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Computing the first few Betti numbers of semi-algebraic sets in single exponential time

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Abstract

In this paper we describe an algorithm that takes as input a description of a semi-algebraic set $S \subset \mathrm{R}^k$, defined by a Boolean formula with atoms of the form $P > 0$, $P < 0$, $P = 0$ for $P \in \mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$, and outputs the first $\ell + 1$ Betti numbers of S, $b_0(S), \ldots, b_\ell(S)$. The complexity of the algorithm is $(sd)^{k^{O(\ell)}}$, where $s = \#(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$, which is singly exponential in k for ℓ any fixed constant. Previously, singly exponential time algorithms were known only for computing the Euler–Poincaré characteristic, the zeroth and the first Betti numbers.

Keywords: Semi-algebraic sets; Betti numbers; Single exponential complexity

1. Introduction

Let R be a real closed field and $S \subset \mathrm{R}^k$ a semi-algebraic set defined by a Boolean formula with atoms of the form $P > 0$, $P < 0$, $P = 0$ for $P \in \mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$ (we call such a set a $\mathcal{P}$-semi-algebraic set and the corresponding formula a $\mathcal{P}$-formula). It is well known (Oleĭnik, 1951; Oleĭnik and Petrovskii, 1949; Milnor, 1964; Thom, 1965; Basu, 1999; Gabrielov and Vorobjov, 2005) that the topological complexity of S (measured by the various Betti numbers of S) is bounded by $(sd)^{O(k)}$, where $s = \#(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$. Note that these bounds are singly exponential in k. More precise bounds on the individual Betti numbers of S appear in Basu (2003). Even though the Betti numbers of S are bounded singly exponentially in k,

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there is no known algorithm for producing a singly exponential sized triangulation of S (which would immediately imply a singly exponential algorithm for computing the Betti numbers of S). In fact, designing a singly exponential time algorithm for computing the Betti numbers of semi-algebraic sets is one of the outstanding open problems in algorithmic semi-algebraic geometry. More recently, determining the exact complexity of computing the Betti numbers of semi-algebraic sets has attracted the attention of computational complexity theorists (Burgisser and Cucker, 2006), who are interested in developing a theory of counting complexity classes for the Blum–Shub–Smale model of real Turing machines.

Doubly exponential algorithms (with complexity $(sd)^{2^{O(k)}}$) for computing all the Betti numbers are known, since it is possible to obtain a triangulation of S in doubly exponential time using cylindrical algebraic decomposition (Collins, 1975). In the absence of singly exponential time algorithms for computing triangulations of semi-algebraic sets, algorithms with single exponential complexity are known only for the problems of testing emptiness (Renegar, 1992; Basu et al., 1996), computing the zeroth Betti number (i.e. the number of semi-algebraically connected components of S) (Grigor'ev and Vorobjov, 1992; Canny, 1993; Gournay and Risler, 1993; Basu et al., 1999), as well as the Euler–Poincaré characteristic of S (Basu, 1999). Very recently a singly exponential time algorithm has been developed for the problem of computing the first Betti number of a given semi-algebraic set (Basu et al., 2005).

In this paper we describe an algorithm, which given a family $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$, a $\mathcal{P}$-formula describing a $\mathcal{P}$-semi-algebraic set $S \subset \mathrm{R}^k$, and a number $\ell, 0 \leq \ell \leq k$ as input, outputs the first ℓ Betti numbers of S. For constant ℓ, the complexity of the algorithm is singly exponential in k. We remark that using Alexander duality, we simultaneously obtain a singly exponential algorithm for computing the top ℓ Betti numbers of S as well. However, the complexity of our algorithm becomes doubly exponential if we want to compute the middle Betti numbers of a semi-algebraic set using it.

There are two main ingredients in our algorithm for computing the first ℓ Betti numbers of a given semi-algebraic set. The first ingredient is a result proved in Basu et al. (2005), which enables us to compute a singly exponential sized cover of the given semi-algebraic set consisting of closed, contractible semi-algebraic sets, in single exponential time. The number and the degrees of the polynomials used to define the sets in this cover are also bounded singly exponentially.

The second ingredient, which is the main contribution of this paper, is an algorithm which uses the covering algorithm recursively and computes in singly exponential time a complex whose cohomology groups are isomorphic to the first ℓ cohomology groups of the input set. This complex is of singly exponential size.

The main result of the paper is the following.

Main result: For any given ℓ, there is an algorithm that takes as input a $\mathcal{P}$-formula describing a semi-algebraic set $S \subset \mathrm{R}^k$, and outputs $b_0(S), \ldots, b_\ell(S)$. The complexity of the algorithm is $(sd)^{k^{O(\ell)}}$, where $s = \#(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$. Note that the complexity is singly exponential in k for every fixed ℓ.

The paper is organized as follows. In Section 2 we recall some basic definitions from algebraic topology and fix notation. In Section 3 we recall a few facts about the Mayer–Vietoris sequence and its associated double complex. In Section 4 we describe the construction of the complexes which allow us to compute the first ℓ Betti numbers of a given semi-algebraic set. In Section 5 we recall the inputs, outputs and complexities of a few algorithms described in detail in Basu et al. (2005), which we use in our algorithm. In Section 6 we describe our algorithm for computing the first ℓ Betti numbers, and prove its correctness as well as the complexity bounds. Finally in Section 7 we comment on issues related to practical implementation.

2. Mathematical preliminaries

In this section, we recall some basic facts about semi-algebraic sets as well as the definitions of complexes and double complexes of vector spaces, and fix notation.

2.1. Semi-algebraic sets and their cohomology groups

Let R be a real closed field. If $\mathcal{P}$ is a finite subset of $\mathrm{R}[X_1, \ldots, X_k]$, we write the set of zeros of $\mathcal{P}$ in R^k as

$$\mathrm{Z}(\mathcal{P}, \mathrm{R}^k) = \left\{ x \in \mathrm{R}^k \mid \bigwedge_{P \in \mathcal{P}} P(x) = 0 \right\}.$$

We denote by $B(0, r)$ the open ball with center 0 and radius r.

Let $\mathcal{Q}$ and $\mathcal{P}$ be finite subsets of $\mathrm{R}[X_1, \ldots, X_k]$, $Z = \mathrm{Z}(\mathcal{Q}, \mathrm{R}^k)$, and $Z_r = Z \cap B(0, r)$. A *sign condition* on $\mathcal{P}$ is an element of $\{0, 1, -1\}^{\mathcal{P}}$. The *realization of the sign condition* σ *over* Z, $\mathcal{R}(\sigma, Z)$, is the basic semi-algebraic set

$$\left\{ x \in \mathrm{R}^k \;\mid\; \bigwedge_{Q \in \mathcal{Q}} Q(x) = 0 \wedge \bigwedge_{P \in \mathcal{P}} \mathrm{sign}(P(x)) = \sigma(P) \right\}.$$

The *realization of the sign condition* σ *over* Z_r, $\mathcal{R}(\sigma, Z_r)$, is the basic semi-algebraic set $\mathcal{R}(\sigma, Z) \cap B(0, r)$. For the rest of the paper, we fix an open ball $B(0, r)$ with center 0 and radius r big enough so that, for every sign condition σ, $\mathcal{R}(\sigma, Z)$ and $\mathcal{R}(\sigma, Z_r)$ are homeomorphic. This is always possible by the local conical structure at infinity of semi-algebraic sets (Bochnak et al., 1987, page 225).

A closed and bounded semi-algebraic set $S \subset \mathrm{R}^k$ is semi-algebraically triangulable (this is a classical fact; see Basu et al. (2006) for example), and we denote by $\mathrm{H}^i(S)$ the ith simplicial cohomology group of S with rational coefficients. The groups $\mathrm{H}^i(S)$ are invariant under semi-algebraic homeomorphisms and coincide with the corresponding singular cohomology groups when $\mathrm{R} = \mathbb{R}$. We denote by $b_i(S)$ the ith Betti number of S (that is, the dimension of $\mathrm{H}^i(S)$ as a vector space), and $b(S)$ the sum $\sum_i b_i(S)$. For a closed but not necessarily bounded semi-algebraic set $S \subset \mathrm{R}^k$, we will denote by $\mathrm{H}^i(S)$ the ith simplicial cohomology group of $S \cap \overline{B(0, r)}$, where r is sufficiently large. The sets $S \cap \overline{B(0, r)}$ are semi-algebraically homeomorphic for all sufficiently large $r > 0$, by the local conical structure at infinity of semi-algebraic sets, and hence this definition makes sense.

The definition of cohomology groups of arbitrary semi-algebraic sets in R^k requires some care and several possibilities exist. In this paper, we follow Basu et al. (2006) and define the cohomology groups of realizations of sign conditions as follows.

Let R denote a real closed field and R' a real closed field containing R. Given a semi-algebraic set S in R^k, the *extension* of S to R', denoted as $\mathrm{Ext}(S, \mathrm{R}')$, is the semi-algebraic subset of R'^k defined by the same quantifier free formula as defines S. The set $\mathrm{Ext}(S, \mathrm{R}')$ is well defined (i.e. it only depends on the set S and not on the quantifier free formula chosen to describe it). This is an easy consequence of the transfer principle (see for example Basu et al. (2006)).

Now, let $S \subset \mathrm{R}^k$ be a $\mathcal{P}$-semi-algebraic set, where $\mathcal{P} = \{P_1, \ldots, P_s\}$ is a finite subset of $\mathrm{R}[X_1, \ldots, X_k]$. Let $\phi(X)$ be a quantifier free formula defining S. Let $P_i = \sum_\alpha a_{i,\alpha} X^\alpha$ where the $a_{i,\alpha} \in \mathrm{R}$. Let $A = (\ldots, A_{i,\alpha}, \ldots)$ denote the vector of variables corresponding to the coefficients of the polynomials in the family $\mathcal{P}$, and let $a = (\ldots, a_{i,\alpha}, \ldots) \in \mathrm{R}^N$ denote the vector of the actual coefficients of the polynomials in $\mathcal{P}$. Let $\psi(A, X)$ denote the formula obtained from $\phi(X)$ by replacing each coefficient of each polynomial in $\mathcal{P}$ by the corresponding variable, so that $\phi(X) = \psi(a, X)$. It follows from Hardt's triviality theorem for semi-algebraic mappings (Hardt, 1980), that there exists $a' \in \mathbb{R}^N_{\mathrm{alg}}$ such that denoting by $S' \subset \mathbb{R}^k_{\mathrm{alg}}$ the semi-algebraic set defined by $\psi(a', X)$, the semi-algebraic set $\mathrm{Ext}(S', \mathrm{R})$ has the same homeomorphism type as S. Here, $\mathbb{R}_{\mathrm{alg}}$ is the field of real algebraic numbers. We define the cohomology groups of S to be the singular cohomology groups of $\mathrm{Ext}(S', \mathbb{R})$. It follows from the Tarski–Seidenberg transfer principle, and the corresponding property of singular cohomology groups, that the cohomology groups defined this way are invariant under semi-algebraic homotopies. It is also clear that this definition is compatible with the simplicial cohomology for closed, bounded semi-algebraic sets, and the singular cohomology groups when the ground field is $\mathbb{R}$. Finally it is clear that the Betti numbers are not changed after extension:

$$b_i(S) = b_i(\mathrm{Ext}(S, \mathrm{R}')).$$

Note that we define the cohomology groups of arbitrary semi-algebraic sets as above in order to treat semi-algebraic sets over arbitrary (possibly non-archimedean) real closed fields R, for which the standard proofs of the homology axioms (in particular the excision axiom) break down for singular homology groups (see Knebusch (1989), page XIII). If one is only interested in the case, $\mathrm{R} = \mathbb{R}$, then singular cohomology groups suffice.

2.2. *Complex of vector spaces*

A sequence $\{\mathrm{C}^p\}$, $p \in \mathbb{Z}$, of $\mathbb{Q}$-vector spaces together with a sequence $\{\delta^p\}$ of homomorphisms $\delta^p : \mathrm{C}^p \to \mathrm{C}^{p+1}$ (called differentials) for which $\delta^{p-1}\ \delta^p = 0$ for all p is called a complex. When it is clear from the context, we will drop the superscripts from the differentials for the sake of readability.

The cohomology groups, $\mathrm{H}^p(\mathrm{C}^\bullet)$, are defined by

$$\mathrm{H}^p(\mathrm{C}^\bullet) = Z^p(\mathrm{C}^\bullet)/B^p(\mathrm{C}^\bullet),$$

where $B^p(\mathrm{C}^\bullet) = \mathrm{Im}(\delta^{p-1})$, and $Z^p(\mathrm{C}^\bullet) = \mathrm{Ker}(\delta^p)$ and we will denote by $\mathrm{H}^*(\mathrm{C}^\bullet)$ the graded vector space $\bigoplus_p \mathrm{H}^p(\mathrm{C}^\bullet)$.

The cohomology groups, $\mathrm{H}^p(\mathrm{C}^\bullet)$, are all $\mathbb{Q}$-vector spaces (finite dimensional if the vector spaces C^p's are themselves finite dimensional). We will henceforth omit reference to the field of coefficients $\mathbb{Q}$ which is fixed throughout the rest of the paper.

2.3. *Homomorphisms of complexes*

Given two complexes, $\mathrm{C}^\bullet = (\mathrm{C}^p, \delta^p)$ and $\mathrm{D}^\bullet = (\mathrm{D}^p, \delta^p)$, a homomorphism of complexes, $\phi^\bullet : \mathrm{C}^\bullet \to \mathrm{D}^\bullet$, is a sequence of homomorphisms $\phi^p : \mathrm{C}^p \to \mathrm{D}^p$ for which $\delta^p\ \phi^p = \phi^{p+1}\ \delta^p$ for all p.

In other words, the following diagram is commutative.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathrm{C}^p & \xrightarrow{\delta^p} & \mathrm{C}^{p+1} & \longrightarrow & \cdots \\ & & \big\downarrow \phi^p & & \big\downarrow \phi^{p+1} & & \\ \cdots & \longrightarrow & \mathrm{D}^p & \xrightarrow{\delta^p} & \mathrm{D}^{p+1} & \longrightarrow & \cdots \end{array}$$

A homomorphism of complexes, $\phi^\bullet : \mathrm{C}^\bullet \to \mathrm{D}^\bullet$, induces homomorphisms, $\phi^i : \mathrm{H}^i(\mathrm{C}^\bullet) \to \mathrm{H}^i(\mathrm{D}^\bullet)$, and we will denote the corresponding homomorphism between the graded vector spaces $\mathrm{H}^*(\mathrm{C}^\bullet), \mathrm{H}^*(\mathrm{D}^\bullet)$ by ϕ^*. The homomorphism $\phi^\bullet$ is called a *quasi-isomorphism* if the homomorphism ϕ^* is an isomorphism.

Given two complexes $\mathrm{C}^\bullet$ and $\mathrm{D}^\bullet$, their direct sum, denoted by $\mathrm{C}^\bullet \oplus \mathrm{D}^\bullet$, is again a complex with its pth term being $\mathrm{C}^p \oplus \mathrm{D}^p$. Moreover, given two homomorphisms of complexes,

$$\phi^\bullet : \mathrm{C}^\bullet \to \bar{\mathrm{C}}^\bullet,$$
$$\psi^\bullet : \mathrm{D}^\bullet \to \bar{\mathrm{D}}^\bullet,$$

their direct sum

$$\phi^\bullet \oplus \psi^\bullet : \mathrm{C}^\bullet \oplus \mathrm{D}^\bullet \to \bar{\mathrm{C}}^\bullet \oplus \bar{\mathrm{D}}^\bullet,$$

is again a homomorphism of complexes defined componentwise. Note that if we specify a basis for the different terms of the complexes $\mathrm{C}^\bullet, \bar{\mathrm{C}}^\bullet, \mathrm{D}^\bullet, \bar{\mathrm{D}}^\bullet$, as well as the matrices for the homomorphisms $\phi^\bullet$, $\psi^\bullet$, then we can write down the matrix for the direct sum homomorphism $\phi^\bullet \oplus \psi^\bullet$ as a sum of block-matrices using elementary linear algebra.

2.4. The nerve lemma and generalizations

We first define formally the notion of a cover of a closed, bounded semi-algebraic set.

Definition 2.1. Let $S \subset \mathrm{R}^k$ be a closed and bounded semi-algebraic set. A cover, $\mathcal{C}(S)$, of S consists of an ordered index set, which by a slight abuse of language we also denote by $\mathcal{C}(S)$, and a map that associates to each $\alpha \in \mathcal{C}(S)$, a closed and bounded semi-algebraic subset $S_\alpha \subset S$, such that $S = \cup_{\alpha\in\mathcal{C}(S)} S_\alpha$.

For $\alpha_0, \ldots, \alpha_p, \in \mathcal{C}(S)$, we associate with the formal product, $\alpha_0 \cdots \alpha_p$, the closed and bounded semi-algebraic set $S_{\alpha_0\cdots\alpha_p} = S_{\alpha_0} \cap \cdots \cap S_{\alpha_p}$.

Recall that the 0th simplicial cohomology group of a closed and bounded semi-algebraic set X, $\mathrm{H}^0(X)$, can be identified with the $\mathbb{Q}$-vector space of $\mathbb{Q}$-valued locally constant functions on X. Clearly, the dimension of $\mathrm{H}^0(X)$ is equal to the number of connected components of X.

For $\alpha_0, \alpha_1, \ldots, \alpha_p, \beta \in \mathcal{C}(S)$, and $\beta \notin \{\alpha_0, \ldots, \alpha_p\}$, let

$$r_{\alpha_0,\ldots,\alpha_p;\beta} : \mathrm{H}^0(S_{\alpha_0\cdots\alpha_p}) \longrightarrow \mathrm{H}^0(S_{\alpha_0\cdots\alpha_p\cdot\beta})$$

be the homomorphism defined as follows. Given a locally constant function, $\phi \in \mathrm{H}^0(S_{\alpha_0\cdots\alpha_p})$, $r_{\alpha_0\cdots\alpha_p;\beta}(\phi)$ is the locally constant function on $S_{\alpha_0\cdots\alpha_p\cdot\beta}$ obtained by restricting ϕ to $S_{\alpha_0\cdots\alpha_p\cdot\beta}$.

We define the generalized restriction homomorphisms

$$\delta^p : \bigoplus_{\alpha_0<\cdots<\alpha_p, \alpha_i\in\mathcal{C}(S)} \mathrm{H}^0(S_{\alpha_0\cdots\alpha_p}) \longrightarrow \bigoplus_{\alpha_0<\cdots<\alpha_{p+1}, \alpha_i\in\mathcal{C}(S)} \mathrm{H}^0(S_{\alpha_0\cdots\alpha_{p+1}})$$

by

$$\delta^p(\phi)_{\alpha_0\cdots\alpha_{p+1}} = \sum_{0\leq i\leq p+1} (-1)^i r_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_{p+1};\alpha_i}(\phi_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_{p+1}}), \tag{1}$$

where $\phi \in \bigoplus_{\alpha_0<\cdots<\alpha_p\in\mathcal{C}(S)} \mathrm{H}^0(S_{\alpha_0\cdots\alpha_p})$ and $r_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_{p+1};\alpha_i}$ is the restriction homomorphism defined previously. The sequence of homomorphisms δ^p gives rise to a complex, $\mathrm{L}^\bullet(\mathcal{C}(S))$, defined by

$$\mathrm{L}^p(\mathcal{C}(S)) = \bigoplus_{\alpha_0<\cdots<\alpha_p,\alpha_i\in\mathcal{C}(S)} \mathrm{H}^0(S_{\alpha_0\cdots\alpha_p}),$$

with the differentials $\delta^p : \mathrm{L}^p(\mathcal{C}(S)) \to \mathrm{L}^{p+1}(\mathcal{C}(S))$ defined in (1). The complex $\mathrm{L}^\bullet(\mathcal{C}(S))$ is often referred to as the *nerve complex* of the cover $\mathcal{C}(S)$.

For any $\ell \geq 0$, we will denote by $\mathrm{L}^\bullet_\ell(\mathcal{C}(S))$ the truncated complex defined by

$$\begin{aligned} \mathrm{L}^p_\ell(\mathcal{C}(S)) &= \mathrm{L}^p(\mathcal{C}(S)), \quad 0 \leq p \leq \ell, \\ &= 0, \quad p > \ell. \end{aligned}$$

Notice that once we have a cover of S, and we identify the connected components of the various intersections, $S_{\alpha_0\cdots\alpha_p}$, we have natural bases for the vector spaces

$$\mathrm{L}^p(\mathcal{C}(S)) = \bigoplus_{\alpha_0<\cdots<\alpha_p,\alpha_i\in\mathcal{C}(S)} \mathrm{H}^0(S_{\alpha_0\cdots\alpha_p})$$

appearing as terms of the nerve complex. Moreover, the matrices corresponding to the homomorphisms δ^p in this basis depend only on the inclusion relationships between the connected components of $S_{\alpha_0\cdots\alpha_{p+1}}$ and those of $S_{\alpha_0\cdots\alpha_p}$.

We say that the cover $\mathcal{C}(S)$ *satisfies the Leray property* if each non-empty intersection $S_{\alpha_0\cdots\alpha_p}$ is contractible. Clearly in this case

$$\begin{aligned} \mathrm{H}^0(S_{\alpha_0\cdots\alpha_p}) &\cong \mathbb{Q}, \quad \text{if } S_{\alpha_0\cdots\alpha_p} \neq \emptyset \\ &\cong 0, \quad \text{if } S_{\alpha_0\cdots\alpha_p} = \emptyset. \end{aligned}$$

It is a classical fact (usually referred to as the *nerve lemma*) that

Theorem 2.2 (*Nerve Lemma*). *Suppose that the cover $\mathcal{C}(S)$ satisfies the Leray property. Then for each $i \geq 0$,*

$$\mathrm{H}^i(\mathrm{L}^\bullet(\mathcal{C}(S))) \cong \mathrm{H}^i(S).$$

Proof. This is classical (see for instance Rotman (1988)). □

Notice that once we have a cover of S satisfying the Leray property, and we are able to test emptiness of the various intersections $S_{\alpha_0\cdots\alpha_p}$, we can use Theorem 2.2 and some basic algorithms from linear algebra to compute the Betti numbers of S.

Now suppose that each individual member, S_{α_0}, of the cover is contractible, but the various intersections $S_{\alpha_0\cdots\alpha_p}$ are not necessarily contractible for $p \geq 1$. Theorem 2.2 does not hold in this case. However, the following is proved in Basu et al. (2005).

Theorem 2.3. *Suppose that each individual member, S_{α_0}, of the cover $\mathcal{C}(S)$ is contractible. Then,*

$$\mathrm{H}^i(\mathrm{L}^\bullet_2(\mathcal{C}(S))) \cong \mathrm{H}^i(S),$$

for $i = 0, 1$.

Proof. See Basu et al. (2005). □

Notice that Theorem 2.3 allows us to compute using linear algebra $b_0(S)$ and $b_1(S)$, once we have a cover by contractible sets, and we have identified the non-empty connected components of the pairwise and triple-wise intersections of the sets in the cover, and their inclusion relationships. It is quite easy to see that if we extend the complex in Theorem 2.3 by one more term, that is consider the complex $\mathrm{L}_3^\bullet(\mathcal{C}(S))$, then the cohomology of the complex does not yield information about $\mathrm{H}^2(S)$. Just consider the cover of the standard sphere $S^2 \subset \mathrm{R}^3$, and the cover $\{H_1, H_2\}$ of S^2 where H_1, H_2 are closed hemispheres meeting at the equator. The corresponding complex, $\mathrm{L}_3^\bullet(\mathcal{C})$, is as follows:

$$0 \to \mathrm{H}^0(H_1) \bigoplus \mathrm{H}^0(H_2) \xrightarrow{\delta^0} \mathrm{H}^0(H_1 \cap H_2) \xrightarrow{\delta^1} 0 \longrightarrow 0$$

Clearly, $\mathrm{H}^2(\mathrm{L}_3^\bullet(\mathcal{C}(S))) \not\cong \mathrm{H}^2(S^2)$, and indeed it is impossible to compute $b_i(S)$ just from the information on the number of connected components of intersections of the sets of a cover by contractible sets for $i \geq 2$. For example, the nerve complex corresponding to the cover of the sphere by two hemispheres is isomorphic to the nerve complex of a cover of the unit segment $[0, 1]$ by the subsets $[0, 1/2]$ and $[1/2, 1]$, but clearly $\mathrm{H}^2(S^2) = \mathbb{Q}$, while $\mathrm{H}^2([0, 1]) = 0$.

In order to deal with covers not satisfying the Leray property, it is necessary to consider a generalization of the nerve complex, namely a double complex arising from the generalized Mayer–Vietoris exact sequence. The construction of this double complex (which is quite classical) in fact motivates the design of our algorithm, which we describe in detail in Section 6.

3. Mayer–Vietoris double complex

3.1. Double complexes

In this section, we recall the basic notions of a double complex of vector spaces and associated spectral sequences. A *double complex* is a bi-graded vector space,

$$\mathrm{C}^{\bullet,\bullet} = \bigoplus_{p,q\in\mathbb{Z}} \mathrm{C}^{p,q},$$

with co-boundary operators $d : \mathrm{C}^{p,q} \to \mathrm{C}^{p,q+1}$ and $\delta : \mathrm{C}^{p,q} \to \mathrm{C}^{p+1,q}$ and such that $d\delta + \delta d = 0$. We say that $\mathrm{C}^{\bullet,\bullet}$ is a first quadrant double complex if it satisfies the condition that $\mathrm{C}^{p,q} = 0$ if either $p < 0$ or $q < 0$. Double complexes lying in other quadrants are defined in an analogous manner.

The complex defined by

$$\mathrm{Tot}^n(\mathrm{C}^{\bullet,\bullet}) = \bigoplus_{p+q=n} \mathrm{C}^{p,q},$$

with differential

$$\mathrm{D}^n = d \pm \delta : \mathrm{Tot}^n(\mathrm{C}^{\bullet,\bullet}) \longrightarrow \mathrm{Tot}^{n+1}(\mathrm{C}^{\bullet,\bullet}),$$

is denoted by $\mathrm{Tot}^\bullet(\mathrm{C}^{\bullet,\bullet})$ and called the *associated total complex of* $\mathrm{C}^{\bullet,\bullet}$.

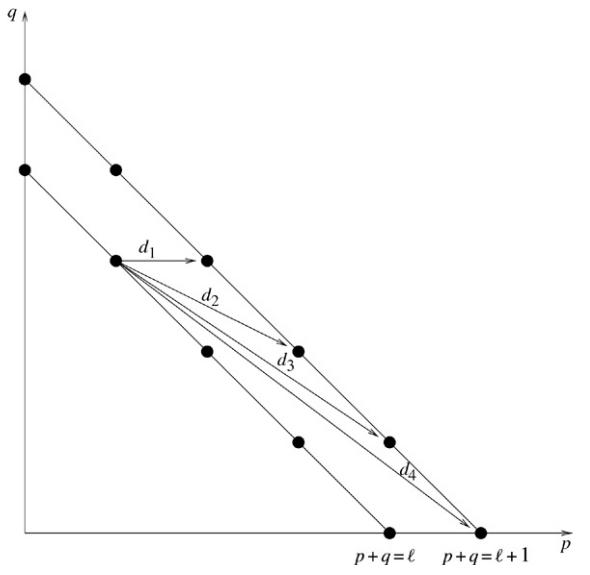

Fig. 1. $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

3.2. Spectral sequences

A *spectral sequence* is a sequence of bi-graded complexes $(E_r, d_r : E_r^{p,q} \to E_r^{p+r,q-r+1})$ (see Fig. 1) such that the complex E_{r+1} is obtained from E_r by taking its cohomology with respect to d_r (that is $E_{r+1} = \mathrm{H}_{d_r}(E_r)$).

There are two spectral sequences, ${}'E_*^{p,q}$, ${}''E_*^{p,q}$ (corresponding to taking row-wise or column-wise filtrations respectively) associated with a first quadrant double complex $\mathrm{C}^{\bullet,\bullet}$, which will be important for us. Both of these converge to $\mathrm{H}^*(\mathrm{Tot}^\bullet(\mathrm{C}^{\bullet,\bullet}))$. This means that the homomorphisms, d_r, are eventually zero, and hence the spectral sequences stabilize, and

$$\bigoplus_{p+q=i} {}'E_\infty^{p,q} \cong \bigoplus_{p+q=i} {}''E_\infty^{p,q} \cong \mathrm{H}^i(\mathrm{Tot}^\bullet(\mathrm{C}^{\bullet,\bullet})),$$

for each $i \geq 0$.

The first terms of these are

$${}'E_1 = \mathrm{H}_d(\mathrm{C}^{\bullet,\bullet}), \qquad {}'E_2 = \mathrm{H}_d\mathrm{H}_\delta(\mathrm{C}^{\bullet,\bullet}),$$

and

$${}''E_1 = \mathrm{H}_\delta(\mathrm{C}^{\bullet,\bullet}), \qquad {}''E_2 = \mathrm{H}_d\mathrm{H}_\delta(\mathrm{C}^{\bullet,\bullet}).$$

Given two (first quadrant) double complexes, $\mathrm{C}^{\bullet,\bullet}$ and $\bar{C}^{\bullet,\bullet}$, a homomorphism of double complexes,

$$\phi^{\bullet,\bullet} : \mathrm{C}^{\bullet,\bullet} \longrightarrow \bar{C}^{\bullet,\bullet},$$

is a collection of homomorphisms, $\phi^{p,q} : \mathrm{C}^{p,q} \longrightarrow \bar{C}^{p,q}$, such that the following diagrams commute.

$$\begin{array}{ccc} \mathrm{C}^{p,q} & \xrightarrow{\delta} & \mathrm{C}^{p+1,q} \\ \downarrow \phi^{p,q} & & \downarrow \phi^{p+1,q} \\ \bar{C}^{p,q} & \xrightarrow{\delta} & \bar{C}^{p+1,q} \end{array}$$

$$\begin{array}{ccc} \mathrm{C}^{p,q} & \xrightarrow{d} & \mathrm{C}^{p,q+1} \\ \downarrow \phi^{p,q} & & \downarrow \phi^{p,q+1} \\ \bar{C}^{p,q} & \xrightarrow{d} & \bar{C}^{p,q+1} \end{array}$$

A homomorphism of double complexes,

$$\phi^{\bullet,\bullet} : \mathrm{C}^{\bullet,\bullet} \longrightarrow \bar{\mathrm{C}}^{\bullet,\bullet},$$

induces a homomorphism of the corresponding total complexes which we will denote by

$$\mathrm{Tot}^{\bullet}(\phi^{\bullet,\bullet}) : \mathrm{Tot}^{\bullet}(\mathrm{C}^{\bullet,\bullet}) \longrightarrow \mathrm{Tot}^{\bullet}(\bar{\mathrm{C}}^{\bullet,\bullet}).$$

It also induces homomorphisms, $'\phi_s : {}'E_s \longrightarrow {}'\bar{E}_s$ (respectively, $''\phi_s : {}''E_s \longrightarrow {}''\bar{E}_s$) between the associated spectral sequences (corresponding either to the row-wise or column-wise filtrations). For the precise definition of homomorphisms of spectral sequences, see Mcleary (2001). We will need the following useful fact (see Mcleary (2001), page 66, Theorem 3.4 for a proof).

Proposition 3.1. *If $'\phi_s$ (respectively, $''\phi_s$) is an isomorphism for some $s \geq 1$, then $'E_r^{p,q}$ and $'\bar{E}_r^{p,q}$ (respectively, $''E_r^{p,q}$ and $''\bar{E}_r^{p,q}$) are isomorphic for all $r \geq s$. In particular, the induced homomorphism*

$$\mathrm{Tot}^{\bullet}(\phi^{\bullet,\bullet}) : \mathrm{Tot}^{\bullet}(\mathrm{C}^{\bullet,\bullet}) \longrightarrow \mathrm{Tot}^{\bullet}(\bar{C}^{\bullet,\bullet})$$

is a quasi-isomorphism.

3.3. *The Mayer–Vietoris double complex*

Let $A_1, \ldots, A_n$ be sub-complexes of a finite simplicial complex A such that $A = A_1 \cup \cdots \cup A_n$. Note that the intersections of any number of the sub-complexes, A_i, is again a sub-complex of A. We will denote by $A_{\alpha_0 \cdots \alpha_p}$ the sub-complex $A_{\alpha_0} \cap \cdots \cap A_{\alpha_p}$.

Let $\mathrm{C}^i(A)$ denote the $\mathbb{Q}$-vector space of i co-chains of A, and $\mathrm{C}^{\bullet}(A)$, the complex

$$\cdots \to \mathrm{C}^{q-1}(A) \xrightarrow{d} \mathrm{C}^q(A) \xrightarrow{d} \mathrm{C}^{q+1}(A) \to \cdots$$

where $d : \mathrm{C}^q(A) \to \mathrm{C}^{q+1}(A)$ are the usual co-boundary homomorphisms. More precisely, given $\omega \in \mathrm{C}^q(A)$, and a $q+1$ simplex $[a_0, \ldots, a_{q+1}] \in A$,

$$\mathrm{d}\omega([a_0, \ldots, a_{q+1}]) = \sum_{0 \leq i \leq q+1} (-1)^i \omega([a_0, \ldots, \hat{a}_i, \ldots, a_{q+1}]) \tag{2}$$

(here and everywhere else in the paper $\hat{}$ denotes omission). Now extend $\mathrm{d}\omega$ to a linear form on all of $C_{q+1}(A)$ by linearity, to obtain an element of $\mathrm{C}^{q+1}(A)$.

The connecting homomorphisms are "generalized" restrictions and are defined below.

The *generalized Mayer–Vietoris sequence* is the following exact sequence of vector spaces:

$$0 \longrightarrow \mathrm{C}^{\bullet}(A) \xrightarrow{r^{\bullet}} \bigoplus_{1 \le \alpha_0 \le n} \mathrm{C}^{\bullet}(A_{\alpha_0}) \xrightarrow{\delta^{0,\bullet}} \bigoplus_{1 \le \alpha_0 < \alpha_1 \le n} \mathrm{C}^{\bullet}(A_{\alpha_0 \cdot \alpha_1}) \xrightarrow{\delta^{1,\bullet}} \cdots$$
$$\bigoplus_{1 \le \alpha_0 < \cdots < \alpha_p \le n} \mathrm{C}^{\bullet}(A_{\alpha_0 \cdots \alpha_p}) \xrightarrow{\delta^{p-1,\bullet}} \bigoplus_{1 \le \alpha_0 < \cdots < \alpha_{p+1} \le n} \mathrm{C}^{\bullet}(A_{\alpha_0 \cdots \alpha_{p+1}}) \xrightarrow{\delta^{p,\bullet}} \cdots$$

where $r^{\bullet}$ is induced by restriction and the connecting homomorphisms $\delta^{p,\bullet}$ are as follows.

Given an $\omega \in \bigoplus_{\alpha_0 < \cdots < \alpha_p} \mathrm{C}^q(A_{\alpha_0 \cdots \alpha_p})$ we define $\delta^{p,q}(\omega)$ as follows:

First note that $\delta^{p,q}\omega \in \bigoplus_{\alpha_0 < \cdots < \alpha_{p+1}} \mathrm{C}^q(A_{\alpha_0 \cdots \alpha_{p+1}})$, and it suffices to define

$$(\delta^{p,q}\omega)_{\alpha_0,\ldots,\alpha_{p+1}}$$

for each $(p+2)$-tuple $1 \le \alpha_0 < \cdots < \alpha_{p+1} \le n$. Note that $(\delta^{p,q}\omega)_{\alpha_0,\ldots,\alpha_{p+1}}$ is a linear form on the vector space, $C_q(A_{\alpha_0 \cdots \alpha_{p+1}})$, and hence is determined by its values on the q-simplices in the complex $A_{\alpha_0 \cdots \alpha_{p+1}}$. Furthermore, each q-simplex, $s \in A_{\alpha_0 \cdots \alpha_{p+1}}$, is automatically a simplex of the complexes

$$A_{\alpha_0 \cdots \hat{\alpha_i} \cdots \alpha_{p+1}}, \quad 0 \le i \le p+1.$$

We define

$$(\delta^{p,q}\omega)_{\alpha_0,\ldots,\alpha_{p+1}}(s) = \sum_{0 \le j \le p+1} (-1)^j \omega_{\alpha_0,\ldots,\hat{\alpha_j},\ldots,\alpha_{p+1}}(s).$$

The fact that the generalized Mayer–Vietoris sequence is exact is classical (see Rotman (1988) or Basu (2003) for example).

We now define the Mayer–Vietoris double complex of the complex A with respect to the sub-complexes A_{α_0}, $1 \le \alpha_0 \le n$, which we will denote by $\mathcal{N}^{\bullet,\bullet}(A)$ (we suppress the dependence of the complex on sub-complexes A_{α_0} in the notation since this dependence will be clear from the context).

Definition 3.2. The Mayer–Vietoris double complex of a simplicial complex A with respect to the sub-complexes A_{α_0}, $1 \le \alpha_0 \le n$, $\mathcal{N}^{\bullet,\bullet}(A)$, is the double complex defined by

$$\mathcal{N}^{p,q}(A) = \bigoplus_{1 \le \alpha_0 < \cdots < \alpha_p \le n} \mathrm{C}^q(A_{\alpha_0 \cdots \alpha_p}).$$

The horizontal differentials are as defined above. The vertical differentials are those induced by the ones in the different complexes, $\mathrm{C}^{\bullet}(A_{\alpha_0 \cdots \alpha_p})$.

$\mathcal{N}^{\bullet,\bullet}(A)$ is depicted in the following figure.

$$\begin{tikzcd}
{} & {} & \\
\bigoplus_{\alpha_0} \mathrm{C}^2(A_{\alpha_0}) \arrow[u] \arrow[r] & \bigoplus_{\alpha_0<\alpha_1} \mathrm{C}^2(A_{\alpha_0\cdot\alpha_1}) \arrow[u] \arrow[r] & \cdots \\
\bigoplus_{\alpha_0} \mathrm{C}^1(A_{\alpha_0}) \arrow[u] \arrow[r] & \bigoplus_{\alpha_0<\alpha_1} \mathrm{C}^1(A_{\alpha_0\cdot\alpha_1}) \arrow[u] \arrow[r] & \cdots \\
\bigoplus_{\alpha_0} \mathrm{C}^0(A_{\alpha_0}) \arrow[u] \arrow[r] & \bigoplus_{\alpha_0<\alpha_1} \mathrm{C}^0(A_{\alpha_0\cdot\alpha_1}) \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

For any $t \geq 0$, we denote by $\mathcal{N}_t^{\bullet,\bullet}(A)$ the following truncated complex:

$$\begin{aligned}
&\mathcal{N}_t^{p,q}(A) = \mathcal{N}^{p,q}(A), \quad 0 \leq p+q \leq t, \\
&\mathcal{N}_t^{p,q}(A) = 0, \quad \text{otherwise.}
\end{aligned}$$

The following proposition is classical (see Rotman (1988) or Basu (2003) for a proof) and follows from the exactness of the generalized Mayer–Vietoris sequence.

Proposition 3.3. *The spectral sequences, ${}'E_r, {}''E_r$, associated with $\mathcal{N}^{\bullet,\bullet}(A)$ converge to $\mathrm{H}^*(A)$ and thus*

$$\mathrm{H}^*(\mathrm{Tot}^\bullet(\mathcal{N}^{\bullet,\bullet}(A))) \cong \mathrm{H}^*(A).$$

Moreover, the homomorphism

$$\psi^\bullet : \mathrm{C}^\bullet(A) \to \mathrm{Tot}^\bullet(\mathcal{N}^{\bullet,\bullet}(A))$$

induced by the homomorphism $r^\bullet$ (in the generalized Mayer–Vietoris sequence) is a quasi-isomorphism.

We denote by $\mathrm{C}^\bullet_{\ell+1}(A)$ the truncation of the complex $\mathrm{C}^\bullet(A)$ after the $(\ell+1)$st term. As an immediate corollary we have that

Corollary 3.4. *For any $\ell \geq 0$, the homomorphism*

$$\psi^\bullet_{\ell+1} : \mathrm{C}^\bullet_{\ell+1}(A) \to \mathrm{Tot}^\bullet(\mathcal{N}^{\bullet,\bullet}_{\ell+1}(A)) \tag{3}$$

induced by the homomorphism $r^\bullet$ (in the generalized Mayer–Vietoris sequence) is a quasi-isomorphism. Hence, for $0 \leq i \leq \ell$,

$$\mathrm{H}^i(\mathrm{Tot}^\bullet(\mathcal{N}^{\bullet,\bullet}_{\ell+1}(A))) \cong \mathrm{H}^i(A).$$

Remark 3.5. Notice that in the truncated Mayer–Vietoris double complex, $\mathcal{N}_t^{\bullet,\bullet}(A)$, the 0th column is a complex having at most $t+1$ non-zero terms, the first column can have at most t non-zero terms, and in general the ith column has at most $t+1-i$ non-zero terms. This observation plays a crucial role in the inductive argument used later in the paper (in the proof of Proposition 4.3).

4. Double complexes associated with certain covers

We begin with a definition.

Definition 4.1. Let $\mathcal{P}$ be a finite subset of $\mathrm{R}[X_1, \ldots, X_k]$. A $\mathcal{P}$-closed formula is a formula constructed as follows:

For each $P \in \mathcal{P}$,

$$P = 0, \quad P \geq 0, \quad P \leq 0,$$

are $\mathcal{P}$-closed formulas.

If Φ_1 and Φ_2 are $\mathcal{P}$-closed formulas, $\Phi_1 \wedge \Phi_2$ and $\Phi_1 \vee \Phi_2$ are $\mathcal{P}$-closed formulas.

Clearly, $\mathcal{R}(\Phi) = \{x \subset \mathrm{R}^k \mid \Phi(x)\}$, the realization of a $\mathcal{P}$-closed formula Φ, is a closed semi-algebraic set and we call such a set a $\mathcal{P}$-closed semi-algebraic set.

For the rest of this section we consider a fixed family of polynomials, $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$, as well as a fixed $\mathcal{P}$-closed and bounded semi-algebraic set, $S \subset \mathrm{R}^k$. We also fix a number, $\ell, 0 \leq \ell \leq k$.

We define below (in Section 4.1) a finite set of indices, $\mathbb{A}_S$, which we call the set of admissible indices, and a map that associates with each $\alpha \in \mathbb{A}_S$ a closed and bounded semi-algebraic subset $X_\alpha \subset S$, which we call an admissible subset. With each $\alpha \in \mathbb{A}_S$, we associate its level, denoted as $\mathrm{level}(\alpha)$, which is an integer between 0 and ℓ. The set $\mathbb{A}_S$ will be partially ordered, and we denote by $\mathrm{ancestors}(\alpha) \subset \mathbb{A}_S$ the set of ancestors of α under this partial order. For $\alpha, \beta \in \mathbb{A}_S$, $\beta \in \mathrm{ancestors}(\alpha)$ implies that $X_\alpha \subset X_\beta$.

For each admissible index $\alpha \in \mathbb{A}_S$, we define a double complex, $\mathcal{M}^{\bullet,\bullet}(\alpha)$, such that

$$\mathrm{H}^i(\mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(\alpha))) \cong \mathrm{H}^i(X_\alpha), \quad 0 \leq i \leq \ell - \mathrm{level}(\alpha).$$

The main idea behind the construction of the double complex $\mathcal{M}^{\bullet,\bullet}(\alpha)$ is as follows. Associated with any cover of X_α there exists a double complex (the Mayer–Vietoris double complex) arising from the generalized Mayer–Vietoris exact sequence (see Definition 3.2). If the individual sets of the cover of X are all contractible, then the first column of the Mayer–Vietoris double complex is zero except at the first row. The cohomology groups of the associated total complex of the Mayer–Vietoris double complex are isomorphic to those of X_α and thus in order to compute $b_0(X_\alpha), \ldots, b_{\ell-\mathrm{level}(\alpha)}(X_\alpha)$, it suffices to compute a suitable truncation of the Mayer–Vietoris double complex. However, computing the Mayer–Vietoris double complex (even the truncated one) directly within a singly exponential time complexity is not possible by any known method, since we are unable to compute triangulations of semi-algebraic sets in singly exponential time. However, making use of the cover construction recursively, we are able to compute another double complex, $\mathcal{M}^{\bullet,\bullet}(\alpha)$, which has much smaller size but whose associated total complex is quasi-isomorphic to the truncated Mayer–Vietoris double complex and hence has isomorphic cohomology groups (see Proposition 4.6 below). The construction of $\mathcal{M}^{\bullet,\bullet}(\alpha)$ is possible in singly exponential time since the covers can be computed in singly exponential time.

Finally, given any closed and bounded semi-algebraic set $X \subset \mathrm{R}^k$, we will denote by $\mathcal{C}'(X)$ a fixed cover of X (we will use the construction in Basu et al. (2005) to compute such a cover).

4.1. Admissible sets and covers

We now define $\mathbb{A}_S$, and for each $\alpha \in \mathbb{A}_S$ a cover $\mathcal{C}(\alpha)$ of X_α obtained by enlarging the cover $\mathcal{C}'(X_\alpha)$.

Definition 4.2 (*Admissible Indices and Covers*). We define $\mathbb{A}_S$ by induction on level.

(1) Firstly, $0 \in \mathbb{A}_S$, $\text{level}(0) = 0$, $X_0 = S$, and $\mathcal{C}(0) = \mathcal{C}'(S)$. The admissible indices of level 1 consist of all formal products, $\beta = \alpha_0 \cdot \alpha_1 \cdots \alpha_{j-1} \cdot \alpha_j$, with $\alpha_i \in \mathcal{C}(0)$ and $0 \le j \le \ell + 1$, and we define the associated semi-algebraic set by

$$X_\beta = X_{\alpha_0} \cap \cdots \cap X_{\alpha_j}.$$

For each $\{\alpha_0, \ldots, \alpha_m\} \subset \{\beta_0, \ldots, \beta_n\} \subset \mathcal{C}(0)$, with $n \le \ell + 1$,

$$\alpha_0 \cdots \alpha_m \in \text{ancestors}(\beta_0 \cdots \beta_n),$$

and $0 \in \text{ancestors}(\beta_0 \cdots \beta_n)$.

(2) We now inductively define the admissible indices at level $i + 1$, in terms of the admissible indices at level $\le i$. For each $\alpha \in \mathbb{A}_S$ at level i, we define $\mathcal{C}(\alpha)$ as follows. Let $\text{ancestors}(\alpha) = \{\alpha_1, \ldots, \alpha_N\}$. Then,

$$\mathcal{C}(\alpha) = \dot{\bigcup_{\beta_i \in \mathcal{C}(\alpha_i), 1 \le i \le N}} \mathcal{C}'(\beta_1 \cdots \beta_N \cdot \alpha),$$

where $\dot{\bigcup}$ denotes the disjoint union. All formal products, $\beta = \alpha_0 \cdot \alpha_1 \cdots \alpha_j$, with $\alpha_i \in \mathcal{C}(\alpha)$ and $0 \le j \le \ell - i + 1$ are in $\mathbb{A}_S$, and we define

$$X_\beta = X_{\alpha_0} \cap \cdots \cap X_{\alpha_j},$$

and $\text{level}(\beta) = i + 1$.

For each $\{\alpha_0, \ldots, \alpha_m\} \subset \{\beta_0, \ldots, \beta_n\} \subset \mathcal{C}(\alpha)$, with $n \le \ell - i + 1$,

$$\alpha_0 \cdots \alpha_m \in \text{ancestors}(\beta_0 \cdots \beta_n),$$

and $\alpha \in \text{ancestors}(\beta_0 \cdots \beta_n)$.

Moreover, for $\alpha' \in \mathcal{C}'(\beta_1 \cdot \cdots \cdot \beta_N \cdot \alpha)$, each β_i is an ancestor of α'. We transitively close the ancestor relation, so that an ancestor of an ancestor is also an ancestor. Moreover, if $\alpha_0 \cdots \alpha_m, \beta_0 \cdots \beta_n \in \mathbb{A}_S$ are such that for every $j \in \{1, \ldots, n\}$ there exists $i \in \{1, \ldots, m\}$ such that α_i is an ancestor of β_j, then $\alpha_0 \cdots \alpha_m$ is an ancestor of $\beta_0 \cdots \beta_n$.

Finally, the set of admissible indices at level $i + 1$ is

$$\dot{\bigcup_{\alpha \in \mathbb{A}_S, \text{level}(\alpha) = i}} \{\alpha_0 \cdot \alpha_1 \cdots \alpha_j \mid \alpha_i \in \mathcal{C}(\alpha), 0 \le j \le \ell - i + 1\}.$$

Observe that by the above definition, if $\alpha, \beta \in \mathbb{A}_S$ and $\beta \in \text{ancestors}(\alpha)$, then each $\alpha' \in \mathcal{C}(\alpha)$ has a unique ancestor in each $\mathcal{C}(\beta)$, which we will denote by $a_{\alpha,\beta}(\alpha')$, and the mapping $a_{\alpha,\beta} : \mathcal{C}(\alpha) \to \mathcal{C}(\beta)$ is injective.

Now, suppose that we have a procedure for computing $\mathcal{C}'(X)$, for any given $\mathcal{P}'$-closed and bounded semi-algebraic set, X, such that the number and the degrees of the polynomials appearing in the descriptions of the semi-algebraic sets, $X_\alpha, \alpha \in \mathcal{C}'(X)$, are bounded by

$$D^{k^{c_1}}, \tag{4}$$

where $c_1 > 0$ is some absolute constant, and $D = \sum_{P \in \mathcal{P}'} \deg(P)$.

Using the above procedure for computing $\mathcal{C}'(X)$, and the definition of $\mathbb{A}_S$, we have the following quantitative bounds on $\#\mathbb{A}_S$ and the semi-algebraic sets $X_\alpha, \alpha \in \mathbb{A}_S$, which is crucial in proving the complexity bound of our algorithm.

Proposition 4.3. *Let $S \subset \mathrm{R}^k$ be a bounded $\mathcal{P}$-closed semi-algebraic set, where $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$ is a family of s polynomials of degree at most d. Then $\#A_S$, as well as the number of polynomials used to define the semi-algebraic sets $X_\alpha, \alpha \in \mathbb{A}_S$ and the degrees of these polynomials, are all bounded by $(sd)^{k^{O(\ell)}}$.*

Proof. Given $\alpha \in \mathbb{A}_S$ with $\mathrm{level}(\alpha) = j$, we first prove by induction on $\mathrm{level}(\alpha)$ that

$$\#\mathrm{ancestors}(\alpha) \leq 2^{\sum_{i=0}^{j}(\ell-i+3)} = 2^{(j+1)(\ell+3)-j(j+1)/2}.$$

The claim is clearly true if $\mathrm{level}(\alpha) = 0$. Otherwise, from the definition of $\mathbb{A}_S$, there exists $\beta \in \mathbb{A}_S$, with $\mathrm{level}(\beta) = j - 1$, such that $\alpha = \gamma_0 \cdots \gamma_m$, $\gamma_i \in \mathcal{C}(\beta)$ and $m \leq \ell - j + 2$.

For each γ_i, we have

$$\mathrm{ancestors}(\gamma_i) = \mathrm{ancestors}(\beta) \cup \{a_{\beta,\theta}(\gamma_i) \mid \theta \in \mathrm{ancestors}(\beta)\},$$

and it follows that

$$\begin{aligned}\mathrm{ancestors}(\alpha) = \mathrm{ancestors}(\beta) \cup \{&a_{\beta,\theta}(\gamma_{i_0}) \cdots a_{\beta,\theta}(\gamma_{i_n}) \mid \\ &\theta \in \mathrm{ancestors}(\beta), \{i_0, \ldots, i_n\} \subset \{1, \ldots, m\}\}.\end{aligned}$$

Hence,

$$\begin{aligned}\#\mathrm{ancestors}(\alpha) &= \#\mathrm{ancestors}(\beta) \cdot 2^m \\ &\leq \#\mathrm{ancestors}(\beta) \cdot 2^{\ell-j+3} \\ &\leq 2^{\sum_{i=0}^{j-1}(\ell-i+3)} \cdot 2^{\ell-j+3} \quad \text{(by induction hypothesis)} \\ &\leq 2^{\sum_{i=0}^{j}(\ell-i+3)} \\ &= 2^{(j+1)(\ell+3)-j(j+1)/2}.\end{aligned}$$

Thus, there exists some absolute constant $c_2 > 0$ such that for any $\alpha \in \mathbb{A}_S$ we have

$$\#\mathrm{ancestors}(\alpha) \leq 2^{c_2\ell^2}.$$

We now prove again by induction on the level that there exists an absolute constant $c > 0$ such that the number of elements of $\mathbb{A}_S$ of level $\leq j$, as well as the number of polynomials needed to define the associated semi-algebraic sets, and the degrees of these polynomials, are all bounded by $(sd)^{k^{cj}}$.

The claim is clear for level 0. Now assume that the claim holds for level $< j$. As before, given $\alpha \in \mathbb{A}_S$ with $\mathrm{level}(\alpha) = j$, there exists $\beta \in \mathbb{A}_S$ with $\mathrm{level}(\beta) = j - 1$, such that $\alpha = \gamma_0 \cdots \gamma_m$, $\gamma_i \in \mathcal{C}(\beta)$ and $m \leq \ell - j + 2$. We have that $\#\mathrm{ancestors}(\beta) \leq 2^{c_2\ell^2}$ by the previous paragraph. Let $\mathrm{ancestors}(\beta) = \{\theta_1, \ldots, \theta_N\}$. Then,

$$\#\mathcal{C}(\theta_i) \leq (sd)^{k^{c(j-1)}},$$

for $1 \leq i \leq N$ by the induction hypothesis.

In order to bound

$$\#\mathcal{C}(\beta) = \# \bigcup_{\beta_i \in \mathcal{C}(\theta_i), 1 \leq i \leq N} \mathcal{C}'(\beta_1 \cdot \cdots \cdot \beta_N \cdot \beta),$$

first observe that $N \leq 2^{c_2 \ell^2}$ and hence the union on the right hand side is over an index set of cardinality bounded by

$$(sd)^{k^{c(j-1)} 2^{c_2 \ell^2}},$$

and each set in the union has cardinality bounded by

$$\begin{aligned} M &= (2^{c_2 \ell^2} (sd)^{k^{c(j-1)}})^{k^{c_1}} \\ &= 2^{c_2 \ell^2 k^{c_1}} (sd)^{k^{cj-(c-c_1)}}, \end{aligned}$$

where c_1 is the constant defined before in (4) above.

Thus, the total number of admissible indices at level j is bounded by the total number of admissible indices at level $j-1$ times $\sum_{0 \leq i \leq \ell - j + 3} \binom{M}{i}$. It follows that if c chosen large enough with respect to the constants c_1, c_2, then for all k large enough, the total number of admissible indices at level j is at most

$$(sd)^{k^{cj}}.$$

The bounds on the number and degrees of polynomials appearing in the description can be proved similarly using the same induction scheme. □

4.2. *Double complex associated with a cover*

Given the different covers described above, we now associate with each $\alpha \in \mathbb{A}_S$ a double complex, $\mathcal{M}^{\bullet,\bullet}(\alpha)$, and for every $\beta \in \mathbb{A}_S$, such that $\alpha \in \text{ancestors}(\beta)$, and $\text{level}(\alpha) = \text{level}(\beta)$, a restriction homomorphism:

$$r^{\bullet,\bullet}_{\alpha,\beta} : \mathcal{M}^{\bullet,\bullet}(\alpha) \to \mathcal{M}^{\bullet,\bullet}(\beta),$$

satisfying the following:

(1)
$$\mathrm{H}^i(\mathrm{Tot}^{\bullet}(\mathcal{M}^{\bullet,\bullet}(\alpha))) \cong \mathrm{H}^i(X_\alpha), \quad \text{for } 0 \leq i \leq \ell - \text{level}(\alpha). \tag{5}$$

(2) The restriction homomorphism
$$r^{\bullet,\bullet}_{\alpha,\beta} : \mathcal{M}^{\bullet,\bullet}(\alpha) \to \mathcal{M}^{\bullet,\bullet}(\beta),$$
induces the restriction homomorphisms between the cohomology groups:
$$r^{*}_{\alpha,\beta} : \mathrm{H}^i(X_\alpha) \to \mathrm{H}^i(X_\beta)$$
for $0 \leq i \leq \ell - \text{level}(\alpha)$ via the isomorphisms in (5).

We now describe the construction of the double complex $\mathcal{M}^{\bullet,\bullet}(\alpha)$ and prove that it has the properties stated above. The double complex $\mathcal{M}^{\bullet,\bullet}(\alpha)$ is constructed inductively using induction on $\text{level}(\alpha)$.

Definition 4.4. The base case is when $\text{level}(\alpha) = \ell$. In this case the double complex, $\mathcal{M}^{\bullet,\bullet}(\alpha)$, is defined by

$$\begin{aligned} \mathcal{M}^{0,0}(\alpha) &= \bigoplus_{\alpha_0 \in \mathcal{C}(\alpha)} \mathrm{H}^0(X_{\alpha_0}), \\ \mathcal{M}^{1,0}(\alpha) &= \bigoplus_{\alpha_0,\alpha_1 \in \mathcal{C}(\alpha)} \mathrm{H}^0(X_{\alpha_0 \cdot \alpha_1}), \\ \mathcal{M}^{p,q}(\alpha) &= 0, \quad \text{if } q > 0 \text{ or } p > 1. \end{aligned}$$

This is shown diagrammatically below.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \\
\bigoplus_{\alpha_0 \in \mathcal{C}(\alpha)} \mathrm{H}^0(X_{\alpha_0}) \arrow[r, "\delta"] \arrow[u] & \bigoplus_{\alpha_0,\alpha_1 \in \mathcal{C}(\alpha)} \mathrm{H}^0(X_{\alpha_0 \cdot \alpha_1}) \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

The only non-trivial homomorphism in the above complex

$$\delta : \bigoplus_{\alpha_0 \in \mathcal{C}(\alpha)} \mathrm{H}^0(X_{\alpha_0}) \longrightarrow \bigoplus_{\alpha_0,\alpha_1 \in \mathcal{C}(\alpha)} \mathrm{H}^0(X_{\alpha_0 \cdot \alpha_1})$$

is defined as follows:

$$\delta(\phi)_{\alpha_0,\alpha_1} = (\phi_{\alpha_1} - \phi_{\alpha_0})|_{X_{\alpha_0 \cdot \alpha_1}} \quad \text{for } \phi \in \bigoplus_{\alpha_0 \in \mathcal{C}(\alpha)} \mathrm{H}^0(X_{\alpha_0}).$$

For every $\beta \in \mathbb{A}_S$ such that $\alpha \in \text{ancestors}(\beta)$, and $\text{level}(\alpha) = \text{level}(\beta) = \ell$, we define $r^{0,0}_{\alpha,\beta} : \mathcal{M}^{0,0}(\alpha) \to \mathcal{M}^{0,0}(\beta)$ as follows.

Recall that $\mathcal{M}^{0,0}(\alpha) = \bigoplus_{\alpha_0 \in \mathcal{C}(\alpha)} \mathrm{H}^0(X_{\alpha_0})$, and $\mathcal{M}^{0,0}(\beta) = \bigoplus_{\beta_0 \in \mathcal{C}(\beta)} \mathrm{H}^0(X_{\beta_0})$.

For $\phi \in \mathcal{M}^{0,0}(\alpha)$ and $\beta_0 \in \mathcal{C}(\beta)$ we define

$$r^{0,0}_{\alpha,\beta}(\phi)_{\beta_0} = \phi_{a_{\beta,\alpha}(\beta_0)}|_{X_{\beta_0}}.$$

We define $r^{1,0}_{\alpha,\beta} : \mathcal{M}^{1,0}(\alpha) \to \mathcal{M}^{1,0}(\beta)$ in a similar manner. More precisely, for $\phi \in \mathcal{M}^{0,0}(\alpha)$ and $\beta_0, \beta_1 \in \mathcal{C}(\beta)$, we define

$$r^{1,0}_{\alpha,\beta}(\phi)_{\beta_0,\beta_1} = \phi_{a_{\beta,\alpha}(\beta_0) \cdot a_{\beta,\alpha}(\beta_1)}|_{X_{\beta_0 \cdot \beta_1}}.$$

(The inductive step.) In general the $\mathcal{M}^{p,q}(\alpha)$ are defined as follows using induction on $\text{level}(\alpha)$ and with $n_\alpha = \ell - \text{level}(\alpha) + 1$.

$$\begin{aligned}
\mathcal{M}^{0,0}(\alpha) &= \bigoplus_{\alpha_0 \in \mathcal{C}(\alpha)} \mathrm{H}^0(X_{\alpha_0}), \\
\mathcal{M}^{0,q}(X) &= 0, && 0 < q, \\
\mathcal{M}^{p,q}(\alpha) &= \bigoplus_{\alpha_0 < \cdots < \alpha_p,\ \alpha_i \in \mathcal{C}(\alpha)} \mathrm{Tot}^q(\mathcal{M}^{\bullet,\bullet}(\alpha_0 \cdots \alpha_p)), && 0 < p,\ 0 < p + q \le n_\alpha, \\
\mathcal{M}^{p,q}(\alpha) &= 0, && \text{else.}
\end{aligned}$$

The double complex $\mathcal{M}^{\bullet,\bullet}(\alpha)$ is shown in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & \cdots & 0 \\
0 \arrow[r] \arrow[u] & \bigoplus\limits_{\alpha_0<\alpha_1} \mathrm{Tot}^{n_\alpha-1}(\mathcal{M}^{\bullet,\bullet}(\alpha_0\cdot\alpha_1)) \arrow[r, "\delta"] \arrow[u] & \cdots & 0 \arrow[u] \\
0 \arrow[r] \arrow[u, "d"] & \bigoplus\limits_{\alpha_0<\alpha_1} \mathrm{Tot}^{n_\alpha-2}(\mathcal{M}^{\bullet,\bullet}(\alpha_0\cdot\alpha_1)) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots & 0 \arrow[u, "d"] \\
\vdots & \vdots & \vdots & \vdots \\
0 \arrow[r] & \bigoplus\limits_{\alpha_0<\alpha_1} \mathrm{Tot}^{2}(\mathcal{M}^{\bullet,\bullet}(\alpha_0\cdot\alpha_1)) \arrow[r, "\delta"] & \cdots & 0 \\
0 \arrow[r] \arrow[u, "d"] & \bigoplus\limits_{\alpha_0<\alpha_1} \mathrm{Tot}^{1}(\mathcal{M}^{\bullet,\bullet}(\alpha_0\cdot\alpha_1)) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots & 0 \arrow[u, "d"] \\
\bigoplus\limits_{X_{\alpha_0}\in\mathcal{C}_X} \mathrm{H}^0(X_{\alpha_0}) \arrow[r, "\delta"] \arrow[u, "d"] & \bigoplus\limits_{\alpha_0<\alpha_1} \mathrm{Tot}^{0}(\mathcal{M}^{\bullet,\bullet}(\alpha_0\cdot\alpha_1)) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots & \bigoplus\limits_{\alpha_0<\cdots<\alpha_{n_\alpha}} \mathrm{Tot}^{0}(\mathcal{M}^{\bullet,\bullet}(\alpha_0\cdots\alpha_{n_\alpha})) \arrow[u, "d"]
\end{tikzcd}$$

The vertical homomorphisms, d, in $\mathcal{M}^{\bullet,\bullet}(\alpha)$ are those induced by the differentials in the various

$$\mathrm{Tot}^{\bullet}(\mathcal{M}^{\bullet,\bullet}(\alpha_0\cdots\alpha_p)), \quad \alpha_i\in\mathcal{C}(\alpha).$$

The horizontal ones are defined by generalized restriction as follows. Let

$$\phi\in\bigoplus_{\alpha_0<\cdots<\alpha_p,\alpha_i\in\mathcal{C}(\alpha)} \mathrm{Tot}^q(\mathcal{M}^{\bullet,\bullet}(\alpha_0\cdots\alpha_p)),$$

with

$$\phi_{\alpha_0,\ldots,\alpha_p}=\bigoplus_{0\le j\le q}\phi^j_{\alpha_0,\ldots,\alpha_p},$$

and

$$\phi^j_{\alpha_0,\ldots,\alpha_p}\in\mathcal{M}^{j,q-j}(\alpha_0\cdots\alpha_p).$$

We define

$$\delta:\bigoplus_{\alpha_0<\cdots<\alpha_p,\alpha_i\in\mathcal{C}(\alpha)}\mathrm{Tot}^q(\mathcal{M}^{\bullet,\bullet}(\alpha_0\cdots\alpha_p))\longrightarrow\bigoplus_{\alpha_0<\cdots<\alpha_{p+1}}\mathrm{Tot}^q(\mathcal{M}^{\bullet,\bullet}(\alpha_0\cdots\alpha_{p+1}))$$

by

$$\delta(\phi)_{\alpha_0,\ldots,\alpha_{p+1}} = \bigoplus_{0\le i\le p+1} (-1)^i \bigoplus_{0\le j\le q} r^{j,q-j}_{\alpha_0\cdots\hat{\alpha_i}\cdots\alpha_{p+1},\alpha_0\cdots\alpha_{p+1}}(\phi^j_{\alpha_0,\ldots,\hat{\alpha_i},\ldots,\alpha_{p+1}}),$$

noting that for each i, $0 \le i \le p+1$, $\alpha_0\cdots\hat{\alpha_i}\cdots\alpha_{p+1}$ is an ancestor of $\alpha_0\cdots\alpha_{p+1}$, and

$$\mathrm{level}(\alpha_0\cdots\hat{\alpha_i}\cdots\alpha_{p+1}) = \mathrm{level}(\alpha_0\cdots\alpha_{p+1}) = \mathrm{level}(\alpha)+1,$$

and hence the homomorphisms $r^{j,q-j}_{\alpha_0\cdots\hat{\alpha_i}\cdots\alpha_{p+1},\alpha_0\cdots\alpha_{p+1}}$ are already defined by induction.

Now let $\alpha, \beta \in \mathbb{A}_S$ with α an ancestor of β and $\mathrm{level}(\alpha) = \mathrm{level}(\beta)$. We define the restriction homomorphism

$$r^{\bullet,\bullet}_{\alpha,\beta} : \mathcal{M}^{\bullet,\bullet}(\alpha) \longrightarrow \mathcal{M}^{\bullet,\bullet}(\beta)$$

as follows.

As before, for $\phi \in \mathcal{M}^{0,0}(\alpha)$ and $\beta_0 \in \mathcal{C}(\beta)$ we define

$$r^{0,0}_{\alpha,\beta}(\phi)_{\beta_0} = \phi_{a_{\beta,\alpha}(\beta_0)}|_{X_{\beta_0}}.$$

For $0 < p$, $0 < p+q \le \ell - \mathrm{level}(\alpha)+1$, we define

$$r^{p,q}_{\alpha,\beta} : \mathcal{M}^{p,q}(\alpha) \to \mathcal{M}^{p,q}(\beta),$$

as follows.

Let $\phi \in \mathcal{M}^{p,q}(\alpha) = \bigoplus_{\alpha_0<\cdots<\alpha_p,\ \alpha_i\in\mathcal{C}(\alpha)} \mathrm{Tot}^q(\mathcal{M}^{\bullet,\bullet}(\alpha_0\cdots\alpha_p))$. We define

$$r^{p,q}_{\alpha,\beta}(\phi) = \bigoplus_{\beta_0<\cdots<\beta_p,\beta_i\in\mathcal{C}(\beta)} \bigoplus_{0\le i\le q} r^{i,q-i}_{a_{\beta,\alpha}(\beta_0\cdots\beta_p),\beta_0\cdots\beta_p}\phi^i_{a_{\beta,\alpha}(\beta_0),\ldots,a_{\beta,\alpha}(\beta_p)},$$

where $a_{\beta,\alpha}(\beta_0\cdots\beta_p) = a_{\beta,\alpha}(\beta_0)\cdots a_{\beta,\alpha}(\beta_p)$. Note that each of the $a_{\beta,\alpha}(\beta_i)$, $0 \le i \le p$, are all distinct and belong to $\mathcal{C}(\alpha)$. Moreover,

$$\mathrm{level}(a_{\beta,\alpha}(\beta_0\cdots\beta_p)) = \mathrm{level}(\beta_0\cdots\beta_p) = \mathrm{level}(\alpha)+1,$$

and hence we can assume that the homomorphisms $r^{\bullet,\bullet}_{a_{\beta,\alpha}(\beta_0\cdots\beta_p),\beta_0\cdots\beta_p}$ used in the definition of $r^{\bullet,\bullet}_{\alpha,\beta}$ are already defined by induction.

It is easy to verify by induction on $\mathrm{level}(\alpha)$ that $\mathcal{M}^{\bullet,\bullet}(\alpha)$, defined as above, is indeed a double complex, that is the homomorphisms d and δ satisfy the equations

$$d^2 = \delta^2 = 0, \qquad d\delta + \delta d = 0.$$

4.3. *Example*

Before proving the main properties of the complexes $\mathcal{M}^{\bullet,\bullet}(\alpha)$ defined above, we illustrate their construction by means of a simple example. We take for the set S, the unit sphere $S^2 \subset \mathrm{R}^3$. Even though this example looks very simple, it is actually illustrative of the main topological ideas behind the construction of the complex $\mathcal{M}^{\bullet,\bullet}(S)$ starting from a cover of S by two closed hemispheres meeting at the equator. Since the intersection of the two hemispheres is a topological circle which is not contractible, Theorem 2.2 is not applicable. Using Theorem 2.3 we can compute $\mathrm{H}^0(S)$, $\mathrm{H}^1(S)$, but it is not enough to compute $\mathrm{H}^2(S)$. The recursive construction of $\mathcal{M}^{\bullet,\bullet}$ described in the last section overcomes this problem and this is illustrated in the example.

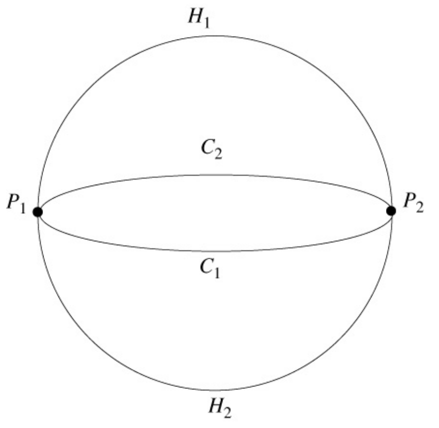

Fig. 2. Example of $S^2 \subset \mathrm{R}^3$.

Example 4.5. We first fix some notation (see Fig. 2). Let H_1 and H_2 denote the closed upper and lower hemispheres respectively. Let $H_{12} = H_1 \cap H_2$ denote the equator, and let $H_{12} = C_1 \cup C_2$, where C_1, C_2 are closed semi-circular arcs. Finally, let $C_{12} = C_1 \cap C_2 = \{P_1, P_2\}$, where P_1, P_2 are two antipodal points.

For the purpose of this example, we will take for the covers $\mathcal{C}'$ the obvious ones, namely:

$$\begin{aligned}
\mathcal{C}'(S) &= \{H_1, H_2\},\\
\mathcal{C}'(H_i) &= \{H_i\}, \quad i = 1, 2,\\
\mathcal{C}'(H_{12}) &= \{C_1, C_2\},\\
\mathcal{C}'(C_i) &= \{C_i\}, \quad i = 1, 2,\\
\mathcal{C}'(C_{12}) &= \{P_1, P_2\},\\
\mathcal{C}'(P_i) &= \{P_i\}, \quad i = 1, 2.
\end{aligned}$$

Note that, in order not to complicate notation further, we are using the same names for the elements of $\mathcal{C}'(\cdot)$, as well as their associated sets. Strictly speaking, we should have defined

$$\mathcal{C}'(S) = \{\alpha_1, \alpha_2\}, \qquad X_{\alpha_1} = H_1, \qquad X_{\alpha_2} = H_2, \ldots.$$

However, since each set occurs at most once, this does not create confusion in this example.

Note that the elements of the sets occurring on the right are all closed, bounded contractible subsets of S. It is now easy to check from Definition 4.2 that the elements of $\mathbb{A}_S$ in order of their levels are as follows.

(1) Level 0:

$$0 \in \mathbb{A}_S, \mathrm{level}(0) = 0,$$

and

$$\mathcal{C}(0) = \{\alpha_1, \alpha_2\}, \qquad X_{\alpha_1} = H_1, \qquad X_{\alpha_2} = H_2.$$

(2) Level 1: The elements of level 1 are

$$\alpha_1, \alpha_2, \alpha_1 \cdot \alpha_2,$$

and

$$\begin{aligned} \mathcal{C}(\alpha_1) &= \{\beta_1\}, \quad X_{\beta_1} = H_1, \\ \mathcal{C}(\alpha_2) &= \{\beta_2\}, \quad X_{\beta_2} = H_2, \\ \mathcal{C}(\alpha_1 \cdot \alpha_2) &= \{\beta_3, \beta_4\}, \quad X_{\beta_3} = C_1, X_{\beta_4} = C_2. \end{aligned}$$

(3) Level 2: The elements of level 2 are $\beta_1, \beta_2, \beta_3, \beta_4, \beta_3 \cdot \beta_4$. We also have

$$\begin{aligned} \mathcal{C}(\beta_i) &= \{\gamma_i\}, \quad X_{\gamma_i} = H_i, \quad i = 1, 2, \\ \mathcal{C}(\beta_i) &= \{\gamma_i\}, \quad X_{\gamma_i} = C_{i-2}, \quad i = 3, 4, \\ \mathcal{C}(\beta_3 \cdot \beta_4) &= \{\gamma_5, \gamma_6\}, \quad X_{\gamma_i} = P_{i-4}, \quad i = 5, 6. \end{aligned}$$

We now display diagrammatically the various complexes, $\mathcal{M}^{\bullet,\bullet}(\alpha)$, for $\alpha \in \mathbb{A}_S$ starting at level 2.

(1) Level 2: For $1 \le i \le 4$, we have

$$\mathcal{M}^{\bullet,\bullet}(\beta_i) = \begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & 0 \\ 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \\ \mathrm{H}^0(X_{\gamma_i}) \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \end{tikzcd}$$

Notice that for $1 \le i \le 4$,

$$\mathrm{H}^0(\mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(\beta_i))) \cong \mathrm{H}^0(X_{\beta_i}) \cong \mathbb{Q}.$$

The complex $\mathcal{M}^{\bullet,\bullet}(\beta_3 \cdot \beta_4)$ is shown below.

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & 0 \\ 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \\ \mathrm{H}^0(P_1) \bigoplus \mathrm{H}^0(P_2) \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \end{tikzcd}$$

Notice that

$$\mathrm{H}^0(\mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(\beta_3 \cdot \beta_4))) \cong \mathrm{H}^0(X_{\beta_3 \cdot \beta_4}) \cong \mathbb{Q} \oplus \mathbb{Q}.$$

(2) Level 1: For $i = 1, 2$, the complex $\mathcal{M}^{\bullet,\bullet}(\alpha_i)$ is as follows.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \\
\mathrm{H}^0(H_i) \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Notice that for $i = 1, 2$ and $j = 0, 1$,

$$\mathrm{H}^j(\mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(\alpha_i))) \cong \mathrm{H}^j(H_i).$$

The complex $\mathcal{M}^{\bullet,\bullet}(\alpha_1 \cdot \alpha_2)$ is shown below.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \\
\mathrm{H}^0(C_1) \bigoplus \mathrm{H}^0(C_2) \arrow[r] \arrow[u] & \mathrm{H}^0(P_1) \bigoplus \mathrm{H}^0(P_2) \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Notice that for $j = 0, 1$,

$$\mathrm{H}^j(\mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(\alpha_1 \cdot \alpha_2))) \cong \mathrm{H}^j(H_{12}).$$

(3) Level 0:

The complex $\mathcal{M}^{\bullet,\bullet}(0)$ is shown below:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & \mathrm{H}^0(P_1) \bigoplus \mathrm{H}^0(P_2) \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \\
\mathrm{H}^0(H_1) \bigoplus \mathrm{H}^0(H_2) \arrow[r, "\delta^{0,0}"] \arrow[u] & \mathrm{H}^0(C_1) \bigoplus \mathrm{H}^0(C_2) \arrow[r] \arrow[u, "d^{1,0}"] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

The matrices for the homomorphisms $\delta^{0,0}$ and $d^{1,0}$ in the obvious bases are both equal to

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

From the fact that the rank of the above matrix is 1, it is not too difficult to deduce that $\mathrm{H}^j(\mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(0))) \cong \mathrm{H}^j(S)$, for $j = 0, 1, 2$, that is

$$\mathrm{H}^0(\mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(0))) \cong \mathbb{Q},$$
$$\mathrm{H}^1(\mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(0))) \cong 0,$$
$$\mathrm{H}^2(\mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(0))) \cong \mathbb{Q}.$$

We now prove properties (1) and (2) of the various $\mathcal{M}^{\bullet,\bullet}(\alpha)$.

Proposition 4.6. *For each $\alpha \in \mathbb{A}_S$ the double complex $\mathcal{M}^{\bullet,\bullet}(\alpha)$ satisfies the following properties:*

(1) $\mathrm{H}^i(\mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(\alpha))) \cong \mathrm{H}^i(X_\alpha)$ *for* $0 \le i \le \ell - \mathrm{level}(\alpha)$.
(2) *For every $\beta \in \mathbb{A}_S$ such that α is an ancestor of β, and* $\mathrm{level}(\alpha) = \mathrm{level}(\beta)$*, the homomorphism $r^{\bullet,\bullet}_{\alpha,\beta} : \mathcal{M}^{\bullet,\bullet}(\alpha) \to \mathcal{M}^{\bullet,\bullet}(\beta)$ induces the restriction homomorphisms between the cohomology groups:*

$$r^* : \mathrm{H}^i(X_\alpha) \longrightarrow \mathrm{H}^i(X_\beta)$$

for $0 \le i \le \ell - \mathrm{level}(\alpha)$ *via the isomorphisms in* (1).

The main idea behind the proof of Proposition 4.6 is as follows. We consider a triangulation $h_0 : \Delta_0 \to S$ such that for any $\alpha \in A_S$, h_0 restricts to a semi-algebraic triangulation,

$h_\alpha : \Delta_\alpha \to X_\alpha$. Note that this implies that if $\beta \in \mathbb{A}_S$ and $\alpha \in \mathrm{ancestors}(\beta)$, then the triangulation $h_\alpha : \Delta_\alpha \to X_\alpha$ restricts to the triangulation $h_\beta : \Delta_\beta \to X_\beta$, and in particular Δ_β is a sub-complex of Δ_α.

For each $\alpha \in \mathbb{A}_S$, we have that $\Delta_\alpha = \cup_{\alpha_0 \in \mathcal{C}(\alpha)} \Delta_{\alpha_0}$, and each Δ_{α_0} for $\alpha_0 \in \mathcal{C}(\alpha)$ is a sub-complex of Δ_α. We denote by $\mathcal{N}^{\bullet,\bullet}(\Delta_\alpha)$ the Mayer–Vietoris double complex of Δ_α with respect to the sub-complexes $\Delta_{\alpha_0}, \alpha_0 \in \mathcal{C}(\alpha)$ (cf. Definition 3.2).

We define $n_\alpha = \ell - \mathrm{level}(\alpha) + 1$. Recall that $\mathcal{N}_{n_\alpha}^{\bullet,\bullet}(\Delta_\alpha)$ is the following truncated complex:

$$\begin{aligned}\mathcal{N}_{n_\alpha}^{p,q}(\Delta_\alpha) &= \mathcal{N}^{p,q}(\Delta_\alpha), \quad 0 \le p+q \le n_\alpha,\\ \mathcal{N}_{n_\alpha}^{p,q}(\Delta_\alpha) &= 0, \quad \text{otherwise.}\end{aligned}$$

By Corollary 3.4 we have that

$$\mathrm{H}^i(\mathrm{Tot}^\bullet(\mathcal{N}_{n_\alpha}^{\bullet,\bullet}(\Delta_\alpha))) \cong \mathrm{H}^i(X_\alpha), \quad 0 \le i \le \ell - \mathrm{level}(\alpha).$$

We then prove by induction on $\mathrm{level}(\alpha)$ that for each $\alpha \in \mathbb{A}_S$ there exists a double complex $D^{\bullet,\bullet}(\alpha)$ and homomorphisms

$$\begin{aligned}&\phi_\alpha^{\bullet,\bullet} : \mathcal{M}^{\bullet,\bullet}(\alpha) \longrightarrow D^{\bullet,\bullet}(\alpha)\\ &\psi_\alpha^\bullet : \mathrm{C}^\bullet(\Delta_\alpha) \longrightarrow \mathrm{Tot}^\bullet(D^{\bullet,\bullet}(\alpha))\end{aligned}$$

such that

$$\mathrm{Tot}^\bullet(\phi_\alpha^{\bullet,\bullet}) : \mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(\alpha)) \longrightarrow \mathrm{Tot}^\bullet(D^{\bullet,\bullet}(\alpha)),$$

as well as $\psi_\alpha^\bullet$ (as shown in the following figure) are quasi-isomorphisms.

$$\begin{tikzcd} & \mathrm{Tot}^\bullet(D^{\bullet,\bullet}(\alpha)) & \\ \mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(\alpha)) \arrow[ur, "\mathrm{Tot}^\bullet(\phi_\alpha^{\bullet,\bullet})"] & & \mathrm{C}^\bullet(\Delta_\alpha) \arrow[ul, "\psi_\alpha^\bullet"'] \end{tikzcd}$$

These quasi-isomorphisms will together imply that

$$\mathrm{H}^i(\mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(\alpha))) \cong \mathrm{H}^i(\mathrm{Tot}^\bullet(D^{\bullet,\bullet}(\alpha))) \cong \mathrm{H}^i(\mathrm{Tot}^\bullet(\mathcal{N}_{n_\alpha}^{\bullet,\bullet}(\Delta_\alpha))) \cong \mathrm{H}^i(X),$$

for $0 \le i \le \ell - \mathrm{level}(\alpha)$.

Proof of Proposition 4.6. The proof of the proposition is by induction on $\mathrm{level}(\alpha)$. When $\mathrm{level}(\alpha) = \ell$, we let $D^{\bullet,\bullet}(\alpha) = \mathcal{N}_{n_\alpha}^{\bullet,\bullet}(\Delta_\alpha)$, and define the homomorphisms $\phi_\alpha^{\bullet,\bullet}$, $\psi_\alpha^\bullet$ as follows. From the definition of $\mathcal{M}^{\bullet,\bullet}(\alpha)$ it is clear that in order to define $\phi_\alpha^{\bullet,\bullet}$, it suffices to define $\phi_\alpha^{0,0}$ and $\phi_\alpha^{0,1}$.

We define

$$\phi_\alpha^{0,0} : \mathcal{M}^{0,0}(\alpha) = \bigoplus_{\alpha_0 \in \mathcal{C}(\alpha)} \mathrm{H}^0(X_{\alpha_0}) \to \bigoplus_{\alpha_0 \in \mathcal{C}(\alpha)} \mathrm{C}^0(\Delta_{\alpha_0}) = \mathcal{N}_1^{0,0}(\Delta_{X_\alpha}),$$

by defining for $\theta \in \bigoplus_{\alpha_0 \in \mathcal{C}(\alpha)} \mathrm{H}^0(X_{\alpha_0})$, and any vertex v of the complex Δ_{α_0}, $\phi_\alpha^{0,0}(\theta)_{\alpha_0}(v)$ to be the value of the locally constant function θ_{α_0} on X_{α_0}.

Similarly, we define

$$\phi_\alpha^{0,1} : \bigoplus_{\alpha_0 < \alpha_1, \alpha_i \in \mathcal{C}(\alpha)} \mathrm{H}^0(X_{\alpha_0 \cdot \alpha_1}) \to \bigoplus_{\alpha_0 < \alpha_1 \alpha_i \in \mathcal{C}(\alpha)} \mathrm{C}^0(\Delta_{\alpha_0 \cdot \alpha_1}),$$

noting that

$$\mathcal{M}^{0,1}(\alpha) = \bigoplus_{\alpha_0<\alpha_1,\alpha_i\in\mathcal{C}(\alpha)} \mathrm{H}^0(X_{\alpha_0\cdot\alpha_1}),$$

and

$$\mathcal{N}_1^{0,0}(\Delta_\alpha) = \bigoplus_{\alpha_0<\alpha_1,\alpha_i\in\mathcal{C}(\alpha)} \mathrm{C}^0(\Delta_{\alpha_0\cdot\alpha_1}),$$

by defining for $\theta \in \bigoplus_{\alpha_0<\alpha_1,\alpha_i\in\mathcal{C}(\alpha)} \mathrm{H}^0(X_{\alpha_0\cdot\alpha_1})$, and any vertex v of the complex $\Delta_{\alpha_0\cdot\alpha_1}$, $\phi_\alpha^{0,1}(\theta)_{\alpha_0,\alpha_1}(v)$ to be the value of the locally constant function $\theta_{\alpha_0,\alpha_1}$ on the connected component of $X_{\alpha_0\cdot\alpha_1}$ containing $h_{\alpha_0\cdot\alpha_1}(v)$.

The homomorphism $\psi_\alpha^\bullet$ is induced by restriction as in the definition of $\psi_{\ell+1}^\bullet$ in Corollary 3.4.

It is now easy to verify that $\mathrm{Tot}^\bullet(\phi_\alpha^{\bullet,\bullet})$ and $\psi_\alpha^\bullet$ are indeed quasi-isomorphisms.

In general for $\alpha \in \mathbb{A}_S$, with $\mathrm{level}(\alpha) < \ell$, we have by induction that for each $\alpha_0,\ldots,\alpha_p,\alpha_{p+1} \in \mathcal{C}(\alpha), 0 \le p \le \ell - \mathrm{level}(\alpha) + 2$, there exists a double complex $D^{\bullet,\bullet}(\alpha_0\cdots\alpha_p)$ and quasi-isomorphisms

$$\begin{aligned}\mathrm{Tot}^\bullet(\phi^{\bullet,\bullet}_{\alpha_0\cdots\alpha_p}) : \mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(\alpha_0\cdots\alpha_p)) &\longrightarrow \mathrm{Tot}^\bullet(D^{\bullet,\bullet}(\alpha_0\cdots\alpha_p))\\ \psi^\bullet_{\alpha_0\cdots\alpha_p} : \mathrm{C}^\bullet_{n_\alpha}(\Delta_\alpha) &\longrightarrow \mathrm{Tot}^\bullet(D^{\bullet,\bullet}(\alpha_0\cdots\alpha_p)).\end{aligned}$$

We now define $D^{\bullet,\bullet}(\alpha)$ by

$$\begin{aligned}D^{p,q}(\alpha) &= \bigoplus_{\alpha_0<\cdots<\alpha_p,\ \alpha_i\in\mathcal{C}(\alpha)} \mathrm{Tot}^q(D^{\bullet,\bullet}(\alpha_0\cdots\alpha_p)), \quad 0\le p+q\le n_\alpha,\\ &= 0, \quad \text{else.}\end{aligned}$$

The homomorphism $\phi_\alpha^{\bullet,\bullet}$ is the one induced by the different $\mathrm{Tot}^\bullet(\phi^{\bullet,\bullet}_{\alpha_0\cdots\alpha_p})$ defined already by induction, that is

$$\phi_\alpha^{p,q} : \mathcal{M}^{p,q}(\alpha) \to D^{p,q}(\alpha),$$

is defined by

$$\phi_\alpha^{p,q} = \bigoplus_{\alpha_0<\cdots<\alpha_p,\ \alpha_i\in\mathcal{C}(\alpha)} \mathrm{Tot}^q(\phi^{\bullet,\bullet}_{\alpha_0\cdots\alpha_p}).$$

In order to define the homomorphism $\psi_\alpha^\bullet$, we first define a homomorphism

$$\rho_\alpha^{\bullet,\bullet} : \mathcal{N}^{\bullet,\bullet}_{n_\alpha}(\Delta_\alpha) \longrightarrow D^{\bullet,\bullet}(\alpha)$$

induced by the different $\psi^\bullet_{\alpha_0\cdots\alpha_p}$.

We define

$$\rho_\alpha^{p,q} : \mathcal{N}^{p,q}_{n_\alpha}(\Delta_\alpha) \to D^{p,q}(\alpha),$$

by

$$\rho_\alpha^{p,q} = \bigoplus_{\alpha_0<\cdots<\alpha_p,\ \alpha_i\in\mathcal{C}(\alpha)} \psi^q_{\alpha_0\cdots\alpha_p}.$$

We now compose the homomorphism

$$\mathrm{Tot}^\bullet(\rho_\alpha^{\bullet,\bullet}) : \mathrm{Tot}^\bullet(\mathcal{N}^{\bullet,\bullet}_{n_\alpha}(\Delta_\alpha)) \longrightarrow \mathrm{Tot}^\bullet(D^{\bullet,\bullet}(\alpha))$$

with the quasi-isomorphism

$$\psi^{\bullet}_{\alpha,n_\alpha} : \mathrm{C}^{\bullet}_{n_\alpha}(\Delta_\alpha) \longrightarrow \mathrm{Tot}^{\bullet}(\mathcal{N}^{\bullet,\bullet}_{n_\alpha}(\Delta_\alpha))$$

(see Proposition 3.3).

Using the induction hypothesis it is easy to see that the homomorphism $\phi^{\bullet,\bullet}_{\alpha}$ induces an isomorphism between the ${}'E_1$ terms of the corresponding spectral sequences. It follows from Proposition 3.1 that $\mathrm{Tot}^{\bullet}(\phi^{\bullet,\bullet}_{\alpha})$ is a quasi-isomorphism. A similar argument shows that $\mathrm{Tot}^{\bullet}(\rho^{\bullet,\bullet}_{\alpha})$ is also a quasi-isomorphism and hence so is $\psi^{\bullet}_{\alpha}$ since it is a composition of two quasi-isomorphisms. This completes the induction. □

5. Algorithmic preliminaries

In this section, we describe some algorithmic results which we need in the main algorithms.

5.1. Computation with complexes

In the description of our algorithm, we compute in a recursive way certain complicated double complexes, whose constructions have already been described in Section 4. The computation of a complex (or a double complex) means computing bases for each term of the complex (or double complex), as well as the matrices representing the differentials in these bases. Given a complex $\mathrm{C}^{\bullet}$ (in terms of some fixed bases), we can compute its homology groups $\mathrm{H}^*(\mathrm{C}^{\bullet})$ using elementary algorithms from linear algebra for computing kernels and images of vector space homomorphisms. Similarly, given a double complex, $D^{\bullet,\bullet}$, we can compute the complex $\mathrm{Tot}^{\bullet}(D^{\bullet,\bullet})$, as well as $\mathrm{H}^*(\mathrm{Tot}^{\bullet}(D^{\bullet,\bullet}))$, using standard algorithms from linear algebra. Since the naive algorithms (using say Gaussian elimination for computing kernels and images of linear maps) run in time polynomial in the dimensions of the vector spaces involved, it is clear that all the above computations involving complexes can be done in time polynomial in the sum of the dimensions of all terms in the input complex. This is sufficient for proving the main result of this paper, and we do not make any attempt to perform these computations in an optimal manner using more sophisticated algorithms.

5.2. Covers by contractible sets

We first recall some results proved in Basu et al. (2005) on constructing a singly exponential sized cover of a given closed semi-algebraic set by closed, contractible semi-algebraic sets. We recall the input, output and the complexity of the algorithms, referring the reader to Basu et al. (2005) for all details including the proofs of correctness.

We say that a finite set of polynomials $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]$ is in *strong ℓ-general position* if any $\ell + 1$ polynomials belonging to $\mathcal{P}$ have no common zeros R^k, and any ℓ polynomials belonging to $\mathcal{P}$ have at most a finite number of zeros in common in R^k.

In our algorithms we will use make use of *infinitesimals*. In order to do so, we will extend the ground field R to $\mathrm{R}\langle\varepsilon\rangle$, the real closed field of algebraic Puiseux series in ε with coefficients in R. The sign of a Puiseux series in $\mathrm{R}\langle\varepsilon\rangle$ agrees with the sign of the coefficient of the lowest degree term in ε. This induces a unique order on $\mathrm{R}\langle\varepsilon\rangle$ which makes ε infinitesimal: ε is positive and smaller than any positive element of R. When $a \in \mathrm{R}\langle\varepsilon\rangle$ is bounded by an element of R, $\lim_\varepsilon(a)$ is the constant term of a, obtained by substituting 0 for ε in a. We will also denote the field $\mathrm{R}\langle\varepsilon_s\rangle\cdots\langle\varepsilon_1\rangle$ by $\mathrm{R}\langle\varepsilon_s\cdots\varepsilon_1\rangle$, where $1 \gg \varepsilon_s \gg \cdots \gg \varepsilon_1 > 0$. More details regarding the use of infinitesimals in algorithms and complexity aspects of their use can be found in (amongst several possible sources) Basu et al. (2006).

5.2.1. *Replacement by closed sets without changing cohomology*

The following algorithm allows us to replace a given semi-algebraic set by a new one which is closed and defined by polynomials in general position and which has the same homotopy type as the given set. This construction is essentially due to Gabrielov and Vorobjov (Gabrielov and Vorobjov, 2005), where it was shown that the sum of the Betti numbers is preserved. The homotopy equivalence property is shown in Basu et al. (2005).

Algorithm 5.1 (*Cohomology Preserving Modification to Closed*).

INPUT. The input consists of

- a finite set of s polynomials

$$\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathrm{R}[X_1, \ldots, X_k], \quad \text{and}$$

- a subset $\Sigma \subset \mathrm{Sign}(\mathcal{P})$, defining a semi-algebraic set X by

$$X = \bigcup_{\sigma \in \Sigma} \mathcal{R}(\sigma).$$

OUTPUT.

A description of a $\mathcal{P}'$-closed and bounded semi-algebraic subset,

$$X' \subset \mathrm{R}\langle \varepsilon, \varepsilon_1, \ldots, \varepsilon_{2s} \rangle^{k+1},$$

with $\mathcal{P}' = \bigcup_{1 \le i \le s, 1 \le j \le 2s} \{P_i \pm \varepsilon_j\} \cup \{\varepsilon^2(X_1^2 + \cdots + X_k^2 + X_{k+1}^2) - 4, X_{k+1}\}$ such that

- $\mathrm{H}^*(X') \cong \mathrm{H}^*(X)$, and
- the family of polynomials $\mathcal{P}'$ is in strong $(k+1)$-general position.

PROCEDURE.
Step 1

- Let ε be an infinitesimal and let $\tilde{X}$ be the intersection of $\mathrm{Ext}(X, \mathrm{R}\langle\varepsilon\rangle)$ with the ball of center 0 and radius $1/\varepsilon$.
- Let $\mathcal{Q} = \mathcal{P} \cup \{\varepsilon^2(X_1^2 + \cdots + X_k^2 + X_{k+1}^2) - 4, X_{k+1}\}$.
- Replace $\tilde{X}$ by the $\mathcal{Q}$-semi-algebraic set S defined as the intersection of the cylinder $\tilde{X} \times \mathrm{R}\langle\varepsilon\rangle$ with the upper hemisphere defined by $\varepsilon^2(X_1^2 + \cdots + X_k^2 + X_{k+1}^2) = 4$, $X_{k+1} \ge 0$.

Step 2 Using the Gabrielov–Vorobjov construction described in Basu et al. (2005), replace S by a $\mathcal{P}'$-closed set S'. Output $\mathcal{P}'$ and the formula describing S'.

Complexity: Let d be the maximum degree among the polynomials in $\mathcal{P}$. The total complexity is bounded by $s^{k+1} d^{O(k)}$ (see Basu et al. (2005)). □

5.2.2. *Algorithm for computing covers by contractible sets*

The following algorithm described in detail in Basu et al. (2005) is used obtain a cover of a given closed and bounded semi-algebraic sets defined by polynomials in general position by closed, bounded and contractible semi-algebraic sets.

Algorithm 5.2 (*Cover by Contractible Sets*).
INPUT. The input consists of

- a finite set of s polynomials $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]$ in strong k-general position, with $\deg(P_i) \leq d$ for $1 \leq i \leq s$, and
- a bounded $\mathcal{P}$-closed semi-algebraic set S, defined by a $\mathcal{P}$-closed formula ϕ.

OUTPUT. A set of formulas $\{\phi_1, \ldots, \phi_M\}$ such that

- each $\mathcal{R}(\phi_i, \mathrm{R}'^k)$ is semi-algebraically contractible, and
- $$\bigcup_{1 \leq i \leq M} \mathcal{R}(\phi_i, \mathrm{R}'^k) = \mathrm{Ext}(S, \mathrm{R}'),$$
  where $\mathrm{R}' = \mathrm{R}\langle \varepsilon_{2s}, \ldots, \varepsilon_1 \rangle$.

Complexity: The total complexity is bounded by $s^{(k+1)^2} d^{O(k^5)}$ (see Basu et al. (2005)). □

6. Algorithm for computing the first ℓ Betti numbers of a semi-algebraic set

We are finally in a position to describe the main algorithm of this paper.

Algorithm 6.1 (*First ℓ Betti Numbers of a $\mathcal{P}$ Semi-algebraic Set*).
INPUT. The input consists of

- a finite set of polynomials $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]$, and
- a formula defining a $\mathcal{P}$ semi-algebraic set S.

OUTPUT. $b_0(S), \ldots, b_\ell(S)$.
PROCEDURE.
Step 1. Using Algorithm 5.1 (Cohomology Preserving Modification to Closed), replace S by a $\mathcal{P}'$-closed and bounded semi-algebraic set, $S' \subset \mathrm{R}'^{k+1}$, where $\mathrm{R}' = \mathrm{R}\langle \varepsilon, \varepsilon_1, \ldots, \varepsilon_{2s} \rangle$.

Step 2. Use Definition 4.2 to compute $\mathbb{A}_{S'}$ using Algorithm 5.2 (Cover by Contractible Sets) for computing the various $\mathcal{C}'(\cdot)$ occurring in the definition of $\mathbb{A}_{S'}$. For each element $\alpha \in \mathbb{A}_{S'}$, we also compute the set of ancestors $\mathrm{ancestors}(\alpha) \subset \mathbb{A}_{S'}$, $\mathcal{C}(\alpha)$, as well as $\mathrm{level}(\alpha)$.

More precisely, we do the following.

(1) (a) Initialize,

$$\mathbb{A}_{S'} \leftarrow \emptyset,$$

(b)

$$\begin{aligned} \mathbb{A}_{S'} &\leftarrow \mathbb{A}_{S'} \cup \{0\}, \\ \mathrm{level}(0) &\leftarrow 0, \\ X_0 &\leftarrow S', \\ \mathcal{C}(0) &\leftarrow \mathcal{C}'(S'), \\ \mathrm{ancestors}(0) &= \{0\}. \end{aligned}$$

Also, maintain a directed graph G with the current set $\mathbb{A}_{S'}$ as its set of vertices representing the ancestor–descendent relationships.

(2) For $i = 0$ to ℓ do the following:

(a) For each $\alpha \in \mathbb{A}_{S'}$ at level i, with ancestors$(\alpha) = \{\alpha_1, \ldots, \alpha_N\}$,

$$\mathcal{C}(\alpha) \leftarrow \bigcup_{\beta_i \in \mathcal{C}(\alpha_i), 1 \leq i \leq N} \mathcal{C}'(\beta_1 \cdots \beta_N \cdot \alpha)$$

using Algorithm 5.2 (Cover by Contractible Sets).

(b) For $0 \leq j \leq \ell - i + 1$ and each $\alpha_0, \ldots, \alpha_j \in \mathcal{C}(\alpha)$,

$$\mathbb{A}_{S'} \leftarrow \mathbb{A}_{S'} \cup \{\alpha_0 \cdot \alpha_1 \cdots \alpha_j\},$$
$$X_{\alpha_0 \cdots \alpha_j} \leftarrow X_{\alpha_0} \cap \cdots \cap X_{\alpha_j},$$
$$\mathrm{level}(\alpha_0 \cdot \alpha_1 \cdots \alpha_j) \leftarrow i + 1.$$

(c) For each $\{\alpha_0, \ldots, \alpha_i\} \subset \{\beta_0, \ldots, \beta_j\} \subset \mathcal{C}(\alpha)$, with $j \leq \ell - i + 1$,
$\quad$ ancestors$(\beta_0 \cdots \beta_j) \leftarrow$ ancestors$(\beta_0 \cdots \beta_j) \cup \{\alpha_0 \cdots \alpha_i\}$,
and update G.

(d) For each $\alpha' \in \mathcal{C}'(\beta_1 \cdot \cdots \cdot \beta_N \cdot \alpha)$,
$\quad$ ancestors$(\alpha') \leftarrow$ ancestors$(\alpha') \cup \{\beta_1, \ldots, \beta_N\}$.
and update G. Use any graph transitive closure algorithm to transitively close G. Accordingly update all the sets ancestors(α), $\alpha \in \mathbb{A}_{S'}$.

Step 3. Using Definition 4.4, compute for each $\alpha \in \mathbb{A}_{S'}$, the complex $\mathcal{M}^{\bullet,\bullet}(\alpha)$ starting with elements $\alpha \in \mathbb{A}_{S'}$ with level$(\alpha) = \ell$. Note that for each $\alpha \in \mathbb{A}_{S'}$, $\mathcal{C}(\alpha)$ has already been computed in Step 2. This allows us to compute matrices corresponding to all the homomorphisms in $\mathcal{M}^{\bullet,\bullet}(\alpha)$ for $\alpha \in \mathbb{A}_{S'}$ with level$(\alpha) = \ell$. The recursive definition of $\mathcal{M}^{\bullet,\bullet}(\alpha)$ implies that we can compute the matrices corresponding to all the homomorphisms in $\mathcal{M}^{\bullet,\bullet}(\alpha)$ for $\alpha \in \mathbb{A}_{S'}$ with level$(\alpha) < \ell$ once we have computed the same for $\mathcal{M}^{\bullet,\bullet}(\beta)$, for all $\beta \in \mathbb{A}_{S'}$ with level$(\beta) >$ level(α). The same is also true for the matrices corresponding to the restriction homomorphisms $r^{\bullet,\bullet}_{\alpha,\beta}$.

Step 4. For each i, $0 \leq i \leq \ell$, compute

$$b_i(S) = \dim_{\mathbb{Q}} \mathrm{H}^i(\mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(0))),$$

using standard linear algebra algorithms for computing dimensions of kernels and images of linear transformations.

Proof of correctness: The correctness of the algorithm is a consequence of the correctness of Algorithms 5.1 (Cohomology Preserving Modification to Closed), Algorithm 5.2 (Cover by Contractible Sets), and Proposition 4.6. $\square$

Complexity analysis: The complexity of Step 1 is bounded by $(sd)^{O(k)}$ using the complexity analysis of Algorithm 5.1 (Cohomology Preserving Modification to Closed). In order to bound the complexity of Step 2, note that the number of calls to Algorithm 5.2 (Cover by Contractible Sets). for computing various covers, $\mathcal{C}'(\cdot)$, is bounded by $\#\mathbb{A}_{S'}$, which in turn is bounded by $(sd)^{k^{O(\ell)}}$ by Proposition 4.3. Moreover, the cost of each such call is also bounded by $(sd)^{k^{O(\ell)}}$. The cost of all other operations, including updating the list of ancestors of elements of $\mathbb{A}_{S'}$, is polynomial in $\#\mathbb{A}_{S'}$. Thus, the total complexity of this step is bounded by $(sd)^{k^{O(\ell)}}$. Finally, the complexity of the computations involving linear algebra in Step 3 is polynomial in the cost of computing the various complexes $\mathcal{M}^{\bullet,\bullet}(\alpha)$, as well their sizes (see Section 5.1). All these are bounded by $(sd)^{k^{O(\ell)}}$ by Proposition 4.3. Thus, the complexity of the whole algorithm is bounded by $(sd)^{k^{O(\ell)}}$. $\square$

7. Implementation and practical aspects

The problem of computing all the Betti numbers of semi-algebraic sets in single exponential time (as well as the related problems of existence of single exponential sized triangulations or even stratifications) is considered a very important question in quantitative real algebraic geometry. The main result of this paper should be considered to be partial progress on this theoretical problem. Since the complexity of Algorithm 5.2 (Cover by Contractible Sets) for computing contractible covers is very high (even though single exponential), the complexity of Algorithm 6.1 is prohibitively expensive for practical implementation. The topological ideas underlying our algorithm have been implemented in a very limited setting in order to compute the first two Betti numbers of sets defined by quadratic inequalities (see Basu and Kettner (2005)). In this implementation, the covering is obtained by means different from Algorithm 5.2. The practical implementation for general semi-algebraic sets remains a formidable challenge.

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