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J. Math. Anal. Appl. 314 (2006) 436–454
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Exponential dichotomy roughness on Banach spaces

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Received 24 June 2004
Available online 23 May 2005
Submitted by Z.S. Athanassov

Abstract

In the present paper we extend existing results on exponential dichotomy roughness for linear ODE systems to infinite dimensional Banach space. We give new conditions for the existence of exponential dichotomy roughness in infinite dimensional space and in the finite interval case. We also improve previous results by indicating the exact values of the dichotomic constants of the perturbed equation.

Keywords: Linear differential equation; Perturbed equation; Exponential dichotomy; Roughness; Banach space

1. Introduction

Throughout this paper, if not specified, E is an infinite dimensional Banach space, and $\mathcal{L}(E)$ is the space of all bounded linear operators acting on E. Consider equation:

$$\frac{dx}{dt} = A(t)x, \tag{1}$$

where $A:\mathcal{I} \to \mathcal{L}(E)$ is a continuous operator function, $\mathcal{I}$ being an interval. We denote by $U(t)$ the Cauchy operator of Eq. (1).

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doi:10.1016/j.jmaa.2005.04.011

We say that (1) is exponential dichotomic if there are two bounded mutually complementary projections P and Q, acting on E and positive constants N_i, ν_i, $i = 1, 2$, such that:

$$\begin{cases} \|U(t)PU^{-1}(s)\| \leqslant N_1 e^{-\nu_1(t-s)}, & \text{for } t \geqslant s, \\ \|U(t)QU^{-1}(s)\| \leqslant N_2 e^{-\nu_2(s-t)}, & \text{for } s \geqslant t. \end{cases} \tag{2}$$

Consider the perturbed equation

$$\frac{dy}{dt} = \big[A(t) + B(t)\big]y, \tag{3}$$

where $B : \mathcal{I} \to \mathcal{L}(E)$ is a bounded and continuous operator function: $\sup_{t\in\mathcal{I}} \|B(t)\| = \delta < \infty$. Roughness means that if Eq. (1) has an exponential dichotomy (2), then for δ small enough, Eq. (3) is still exponential dichotomic, with some mutually complementary projections $\tilde{P}$ and $\tilde{Q}$, and dichotomic constants $\tilde{N}_i$, $\tilde{\nu}_i$, $i = 1, 2$:

$$\begin{cases} \|V(t)\tilde{P}V^{-1}(s)\| \leqslant \tilde{N}_1 e^{-\tilde{\nu}_1(t-s)}, & \text{for } t \geqslant s, \\ \|V(t)\tilde{Q}V^{-1}(s)\| \leqslant \tilde{N}_2 e^{-\tilde{\nu}_2(s-t)}, & \text{for } s \geqslant t. \end{cases} \tag{4}$$

Here $V(t)$ stands for the Cauchy operator of Eq. (3). We present a short historical of this problem. Notice that the results below were exposed for finite dimensional space.

In [2, Proposition 1, p. 34], it is shown that for $N_1 = N_2 = N$, $\nu_1 = \nu_2 = \nu$, and $\delta < \nu/(4N^2)$, exponential dichotomy for Eq. (3) is preserved, with dichotomic constants $\tilde{N}_1 = \tilde{N}_2 = 5N^2/2$, $\tilde{\nu}_1 = \tilde{\nu}_2 = \nu - 2N\delta$. Moreover, P and $\tilde{P}$ are similar.

In [2, Proposition 1, p. 42] it is proved that for any interval (finite or infinite), $\delta < \nu/(36N^5)$, $N_1 = N_2 = \nu$ and $\nu_1 = \nu_2 = \nu$, Eq. (3) has an exponential dichotomy (4) with $\tilde{\nu}_1 = \tilde{\nu}_2 = \tilde{\nu} = \nu - 6N^3\delta$, $\tilde{N}_1 = \tilde{N}_2 = \tilde{N} = 12N^3$. To prove this result Coppel used a special reducibility principle (Lemmas 1–3 in [2, pp. 39–41]).

[8] deals with the general case $N_1 \neq N_2$, $\nu_1 \neq \nu_2$. It is shown that for any δ satisfying

$$\delta\left(\frac{N_1}{\nu_1} + \frac{N_2}{\nu_2}\right) < \frac{1}{2},$$

the perturbed equation (3) is still exponential dichotomic, but the estimation of dichotomic constants $\tilde{N}_i$ is not very accurate (see [8, Theorem 2, p. 568]).

A real progress in estimation of δ is made in [6]. It is proved that if δ satisfies:

$$\delta\left(\frac{N_1}{\nu_1} + \frac{N_2}{\nu_2}\right) < 1, \tag{5}$$

then Eq. (3) exhibits an exponential dichotomy (4), with dichotomic constants $\tilde{\nu}_1 = \tilde{\nu}_2 = \tilde{\nu}$ and $\tilde{N}_1 = \tilde{N}_2 = \tilde{N}$.

Other papers like [5] and [13], investigate on exponential trichotomy roughness.

Exact bounds for dichotomic constants are exposed in [9,14–16], but under stronger conditions imposed to δ.

When turning to infinite dimensional Banach space, a major problem arises: the existence of a complement (or equivalent a corresponding bounded projection) for the subspace of initial values of bounded solutions of (3), needs to be proved. This is the main difference between the finite and infinite dimensional case. For this reason, Propositions 1–4 in [2, p. 22], do not apply to Banach spaces in absence of hypothesis above, as the reader may easily observe when lecturing [1, p. 170, Lemma 3.3, p. 171, Theorem 3.3 and p. 174, Theorem 3.3’].

Therefore the main results in [8] and [6], where Propositions 1–4 in [2, p. 22] were used, are not applicable to infinite dimensional Banach space. We especially refer to Theorem 1 in [8, p. 565], Theorem 2 in [8, p. 568], Theorem 3 in [8, p. 570] and also to Theorem 3.1 in [6, p. 45], as well as Theorem 3.2 in [6, p. 48].

To avoid these complications, throughout this paper we use a different method, in order to prove the existence of the named complement. The construction exposed in Section 3 is inspired by the arguments in the proof of Proposition 1 in [2, p. 34] and is based on the Contraction Mapping Theorem. Otherwise this type of argument is completely absent in [8] and [6].

In Section 4 we extend Theorem 3.1 from [6, p. 45] to infinite dimensional Banach space and also improve the existing result by giving the exact values of the new dichotomic constants.

In Section 5 we investigate on the existence of exponential dichotomy for the perturbed equation (3), defined on all $\mathbb{R}$. We also prove that condition (5) needs to be changed when in infinite dimensional.

In Section 6 we deal with the case of the finite interval, using a different method than that in [2, p. 42, Proposition 1]. This is because Lemmas 1–3 in [2, p. 39–41], that were used in the proof of proposition we refer, have not been proved for Banach space (but they still hold in Hilbert space, as one may notice from [1, p. 220, Theorem 1] or [3, p. 154, Theorem 1.2]). For example, works as [12] speculated exactly on this issue.

The reader will also observe the importance of our new dichotomic inequalities exposed in Section 2.

We consider as our main results: Lemma 2.1 in Section 2, Theorem 4.1 in Section 4, Theorems 5.3, 5.6 in Section 5, as well as Theorems 6.2, 6.3 and Corollary 6.4 in the last section.

2. Preliminaries

The main tool we use in this section is Lemma 2.1 in [3, p. 105]. Let N_i, ν_i, $i = 1, 2$, and δ be positive constants. Consider functions $x : [s, \infty) \to \mathbb{R}_+$, $y : (-\infty, s] \to \mathbb{R}_+$, supposed to be bounded and continuous, satisfying inequalities:

$$x(t) \leqslant N_1 e^{-\nu_1(t-s)} + \delta N_1 \int_s^t e^{-\nu_1(t-u)} x(u)\, du + \delta N_2 \int_t^\infty e^{-\nu_2(u-t)} x(u)\, du, \tag{6}$$

$$y(t) \leqslant N_2 e^{-\nu_2(s-t)} + \delta N_1 \int_{-\infty}^t e^{-\nu_1(t-u)} y(u)\, du + \delta N_2 \int_t^s e^{-\nu_2(u-t)} y(u)\, du. \tag{7}$$

Lemma 2.1. *If δ satisfy inequality* (5), *then there exist positive constants K_i, $i = 1, 2$, and $\tilde{\nu}$ such that*:

$$x(t) \leqslant K_1 e^{-\tilde{\nu}(t-s)}, \quad \textit{for } t \geqslant s,$$
$$y(t) \leqslant K_2 e^{-\tilde{\nu}(s-t)}, \quad \textit{for } s \geqslant t.$$

For constants K_i, $i = 1, 2$, *and* $\tilde{\nu}$ *we have estimations*:

$$\tilde{\nu} = \frac{1}{2}\Bigg[\delta(N_2 - N_1) + \nu_1 - \nu_2 + \sqrt{\big[\delta(N_2 - N_1) + \nu_1 - \nu_2\big]^2 + 4\nu_1\nu_2\left[1 - \delta\left(\frac{N_1}{\nu_1} + \frac{N_2}{\nu_2}\right)\right]}\,\Bigg], \tag{8}$$

$$K_1 = \frac{N_1(\tilde{\nu} + \nu_2)}{\tilde{\nu} + \nu_2 - \delta N_2}, \tag{9}$$

$$K_2 = \frac{N_2(\tilde{\nu} + \nu_1)}{\tilde{\nu} + \nu_1 - \delta N_1}. \tag{10}$$

Proof. Since δ verify (5), then Lemma 2.1 in [3, p. 105] is applicable. It follows that

$$x(t) \leqslant \Phi(t), \qquad y(t) \leqslant \Psi(t),$$

where the continuous and bounded functions $\Phi(t)$ and $\Psi(t)$ are defined by integral equations

$$\Phi(t) = N_1 e^{-\nu_1(t-s)} + \delta N_1 \int_s^t e^{-\nu_1(t-u)} \Phi(u)\,du + \delta N_2 \int_t^\infty e^{-\nu_2(u-t)} \Phi(u)\,du, \quad t \geqslant s, \tag{11}$$

$$\Psi(t) = N_2 e^{-\nu_2(s-t)} + \delta N_1 \int_{-\infty}^t e^{-\nu_1(t-u)} \Psi(u)\,du + \delta N_2 \int_t^s e^{-\nu_2(u-t)} \Psi(u)\,du, \quad s \geqslant t. \tag{12}$$

Elementary calculations show that both Φ and Ψ verify differential equation:

$$z'' + z'\big[\delta(N_2 - N_1) + \nu_1 - \nu_2\big] - z\nu_1\nu_2\left[1 - \delta\left(\frac{N_1}{\nu_1} + \frac{N_2}{\nu_2}\right)\right] = 0. \tag{13}$$

Set $\tilde{\nu} = -r_-$, where r_- is the negative root of the corresponding characteristic equation; this yields (8).

Put $\Phi(t) = K_1 e^{-\tilde{\nu}(t-s)}$ and $\Psi(t) = K_2 e^{-\tilde{\nu}(s-t)}$. Substituting Φ (respectively, Ψ) in (11) (respectively, (12)), we have

$$K_1 = N_1 + K_1 \frac{\delta N_2}{\tilde{\nu} + \nu_2},$$
$$K_2 = N_2 + K_2 \frac{\delta N_1}{\tilde{\nu} + \nu_1},$$

which give us estimations (9) (respectively, (10)). □

Corollary 2.2. *When replacing the symbols ∞, respectively $-\infty$, in inequalities* (6), *respectively* (7), *by a finite number all the statements in above lemma remain valid.*

Proof. Suppose $x : [s, b] \to \mathbb{R}_+$ is a continuous function, satisfying inequality (6′), which is obtained from (6) replacing $+\infty$ by $b \in \mathbb{R}$. Then we extend x to $[s, +\infty)$ as follows:

$$\tilde{x}(t) = \begin{cases} x(t), & \text{if } t \in [s, b], \\ -t\frac{x(b)}{\varepsilon} + \frac{x(b)}{\varepsilon}(b + \varepsilon), & \text{if } t \in (b, b + \varepsilon), \\ 0, & \text{if } t \in [b + \varepsilon, +\infty). \end{cases}$$

It easy to see that $\tilde{x}$ verifies inequality (6). According to Lemma 2.1, $\tilde{x}$ verifies the first inequality in this lemma, and therefore x. □

Comparing estimations (8)–(10) with their correspondents in Lemmas 1 and 2 in [8, pp. 561–564], we see that they are qualitative superior. In addition the proof is more simple.

3. Basic constructions

Let us denote by $\Gamma(t, s)$ the Green function of Eq. (1):

$$\Gamma(t, s) = \begin{cases} U(t)PU^{-1}(s), & \text{if } t \geqslant s, \\ U(t)QU^{-1}(s), & \text{if } s \geqslant t. \end{cases}$$

Set $I_+ = \{(t, s)\colon\ t \geqslant s,\ t, s \in \mathcal{I}\}$ and $I_- = \{(t, s)\colon\ s \geqslant t,\ t, s \in \mathcal{I}\}$.

Consider the Banach spaces

$$\mathcal{B}_+(\mathcal{I}) = \big\{X : I_+ \to \mathcal{L}(E)\colon\ X \text{ is continuous and bounded}\big\},$$
$$\mathcal{B}_-(\mathcal{I}) = \big\{Y : I_- \to \mathcal{L}(E)\colon\ Y \text{ is continuous and bounded}\big\}.$$

They are endowed with the supremum norm:

$$\|X\|_C = \sup_{(t,s)\in I_+} \big\|X(t, s)\big\|,$$
$$\|Y\|_C = \sup_{(t,s)\in I_-} \big\|Y(t, s)\big\|.$$

Define operator $K : \mathcal{B}_+(\mathcal{I}) \to \mathcal{B}_+(\mathcal{I})$ by

$$(KX)(t, s) = U(t)PU^{-1}(s) + \int_s^\infty \Gamma(t, u)B(u)X(u, s)\,du \tag{14}$$

and also $L:\mathcal{B}_- \to \mathcal{B}_-$,

$$(LY)(t,s) = U(t)QU^{-1}(s) + \int_{-\infty}^{s} \Gamma(t,u)B(u)Y(u,s)\,du. \tag{15}$$

We notify that in the definition of operator K (respectively, L) the interval $\mathcal{I}$ is supposed to be a neighborhood of $+\infty$ (respectively, $-\infty$), and $B:\mathcal{I} \to \mathcal{L}(E)$ is a continuous and bounded operator function such that $\sup_{t\in\mathcal{I}} \|B(t)\| = \delta < \infty$.

Notice that if δ satisfies (5), then both K and L are contractions:

$$\|KX_1 - KX_2\|_C \leqslant \theta\|X_1 - X_2\|_C,$$
$$\|LY_1 - LY_2\|_C \leqslant \theta\|Y_1 - Y_2\|_C.$$

Here $\theta = \delta(N_1/\nu_1 + N_2/\nu_2)$.

Lemma 3.1. *Operators K, L have unique fixed points $\tilde{X} \in \mathcal{B}_+(\mathcal{I})$, respectively $\tilde{Y} \in \mathcal{B}_-(\mathcal{I})$ that satisfy inequalities*:

$$\bigl\|\tilde{X}(t,s)\bigr\| \leqslant K_1 e^{-\tilde{\nu}(t-s)}, \quad \text{if } t \geqslant s, \tag{16}$$
$$\bigl\|\tilde{Y}(t,s)\bigr\| \leqslant K_2 e^{-\tilde{\nu}(s-t)}, \quad \text{if } s \geqslant t. \tag{17}$$

Moreover, for each fixed s, both $\tilde{X}$ and $\tilde{Y}$ are solutions of differential operator equation

$$\frac{dZ}{dt} = \bigl[A(t) + B(t)\bigr]Z$$

and constants in (16) *and* (17) *are given by* (8)–(10).

This result can be easily proved if we put $x(t) = \|\tilde{X}(t,s)\|$, respectively $y(t) = \|\tilde{Y}(t,s)\|$, in Lemma 2.1. Operator functions $\tilde{X}$ and $\tilde{Y}$ have another interesting property, illustrated in the following lemma:

Lemma 3.2. *For each $\tau \geqslant t \geqslant s$ we have identities*:

(i) $\tilde{X}(\tau,t)\tilde{X}(t,s) = \tilde{X}(\tau,s)$;
(ii) $\tilde{Y}(s,t)\tilde{Y}(t,\tau) = \tilde{Y}(s,\tau)$.

Proof. Fix $\tau \geqslant t \geqslant s$.

$$\begin{aligned}\tilde{X}(\tau,t)\tilde{X}(t,s) &= \Biggl[U(\tau)PU^{-1}(t) + \int_t^{\tau} U(\tau)PU^{-1}(u)B(u)\tilde{X}(u,t)\,du \\ &\quad - \int_{\tau}^{\infty} U(\tau)QU^{-1}(u)B(u)\tilde{X}(u,t)\,du\Biggr]\tilde{X}(t,s) \\ &= U(\tau)PU^{-1}(s) + \int_s^{t} U(\tau)PU^{-1}(u)B(u)\tilde{X}(u,s)\,du\end{aligned}$$

$$+\int_t^\tau U(\tau)PU^{-1}(u)B(u)\tilde{X}(u,t)\tilde{X}(t,s)\,du$$
$$-\int_\tau^\infty U(\tau)QU^{-1}(u)B(u)\tilde{X}(u,t)\tilde{X}(t,s)\,du.$$

Furthermore:

$$\tilde{X}(\tau,t)\tilde{X}(t,s)-\tilde{X}(\tau,s)=\int_t^\tau U(\tau)PU^{-1}(u)B(u)\big[\tilde{X}(u,t)\tilde{X}(t,s)-\tilde{X}(u,s)\big]\,du$$
$$-\int_\tau^\infty U(\tau)QU^{-1}(u)B(u)\big[\tilde{X}(u,t)\tilde{X}(t,s)-\tilde{X}(u,s)\big]\,du.$$

Consider function $\Psi:[t,\infty)\to\mathcal{L}(E)$ defined by $\Psi(u)=\tilde{X}(u,t)\tilde{X}(t,s)-\tilde{X}(u,s)$. We have:

$$\Psi(\tau)=\int_t^\tau U(\tau)PU^{-1}(u)B(u)\Psi(u)\,du-\int_\tau^\infty U(\tau)QU^{-1}(u)B(u)\Psi(u)\,du.$$

As $\theta<1$ and Ψ is bounded, the contraction mapping theorem yields $\Psi\equiv 0$.

Similar arguments lead us to the conclusion in (ii). □

An immediate consequence of above lemma is that $\tilde{X}(t,t)$ and $\tilde{Y}(t,t)$ are projections for each $t\in\mathcal{I}$. Moreover, if we denote $P(t)=U(t)PU^{-1}(t)$ and $Q(t)=U(t)QU^{-1}(t)$, then uniqueness of fixed points $\tilde{X}$, respectively, $\tilde{Y}$ of operators K, respectively L implies:

$$\begin{aligned}&\tilde{X}(t,t)P(t)=\tilde{X}(t,t), &&P(t)\tilde{X}(t,t)=P(t),\\ &\tilde{Y}(t,t)Q(t)=\tilde{Y}(t,t), &&Q(t)\tilde{Y}(t,t)=Q(t).\end{aligned}\tag{18}$$

Set $P_+=\tilde{X}(0,0)$ and $Q_-=\tilde{Y}(0,0)$. Using identities above we obtain the following relations:

$$PP_+=P,\qquad P_+P=P_+\tag{18$'$}$$

and

$$QQ_-=Q,\qquad Q_-Q=Q_-.$$

Denote by $Q_+=I-P_+$, $P_-=I-Q_-$, $Q_+(s)=V(s)Q_+V^{-1}(s)$, etc.

Eventually using the arguments in [8, pp. 567–568] we obtain:

$$\begin{aligned}V(t)P_+V^{-1}(s)&=U(t)PU^{-1}(s)P_+(s)\\&\quad+\int_s^\infty \Gamma(t,u)B(u)V(u)P_+V^{-1}(s)\,du,\quad t\geqslant s,\\ V(t)Q_+V^{-1}(s)&=U(t)QU^{-1}(s)Q_+(s)\end{aligned}\tag{19}$$

$$+\int_0^s \Gamma(t,u)B(u)V(u)Q_+V^{-1}(s)\,du, \quad s \geqslant t \geqslant 0,$$

and also,

$$\begin{aligned} V(t)Q_-V^{-1}(s) &= U(t)QU^{-1}(s)Q_-(s) \\ &\quad + \int_{-\infty}^s \Gamma(t,u)B(u)V(u)Q_-V^{-1}(s)\,du, \quad s \geqslant t, \\ V(t)P_-V^{-1}(s) &= U(t)PU^{-1}(s)P_-(s) \\ &\quad + \int_s^0 \Gamma(t,u)B(u)V(u)P_-V^{-1}(s)\,du, \quad 0 \geqslant t \geqslant s. \end{aligned} \tag{20}$$

Consider operator $\tilde{K}:\mathcal{B}_+(\mathcal{I}) \to \mathcal{B}_+(\mathcal{I})$ defined by

$$(\tilde{K}X)(t,s) = U(t)PU^{-1}(s)P_+(s) + \int_s^\infty \Gamma(t,s)B(u)X(u,s)\,du.$$

If δ satisfies (5), then $\tilde{K}$ is a contraction and it's unique fixed point is $X(t,s) = V(t)P_+V^{-1}(s)$, $t \geqslant s \geqslant 0$. Multiplying relation

$$\tilde{X}(t,s) = (K\tilde{X})(t,s) \quad \text{(see Lemma 3.1)}$$

by $P_+(s)$, we obtain that $\tilde{X}(t,s)P_+(s)$ is also a fixed point of operator $\tilde{K}$, which yields

$$V(t)P_+V^{-1}(s) = \tilde{X}(t,s)P_+(s).$$

According to relation (16) we have:

$$\left\| V(t)P_+V^{-1}(s) \right\| \leqslant K_1 e^{-\tilde{\nu}(t-s)} \left\| P_+(s) \right\|, \quad \text{for } t \geqslant s. \tag{21}$$

Using the same type of argument, one may prove that:

$$\left\| V(t)Q_+V^{-1}(s) \right\| \leqslant K_2 e^{-\tilde{\nu}(s-t)} \left\| Q_+(s) \right\|, \quad \text{for } s \geqslant t \geqslant 0, \tag{22}$$

$$\left\| V(t)P_-V^{-1}(s) \right\| \leqslant K_1 e^{-\tilde{\nu}(t-s)} \left\| P_-(s) \right\|, \quad \text{for } 0 \geqslant t \geqslant s, \tag{23}$$

$$\left\| V(t)Q_-V^{-1}(s) \right\| \leqslant K_2 e^{-\tilde{\nu}(s-t)} \left\| Q_-(s) \right\|, \quad \text{for } s \geqslant t. \tag{24}$$

Remark 3.3. Let us observe that the subspace of initially bounded solutions of Eq. (3) is complemented, when $\mathcal{I} = [0,+\infty)$ being P_+E, and when $\mathcal{I} = (-\infty, 0]$ this space is Q_-E. Notice that in absence of this condition, the statements in Propositions 1–3 in [2, p. 22] may not hold in infinite dimensional Banach spaces (see also Theorem 3.3 in [3, p. 171]).

Remark 3.4. Suppose that Eq. (1) is defined on the whole real axis. Then relations (21), respectively (24) show that the solutions starting at $t = 0$ from P_+E, respectively from Q_-E, are unbounded on $(-\infty, 0]$, respectively on $[0,\infty)$. This means that condition (5) assures that $P_+E \cap Q_-E = \{0\}$, and further the absence of nontrivial bounded solutions on the whole line.

Remark 3.5. Projections P and P_+ are similar, and so are Q and Q_-. Indeed, since $P_+P = P_+$, $PP_+ = P$ (relation (18′)) the operator $T = I - P + P_+$ is invertible, with inverse $T^{-1} = I + P - P_+$, which yield rapidly $P_+ = TPT^{-1}$.

Notice that in [8, Theorem 1, p. 565] it is proved by using much more complicate calculations, that $\dim P_+E = \dim PE$, when E is finite dimensional. This fact is directly used in [6, Theorem 3.1, p. 45].

4. The case of the semi infinite interval

Throughout this section, we will assume that interval $\mathcal{I}$ is either $(-\infty, 0]$ or $[0, +\infty)$.

Let $\mathcal{C}(\mathcal{I}, E)$ be the space of E-valued, bounded and continuous maps acting on $\mathcal{I}$, and let $\mathcal{L}(\mathcal{I}, E)$ be the space of Bochner integrable, E-valued maps, acting on $\mathcal{I}$. In finite dimension the concept of Bochner integral will automatically be replaced by Lebesgue integral. They are Banach spaces, endowed with norms: $\|x\|_c = \sup_{u\in\mathcal{I}} \|x(u)\|$, respectively, $\|f\|_L = \int_{\mathcal{I}} \|f(u)\|\,du$. The following construction is used in [8, Lemma 8, p. 564] and [6, Theorem 3.1, p. 45]:

For each fixed $f \in \mathcal{L}(\mathcal{I}, E)$, consider the function $\mathcal{T}: \mathcal{C}(\mathcal{I}, E) \to \mathcal{C}(\mathcal{I}, E)$, defined by

$$(\mathcal{T}x)(t) = \int_{\mathcal{I}} \Gamma(t,u)B(u)x(u)\,du + \int_{\mathcal{I}} \Gamma(t,u)f(u)\,du. \tag{25}$$

If $\theta < 1$, $\mathcal{T}$ becomes a contraction and it's fixed point x is a bounded solution of inhomogeneous equation

$$\frac{dz}{dt} = \bigl[A(t) + B(t)\bigr]z + f(t). \tag{26}$$

Moreover, using (25) we obtain estimation:

$$\|x\|_c \leqslant \frac{N}{1-\theta}\|f\|_L, \tag{27}$$

where

$$N = \max\{N_1, N_2\}. \tag{28}$$

Remark that equality $x = \mathcal{T}x$ defines a bounded and linear operator $\mathcal{L}(\mathcal{I}, E) \ni f \to x \in \mathcal{C}(\mathcal{I}, E)$, with norm less then $N/(1-\theta)$.

As $P_+P = P_+$, we have $Q_+Q = Q$, and $x(0) \in QE$ implies $x(0) \in Q_+E$. Therefore $x(t)$ is the unique bounded solution of Eq. (26), starting at $t = 0$ from the subspace Q_+E, when $\mathcal{I} = [0, +\infty)$. Similarly, if $\mathcal{I} = (-\infty, 0]$, then $x(t)$ starts at $t = 0$ from P_-E. From [2, Proposition 1, p. 22] it follows that projections $P_+(t)$, $Q_-(t)$ are subject to estimates:

$$\|P_+(t)\| \leqslant \frac{N}{1-\theta}, \quad \text{if } t \geqslant 0,$$

$$\|Q_-(t)\| \leqslant \frac{N}{1-\theta}, \quad \text{if } t \leqslant 0.$$

Now we are able to expose the main result of this section, which is valid in any Banach space.

Theorem 4.1. *If Eq.* (1) *has an exponential dichotomy* (2)*, then for any δ satisfying* (5)*, the perturbed equation* (3) *exhibits an exponential dichotomy* (4)*, with projection $\tilde{P} = P_+$ if $\mathcal{I} = [0, +\infty)$, respectively, $\tilde{P} = I - Q_-$ if $\mathcal{I} = (-\infty, 0]$.*

Dichotomic constants are: $\tilde{\nu}_1 = \tilde{\nu}_2 = \tilde{\nu}$*, given by* (8)*, $\tilde{N}_i = K_i N/(1-\theta)$, $i = 1, 2$, K_i given by* (9)*,* (10) *and N by* (28)*.*

Moreover, $\tilde{P}$ is similar to P and we have

$$\|\tilde{P}(t) - P(t)\| \leqslant \frac{N(N_1 + N_2)}{1-\theta} \quad \text{(\textit{see also Theorem} 3.1 \textit{in} [6, p. 45])}. \tag{29}$$

5. The roughness on all $\mathbb{R}$

From the last section we have that if $\theta < 1$, the perturbed equation (3) remains exponential dichotomic on $[0, +\infty)$ with projection P_+, and on $(-\infty, 0]$ with projection $P_- = I - Q_-$.

From Remark 3.4 it follows that for $\theta < 1$ the perturbed equation (3) does not have nontrivial bounded solutions on all $\mathbb{R}$. Consequently $P_+E \cap Q_-E = \{0\}$.

It remains to show that Eq. (3) has an exponential dichotomy on both half lines with the same projection. This last problem was studied in a lot of papers as for example [4,5,7,10, 11,13].

The best existing result, at our knowledge, for the roughness on all $\mathbb{R}$, in finite dimensional, seems to be that in [6, p. 48, Theorem 10.2]. In fact the authors showed that perturbed equation (3) is exponential dichotomic on both $\mathbb{R}_+$, $\mathbb{R}_-$ and has no nontrivial bounded solutions on $\mathbb{R}$, concluding that (3) has an exponential dichotomy on the whole line.

The next example shows that this type of argument does not suffice to prove the dichotomy on $\mathbb{R}$.

Example 5.1. Equation $\frac{dx}{dt} = 2tx$, with Cauchy operator $U(t) = e^{t^2} I$, has an exponential dichotomy on both $\mathbb{R}_+$, $\mathbb{R}_-$, with projections $P_+ = 0$, $P_- = I$ and has no nontrivial bounded solutions on $\mathbb{R}$. (This type of dichotomy was called β-exponential trichotomy and was introduced in [13].)

Next lemma will be crucial in the sequel, showing exactly where and why finite and infinite dimensional situation differ.

Lemma 5.2. *Let E be a Banach space. We consider* 3 *couples of bounded complementary projections*: *P and Q, P_+ and Q_+, P_- and Q_-. Suppose further that*: $P_+P = P_+$, $PP_+ = P$, $P_-P = P$, $PP_- = P_-$.

The following statements are equivalent:

(i) $E = P_+E \oplus Q_-E$ (*direct sum*);
(ii) *Operator $S = P_+ + Q_-$ is invertible.*

Proof. (i) $\Rightarrow$ (ii). If $Sx = 0$, then $P_+x + Q_-x = 0$, so $P_+x = -Q_-x \in P_+E \cap Q_-E \Rightarrow P_+x = Q_-x = 0$.

We rapidly obtain $Px = Qx = 0$, and finally $x = 0$. This proves that S is one-to-one.

To prove that S is surjective take $y \in E$. As $E = P_+E \oplus Q_-E$, there exist (unique) $y_1 \in P_+E$, $y_2 \in Q_-E$ such that $y = y_1 + y_2$. Put $x = y_2 + Py_1 - Py_2$. Observe first that $Q_-P = (I - P_-)P = 0$.

$$\begin{aligned} Sx &= (P_+ + Q_-)(y_2 + Py_1 - Py_2) \\ &= P_+y_2 + P_+y_1 - P_+y_2 + Q_-y_2 = y_1 + y_2 = y. \end{aligned}$$

So, S is surjective. As S is bijective, according to Banach theorem, it is invertible.

(ii) $\Rightarrow$ (i). Put

$$\tilde{P} = SPS^{-1}. \tag{30}$$

Then using the arguments following Proposition 1 in [2, pp. 34–35] and relations (18′), we have that $E = \tilde{P}E \oplus (I - \tilde{P})E = P_+E \oplus Q_-E$. □

Theorem 5.3. *A necessary and sufficient condition for the existence of the exponential dichotomy for Eq.* (3)*, on whole $\mathbb{R}$, whenever δ satisfies condition* (5)*, is that operator $S = P_+ + Q_-$ be invertible. In this case the structural projection is $\tilde{P} = SPS^{-1}$.*

Proof. As δ satisfies (5), Eq. (3) has an exponential dichotomy on both half lines. Then exponential dichotomy on whole line is equivalent to $E = P_+E \oplus Q_-E$, which is equivalent to the invertibility of S. □

Corollary 5.4. *When E is finite dimensional and δ satisfies* (5)*, as $P_+E \cap Q_-E = \{0\}$, linear operator S is injective, so invertible. Therefore, condition* (5) *in finite dimensional guarantees that* (3) *is exponential dichotomic on $\mathbb{R}$, with projection $\tilde{P}$.*

Next theorem improves the results from Theorem 10.2 in [6, p. 48], giving exact estimations for dichotomic constants of perturbed equation, in finite dimensional space. Meanwhile, we claim we complete its proof, as exposed in [6].

Theorem 5.5 (Roughness on $\mathbb{R}$ in finite dimensional space). *Suppose that E is finite dimensional and Eq.* (1) *has an exponential dichotomy* (2) *on all $\mathbb{R}$. Then for $\theta < 1$ the perturbed equation* (3) *possess an exponential dichotomy* (4)*, with projection $\tilde{P}$ given by* (30)*. Moreover, estimations on dichotomic constants of Eq.* (3) *in Theorem* 4.1*, and inequality* (29) *remain valid.*

Proof. The existence of exponential dichotomy on whole $\mathbb{R}$, for Eq. (3), with projection $\tilde{P}$, follows directly from Corollary 5.4.

To obtain the required estimation for dichotomic constants, we put $\mathcal{I} = \mathbb{R}$ in Sections 2 and 3, and relations (8)–(10) hold true. If $\mathcal{I}$ is replaced by $\mathbb{R}$ in Section 4 and $\theta < 1$, then for any fixed $f \in \mathcal{L}(\mathbb{R}, E)$, there exist an unique $x \in \mathcal{C}(\mathbb{R}, E)$ which is a fixed point of operator $\mathcal{T}$. Therefore x is the unique solution of inhomogeneous equation (26), that is bounded on all $\mathbb{R}$. If we denote by $\tilde{\Gamma}(t,s)$ the Green function of Eq. (3) and choose $f \in \mathcal{L}(\mathbb{R}, E)$, a map vanishing outside an arbitrary interval $(-\varepsilon; \varepsilon)$, we have that function

$$y(t) = \int_{-\varepsilon}^{\varepsilon} \tilde{\Gamma}(t,u) f(u)\, du$$

is a solution of Eq. (26) which is bounded on all $\mathbb{R}$.

Using the same type of argument as in Section 4 and in [2, p. 23], we easily obtain

$$\|\tilde{\Gamma}(t,s)\| \leqslant \frac{N}{1-\theta}, \text{ etc.} \qquad \square$$

The situation when E is infinite dimensional is certainly more complicated, because under supposition $\theta < 1$, the operator $S = P_+ + Q_-$ may be only injective, but not necessarily surjective (or equivalent $E = P_+E \oplus Q_-E$ as a direct sum). As $S = I + P_+ - P_-$, a sufficient condition for the invertibility of S is that the spectral radius of operator $P_+ - P_-$ is less than one. A stronger condition is that the norm of $P_+ - P_-$ is less than one and it was used in [2, p. 34]. We prefer using the first condition in order to obtain a better condition for δ, as it was given in [2]. Let us estimate this spectral radius. Put first:

$$A = -\int_0^{\infty} QU^{-1}(u)B(u)V(u)P_+, \quad \text{then} \quad P_+ = P + A \quad (\text{in (19) put } t = s = 0),$$

$$B = \int_{-\infty}^{0} PU^{-1}(u)B(u)V(u)Q_-, \quad \text{then} \quad P_- = P - B \quad (\text{in (20) put } t = s = 0).$$

As $P_+Q = P_+(I-P) = 0$ and $Q_-P = (I - P_-)P = 0$, it follows $A^2 = B^2 = 0$.

If we put $s = 0$ in the first equation (19) and note $x(t) = \|V(t)P_+\|$, then it is easy to see that $x(t)$ verifies inequality (6), with $s = 0$. Using Lemma 2.1 we obtain $x(t) \leqslant K_1 e^{-\tilde{\nu} t}$, and therefore $\|V(u)P_+\| \leqslant K_1 e^{-\tilde{\nu} u}$. Using this type of argument, from the expressions of A and B above, we obtain estimations:

$$\|A\| \leqslant a = \frac{\delta N_1 N_2}{\tilde{\nu} + \nu_2 - \delta N_2}, \qquad \|B\| \leqslant b = \frac{\delta N_1 N_2}{\tilde{\nu} + \nu_1 - \delta N_1}.$$

As $A^2 = B^2 = 0$ we have:

$$(P_+ - P_-)^{2n} = (AB)^n + (BA)^n,$$
$$(P_+ - P_-)^{2n+1} = (AB)^n A + (BA)^n B,$$

for any $n \in \mathbb{N}$.

This yields:

$$\sqrt[2n]{\|(P_+ - P_-)^{2n}\|} \leqslant \sqrt[2n]{2} \cdot \sqrt{ab}$$

and also

$$\sqrt[2n+1]{\|(P_+ - P_-)^{2n+1}\|} \leqslant \sqrt[2n+1]{a+b} \cdot \sqrt{ab}.$$

If r is the spectral radius of $P_+ - P_-$ then we have estimation

$$r \leqslant \sqrt{ab}.$$

Therefore if $\sqrt{ab} < 1$, the operator S is invertible. This is certainly satisfied for any δ verifying

$$\frac{\delta^2 N_1^2 N_2^2}{(\tilde{\nu} + \nu_1 - \delta N_1)(\tilde{\nu} + \nu_2 - \delta N_2)} < 1. \tag{31}$$

Our main theorem, that will be exposed below, shows that condition (5) imposed to δ, needs to be sharped in infinite dimensional Banach space.

Theorem 5.6 (The roughness on all $\mathbb{R}$ in Banach space). *Suppose that E is a Banach space and Eq.* (1) *has an exponential dichotomy* (2) *on* $\mathbb{R}$. *Then, for δ satisfying* (31), *the perturbed equation* (3) *has an exponential dichotomy* (4) *on* $\mathbb{R}$, *with projection $\tilde{P}$ similar to P, and estimations of dichotomic constants in Theorem* 5.5 *remain valid.*

Corollary 5.7. *If E is a Banach space, $\mathcal{I} = \mathbb{R}$, and* (1) *has an exponential dichotomy* (2) *with constants $N_1 = N_2 = N$, $\nu_1 = \nu_2 = \nu$, then for any δ verifying*

$$\delta < \frac{2\nu}{(N+1)^2} \tag{32}$$

Eq. (3) *is still exponential dichotomic.*

Observe that the estimation (32) is better than that obtained by Coppel in [2, p. 34, Proposition 1], in finite dimensional space.

Corollary 5.8. *Let E be a Banach space and $\mathcal{I}$ an arbitrary interval. If Eq.* (1) *is uniformly asymptotically stable on $\mathcal{I}$, i.e., for some positive constants N and ν, we have*

$$\left\|U(t)U^{-1}(s)\right\| \leqslant Ne^{-\nu(t-s)}, \quad \text{for } t \geqslant s,$$

then for $\delta < \nu/N$, the perturbed equation (3) *is still uniformly asymptotically stable on $\mathcal{I}$*:

$$\left\|V(t)V^{-1}(s)\right\| \leqslant Ne^{-(\nu - N\delta)(t-s)}, \quad \text{for } t \geqslant s.$$

The statements above holds true if Eq. (1) is supposed to be uniformly asymptotically unstable on $\mathcal{I}$. More exactly if

$$\left\|U(t)U^{-1}(s)\right\| \leqslant Ne^{-\nu(s-t)}, \quad \text{for } s \geqslant t,$$

then if $\delta < \nu/N$

$$\left\|V(t)V^{-1}(s)\right\| \leqslant Ne^{-(\nu - N\delta)(s-t)}, \quad \text{for } s \geqslant t.$$

6. The case of the finite interval

We believe that this case also needs a special attention. At our knowledge the only result for this particular situation, clearly exposed, can be found in [2, Proposition 1, p. 42], but for E finite dimensional and $N_1 = N_2$, $\nu_1 = \nu_2$ (see Section 1).

We need to remember that the result above cannot be extended to infinite dimensional Banach space, as the author of [2] used reducibility lemmas in the proof, as already commented in Introduction. Therefore, we are obliged to use a different method.

Firstly, observe that Lemma 2.1 in [3, p. 105] is also applicable for $\mathcal{I} = (a, b)$ being a finite interval and δ satisfying (5).

Secondly, we have to consider two cases: when $\mathcal{I} = [0, b)$ or $\mathcal{I} = (a, 0]$, and $\mathcal{I} = (a, b)$. For example equation

$$\frac{dx}{dt} = \sqrt{\frac{t}{1-t}}x$$

is defined on $\mathcal{I} = [0, 1)$, meanwhile equation

$$\frac{dy}{dt} = \frac{1}{\sqrt{1-t^2}}y$$

is defined on $\mathcal{I} = (-1, 1)$.

Furthermore, equalities (11) and (12) hold true when replacing $+\infty$ by b (respectively, $-\infty$ by a), and Eq. (13) is also valid. Put r_-, r_+ the roots of the corresponding characteristic equation of (13), and set

$$\Phi(t) = \alpha_1(s)e^{r_-(t-s)} + \alpha_2(s)e^{r_+(t-s)},$$
$$\Psi(t) = \beta_1(s)e^{r_-(t-s)} + \beta_2(s)e^{r_+(t-s)}.$$

$\alpha_1(s)$ and $\alpha_2(s)$ are uniquely determined as solutions of an algebraic linear system obtained by substituting $\Phi(t)$ in (13), putting $t = s$, then $t = b$. It is easy to see that

$$\alpha_1 = \sup_{s\in\mathcal{I}} \alpha_1(s) < \infty, \qquad \alpha_2 = \sup_{s\in\mathcal{I}} \alpha_2(s) < \infty.$$

Similarly we obtain constants β_1 and β_2, using (12).

If we denote by:

$$K_1 = \alpha_2 e^{(r_+-r_-)(b-a)} + \alpha_1, \tag{33}$$
$$K_2 = \beta_2 e^{(r_+-r_-)(b-a)} + \beta_1, \tag{34}$$

then we see that Lemma 2.1 in Section 2 holds true for K_1, K_2 above and $\tilde{\nu} = -r_-$.

Remark that operator $\mathcal{T}$ in (25) becomes a contraction if $\theta < 1$, and linear operator $\mathcal{L}(\mathcal{I}, E) \ni f \to x \in \mathcal{C}(\mathcal{I}, E)$ is bounded, with norm less then $N/(1-\theta)$ (see also relation (27)).

Notice that all constructions in Sections 2 and 3 remain valid, when $\mathcal{I}$ is either $[0, b)$ or $(a, 0]$. Projections P_+ and Q_- are obtained by using (18).

Replacing s by 0 in the first equality (19) and observing that

$$\frac{d}{du}\left[U^{-1}(u)V(u)\right] = U^{-1}(u)B(u)V(u) \tag{35}$$

we obtain:

$$
\begin{aligned}
V(t)P_+ &= U(t)P + \int_0^t U(t)PU^{-1}(u)B(u)V(u)P_+\,du \\
&\quad - \int_t^b U(t)QU^{-1}(u)B(u)V(u)P_+\,du \\
&= U(t)P + U(t)PU^{-1}(u)V(u)P_+\big|_{u=0}^{u=t} + U(t)QU^{-1}(u)V(u)P_+\big|_{u=b}^{u=t} \\
&= U(t)P + P(t)V(t)P_+ - U(t)P \\
&\quad + Q(t)V(t)P_+ - U(t)QU^{-1}(b)V(b)P_+.
\end{aligned}
$$

These considerations lead us to

$$QU^{-1}(b)V(b)P_+ = 0 \quad \text{(as a limit).} \tag{36}$$

Using the first equality (20), the same type of argument yield

$$PU^{-1}(a)V(a)Q_- = 0 \quad \text{(as a limit).} \tag{37}$$

Suppose that $\mathcal{I} = [0, b)$, set $\tilde{P} = P_+$, $\tilde{Q} = I - P_+$ and let $\tilde{\Gamma}$ be the Green function of Eq. (3). Take $\varepsilon \in (0, b)$ and consider $f \in \mathcal{L}(\mathcal{I}, E)$ a map vanishing outside the interval $[0, \varepsilon]$.

Lemma 6.1. *The bounded function*

$$y(t) = \int_{\mathcal{I}} \tilde{\Gamma}(t,u) f(u)\,du \tag{38}$$

is exactly the fixed point of $\mathcal{T}$ *in* (25).

Proof. Indeed, using when necessary (36), we successively have:

$$
\begin{aligned}
&\int_{\mathcal{I}} \Gamma(t,u)B(u)y(u)\,du \\
&\quad = \int_0^b \Gamma(t,u)B(u)\left(\int_0^u V(u)\tilde{P}V^{-1}(s)f(s)\,ds - \int_u^b V(u)\tilde{Q}V^{-1}(s)f(s)\,ds\right)du \\
&\quad = \int_0^t U(t)PU^{-1}(u)B(u)\int_0^u V(u)\tilde{P}V^{-1}(s)f(s)\,ds\,du \\
&\qquad - \int_0^t U(t)PU^{-1}(u)B(u)\int_u^b V(u)\tilde{Q}V^{-1}(s)f(s)\,ds\,du
\end{aligned}
$$

$$
\begin{aligned}
&- \int_t^b U(t)QU^{-1}(u)B(u)\int_0^u V(u)\tilde{P}V^{-1}(s)f(s)\,ds\,du \\
&+ \int_t^b U(t)QU^{-1}(u)B(u)\int_u^b V(u)(\tilde{Q})V^{-1}(s)f(s)\,ds\,du \\
= {}&\int_0^t U(t)P\big[U^{-1}(u)V(u)\big]'\int_0^u \tilde{P}V^{-1}(s)f(s)\,ds\,du \\
&- \int_0^t U(t)P\big[U^{-1}(u)V(u)\big]'\int_u^b \tilde{Q}V^{-1}(s)f(s)\,ds\,du \\
&- \int_t^b U(t)Q\big[U^{-1}(u)V(u)\big]'\int_0^u \tilde{P}V^{-1}(s)f(s)\,ds\,du \\
&+ \int_t^b U(t)Q\big[U^{-1}(u)V(u)\big]'\int_u^b \tilde{Q}V^{-1}(s)f(s)\,ds\,du \\
= {}&U(t)PU^{-1}(u)V(u)\int_0^u \tilde{P}V^{-1}(s)f(s)\,ds\Big|_{u=0}^{u=t} - \int_0^t U(t)PU^{-1}(u)\tilde{P}(u)f(u)\,du \\
&+ U(t)PU^{-1}(u)V(u)\int_u^b \tilde{Q}V^{-1}(s)f(s)\,ds\Big|_{u=t}^{u=0} \\
&- \int_0^t U(t)PU^{-1}(u)\tilde{Q}(u)f(u)\,du \\
&+ U(t)QU^{-1}(u)V(u)\int_0^u \tilde{P}V^{-1}(s)f(s)\,ds\Big|_{u=b}^{u=t} \\
&+ \int_t^b U(t)QU^{-1}(u)\tilde{P}(u)f(u)\,du \\
&+ U(t)QU^{-1}(u)V(u)\int_u^b \tilde{Q}V^{-1}(s)f(s)\,ds\Big|_{u=t}^{u=b} \\
&+ \int_t^b U(t)QU^{-1}(u)\tilde{Q}(u)f(u)\,du
\end{aligned}
$$

$$
\begin{aligned}
&= P(t)\int_0^t V(t)\tilde{P}V^{-1}(s)f(s)\,ds - \int_0^t U(t)PU^{-1}(u)\tilde{P}(u)f(u)\,du \\
&\quad - P(t)\int_t^b V(t)\tilde{Q}V^{-1}(s)f(s)\,ds \\
&\quad - \int_0^t U(t)PU^{-1}(u)\tilde{Q}(u)f(u)\,du + Q(t)\int_0^t V(t)\tilde{P}V^{-1}(s)f(s)\,ds \\
&\quad + \int_t^b U(t)QU^{-1}(u)\tilde{P}(u)f(u)\,du \\
&\quad - Q(t)\int_t^b V(t)\tilde{Q}V^{-1}(s)f(s)\,ds + \int_t^b U(t)QU^{-1}(u)\tilde{Q}(u)f(u)\,du \\
&= \int_0^t V(t)\tilde{P}V^{-1}(s)f(s)\,ds - \int_t^b V(t)\tilde{Q}V^{-1}(s)f(s)\,ds \\
&\quad - \int_0^t U(t)PU^{-1}(u)f(u)\,du + \int_t^b U(t)QU^{-1}(u)f(u)\,du \\
&= \int_{\mathcal{I}} \tilde{\Gamma}(t,s)f(s)\,ds - \int_{\mathcal{I}} \Gamma(t,u)f(u)\,du \\
&= y(t) - \int_{\mathcal{I}} \Gamma(t,u)f(u)\,du. \qquad \square
\end{aligned}
$$

Using now the same kind of argument as in [2, p. 210], we obtain that

$$
\|\tilde{P}(t)\| \leqslant \frac{N}{1-\theta}, \qquad \|\tilde{Q}(t)\| \leqslant \frac{N}{1-\theta}.
$$

Theorem 6.2. *If $\mathcal{I} = [0,b)$ or $\mathcal{I} = (a,0]$,* (1) *has an exponential dichotomy* (2) *and δ verify* (5)*, then Eq.* (3) *has an exponential dichotomy* (4) *with dichotomic constants*: *$\tilde{\nu}$ given by* (8)*, $\tilde{N}_i = NK_i/(1-\theta)$, $i = 1, 2$, K_i given by* (33) *and* (34)*. If $\mathcal{I} = [0,b)$, then $\tilde{P} = P_+$, and if $\mathcal{I} = (a,0]$, then $\tilde{P} = I - Q_-$.*

When $\mathcal{I} = (a,b)$, then the situation is more complicated for two reasons:

1. Equation (1) may have or may have not nontrivial bounded solutions;
2. The space E may be finite or infinite dimensional.

Anyway, the arguments preceding Theorem 6.2 are still valid replacing 0 by a, and using both (36) and (37). These considerations lead us to the following result:

Theorem 6.3. *Suppose that* $\mathcal{I} = (a, b)$ *and Eq.* (1) *has an exponential dichotomy* (2).

(i) *If* E *is finite dimensional and Eq.* (1) *does not have nontrivial bounded solutions, then for any* δ *satisfying* (5), *the perturbed equation* (3) *has an exponential dichotomy* (4) *with projection* $\tilde{P} = SPS^{-1}$ $(S = P_+ + Q_-)$, *and dichotomic constants as in Theorem* 6.2.

(ii) *If* E *is infinite dimensional or Eq.* (1) *has nontrivial bounded solutions, then for* δ *satisfying*

$$\frac{\delta^2 N_1 N_2 K_1 K_2}{(\tilde{\nu} + \nu_1)(\tilde{\nu} + \nu_2)} < 1, \tag{39}$$

Eq. (3) *is exponential dichotomic as in* (i) *above.*

Using Corollary 2.2, and also Lemma 7 in [8, p. 568], from all the arguments preceding Theorems 6.2 and 6.3, we easily obtain:

Corollary 6.4. *All the statements in the above theorems remain valid when estimations for constants* K_i *in* (33)–(34) *are replaced by those in* (9)–(10), *and condition* (39) *changes in* (31).

Remark 6.5. In any interval $\mathcal{I}$, for $N_1 = N_2 = N$, $\nu_1 = \nu_2 = \nu$, in any Banach space condition (32) imposed to δ, assures the existence of exponential dichotomy for Eq. (3).

Notice that this result improves substantially Proposition 1 in [2, p. 42].

Remark 6.6. Condition (5) in all above results, can be weakened as follows:

$$\delta\left[\frac{N_1}{\nu_1}\left(1 - e^{-\nu_1(b-a)}\right) + \frac{N_2}{\nu_2}\left(1 - e^{-\nu_2(b-a)}\right)\right] < 1. \tag{40}$$

Indeed, when replacing ∞ by b, and $-\infty$ by a in Section 3, condition (40) assures that operators K and L are contractions. This fact shows that the admissible perturbations of Eq. (1) depend on the length of the definition interval.

Final remark. Throughout this paper we consider $U(0) = V(0) = I$, but instead of 0 we can choose any fixed $t_0 \in \mathcal{I}$, as $U(t)$ can be replaced by $U(t)U^{-1}(t_0)$, respectively $V(t)$ by $V(t)V^{-1}(t_0)$. The only difference is that the constants N_i and $\tilde{N}_i$, $i = 1, 2$, may be different.

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