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A New Rabin-type Trapdoor Permutation Equivalent to Factoring

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Abstract

Public key cryptography has been invented to overcome some key management problems in open networks. Although nearly all aspects of public key cryptography rely on the existence of trapdoor one-way functions, only a very few candidates for this primitive have been observed yet. In this paper, we introduce a new trapdoor one-way permutation based on the hardness of factoring integers of p^2q-type. We point out that there are some similarities between Rabin's trapdoor permutation and our proposal. Although our function is less efficient, it possesses a nice feature which is not known for modular squaring, namely there is a variant with a different and easy-to-handle domain. Thus it provides some advantages for practical applications. To confirm this statement, we develop a simple hybrid encryption scheme based on our proposed trapdoor permutation that is CCA-secure in the random oracle model.

Keywords: trapdoor one-way permutations, hybrid encryption, Tag-KEM/DEM framework

1 Introduction

Informally, a *one-way permutation* is a bijective function that is "easy" to compute but "hard" to invert. If there is some token of information that makes the inversion also an easy task, then we call the function *trapdoor*. Trapdoor one-way permutations are used as building blocks for various kind of cryptographic schemes, *e. g.* asymmetric encryption, digital signatures, and private information retrieval. There is no doubt that the concept of trapdoor one-way permutations is of particular importance especially in public

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key cryptography. Nevertheless, just a relatively small number of promising candidates can be found in the literature. Promising here means that the one-wayness can be reduced to a presumed hard problem such as the integer factorization problem. As not even the pure existence of one-way functions can be proven today [2], this kind of *provable secure* trapdoor permutations is the best alternative solution at present.

1.1 Previous Work

The oldest and still best known candidate trapdoor permutation is the RSA function, *i. e.* modular exponentiation with exponents coprime to the order of the multiplicative residue group [21]. The factors of the modulus can serve as a trapdoor to invert the RSA function, but the opposite direction is unknown. Thus RSA is not provably equivalent to factoring, and there are serious doubts that this equivalence holds indeed [3]. Anyway, as the RSA problem has been extensively studied for decades, nowadays inverting the RSA function is widely accepted as a hard problem itself. Slightly later, M. O. Rabin observed that the special case of modular squaring *can* be reduced to factoring [20]. Modular squaring, however, is not a permutation, it is 4-to-1 (if a two-factor modulus is used). This can be overcome: squaring modulo a Blum integer [3] n is a permutation of the quadratic residues modulo n. The resulting trapdoor permutation is referred to as Blum-Williams function in the literature, and an extension (exponent $2e$, where e is coprime to $\lambda(n)$) is denoted Rabin-Williams function. More factorization-based trapdoor permutations were proposed by Kurosawa et al [11], Paillier [17,18], and Galindo *et al.* [9]. A survey on trapdoor permutations including some less established candidates can be found in [19].

1.2 Our Contribution

In this paper, we introduce a rather simple trapdoor one-way permutation equivalent to factoring integers of the shape $n = p^2q$. As many previous candidates, our proposed trapdoor function is also based on modular exponentiation, namely in our case the public exponent is the same as the modulus $n = p^2q$. With the domain $\mathbb{Z}_n^\times$ the function $x \mapsto x^n \bmod n$ is p-to-one, but restricted to the subgroup of n-th residues modulo n, it is indeed a permutation. This property is similar to the Blum-Williams function (where n-th residues are replaced by quadratic residues). Analogical to the quadratic residuosity

[2] Interestingly, the current knowledge in complexity theory does not even allow to prove the existence of one-way functions assuming $\mathcal{P} \neq \mathcal{NP}$.

[3] A *Blum integer* is a product of two distinct primes each congruent to 3 modulo 4.

assumption, we assume that without knowledge of the factorization of n, it is hard to distinguish n-th residues from non-residues, whereas it is efficient if the factors of n are known. However, the restricted domain has some shortcomings that also apply to Blum-Williams and Rabin-Williams functions: in practical applications, the data has to be preprocessed into the set of n-th resp. quadratic residues. But fortunately, we can prove that for $n = p^2q$ the set of n-th residues is isomorphic to $\mathbb{Z}_{pq}^\times$, thus our proposed trapdoor function is also a bijection between the easy-to-handle domain $\mathbb{Z}_{pq}^\times$ and the set of n-th residues. No such property is known for Rabin-type functions. Indeed, we can show that our proposed trapdoor permutation easily provides practical applications by constructing a hybrid encryption scheme based on Abe *et al.*'s Tag-KEM/DEM framework [1].

2 Trapdoor One-way Permutations Equivalent to Factoring

In this section, we introduce a new trapdoor one-way permutation. We also give a short account on its mathematical background in order to deepen the understanding about the special properties of the group $\mathbb{Z}_n^\times$ for n of p^2q-type.

2.1 Notations and definitions

Let n be a positive integer. We write $\mathbb{Z}_n$ for the ring of residue classes modulo n, and $\mathbb{Z}_n^\times$ for its multiplicative group, *i. e.* the set of invertible elements modulo n. For $x \in \mathbb{Z}_n^\times$, $\mathrm{ord}_n(x)$ denotes the multiplicative order of x modulo n, *i. e.* the smallest positive integer k with $x^k = 1 \bmod n$. Furthermore, $\varphi : \mathbb{N} \to \mathbb{N}$ means Euler's totient function.

For any homomorphism h, we denote the kernel and the image with $\ker(h)$ and $\mathrm{im}(h)$, respectively.

As usual, a probability $\Pr(k)$ is called *negligible* if $\Pr(k)$ decreases faster than the reciprocal of any polynomial in k, *i. e.* $\forall c \exists k_c (k > k_c \Rightarrow \Pr(k) < k^{-c})$.

Unless indicated otherwise, all algorithms are randomized, but we don't mention the random coins as an extra input. If A is a probabilistic algorithm, then $A(y_1, \ldots, y_n)$ refers to the probability space which to the string x assigns the probability that A on input $y_1, \ldots, y_n$ outputs x. For any probability space S, the phrase $x \leftarrow\!\shortmid S$ denotes that x is selected at random according to S. In particular, if S is a finite set, then $x \leftarrow\!\shortmid S$ is the operation of picking x uniformly at random from S.

We write $|n|_2$ for the bit-length of the integer n.

For the sake of completeness, we formally define the notion of trapdoor one-way permutation.

Definition 2.1 [Collection of trapdoor one-way permutations] Let I be a set of indices such that for each $i \in I$ the sets D_i and $\tilde{D}_i$ are finite and of the same order. Let $\mathcal{F} = \{f_i | f_i : D_i \to \tilde{D}_i\}_{i \in I}$ be a family of bijections. Then $\mathcal{F}$ is said to be a *collection of trapdoor one-way permutations* if

(i) There exists a polynomial p and a probabilistic polynomial time key generator KeyGen such that KeyGen on input 1^k (the security parameter) outputs a pair (i, t_i) where $i \in \{0,1\}^k \cap I, |t_i|_2 < p(k)$. The data t_i is denoted the trapdoor information of f_i.

(ii) The domains D_i are samplable: There exists a probabilistic polynomial time sampling algorithm S that on input $i \in I$ outputs $x \in D_i$ uniformly chosen at random.

(iii) The members of $\mathcal{F}$ are easy to evaluate: There exists a deterministic polynomial time evaluator Eval that on input $i \in I, x \in D_i$ outputs $f_i(x)$.

(iv) Inverting the members of $\mathcal{F}$ is easy if the trapdoor information is known: There exists a deterministic polynomial time inverter Inv such that for all $x \in D_i$ we have $\mathsf{Inv}(t_i, f_i(x)) = x$.

(v) Inverting the members of $\mathcal{F}$ is hard if the trapdoor information is unknown: For every probabilistic polynomial time algorithm $\mathcal{A}_I$ the following probability is negligible in k:

$$\Pr[(i, t_i) \hookleftarrow 1^k; x \hookleftarrow D_i : \mathcal{A}_I(f_i(x)) = x].$$

Note that in contrast to strictly mathematical parlance we do not require that permutations are maps onto itself.

2.2 Our proposed trapdoor one-way permutations

Throughout this section, let p, q be primes with $p \nmid q-1$ and $q \nmid p-1$ and define $n = p^2 q$.

All of our constructions are based on the following group homomorphism:

Definition 2.2 [The homomorphism h] We define:

$$\begin{aligned} h : \mathbb{Z}_n^\times &\longrightarrow \mathbb{Z}_n^\times \\ x &\mapsto x^n \bmod n \end{aligned}$$

The reason why we do not use standard RSA moduli is the observation that in $\mathbb{Z}_n^\times$ with $n = p^2 q$ there are elements of order p:

Lemma 2.3 *Define the set $\mathcal{S}$ as*

$$\mathcal{S} := \{x \in \mathbb{Z}_n^\times \mid x = 1 + kpq \text{ for an integer } k, 0 < k < p\}.$$

Then $\mathcal{S}$ consists of exactly the elements of multiplicative order p in $\mathbb{Z}_n^\times$.

Proof. See Appendix A. □

From Lemma 2.3 we can easily deduce that each element of order p in $\mathbb{Z}_n^\times$ reveals the factorization of n. On this fact we will base the one-wayness of our proposed trapdoor permutations. Next, we analyze the relationship between the homomorphism h and the set $\mathcal{S}$:

Lemma 2.4 *Let h and $\mathcal{S}$ be defined as above. Then we have*

$$\ker(h) = \{1\} \cup \mathcal{S}.$$

Proof. See Appendix A □

As the magnitude of the kernel of h is exactly p, we conclude that the homomorphism h as defined above is p-to-1. The following theorem identifies the elements mapping to the same value.

Theorem 2.5 *For $x, y \in \mathbb{Z}_n^\times$ we have*

$$h(x) = h(y) \iff x = y \bmod pq.$$

Proof.

"if": Let $y = x + kpq$ for $k \in \mathbb{Z}$. Then, $(x + kpq)^n = x^n + nx^{n-1}kpq = x^n \bmod n$.

"only if": $x^n = y^n \bmod n$ leads to $xy^{-1} \in \ker(h)$, consequently $xy^{-1} = 1 \bmod pq$ using Lemma 2.3 and Lemma 2.4.

□

Hence we conclude that the homomorphism h is collision-resistant if factoring integers of the shape p^2q is hard. In [22], this fact has been exploited to construct a new fail-stop signature scheme. With little additional effort we also derive two trapdoor permutations from h. For this reason, we introduce the set of n-th residues modulo n.

Definition 2.6 [N-R(n)] Let N-R(n) $= \{x \in \mathbb{Z}_n^\times | x = y^n \bmod n \text{ for a } y \in \mathbb{Z}_n^\times\} = \mathrm{im}(h)$ denote the set of the n-th residues modulo n.

N-R(n) is a subgroup of $\mathbb{Z}_n^\times$ of order $(p-1)(q-1)$ (as there are exactly $\varphi(pq) = (p-1)(q-1)$ pairwise different n-th residues modulo n, namely the elements $\{x^n \bmod n | x \in \mathbb{Z}_{pq}^\times\}$).

Now we can state the main results of this section:

Theorem 2.7 (i) *Let* $I = \{n | n = p^2 q, |p|_2 = |q|_2, p \nmid q-1, q \nmid p-1\}$ *be a set of indices. The family* $\mathcal{F}_{\text{N-R}} = \{f^{(n)}_{\text{N-R}}\}_{n \in I}$ *is a collection of trapdoor one-way permutations, where* $f^{(n)}_{\text{N-R}}$ *is defined as*

$$\begin{aligned} f^{(n)}_{\text{N-R}} : \text{N-R}(n) &\to \text{N-R}(n) \\ x &\mapsto x^n \bmod n. \end{aligned}$$

(ii) *Let* I *be defined as above. The family* $\mathcal{F}_{pq} = \{f^{(n)}_{pq}\}_{n \in I}$ *is a collection of trapdoor one-way permutations, where* $f^{(n)}_{pq}$ *is defined as*

$$\begin{aligned} f^{(n)}_{pq} : \mathbb{Z}^\times_{pq} &\to \text{N-R}(n) \\ x &\mapsto x^n \bmod n. \end{aligned}$$

In both cases, the trapdoor is the factorization of n *and the one-wayness is based on the factorization assumption. For individual members, we omit the superscript* (n) *whenever it is clear from the context.*

Proof.

(i) We first show that the $f_{\text{N-R}}$ are indeed permutations. Define $d = n^{-1} \bmod \varphi(pq)$ (note that $\gcd(n, \varphi(pq)) = 1$). Let x be an element of N-R(n), *i. e.* $x = y^n \bmod n$ for an appropriate $y \in \mathbb{Z}^\times_n$. Then we have $(x^n)^d = y^{n^2 d} = x \bmod n$, because of $n^2 d = n \bmod \varphi(n)$ (equality holds modulo p and modulo $\varphi(pq)$). Thus, $x \mapsto x^n \bmod n$ is a permutation of N-R(n). Properties (i) to (iii) of Definition 2.1 are obviously fulfilled. It is clear that d (resp. the factorization of n) can be used as a trapdoor to invert $f_{\text{N-R}}$. The one-wayness (property (v)) is a consequence of Theorem 2.5: To factor n with access to an oracle that inverts $f_{\text{N-R}}$, we choose an element $x \in \mathbb{Z}^\times_n$ at random and query the oracle on $h(x) = x^n \bmod n$. With probability $1 - 1/p$ we have $x \notin \text{N-R}(n)$ and the oracle will answer $x' \in \text{N-R}(n)$ with $x \neq x' \bmod n$ such that x and x' collide under h. Hence $\gcd(x - x', n) = pq$ reveals the factorization of n.

(ii) Define d as above. Then it is easy to see that $(f_{pq}(x))^d = x \bmod pq$ holds for all $x \in \mathbb{Z}_{pq}$. Thus f_{pq} is a bijection. The remaining properties can be shown along the lines of the proof given above.

□

Remark 2.8 The fact that modular exponentiation with $n = p^2 q$ can be inverted uniquely modulo pq has been implicitly exploited in [18], where P. Paillier introduced a trapdoor permutation based on the Okamoto-Uchiyama trapdoor mechanism. However, the results and the proof techniques used in [18] are substantial different from our proposal.

We want to point out the similarities among exponentiation modulo $n = p^2q$ and Rabin-type modular squaring. In both cases, we have a group homomorphism with a non-trivial kernel. Moreover, one-wayness holds because each non-trivial kernel element reveals the factorization of the modulus. Obviously, the Rabin-Williams permutation on quadratic residues corresponds to our permutation $f_{\text{N-R}}$ on n-th residues. In the case of modular squaring, however, there is no analogue to the bijection f_{pq}. The latter is interesting for practical applications, as no preprocessing into the set of n-th residues is necessary. In particular, f_{pq} can be used to encrypt *arbitrary* strings like keys. We provide an application in Section 3. Further advantages of our proposal are due to the fact that the magnitude of the kernel is larger. For instance, it is possible to construct fail-stop signatures [22] and trapdoor commitments [24] from the homomorphism h. To emphasize the analogy to modular squaring even more, we assume that without knowledge of the factors of n distinguishing $\text{N-R}(n)$ from $\mathbb{Z}_n^\times$ is hard [4] (*cf.* the well-known quadratic residuosity assumption). Given p and q, however, deciding n-th residuosity is efficient.

Theorem 2.9 *For all $x \in \mathbb{Z}_n^\times$ we have*

$$x \in \text{N-R}(n) \iff x^{p-1} = 1 \bmod p^2.$$

Proof. First we show an auxiliary proposition:

$$x \in \text{N-R}(n) \iff x^{(p-1)(q-1)} = 1 \bmod n.$$

From $p \nmid q-1, q \nmid p-1$ we deduce $\gcd((p-1)(q-1), n) = 1$. Hence there exists $z \in \mathbb{Z}$ with $z(p-1)(q-1) = 1 \bmod n$, leading to $-z(p-1)(q-1)+1 = kn$ for a suitable $k \in \mathbb{Z}$. Thus we have

$$\begin{aligned} x^{(p-1)(q-1)} = 1 \bmod n &\Rightarrow x^{-z(p-1)(q-1)+1} = x \bmod n \\ &\Rightarrow x^{kn} = x \bmod n. \end{aligned}$$

This finishes the proof of the auxiliary proposition, as the opposite direction is straightforward.

[4] In case of RSA modulus n, this assumption is known as *Decisional Composite Residuosity Assumption*, and it is the basis for the semantic security of Paillier's homomorphic encryption scheme [17].

Therefore we have the following for $x \in \mathbb{Z}_n^\times$:

$$\begin{aligned} x \in \text{N-R}(n) &\iff x^{(p-1)(q-1)} = 1 \bmod n \\ &\iff x^{(p-1)(q-1)} = 1 \bmod p^2 \text{ and } x^{(p-1)(q-1)} = 1 \bmod q \quad (1) \\ &\iff x^{(p-1)(q-1)} = 1 \bmod p^2 \quad (2) \\ &\iff x^{(p-1)} = 1 \bmod p^2 \quad (3) \end{aligned}$$

Note that (2) $\Rightarrow$(1) holds because $x^{(p-1)(q-1)} = 1 \bmod q$ is true for all $x \in \mathbb{Z}_n^\times$. (2) $\Rightarrow$(3) is deduced from $\gcd(q-1, \varphi(p^2) = 1)$. □

3 An Exemplary Application: Hybrid Encryption

In this section, we will prove that our proposed trapdoor function is not only of theoretical interest by constructing a simple chosen-ciphertext secure hybrid encryption scheme as an exemplary application. In particular, we show that our novel scheme offers notable advantages compared to the members of the well-known EPOC family [6,14]. We choose these schemes as a candidate because they all rely on the Okamoto-Uchiyama trapdoor mechanism that like ours is based on the hardness of factoring integers $n = p^2q$ [16]. However, as EPOC-1 has a worse security reduction than EPOC-2 and a similar performance, we focus on EPOC-2 and EPOC-3.

3.1 The Okamoto-Uchiyama trapdoor mechanism and EPOC-2/3

For the sake of self-containedness of this paper, we briefly sketch the Okamoto-Uchiyama trapdoor mechanism (see [16] for details). Let n be of the shape $n = p^2q$ for two large primes p, q. Consider the Sylow group $\Gamma_p = \{x \in \mathbb{Z}_{p^2} | x = 1 \bmod p\}$ of $\mathbb{Z}_{p^2}^\times$. The crucial observation is that the L-function defined on Γ_p as $L(x) = (x-1)/p$ provides additive homomorphic properties. For fixed $h \in \text{N-R}(n)$ and $g \in \mathbb{Z}_n^\times$ with $p | \operatorname{ord}_{p^2}(g)$ the Okamoto-Uchiyama encryption of $m \in \{0, 1, \ldots, p-1\}$ is as follows: choose randomness $r \in \mathbb{Z}_n$ and compute $c = g^m h^r \bmod n$. If p is known, then m can be recovered from c in the following way: $c' = c^{p-1} \bmod p^2, g_p = g^{p-1} \bmod p^2, m = L(c')L(g_p)^{-1} \bmod p$. The correctness is deduced from the additive homomorphic properties of the L-function because we have $c' = g_p^m \bmod p^2$ and $g_p \in \Gamma_p$. In [16] it is shown that breaking the one-wayness of this encryption scheme is as hard as factoring the modulus.

EPOC-3 is obtained by applying the REACT-conversion [15] to the Okamoto-Uchiyama encryption. The REACT-conversion builds an CCA-secure (in the random oracle model) hybrid encryption scheme from any one-way-PCA secure asymmetric encryption scheme combined with a symmetric cryptosystem semantically secure against passive attacks. Here, PCA denotes plaintext-checking attack. In this model, the adversary has access to an oracle that on input a message m and a ciphertext c answers if c is a possible encryption of m. Of course, this oracle is only helpful if the encryption is probabilistic, otherwise the adversary can answer the queries himself. Thus, in the deterministic scenario, one-wayness-PCA is equivalent to one-wayness under the weakest attack, *i. e.* chosen-plaintext-attack (CPA). In the case of EPOC-3, note that although the one-wayness of the Okamoto-Uchiyama encryption is equivalent to factoring integers p^2q, the security of the converted scheme is only based on the probably stronger Gap-High-Residuosity assumption. This is due to the fact that Okamoto-Uchiyama is probabilistic and thus one-way-PCA is not equivalent to one-way-CPA [5] .

EPOC-2 is the outcome of combining the Okamoto-Uchiyama encryption and a semantically secure (against passive adversaries) symmetric encryption scheme using the Fujisaki-Okamoto conversion technique [7]. In contrast to EPOC-3, EPOC-2 is CCA secure under the p^2q-factoring assumption in the random oracle model. Although in general the security reduction of the Fujisaki-Okamoto conversion technique is not very tight, Fujisaki observed a tight reduction proof tailored to the special application EPOC-2 [8]. A disadvantage of EPOC-2 is that in the decryption phase a re-encryption is necessary as an integrity check. For efficiency reasons, this re-encryption is only performed modulo q instead modulo n (accepting a small error probability). Nevertheless, the decryption is less efficient than in case of EPOC-3. There is also a second drawback due to the re-encryption: poor implementation makes EPOC-2 vulnerable against reject-timing attacks [25,5]. In this attack, the adversary can find the secret key if he is able to distinguish the two different kinds of rejections of invalid ciphertexts (if the enciphered text does not meet length restrictions on the one hand, or if the re-encryption test fails on the other hand). As the re-encryption involves the costly public key operations and hence takes a suitable amount of time, careless implementation makes it possible for an adversary to distinguish between the two cases by measuring the time of rejection.

[5] One could ask why the randomization is not removed before applying REACT (this would lead to the enciphering $c = g^m \bmod n$, and the same decryption as in the original scheme). But note that in this case, we cannot reduce one-wayness to factoring as before, because the distributions of $\{g^m \bmod n | m > p\}$ and $\{g^m \bmod n | m < p\}$ are not necessarily the same.

3.2 The Tag-KEM/DEM framework for hybrid encryption

Beside the technique of applying specific generic constructions to suitable asymmetric and symmetric primitives, a more general solution of hybrid encryption has been introduced by Cramer and Shoup in [4]. In this paper, Cramer and Shoup formalize the so-called KEM/DEM framework where KEM is a probabilistic asymmetric *key-encapsulation mechanism*, and DEM is a symmetric encryption scheme (a *data encapsulation mechanism*) used to encrypt messages of arbitrary length with the key given by the KEM. Needless to say, such combinations of public and secret key schemes have been folklore for years, but Cramer and Shoup for the first time gave a rigorously analyzed formal treatment of this subject. Note that a KEM is not the same as a key agreement protocol: the encapsulated key is designated to be used once only, therefore the DEM is only required to be secure in the one-time scenario. For more details on security definitions and requirements the reader is referred to [4]. Roughly speaking, if both the KEM and the DEM part are CCA secure, then the same holds for the whole KEM/DEM scheme.

At this year's Eurocrypt, Abe *et al.* enhanced Cramer and Shoup's framework by introducing the notion of a *Tag*-KEM, which is a KEM equipped with a special piece of information, the tag [1]. In their novel framework for hybrid encryption, this tag as part of an CCA-secure Tag-KEM is assigned to protect the non-malleability of the DEM part. Consequently, for the CCA-security of the whole Tag-KEM/DEM hybrid scheme with a CCA-secure Tag-KEM, it is only required that the DEM part is secure against *passive* adversaries. This is an obvious improvement compared to the KEM/DEM framework, but the flip-side of the coin is that the security proof of a Tag-KEM is somewhat more involved than the analogue proof for a "plain" KEM.

In the following, we construct a new Tag-KEM based on our proposed trapdoor permutation and prove its CCA-security in the random oracle model. Then we show how this leads to a CCA-secure hybrid encryption scheme in the Tag-KEM/DEM framework. Finally, we compare this novel scheme with EPOC-2/3.

3.3 The proposed Tag-KEM

In [1], the notion of Tag-KEM is formally defined. Here – to prevent redundancy – we only give the concrete description of our proposed Tag-KEM.

TKEM.Gen(1^k): Let k be a security parameter. Choose two distinct k bit primes p, q with $p \nmid q-1, q \nmid p-1$ such that each of $p-1, q-1$ has a large prime factor[6]. Build the product $n = p^2q$, compute $d = n^{-1} \bmod \varphi(pq)$ and define $rLen = 2k - 2$. Select a key derivation function KDF that maps bit-strings into the key-space of the designated DEM and a hash-function H, which outputs bit-strings of length $hashLen$. Return a pair $(\mathrm{pk}, \mathrm{sk})$ of public and secret key, where $\mathrm{pk} = (n, rLen, \mathrm{KDF}, H)$ and $\mathrm{sk} = (d, p, q)$.

TKEM.Key(pk): Choose $\omega \in \{0, 1, \ldots, 2^{rLen} - 1\}$ uniformly at random, compute $\mathrm{dk} = \mathrm{KDF}(\omega)$ and return (ω, dk).

TKEM.Enc(ω, τ): Given the key carrier ω and a tag τ, compute $c_1 = \omega^n \bmod n, c_2 = H(\omega, \tau)$ and return $\Psi = (c_1, c_2)$.

TKEM.Dec$_{\mathrm{sk}}(\Psi, \tau)$: Given the encapsulated key Ψ and a tag τ, parse Ψ to c_1, c_2 and compute $r = c_1^d \bmod pq$. If $|r|_2 > rLen$ or $H(r, \tau) \neq c_2$, then return $\perp$, return $\mathrm{KDF}(r)$, otherwise.

In the first step, a key pair is generated. Then a one-time key dk for the DEM part is constructed by applying a key derivation function to a random bit string. In the security proof, both of KDF and H are modeled as random oracles [2]. In the step TKEM.Enc, the one-time key (which in some sense is embedded in ω) is encrypted together with the tag τ. Finally, using TKEM.Dec$_{\mathrm{sk}}$ the one-time key dk can be recovered from the encapsulation Ψ and the tag τ.

Remark 3.1 In the decapsulation procedure, it is necessary to check if r indeed meets the length requirements ($|r|_2 \leq rLen = 2k - 2$), because otherwise a simple chosen-ciphertext attack can be mounted to obtain the secret factor pq by binary search [10].

CCA-security of a Tag-KEM requires that an adversary with adaptive oracle access to TKEM.Dec$_{\mathrm{sk}}$ has no chance to distinguish whether a given one-time key dk is encapsulated in a challenge (Ψ, τ) or not, even if the tag τ is chosen by the adversary himself. As usual, this is defined via an appropriate game. It is notable that although the adversary is restricted not to query the decapsulation oracle on the challenge (Ψ, τ), queries $(\Psi, \tilde{\tau})$ for $\tau \neq \tilde{\tau}$ are permitted. Hence a secure Tag-KEM provides non-malleability of the tag. For details see [1].

Due to space restrictions, the lengthy proof of the following theorem is presented in the full version of this paper [23]:

Theorem 3.2 *If factoring integers of the shape $n = p^2q$ is hard, then the Tag-KEM defined above is CCA-secure in the random oracle model.*

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[6] meaning that the bit-length of $p - 1$ (resp. $q - 1$) divided through its largest prime factor is $\mathcal{O}(\log k)$

More formally: If there exists an adversary $\mathcal{A}_T$ attacking the proposed Tag-KEM in the random oracle model

- *in time t,*
- *with advantage ϵ,*
- *querying the random oracle representing the key derivation function at most q_K times,*
- *querying the random oracle representing the hash function at most q_H times,*
- *invoking the decapsulation oracle at most q_D times,*

then there exists an adversary $\mathcal{A}_{Fact}$ who factors $n = p^2q$ in time t' and with advantage ϵ', where

$$t' \leq t + t_{\mathrm{gcd}}(q_H) + t_{f_{pq}} q_D,$$
$$\epsilon' \geq \left(\epsilon - \frac{q_K}{KLen} - \frac{2q_D}{HashLen} - \frac{q_D}{n + HashLen}\right)\left(1 - \frac{1}{p}\right),$$

where t_{gcd} is the time needed to perform a gcd *computation with inputs $\mathcal{O}(n)$ and $t_{f_{pq}}$ is the time needed to evaluate f_{pq}.*

3.4 The proposed hybrid encryption scheme

As before, to avoid lengthy recurrences, we do not review the generic Tag-KEM/DEM framework, but we only describe the concrete hybrid encryption scheme that is obtained when combining our proposed Tag-KEM with an appropriate DEM. The interested reader is referred to [1] for the general treatment. Let $(\mathcal{E}_K^{sym}, \mathcal{D}_K^{sym})$ be any symmetric cryptosystem with key K that is one-time secure (roughly speaking, this means that $(\mathcal{E}_K^{sym}, \mathcal{D}_K^{sym})$ is semantically secure against passive adversaries when K is used once only). Assume that the message space of $\mathcal{E}_K^{sym}$ is given as $\{0,1\}^{mLen}$.

Key Generation: The key generation is the same as in TKEM.Gen(.).

Encryption and **decryption** is performed as follows:

$\mathcal{E}_{\mathrm{pk}}(m)$:	$\mathcal{D}_{\mathrm{sk}}(\Psi, \tau)$:
$\omega \leftarrow\!\!\dashv \{0,1\}^{RLen}$	$(c_1, c_2) := \Psi$
$\mathrm{dk} := \mathrm{KDF}(\omega)$	$r := c_1^d \bmod pq$
$\tau \leftarrow\!\!\dashv \mathcal{E}_{\mathrm{dk}}^{sym}(m)$	if $\lvert r\rvert_2 > RLen$ or $H(r,\tau) \neq c_2$ return $\perp$
$\Psi := (\omega^n \bmod n, H(\omega, \tau))$	$m := \mathcal{D}_{\mathrm{KDF}(r)}^{sym}(\tau)$
Return (Ψ, τ)	Return m

Note that the DEM ciphertext of the message m encrypted with the encapsulated one-time key serves as the tag. Thus non-malleability of the DEM part is intuitively fulfilled because a CCA-secure Tag-KEM provides integrity of the tag. From the results of [1] we derive

Theorem 3.3 *If factoring integers of the type p^2q is hard and if $(\mathcal{E}_K^{sym}, \mathcal{D}_K^{sym})$ is one-time secure, then the proposed hybrid encryption scheme is CCA-secure in the random oracle model. More precisely, we have $\epsilon_{hy} \leq 2\epsilon_{KEM} + \epsilon_{DEM}$, where ϵ_{hy}, ϵ_{KEM} and ϵ_{DEM} denote the maximum advantage of a polynomial time attack against the CCA security of proposed hybrid encryption scheme, against the CCA security of our new Tag-KEM, resp. against the one-time security of $(\mathcal{E}_K^{sym}, \mathcal{D}_K^{sym})$.*

In the above theorem, CCA security of hybrid encryption is defined in the standard sense, *i. e.* indistinguishability of ciphertexts under adaptive chosen-ciphertext attacks. Note that the reduction to factoring is tight.

3.5 Comparison

In this section, we give a brief comparison of EPOC-2/3 and our proposed hybrid encryption scheme.Table 1 summarizes the most important parameters regarding security and performance. The efficiency of encryption and decryption is measured in modular multiplications, where $MM(k)$ denotes a modular multiplication modulo a k-bit number. We do not distinguish between multiplications and squarings, and we assume that a modular exponentiation with a k bit exponent takes approximately $3k/2$ modular multiplications, whilst a double exponentiation as necessary for performing the Okamoto-Uchiyama encryption takes approximately $7k/4$ modular multiplications using standard techniques. We have not considered exponent recoding techniques, but Chinese remaindering is taken into account if possible. Hashing, evaluations of the key derivation function and the symmetric key operations are not measured, because these magnitudes are comparable in all schemes. For evaluating the public key sizes, we compare n, g, h on the EPOC-2/3 side with n in our proposed scheme. The secret key sizes are the same (namely $3k$ referring to p, g_p for EPOC-2/3, resp. p, d for our proposed scheme). All quantities are measured in terms of the security parameter k (*i. e.* the bit-length of the prime factors p, q). In case of EPOC-2/3, we assume that $rLen = k$ and $HashLen \geq 2k$ hold (these are the values determining the exponent sizes).
As modular multiplication is quadratic in the length of the modulus, we conclude that our scheme is the most efficient one in decryption, whilst in encryption it is slightly less efficient than EPOC-2/3. Furthermore, the public key is 3 times shorter in our proposed scheme, and the underlying security assumption is optimal (as it is the case for EPOC-2). Another advantage of our scheme is the following: If one-time pad is used for the symmetric part,

Scheme	Assumption	encrypt	decrypt	pk
EPOC-2	FACT	$\geq 7k/2\ MM(3k)$	$\approx 3k/2\ MM(2k) + 7k/4\ MM(k)$	$9k$
EPOC-3	Gap-HR	$\geq 7k/2\ MM(3k)$	$\approx 3k/2\ MM(2k)$	$9k$
Proposed	FACT	$\approx 9k/2\ MM(3k)$	$\approx 3k\ MM(k)$	$3k$

Table 1
Comparison of important parameters

then the message length in our scheme is $2k$ compared to k in EPOC-2/3. This is because the bandwidth of f_{pq} is twice as large as the bandwidth of the Okamoto-Uchiyama encryption. Moreover, as in decryption there is only one hashing to be computed between the two potential rejection events, our scheme is more resistant against reject timing attacks [25,5] than EPOC-2, where a re-encryption is performed.

4 Conclusion

In this paper we introduced a new simple trapdoor one-way permutation based on the hardness of factoring. As provable secure trapdoor one-way permutations are so rare and nevertheless of outstanding importance in public key cryptography, the development of new candidates is a fundamental issue on its own. Moreover, to constitute the claim that our proposed trapdoor function is not only of theoretical interest, we constructed a novel CCA-secure hybrid encryption scheme as an exemplary application. To do so, we made use of the recently published Tag-KEM/DEM framework for hybrid encryption. We were able to show that already our proposed ad-hoc construction compares favorably with the members of the well-known EPOC family which are based on the same intractability assumption as our proposal.

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A Some Proofs

The following proofs have already been published in [22], therefore we put them to the appendix.

Proof of Lemma 2.3

Let x be an element of multiplicative order p in $\mathbb{Z}_n^\times$. Then we have

$$\begin{aligned} x^p = 1 \bmod n &\Rightarrow (x^p = 1 \bmod p \wedge x^p = 1 \bmod q) \\ &\Rightarrow (x = 1 \bmod p \wedge x = 1 \bmod q). \end{aligned}$$

Hence $pq | x - 1$ must hold, and we conclude $x \in \mathcal{S}$.
On the other hand, from the binomial expansion formula it is obvious that for all $x \in \mathcal{S}$ we have $x^p = 1 \bmod n \wedge x \neq 1$, thus the assertion follows. □

Proof of Lemma 2.4

Note that as p is the only non-trivial common factor of n and $\varphi(n) = p(p-1)(q-1)$, we must have

$$x^n = 1 \bmod n \iff x = 1 \vee \mathrm{ord}_n(x) = p$$

Hence the kernel of h consists of 1 and exactly the elements of multiplicative order p in $\mathbb{Z}_n^\times$, i.e the elements of $\mathcal{S}$ as defined in Lemma 2.3. □