fig. 7. There are \( n/2 \) points with isotropy groups conjugate to each of \( D_2^{(1)} \) and \( D_2^{(2)} \) arranged alternately round the single \( Z_2^{(3)} \) circle. Passing through each \( D_2^{(1)} \) point are two more circles with different isotropy
subgroups conjugate to $Z_2^{(j)}$. These two circles also intersect at the “opposite” $D_2^{(j)}$ point.

The similarity with Example 2 should be apparent. The difference is that in the present case the $Z_2^{(j)}$ circle has only a finite number of points on it with $D_2$ isotropy, and there are only a finite number of $Z_2$ circles. The action of $D_n$ on this “complex” interchanges opposite $D_2^{(j)}$ points and the two circles through those points. It also “rotates” the two circles associated with one pair of $D_2^{(j)}$ to those associated with any other pair (with the same $j$). It follows that, if, for either $j=1$ or $2$, the $D_2^{(j)}$ points are the only equilibrium points on the $Z_2^{(j)}$ circles then these circles form heteroclinic connections between the $D_2^{(j)}$ equilibria.

If the $D_2^{(j)}$ points are the only equilibria on any of the $Z_2$ circles then the flow must be essentially as shown in Fig. 7. This has heteroclinic cycles between opposite $D_2^{(j)}$ points, and one set of heteroclinic cycles (the $D_2^{(1)}$’s) are unstable to the other (the $D_2^{(2)}$’s). Suppose, in the flow in Fig. 7, that the stable and unstable eigenvalues of the $D_2^{(2)}$ equilibrium points in the directions of the heteroclinic connections are $-s$ and $u$, respectively, with

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**Fig. 6.** Subconjugacy relations between conjugacy classes of subgroups of $D_n$ isomorphic to $D_2$, $Z_2$, and 1, when $n \equiv 0 \mod 4$.

**Fig. 7.** Schematic illustration of the fixed point sets of the subgroups of $D_n$ isomorphic to $Z_2$ and $D_2$ when $n = 4$. For clarity only parts of the $Z_2^{(1)}$ and $Z_2^{(2)}$ circles are shown. A “typical” flow is also shown.
Then the theorem of dos Reis [9] used in Example 1 may be extended (see [20]) to deduce that the $D^{(2)}_n$ heteroclinic connections are asymptotically stable if $s > u$ and unstable if $s < u$.

In a $D_n$-equivariant flow on $SO(3)/\mathbb{T}$ which is obtained by a small perturbation of an $O(2)$-equivariant flow (studied in Example 2) the eigenvalues of the $D^{(1)}_2$ equilibria will be close to those of the $D^{(2)}_2$ equilibria. It follows that in any such flow without extra equilibrium points on the $\mathbb{Z}_2$ circles one set of heteroclinic cycles will be asymptotically stable in either that flow or its time reversal.

$n \equiv 2 \mod 4$. This is very similar to the $n \equiv 0 \mod 4$ case, except that now all the subgroups isomorphic to $D_2$ are conjugate in $D_n$; the subconjugacy relations are shown in Fig. 8.

The fixed point sets of the subgroups isomorphic to $\mathbb{Z}_2$ and $D_2$ are illustrated schematically in Fig. 9, for $n = 6$. There are again two $D_n$ orbits of points with $D_2$ isotropy, but now the $\mathbb{Z}_2$ circles connect $D_2$ points in different $D_n$ orbits. Thus the stabilities of the two points of intersection of two circles need not be the same. Note also that two such intersecting
circles have non conjugate isotropy subgroups. Thus the flows on them will not in general be the same.

To discuss possible flows on this complex it is convenient to take its quotient by the action of the subgroup $\mathbb{Z}_{n/2}$ in $D_n$. This acts freely on a neighborhood of the complex and so the quotient space is non singular. The resulting invariant set is shown in Fig. 10 with two “simple” flows without extra equilibrium points. The $n$ $D_2$-points have been identified to just two points and passing through these are three circles, the quotients of the $\mathbb{Z}_2^n$ circle, the $n/2 \mathbb{Z}_2^{(1)}$ circles and the $n/2 \mathbb{Z}_2^{(2)}$ circles. The two types of flow illustrated are, up to time reversal and interchanging isotropy labels on the circles, the only ones possible without extra points on the circles.

The flow illustrated in Fig. 10(a) simply results in one orbit of $D_2$ equilibrium points being asymptotically stable. The flow in Fig. 10(b) is more interesting. Note that one equilibrium point has a two dimensional unstable manifold and the other a two dimensional stable manifold. In general these will only intersect in lines (such as the $\mathbb{Z}_2^n$ and $\mathbb{Z}_2^{(1)}$ circles) so there will be trajectories which leave a neighborhood of the complex. Thus in general we should not expect this complex to be asymptotically stable. However, a result of Melbourne [20] shows that such flows do contain heteroclinic cycles that are “essentially stable.” More precisely,
in any neighborhood $U$ of the cycle there is a closed subset $C$ and an open neighborhood $V$ of the cycle such that trajectories starting in $V \setminus C$ remain in $U \setminus C$ in forward time, and are asymptotic to the heteroclinic cycle. Moreover, by taking $V$ sufficiently small the measure of $V \setminus C$ in $V$ can be made to be as close to full measure as required. Thus “most” nearby trajectories are asymptotic to the cycle in a very strong sense.

There are two types of heteroclinic cycles in the flow of Fig. 10(b) that Melbourne's work can be applied to. In both cases the outward trajectories from the $D_2$ equilibrium on the right is along points with $Z_2^{(2)}$ isotropy. However, the return trajectories can have either $Z_2^{(1)}$ or $Z_2^\pi$ isotropy. In the quotient by $\mathbb{Z}_{n/2}$ these two possibilities look much the same, but back up in $SO(3)/T$ their behaviour is quite different. A trajectory near the $Z_2^{(2)} - Z_2^{(1)}$ cycle will switch back and forth between two “opposite” $D_2$ equilibria, while a trajectory near the $Z_2^\pi - Z_2^n$ cycle will visit each $D_2$ equilibrium “cyclically” in turn.

Melbourne's Theorem gives sufficient conditions for these cycles to be essentially asymptotically stable. If the eigenvalues of the $D_2$ equilibria are as labelled in Fig. 10(b), with $s_j, u_j > 0$ for $j = 0, 1,$ and $2$, then the $Z_2^{(2)} - Z_2^{(1)}$ cycles are stable if:

(a) $s_1 s_2 > u_1 u_2$
(b) $s_1 (u_1 - u_0) > u_1 u_2$
(c) $s_0 > s_1$

and the $Z_2^{(2)} - Z_2^n$ cycles are stable if these conditions hold when the subscripts 0 and 1 are interchanged. Condition (a) is just the dos Reis condition. Conditions (b) and (c) imply that $u_1 > u_0$ and $s_1 < s_0$, each of which holds (generically) for precisely one of the two types of heteroclinic cycles.

If there are extra equilibrium points on any of the $Z_2$ circles then these prevent that circle becoming part of a heteroclinic cycle. For example $Z_2^{(1)} - Z_2^{(2)}$ heteroclinic cycles exist if and only if there are no equilibrium points with either $Z_2^{(1)}$ or $Z_2^{(2)}$ isotropy. Similar statements hold for $Z_2^{(1)} - Z_2^n$ and $Z_2^{(2)} - Z_2^n$ heteroclinic cycles.

We end with two questions on the dynamics of the flow shown in Fig. 10(b).

1) Can there exist open regions of initial conditions for which the trajectories are asymptotic to other sequences of $Z_2^{(1)}, Z_2^{(2)}$ and $Z_2^n$ connections?

2) What happens to the asymptotically stable attractor of the $O(2)$-invariant system on $SO(3)/T$ described in Example 2 when the flow is perturbed to one with $D_n$ symmetry where $n \equiv 2 \mod 4$?
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