Corrigendum


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Abstract

A theorem and a corollary in the paper cited in the title were stated incorrectly, as was pointed out by Christopher Hammond. We now state correctly and prove both of them. These results still generalize and explain the geometric meaning of the Cowen–Hurst norm formula. We also include additional references and provide an example relevant for further study.

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In [8] we established several norm estimates and exact norm formulas for composition operators acting on a general Bergman space $A^p$ of the unit disk, easy to adapt to the weighted Bergman or Hardy spaces. In particular, we were concerned with a generalization and geometric meaning of the well-known formula due to Cowen [2] and Hurst [7] for the norm of a composition operator induced by an affine map.

Christopher Hammond has observed that both Theorem 6 and Corollary 9 in [8] were incorrect as stated (for linear fractional maps). However, it is not difficult to verify that they remain true for linear symbols, as in [2] or [7]. Our geometric interpretation is thus still


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valid and the proofs are simple. Having the result for $A^p$ instead of just $A^2$ also appears to be a true generalization. Namely, even though knowing the norm of $C_{\varphi}$ as an $H^2$ operator readily yields an analogous formula for $H^p$ because of the properties of Blaschke products (see, for example, Proposition 2.15 of [4]), this phenomenon does not carry over directly to the Bergman spaces in any obvious way.

We first state and prove the correct version of Theorem 6 from [8], using the same notation as in [2], in Chapter 9 of [3], or in [7]. It should be noted that

$$\left(1 + |s|^2 - |t|^2\right)^2 - 4|s|^2 = \left(1 - |s|^2 + |t|^2\right)^2 - 4|t|^2,$$

so the quantity in our formula coincides with the expression that appears in the three sources cited.

**Theorem.** Let $\varphi(z) = sz + t$, $|s| + |t| < 1$; in this case, $\varphi(\mathbb{D})$ is the Euclidean disk $D(t, |s|)$ that coincides with some pseudo-hyperbolic disk $\Delta(a, r)$, $a \in \mathbb{D}$, $0 < r < 1$. Then the norm of $C_{\varphi}$ as an operator on the Bergman space $A^p$, $1 \leq p < \infty$, equals

$$\|C_{\varphi}\| = \left(\frac{r}{|s|}\right)^{2/p} = \left(\frac{2}{1 + |s|^2 - |t|^2 + \sqrt{(1 + |s|^2 - |t|^2)^2 - 4|s|^2}}\right)^{2/p}.$$  \hspace{1cm} (1)

**Proof.** After the change of variable $w = sz + t$, we get

$$\int_{\mathbb{D}} |f(\varphi(z))|^p dA(z) = \frac{1}{|s|^2} \int_{D(t, |s|)} |f(w)|^p dA(w) = \frac{1}{|s|^2} \int_{\Delta(a, r)} |f|^p dA \leq \left(\frac{r}{|s|}\right)^2 \int_{\mathbb{D}} |f|^p dA,$$

with equality when $f = I_a(1) = (-\varphi'_a)^{2/p}$, in view of Lemma 5 of [8] and the basic property of the isometries $I_a$. \hfill $\Box$

The error in the earlier proof occurred in the change of variable. The problem lies in the fact that even though $\varphi_a$ is its own inverse and $\varphi'_a = \varphi'$, rewriting the Jacobian as was done in [8] was not justified.

We should also point out that Theorem 6 as announced in [8] does not hold in full generality. Namely, the initial incorrect statement suggested that the norms of two operators $C_{\varphi}$ and $C_{\psi}$ induced by linear fractional maps should be equal as long as the image disks $\varphi(\mathbb{D})$ and $\psi(\mathbb{D})$ coincide. This is false as shown by the following example due to Hammond [6]. Consider the linear fractional self maps of the disk given by the formulas

$$\varphi(z) = \frac{z + 1}{z + 2}, \quad \psi(z) = \frac{z + 1}{3}.$$  \hspace{1cm} \hspace{1cm}

Then $\varphi(\mathbb{D}) = \psi(\mathbb{D}) = D(1/3, 1/3)$. However, the norms of the two operators acting on $A^2$ are different! According to, e.g., Theorem 1 of [8], on $A^2$ we get

$$\|C_{\varphi}\| \geq \frac{1}{1 - |\varphi(0)|^2} = \frac{4}{3},$$
while, according to the theorem above (in the Cowen–Hurst case $p = 2$, with $s = t = 1/3$) we obtain
\[ \|C_\psi\| = \frac{2}{1 + \sqrt{5}/9} < \frac{4}{3}. \]

It thus remains an interesting open question to find the operator norm of $C_\psi$ acting on $A^p$ (or $H^p$) when $\psi$ is an arbitrary linear fractional (but not affine) map and, in particular, knowing how the value $\psi(0)$ influences the norm formula. Explicit computation should still be possible at least for certain very special types of linear fractional maps.

The error in Corollary 9 of [8] was a careless one: the proof was given only in the case of maps $\psi$ for which the disk $\psi(\mathbb{D})$ touches the unit circle at the point $z = a/|a|$ and this happens rarely, so the formula again applies essentially only to the affine maps. The correct statement is as follows.

**Corollary.**

(a) Let $\psi$ be an affine self-map of $\mathbb{D}$ for which the disks $\mathbb{D}$ and $\psi(\mathbb{D})$ touch at one point, that is, $\psi(z) = sz + t$, $|s| + |t| = 1$, $s, t \neq 0$. Then, viewing $C_\psi$ as an $A^p$ operator,
\[ \|C_\psi\| = \left(\frac{1}{|s|}\right)^{2/p}. \]

(b) Let $\psi$ be a disk automorphism: $\psi(z) = \lambda \varphi_a(z)$, $|\lambda| = 1$, $a \in \mathbb{D}$. Then
\[ \|C_\psi\| = \left(\frac{1 + |a|}{1 - |a|}\right)^{2/p}. \]

**Proof.** (a) The tangential contact occurs at the point $\frac{t|s|}{s|t|}$. Observe that
\[ |\psi'(\xi)| = \begin{cases} 
|s|, & \text{when } \xi = \frac{t|s|}{s|t|}, \\
\infty, & \text{otherwise},
\end{cases} \]
while $\psi'(z) \equiv s$ in $\mathbb{D}$. Thus, $\inf_{\mathbb{D}} |\psi'| = \min_{\mathbb{T}} |\psi'| = |s|$ and we may apply Theorem 8 of [8] to get the desired conclusion.

(b) In this case,
\[ |\psi'(\xi)| = \begin{cases} 
\infty, & \text{when } \xi \neq a/|a|, \\
\frac{1 + |a|}{1 - |a|}, & \text{when } \xi = a/|a|,
\end{cases} \]
while inside the disk we have
\[ |\psi'(z)| = |\varphi'_a(z)| = \frac{1 - |a|^2}{|1 - \overline{a}z|^2} \geq \frac{1 - |a|}{1 + |a|} \]
and this lower bound is achieved as $z \to -a/|a|$. □

Note that in part (a) we get the limit case of the above theorem as $r \to 1$, as was to be expected, thus completing the norm computation for the remaining cases of affine symbols.

Finally, let us mention that in the most recent works [1,4,5] the same question was studied but from a slightly different point of view.
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References