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Null-controllability of some reaction–diffusion systems with one control force

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Abstract

This work is concerned with the null-controllability of semilinear parabolic systems by a single control force acting on a subdomain.

Keywords: Controllability; Semilinear parabolic systems; Carleman inequalities

1. Introduction

We consider a general reaction–diffusion system which arises in mathematical biology:

$$\begin{aligned}
&\psi_t = \Delta\psi + f_1(\psi, w) \quad \text{in } Q_T = \Omega \times (0, T), \\
&w_t = \Delta w + f_2(\psi, w) + \chi_\omega g \quad \text{in } Q_T, && (1)\\
&\psi = w = 0 \quad \text{on } \Sigma_T = \partial\Omega \times (0, T), \\
&\psi(x, 0) = \psi_0, \quad w(x, 0) = w_0, \quad x \in \Omega, && (2)
\end{aligned}$$

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where Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, f_i $(i=1,2)$ are smooth real functions (let us say C^2 functions) and g is a control in $L^2(Q_T)$. Let g^* in $L^2(Q_T)$ (with $Q_T=\Omega\times]0,T[$), and $(\psi_0^*,w_0^*)\in L^2(\Omega)^2$. Suppose that there exists a (ψ^*,w^*) satisfying (1) in $C(]0,T]\times L^2(\Omega))^2$ with $(\psi(0),w(0))=(\psi_0^*,w_0^*)$. Therefore, by setting:

$$\psi=\bar\psi-\psi^*,\qquad w=\bar w-w^*,$$

where $(\bar\psi,\bar w,\bar g)$ satisfies (1), one gets:

$$\begin{aligned}
&\psi_t=\Delta\psi+f_1(\bar\psi,\bar w)-f_1(\psi^*,w^*)\quad \text{in } Q_T,\\
&w_t=\Delta w\ +f_2(\bar\psi,\bar w)-f_2(\psi^*,w^*)+\chi_\omega g\quad \text{in } Q_T,\\
&\psi=w=0\quad \text{on } \Sigma_T,\\
&\psi(x,0)=\psi_0,\quad w(x,0)=w_0,\quad x\in\Omega,
\end{aligned}$$

where $g=\bar g-g^*$. We write this last system in the following form:

$$\begin{aligned}
&\psi_t=\Delta\psi+a(\psi,w)\psi+b(\psi,w)w\quad \text{in } Q_T,\\
&w_t=\Delta w+c(\psi,w)\psi+d(\psi,w)w+\chi_\omega g\quad \text{in } Q_T, &&(3)\\
&\psi=w=0\quad \text{on } \Sigma_T,\\
&\psi(x,0)=\psi_0,\quad w(x,0)=w_0,\quad x\in\Omega, &&(4)
\end{aligned}$$

where:

$$\begin{aligned}
a(\psi,w)&=\int_0^1\frac{\partial f_1}{\partial\psi}(s\psi+\psi^*,sw+w^*)\,ds,\\
b(\psi,w)&=\int_0^1\frac{\partial f_1}{\partial w}(s\psi+\psi^*,sw+w^*)\,ds,\\
c(\psi,w)&=\int_0^1\frac{\partial f_2}{\partial\psi}(s\psi+\psi^*,sw+w^*)\,ds,\\
d(\psi,w)&=\int_0^1\frac{\partial f_2}{\partial w}(s\psi+\psi^*,sw+w^*)\,ds.
\end{aligned}$$

Our aim is, for any (ψ_0,w_0) belonging to a suitable space, to find a control $g\in L^2(Q_T)$ such that the associated solution of (3)–(4) satisfies

$$\psi(T)=w(T)=0\quad \text{on } \Omega.$$

This is the *null-controllability* or the *controllability to the trajectories* property.

For reaction–diffusion systems, this question has been considered in Anita–Barbu [4] with

$$f_i(x,\psi,w)=\alpha_i a(x)\psi w,\quad i=1,2,$$

where the α_i are positive constants and $a \in L^\infty(\Omega)$ is such that $a \geqslant a_0 > 0$ a.e. in Ω. The authors proved a local exact controllability with two localized (in space) control functions (one for each equation). Another connected question is tackled by Barbu–Wang [6]. In their paper, these last authors prove, by way of direct techniques, the stabilization of system (1). As they pointed out, local null-controllability implies (local) stabilization but the first property is still an open problem. Our work is just concerned by this problem (null-controllability). We prove in this paper that, under an assumption (which does not seem very restrictive (see (34) below in Theorem 6), this property holds for system (1). Our approach is based on earlier works on the local and global null-controllability of phase-field systems and abstract parabolic-like systems (see [2,3]).

This paper is in keeping with the idea of controlling or stabilizing systems using the least control forces possible: works in this direction dealing with various systems governed by partial differential equations or equations in an abstract framework can be found in [1,7].

The paper is organized as follows. We set and prove the local null-controllability of system (1) in Section 4. Before this, we first prove in Section 2 a crucial observability estimate for the linearized problem (see Theorem 3 below). In Section 3, we use this estimate to prove the null-controllability of a linearized system derived from (3).

2. Observability estimate

We consider in this section the problem:

$$\begin{aligned}
&u_t = \Delta u + au + bv \quad \text{in } Q_T,\\
&v_t = \Delta v + cu + dv + \chi_\omega g \quad \text{in } Q_T, \qquad (5)\\
&u = v = 0 \quad \text{on } \Sigma_T,\\
&u(x,0) = u_0, \quad v(x,0) = v_0, \quad x \in \Omega, \qquad (6)
\end{aligned}$$

and its adjoint problem:

$$\begin{aligned}
&-\varphi_t = \Delta\varphi + a\varphi + cw \quad \text{in } Q_T,\\
&-w_t = \Delta w + b\varphi + dw \quad \text{in } Q_T, \qquad (7)\\
&\varphi = w = 0 \quad \text{on } \Sigma_T,\\
&\varphi(x,T) = \varphi_0, \quad w(x,T) = w_0, \quad x \in \Omega, \qquad (8)
\end{aligned}$$

where $a, b, c, d \in L^\infty(Q_T)$.

Following [9], let us introduce some notations. Let $\omega' \Subset \omega$ be a subdomain of ω and let β be a $C^2(\bar{\Omega})$ function such that

$$\min\{|\nabla\beta(x)|,\ x \in \overline{\Omega\backslash\omega'}\} > 0 \quad \text{and} \quad \frac{\partial\beta}{\partial n} \leqslant 0 \quad \text{on } \partial\Omega, \qquad (9)$$

where n denotes the outward unit normal to $\partial\Omega$. Moreover, we can always assume that β satisfies

$$\min\{\beta(x),\ x \in \bar{\Omega}\} \geqslant \max\left(\frac{3}{4}\|\beta\|_{L^\infty(\Omega)}, \ln(3)\right). \qquad (10)$$

Finally, we introduce the following functions with parameters $\lambda > 0$ and $\tau > 0$:

$$\rho(t,x) := \frac{e^{\lambda\beta(x)}}{t(T-t)}, \quad (t,x) \in Q_T, \tag{11}$$

$$\alpha(t,x) := \tau \frac{e^{\frac{4}{3}\lambda\|\beta\|_{L^\infty(\Omega)}} - e^{\lambda\beta(x)}}{t(T-t)}, \quad (t,x) \in Q_T. \tag{12}$$

Note in particular that $\rho > 4/T^2$.

Then the following result holds (Carleman estimate):

Theorem 1. [9, Theorem 7.1, p. 288] *There exist $\lambda_0 > 0$, $\tau_0 > 0$ and a positive constant C such that $\forall\lambda \geqslant \lambda_0$, $\forall\tau \geqslant \tau_0$ and $\forall s \geqslant -3$ the inequality*

$$\int_{Q_T} \left(\frac{1}{\lambda}|z_t|^2 + \frac{1}{\lambda}\left|D_x^2 z\right|^2 + \lambda\tau^2\rho^2|\nabla z|^2 + \lambda^4\tau^4\rho^4 z^2\right)\rho^{2s-1}e^{-2\alpha}\,dx\,dt$$

$$\leqslant C\left(\tau\int_{Q_T} |z_t \pm \Delta z|^2\rho^{2s}e^{-2\alpha}\,dx\,dt + \lambda^4\tau^4\int_0^T\int_{\omega'} z^2\rho^{2s+3}e^{-2\alpha}\,dx\,dt\right), \tag{13}$$

holds for any function z satisfying homogeneous Dirichlet condition and such that the right-hand-side of (13) *is finite. Moreover, the constants C and λ_0 depend only on Ω and ω'. The constant τ_0 is of the form*

$$\tau_0 = c_0(\Omega,\omega')\left(T + T^2\right).$$

The explicit dependence in time of the constants is not given in [9]. We refer to [10] where the above formula for τ_0 is obtained.

In the sequel, the symbol C will stand for various constants independent of T and a, b, c, d.

Let us introduce the following notation: for given λ and τ as in Theorem 1, we set $\delta = \tau\rho$ and consider the functional

$$I(s,z) = \int_{Q_T}\left(\frac{1}{\lambda}|z_t|^2 + \frac{1}{\lambda}|\Delta z|^2 + \lambda\delta^2|\nabla z|^2 + \lambda^4\delta^4 z^2\right)\delta^{2s-1}e^{-2\alpha}\,dx\,dt. \tag{14}$$

On the other hand, we set

$$\|a,b,c,d\|_\infty = \left(\|a\|_\infty^2 + \|b\|_\infty^2 + \|c\|_\infty^2 + \|d\|_\infty^2\right)^{1/2}.$$

Lemma 2. *Let $\lambda_0 > 1$, C being the constant given in Theorem* 1. *Then $\forall\lambda \geqslant \lambda_0$, $\forall\tau \geqslant \tau_1 = \frac{T^2}{4}(\frac{4C}{\lambda^4})^{1/3}\|a,b,c,d\|_\infty^{2/3}$ and $\forall s \geqslant -3$, the solution (φ, w) of* (7)–(8) *satisfies the estimate*:

$$I(s,\varphi) + I(s,w) \leqslant C\lambda^4\int_0^T\int_{\omega'}\left(\varphi^2 + w^2\right)\delta^{2s+3}e^{-2\alpha}\,dx\,dt. \tag{15}$$

As a consequence, we get

$$I\left(-\frac{3}{2},\varphi\right)+I\left(-\frac{3}{2},w\right)\leqslant C\lambda^4\int_0^T\int_{\omega'}(\varphi^2+w^2)e^{-2\alpha}\,dx\,dt. \tag{16}$$

Proof. Applying Theorem 1, we get with our notations:

$$\begin{aligned}&I(s,\varphi)+I(s,w)\\&\quad\leqslant 2C\bigg[\int_{Q_T}\big(\big(\|a\|_\infty^2+\|b\|_\infty^2\big)\varphi^2+\big(\|c\|_\infty^2+\|d\|_\infty^2\big)w^2\big)\delta^{2s}e^{-2\alpha}\,dx\,dt\\&\qquad+\lambda^4\int_0^T\int_{\omega'}(\varphi^2+w^2)\delta^{2s+3}e^{-2\alpha}\,dx\,dt\bigg].\end{aligned}$$

Choosing then

$$\tau\geqslant\tau_1=\frac{T^2}{4}\left(\frac{4C}{\lambda^4}\right)^{1/3}\|a,b,c,d\|_\infty^{2/3},$$

which implies

$$2C\|a,b,c,d\|_\infty^{2/3}\leqslant\frac{1}{2}\lambda^4\delta^3,$$

we deduce (15) and therefore (16). □

This lemma already implies the null-controllability of (5) by two control forces (i.e., in the case where a second force $\chi_\omega f$ occurs in the first equation of (5)). So, now, the problem is to get rid of the term $\int_0^T\int_{\omega'}\varphi^2e^{-2\alpha}\,dx\,dt$ in the right-hand side of (16) and the main idea is to use the second equation in (7) to estimate this integral in terms of $\int_{Q_{T,\omega}}e^{-r\alpha}w^2\,dx\,dt$. The construction of the functional (21) below turns around this idea. Our crucial result is the following:

Theorem 3. *With the hypotheses of Lemma* 2*, assume moreover that there exist a constant* $b_0>0$ *and a domain* ω_b *such that*

$$\omega_b\Subset\omega, \tag{17}$$

$$|b|\geqslant b_0\quad\text{in }\omega_b\times(0,T_0), \tag{18}$$

for some $T_0>0$*. Then for all* $r\in[0,2)$ *there exists a constant* $C=C_{r,T}$ *such that*

$$\int_0^T\int_{\omega'}(\varphi^2+w^2)e^{-2\alpha}\,dx\,dt\leqslant C\int_0^T\int_{\omega}e^{-r\alpha}w^2\,dx\,dt, \tag{19}$$

for all ω' *such that* $\omega'\Subset\omega_b\Subset\omega$.

As an immediate consequence, it follows that for all $r \in [0, 2)$ there exists a constant $C = C_{r,T}$ such that

$$I(-3/2, \varphi) + I(-3/2, w) \leqslant C \int_0^T \int_\omega e^{-r\alpha} w^2 \, dx \, dt. \tag{20}$$

Remark 1. We will see later that it is important to be able to choose $r > 1$. It seems that r cannot be equal to 2: this is the "cost" of the control by a single force.

Remark 2. Of course, assumption (18) will imply a restriction on (ψ^*, w^*, g^*) satisfying (1) (see (34)).

Proof. The main idea is to estimate $\int_0^T \int_{\omega'} \varphi^2 e^{-2\alpha} \, dx \, dt$ by $\int_0^T \int_\omega e^{-r\alpha} w^2 \, dx \, dt$ for some $r \in [0, 2)$ using the second equation of (7). To do this, we first localize the system in space, multiply the second equation by $-\beta_0 e^{-2\alpha} \eta\varphi$ and manage the "bad" terms appearing (see $\Lambda(t)$ in (21) and [3] for the construction of this function in an abstract setting).

Let $\xi \in C^\infty(R^n)$ be a truncation function satisfying

$$\begin{cases} \xi(x) = 1, & \forall x \in \omega', \\ 0 < \xi(x) \leqslant 1, & \forall x \in \omega'', \\ \xi(x) = 0, & \forall x \in \mathbb{R}^n \backslash \omega'', \end{cases}$$

where $\omega' \Subset \omega'' \Subset \omega_b \Subset \omega \Subset \Omega$.

Assume for example that $b \geqslant b_0 > 0$ in $\omega_b \times (0, T)$ and introduce the function $\eta := \xi^6$. For real numbers $\beta_0, \beta_1, p, q > 0$, which will be chosen below, set

$$\Lambda(t) = \int_\Omega \left(e^{-p\alpha} \eta^{4/3} \varphi^2 - \beta_0 e^{-2\alpha} \eta\varphi w + \beta_1 e^{-q\alpha} \eta^{2/3} w^2\right) dx, \tag{21}$$

and remark that $\Lambda(T) = \Lambda(0) = 0$.

Note that if instead of $b \geqslant b_0 > 0$ we have $-b \geqslant b_0 > 0$ in (17) then the expression of Λ must be modified by taking $(\beta_0 e^{-2\alpha} \eta\varphi w)$ instead of $(-\beta_0 e^{-2\alpha} \eta\varphi w)$ for its second term.

Differentiating Λ with respect to t and replacing φ_t and w_t by their expressions given by (7), we obtain

$$\begin{aligned} \Lambda' = {} & \int_\Omega \left(-p e^{-p\alpha} \eta^{4/3} \varphi^2 + 2\beta_0 e^{-2\alpha} \eta\varphi w - \beta_1 q \, e^{-q\alpha} \eta^{2/3} w^2\right) \alpha_t \, dx \\ & - \int_\Omega 2e^{-p\alpha} \eta^{4/3} \varphi(\Delta\varphi + a\varphi + cw) \, dx \\ & + \beta_0 \int_\Omega e^{-2\alpha} \eta\left[\varphi(\Delta w + b\varphi + dw) + w(\Delta\varphi + a\varphi + cw)\right] dx \\ & - \int_\Omega 2\beta_1 \, e^{-q\alpha} \eta^{2/3} w(\Delta w + b\varphi + dw) \, dx. \end{aligned} \tag{22}$$

Let us, at this level, throw light on the construction of Λ. The main point is contained in the third integral of (22): we have multiplied the second equation in (7) by $\beta_0 e^{-2\alpha}\eta\varphi$ (in order to get $\int_{Q_T} e^{-2\alpha}\eta b\varphi^2\,dx$) and the first equation in (7) by $\beta_0 e^{-2\alpha}\eta w$. After integrating by parts (in space) this integral, some terms containing in particular $\nabla\varphi$ and ∇w appear and in order to manage these "bad" terms, we have again multiplied the two equations of (7) by $e^{-p\alpha}\eta^{4/3}\varphi$ and $e^{-q\alpha}\eta^{2/3}w$ respectively getting terms of the form $|\nabla\varphi|^2$ and $|\nabla w|^2$ with good signs. The three other integrals in (22) just come from these computations. Of course, in the forthcoming lines, we verify that this "strategy" works.

Coming back to (22) and recalling that $\Lambda(0) = \Lambda(T) = 0$, the integration of (22) over $(0, T)$ yields

$$\begin{aligned}
&\beta_0 \int\limits_{Q_T} e^{-2\alpha}\eta b\varphi^2\,dx \\
&\quad= \int\limits_{Q_T} \bigl\{(p\alpha_t + 2a)e^{-p\alpha}\eta^{4/3}\varphi^2 + \bigl[\beta_1(q\alpha_t + 2d)e^{-q\alpha}\eta^{2/3} - \beta_0 e^{-2\alpha}\eta c\bigr]w^2 \\
&\qquad - \bigl[\beta_0(2\alpha_t + a + d)e^{-2\alpha}\eta - 2\beta_1 e^{-q\alpha}\eta^{2/3}b - 2e^{-p\alpha}\eta^{4/3}c\bigr]\varphi w\bigr\}\,dx\,dt \\
&\qquad + 2\int\limits_{Q_T} e^{-p\alpha}\eta^{4/3}\varphi\Delta\varphi\,dx\,dt - \beta_0\int\limits_{Q_T} e^{-2\alpha}\eta(\varphi\Delta w + w\Delta\varphi)\,dx\,dt \\
&\qquad + 2\beta_1\int\limits_{Q_T} e^{-q\alpha}\eta^{2/3}w\Delta w\,dx\,dt \\
&\quad= J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{23}$$

Here appears clearly why we need the assumption (18). Now we estimate each of the four terms $J_1, \ldots, J_4$.

For J_1, in order to estimate $\beta_0(2\alpha_t + a + d)e^{-2\alpha}\eta\varphi w$ in terms of $e^{-2\alpha}\eta\varphi^2$, we need an upper-bound for $(2\alpha_t + a + d)^2 e^{-2\alpha}\eta^{2/3}w^2$. So, since $\alpha_t \notin L^\infty(Q_T)$, we introduce $r \in [0, 2)$ and write $e^{-2\alpha} = e^{-(2-r)\alpha}e^{-r\alpha}$. Assuming that

$$p > 2, \quad q > 1 + \frac{r}{2}, \quad r < 2, \tag{24}$$

and that $\beta_0, \beta_1 \geqslant 1$ and using the Cauchy–Schwarz inequality together with the fact that $\|\eta\|_\infty \leqslant 1$ and $T \leqslant C\tau$, we get

$$\begin{aligned}
J_1 &\leqslant \left(\frac{1}{2} + \bigl\|(p\alpha_t + 2a)e^{-(p-2)\alpha}\eta^{1/3}\bigr\|_\infty\right)\int\limits_{Q_T} e^{-2\alpha}\eta\varphi^2\,dx\,dt \\
&\quad + \Biggl\{\frac{1}{2}\bigl\|\beta_0(2\alpha_t + a + d)e^{-(1-\frac{r}{2})\alpha}\eta^{1/3} - 2e^{-(p-1-\frac{r}{2})\alpha}\eta^{2/3}c \\
&\quad - 2\beta_1 e^{-(q-1-\frac{r}{2})\alpha}b\bigr\|_\infty^2 \\
&\quad + \bigl\|\beta_1(q\alpha_t + 2d)e^{-(q-r)\alpha}\eta^{1/3} - \beta_0 e^{-(2-r)\alpha}\eta^{2/3}c\bigr\|_\infty\Biggr\}\int\limits_{Q_T} \eta^{1/3}e^{-r\alpha}w^2\,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\bigg[\big(1+\|a\|_\infty+\big\|\alpha_t e^{-(p-2)\alpha}\big\|_\infty\big)\int\limits_{Q_T} e^{-2\alpha}\eta\varphi^2\,dx\,dt \\
&\quad+\big(\big(1+\|a+d\|_\infty^2+\|c\|_\infty^2+\big\|\alpha_t e^{-(1-\frac{r}{2})\alpha}\big\|_\infty^2\big)\beta_0^2 \\
&\quad+\big(1+\|b\|_\infty^2+\|d\|_\infty^2+\big\|\alpha_t e^{-(q-r)\alpha}\big\|_\infty^2\big)\beta_1^2\big)\int\limits_{Q_T}\eta^{1/3}e^{-r\alpha}w^2\,dx\,dt\bigg] \\
&\leqslant C\bigg[\bigg(1+\|a\|_\infty+\frac{\tau^2}{T^4}\bigg)\int\limits_{Q_T} e^{-2\alpha}\eta\varphi^2\,dx\,dt \\
&\quad+|\beta|^2\bigg(1+\|Df_*\|_\infty^2+\frac{\tau^4}{T^8}\bigg)\int\limits_{Q_T}\eta^{1/3}e^{-r\alpha}w^2\,dx\,dt\bigg]
\end{aligned}
\tag{25}
$$

where $C=C(p,q,\|\eta\|_{L^\infty(\Omega)})$, $\|Df_*\|_\infty^2=\|a,b,c,d\|_\infty^2=\|a\|_\infty^2+\|b\|_\infty^2+\|c\|_\infty^2+\|d\|_\infty^2$ and $|\beta|^2=\beta_0^2+\beta_1^2$.

Concerning J_2, we have that

$$
\begin{aligned}
J_2&=2\int\limits_{Q_T} e^{-p\alpha}\eta^{4/3}\varphi\Delta\varphi\,dx\,dt \\
&=-2\int\limits_{Q_T} e^{-p\alpha}\eta^{4/3}|\nabla\varphi|^2\,dx\,dt-2\int\limits_{Q_T}\varphi\nabla\varphi.\nabla\big(e^{-p\alpha}\eta^{4/3}\big)\,dx\,dt \\
&=-2\int\limits_{Q_T} e^{-p\alpha}\eta^{4/3}|\nabla\varphi|^2\,dx\,dt+\int\limits_{Q_T}\varphi^2\Delta\big(e^{-p\alpha}\eta^{4/3}\big)\,dx\,dt \\
&=-2\int\limits_{Q_T} e^{-p\alpha}\eta^{4/3}|\nabla\varphi|^2\,dx\,dt+\int\limits_{Q_T}\big(e^{-2\alpha}\eta\varphi^2\big)e^{2\alpha}\eta^{-1}\Delta\big(e^{-p\alpha}\eta^{4/3}\big)\,dx\,dt.
\end{aligned}
$$

We have

$$
\begin{aligned}
\Delta\big(e^{-p\alpha}\eta^l\big)=e^{-p\alpha}\big[&\big(p^2|\nabla\alpha|^2-p\Delta\alpha\big)\eta^l+(\Delta\eta-2p\nabla\alpha.\nabla\eta)\eta^{l-1} \\
&+p(p-1)|\nabla\eta|^2\eta^{l-2}\big],
\end{aligned}
$$

so

$$
\begin{aligned}
e^{2\alpha}\eta^{-1}\Delta\big(e^{-p\alpha}\eta^{4/3}\big)=e^{-(p-2)\alpha}\big[&\big(p^2|\nabla\alpha|^2-p\Delta\alpha\big)\eta^{1/3} \\
&+\frac{4}{3}(\Delta\eta-2p\nabla\alpha.\nabla\eta)\eta^{-2/3}+p(p-1)|\nabla\eta|^2\eta^{-5/3}\big].
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\frac{\nabla\eta}{\eta^{5/6}}=6\nabla\xi\in L^\infty(\Omega),\qquad \frac{\nabla\eta}{\eta^{2/3}}=6\xi\nabla\xi\in L^\infty(\Omega), \\
&\frac{\Delta\eta}{\eta^{2/3}}=30|\nabla\xi|^2+6\xi\Delta\xi\in L^\infty(\Omega).
\end{aligned}
$$

It follows from this last computation and $p > 2$ that

$$\left\| e^{2\alpha}\eta^{-1}\Delta\big(e^{-p\alpha}\eta^{4/3}\big)\right\|_\infty \leqslant C\left(1+\frac{\tau^2}{T^4}\right)$$

where $C = C(p, \|\eta\|_{L^\infty(\Omega)})$. Coming back to J_2, we get

$$J_2 \leqslant -2\int\limits_{Q_T} e^{-p\alpha}\eta^{4/3}|\nabla\varphi|^2\,dx\,dt + C\left(1+\frac{\tau^2}{T^4}\right)\int\limits_{Q_T} e^{-2\alpha}\eta\varphi^2\,dx\,dt. \tag{26}$$

We now estimate J_3. We have

$$\begin{aligned} J_3 &= -\int\limits_{Q_T} \beta_0 e^{-2\alpha}\eta(\varphi\Delta w + w\Delta\varphi)\,dx\,dt \\ &= -\int\limits_{Q_T} \beta_0 e^{-2\alpha}\eta\big(\Delta(\varphi w) - 2\nabla\varphi.\nabla w\big)\,dx\,dt \\ &= -\beta_0\int\limits_{Q_T} \Delta\big(e^{-2\alpha}\eta\big)\varphi w\,dx\,dt + 2\int\limits_{Q_T} \beta_0 e^{-2\alpha}\eta\nabla\varphi.\nabla w\,dx\,dt. \end{aligned}$$

Proceeding as previously, thanks to the assumption $r < 2$, we get

$$\begin{aligned} &\left|\beta_0\int\limits_{Q_T} \Delta\big(e^{-2\alpha}\eta\big)\varphi w\,dx\,dt\right| \\ &\quad\leqslant C\left(\int\limits_{Q_T} e^{-2\alpha}\eta\varphi^2\,dx\,dt + \beta_0^2\left(1+\frac{\tau^4}{T^8}\right)\int\limits_{Q_T} \eta^{1/3}e^{-r\alpha}w^2\,dx\,dt\right). \end{aligned}$$

Thus it appears that

$$\begin{aligned} J_3 &\leqslant C\left(\int\limits_{Q_T} e^{-2\alpha}\eta\varphi^2\,dx\,dt + \beta_0^2\left(1+\frac{\tau^4}{T^8}\right)\int\limits_{Q_T} \eta^{1/3}e^{-r\alpha}w^2\,dx\,dt\right) \\ &\quad + 2\int\limits_{Q_T} \beta_0 e^{-2\alpha}\eta\nabla\varphi.\nabla w\,dx\,dt. \end{aligned} \tag{27}$$

Finally, we estimate J_4:

$$\begin{aligned} J_4 &= 2\beta_1\int\limits_{Q_T} e^{-q\alpha}\eta^{2/3}w\Delta w\,dx\,dt \\ &= -2\beta_1\int\limits_{Q_T} e^{-q\alpha}\eta^{2/3}|\nabla w|^2\,dx\,dt + \beta_1\int\limits_{Q_T} \Delta\big(e^{-q\alpha}\eta^{2/3}\big)w^2\,dx\,dt. \end{aligned}$$

Again, in the same way, we get using the condition $q > r$ and the definition of η:

$$\left|\int\limits_{Q_T} \Delta\big(e^{-q\alpha}\eta^{2/3}\big)w^2\,dx\,dt\right| \leqslant C\left(1+\frac{\tau^2}{T^4}\right)\int\limits_{Q_T} \eta^{1/3}e^{-r\alpha}w^2\,dx\,dt.$$

Thus

$$J_4 \leqslant -2\beta_1 \int\limits_{Q_T} e^{-q\alpha}\eta^{2/3}|\nabla w|^2\,dx\,dt + C\beta_1\left(1+\frac{\tau^2}{T^4}\right)\int\limits_{Q_T} \eta^{1/3}e^{-r\alpha}w^2\,dx\,dt. \tag{28}$$

Then from (22) and (25)–(28), we get with $\tau \geqslant \tau_0$, conditions (24) and $\beta_1 \geqslant 1$:

$$\begin{aligned}
&\beta_0 b_0 \int\limits_{Q_T} e^{-2\alpha}\eta\varphi^2\,dx\,dt \\
&\quad\leqslant C\left(1+\|a\|_\infty+\frac{\tau^2}{T^4}\right)\int\limits_{Q_T} e^{-2\alpha}\eta\varphi^2\,dx\,dt \\
&\qquad + C|\beta|^2\left(1+\|Df_*\|_\infty^2+\frac{\tau^4}{T^8}\right)\int\limits_{Q_T} e^{-r\alpha}\eta^{1/3}w^2\,dx\,dt \\
&\qquad - 2\int\limits_{Q_T} e^{-p\alpha}\eta^{4/3}|\nabla\varphi|^2\,dx\,dt - 2\beta_1\int\limits_{Q_T} e^{-q\alpha}\eta^{2/3}|\nabla w|^2\,dx\,dt \\
&\qquad + 2\beta_0\int\limits_{Q_T} e^{-2\alpha}\eta\nabla\varphi.\nabla w\,dx\,dt.
\end{aligned}$$

But fixing now $\beta_0 = \frac{2C}{b_0}(1+\|a\|_\infty+\frac{\tau^2}{T^4})$, we obtain

$$\begin{aligned}
\int\limits_{Q_T} e^{-2\alpha}\eta\varphi^2\,dx &\leqslant C_{T,\|a\|_\infty}\int\limits_{Q_T} \eta^{1/3}e^{-r\alpha}w^2\,dx\,dt - \frac{2}{\beta_0}\int\limits_{Q_T} e^{-p\alpha}\eta^{4/3}|\nabla\varphi|^2\,dx\,dt \\
&\quad - \frac{2\beta_1}{\beta_0}\int\limits_{Q_T} e^{-q\alpha}\eta^{2/3}|\nabla w|^2\,dx\,dt + 2\int\limits_{Q_T} e^{-2\alpha}\eta\nabla\varphi.\nabla w\,dx\,dt.
\end{aligned}$$

Consider the three last terms in the right-hand side of the previous inequality. Assume moreover that

$$p \leqslant 4-q.$$

With this assumption, for $\beta_1 \geqslant \frac{1}{2}\beta_0^2$, we have that

$$\left(\beta_0 e^{-2\alpha}\eta\right)^2 \leqslant 2\left(e^{-p\alpha}\eta^{4/3}\right)\left(\beta_1 e^{-q\alpha}\eta^{2/3}\right) \quad \text{on } Q_T,$$

which implies

$$\begin{aligned}
&-\frac{2}{\beta_0}\int\limits_{Q_T} e^{-p\alpha}\eta^{4/3}|\nabla\varphi|^2\,dx\,dt - \frac{2\beta_1}{\beta_0}\int\limits_{Q_T} e^{-q\alpha}\eta^{2/3}|\nabla w|^2\,dx\,dt \\
&\quad + 2\int\limits_{Q_T} e^{-2\alpha}\eta\nabla\varphi.\nabla w\,dx\,dt \leqslant 0.
\end{aligned}$$

To summarize, we have

$$\int_{Q_T} e^{-2\alpha}\eta\varphi^2\,dx\,dt \leqslant C\frac{\left(1+\|Df_*\|_\infty^2+\frac{\tau^4}{T^8}\right)(\beta_1^2+\beta_0^2)}{\beta_0}\int_{Q_T}\eta^{1/3}e^{-r\alpha}w^2\,dx\,dt$$

and all the computations are valid with the following conditions:

$$\begin{aligned}
&r<2, \qquad p>2, \qquad q>1+\frac{r}{2}, \qquad p\leqslant 4-q,\\
&\beta_0=\frac{2c}{b_0}\left(1+\frac{\tau^2}{T^4}+\|a\|_\infty^2\right), \qquad \beta_1\geqslant\frac{1}{2}\beta_0^2, \qquad \tau\geqslant\tau_1.
\end{aligned}\tag{29}$$

It is clear that there is a nonempty set of (p,q,r) satisfying all the conditions in (29): for instance, $(2+\frac{1}{16},2-\frac{1}{8},\frac{3}{2})$ satisfies this condition. With may be a modified constant C, we get the following final estimate:

$$\int_{Q_{T,\omega'}} e^{-2\alpha}\varphi^2\,dx\,dt\leqslant C\int_{Q_{T,\omega}} e^{-r\alpha}w^2\,dx\,dt,$$

where $Q_{T,\omega}=(0,T)\times\omega$ and $C=C(T,\|Df_*\|_\infty)$. This final estimate ends the proof of the theorem. $\square$

3. Null controllability of (5)

For $\varepsilon>0$ and $r\in(0,2)$, we define:

$$J_\varepsilon(g)=\frac{1}{2}\int_{Q_T}e^{r\alpha}g^2\,dx\,dt+\frac{1}{2\varepsilon}\left\|(u,v)(T)\right\|_{L^2(\Omega)}^2,$$

where $g\in L^2(Q_T)$ and (u,v) is the associated solution of (5) with given $X_0=(u_0,v_0)\in H_0^1(\Omega)\times H_0^1(\Omega)$. Introduce also the dual functional (see [8]):

$$J_\varepsilon^*(Y_0)=\frac{1}{2}\int_0^T\int_\omega e^{-r\alpha}w^2\,dx\,dt+\frac{\varepsilon}{2}\|Y_0\|_{L^2(\Omega)}^2+\int_\Omega Y(0).X_0\,dx,$$

where $Y=(\varphi,w)$ is the solution of the backward linear system (7) with data $Y_0=(\varphi_0,w_0)\in L^2(\Omega)\times L^2(\Omega)$.

By classical arguments, the minimization problems

$$\min_g J_\varepsilon(g) \quad\text{and}\quad \min_{Y_0} J_\varepsilon^*(Y_0),$$

have both exactly one solution g_ε and $Y_{0\varepsilon}$, respectively. Moreover, by the maximum principle (or see for instance [8]):

$$g_\varepsilon=\chi_\omega e^{-r\alpha}w_\varepsilon \quad\text{on } Q_T; \qquad Y_{0\varepsilon}=-\frac{1}{\varepsilon}(u_\varepsilon,v_\varepsilon)(T)\quad\text{on } \Omega,\tag{30}$$

where $(u_\varepsilon, v_\varepsilon)$ (respectively $(\varphi_\varepsilon, w_\varepsilon)$) is the solution of (5) (respectively (7)) associated with g_ε (respectively $Y_{0\varepsilon}$). Since $J_\varepsilon^*(Y_{0\varepsilon}) \leqslant 0$, we get

$$\frac{1}{2}\int_0^T\!\!\int_\omega e^{-r\alpha} w_\varepsilon^2\,dx\,dt + \frac{1}{2\varepsilon}\big\|(u_\varepsilon, v_\varepsilon)(T)\big\|^2_{L^2(\Omega)} \leqslant \big\|(\varphi_\varepsilon, w_\varepsilon)(0)\big\|_{L^2(\Omega)}.\|X_0\|_{L^2(\Omega)}. \tag{31}$$

To obtain an uniform estimate, we will need the following results:

Lemma 4. *With the hypotheses of Theorem* 3*, for* $r \in (0, 2)$*, any solution pair of* (7) *satisfies the estimate*

$$\big\|(\varphi, w)(0)\big\|^2_{L^2(\Omega)} \leqslant C_T \int_0^T\!\!\int_\omega e^{-r\alpha} w^2\,dx\,dt,$$

with

$$C_T = \exp\left(C\left(1 + \frac{1}{T} + \big(1 + \big\|(a,b,c,d)\big\|_\infty\big)T + \big\|(a,b,c,d)\big\|_\infty^{4/3}\right)\right),$$

where $\|(a,b,c,d)\|_\infty = (\|a\|^2_\infty + \|b\|^2_\infty + \|c\|^2_\infty + \|d\|^2_\infty)^{1/2}$.

Proof. The proof of this lemma is by now classical and very similar to the equivalent lemma in [2]. □

For $N \geqslant 1$, let q_N such that

$$\frac{N+2}{2} < q_N < 2\frac{N+2}{N-2} \quad \text{if } N \geqslant 3, \qquad q_N \in (2, +\infty) \quad \text{if } N = 1, 2. \tag{32}$$

Lemma 5. *With the hypotheses of Lemma* 4*, for any* $X_0 = (u_0, v_0) \in (H_0^1(\Omega) \cap W^{2(1-\frac{1}{q_N}), q_N}(\Omega))^2$*, there exists* $((u,v), g) \in (L^2(0,T; H_0^1(\Omega)) \cap W^{2,1}_{q_N}(Q_T))^2 \times L^{q_N}(Q_{T,\omega})$ *satisfying* (5) *and*

$$(u,v)(T) = 0 \quad \text{on } \Omega,$$
$$\|\chi_\omega g\|^2_{L^{q_N}(Q_T)} \leqslant C_T \|X_0\|^2_{L^2(\Omega)},$$

where C_T *is defined in Lemma* 4.

Proof. From (31) and (32), we get for all $\varepsilon > 0$:

$$\frac{1}{2}\int_0^T\!\!\int_\omega e^{-r\alpha} w_\varepsilon^2\,dx\,dt + \frac{1}{2\varepsilon}\big\|(u_\varepsilon, v_\varepsilon)(T)\big\|^2_{L^2(\Omega)} \leqslant C_T \|X_0\|^2_{L^2(\Omega)}.$$

We should obtain from this last estimate a control in $L^2(Q_T)$ just by passing to the limit in ε. But we will prove that our control is in $L^{q_N}(Q_T)$ because we will need this property in the following section. So let us introduce $\zeta_\varepsilon = e^{-r\alpha} w_\varepsilon$. It satisfies by (7):

$$\begin{cases} (\zeta_\varepsilon)_t + \Delta\zeta_\varepsilon = f_\varepsilon & \text{in } (0,T)\times\Omega = Q_T, \\ \zeta_\varepsilon = 0 & \text{on } (0,T)\times\partial\Omega = \Sigma_T, \\ \zeta_\varepsilon(T) = 0 & \text{in } \Omega, \end{cases}$$

with

$$f_\varepsilon = -2r\nabla\alpha.\big(e^{-r\alpha}\nabla w_\varepsilon\big) + \big(\Delta\big(e^{-r\alpha}\big) + \big(e^{-r\alpha}\big)_t - de^{-r\alpha}\big)w_\varepsilon - be^{-r\alpha}\varphi_\varepsilon.$$

By parabolic regularity, we have

$$\|\zeta_\varepsilon\|_{W_2^{2,1}(Q_T)} \leqslant C\|f_\varepsilon\|_{L^2(Q_T)}.$$

On the other hand, setting

$$I_1 = \int\limits_{Q_T} e^{-2r\alpha}\varphi_\varepsilon^2\,dx\,dt$$

we have, using (20) in Theorem 3,

$$I_1 = \int\limits_{Q_T} \big(e^{-2(r-1)\alpha}\big)\big(e^{-2\alpha}\varphi_\varepsilon^2\big)\,dx\,dt \leqslant \big\|e^{-2(r-1)\alpha}\big\|_\infty \int\limits_{Q_T} e^{-2\alpha}\varphi_\varepsilon^2\,dx\,dt$$
$$\leqslant C_T \int\limits_0^T\!\!\int\limits_\omega e^{-r\alpha}w_\varepsilon^2\,dx\,dt.$$

Of course, $\|e^{-2(r-1)\alpha}\|_\infty$ is finite if we assume

$$r \geqslant 1$$

and the same remark holds in the sequel. In the same way, setting

$$I_2 = \int\limits_{Q_T} \big|\nabla\alpha.\big(e^{-r\alpha}\nabla w_\varepsilon\big)\big|^2\,dx\,dt,$$
$$I_3 = \int\limits_{Q_T} \big|\big(\Delta\big(e^{-r\alpha}\big) + \big(e^{-r\alpha}\big)_t - de^{-r\alpha}\big)w_\varepsilon\big|^2\,dx\,dt,$$

we prove by the same kind of computations that

$$I_2, I_3 \leqslant C_T \int\limits_0^T\!\!\int\limits_\omega e^{-r\alpha}w_\varepsilon^2\,dx\,dt.$$

It follows from these last inequalities and (33) that

$$\|\zeta_\varepsilon\|^2_{W_2^{2,1}(Q_T)} \leqslant C_T \int\limits_0^T\!\!\int\limits_\omega e^{-r\alpha}w_\varepsilon^2\,dx\,dt.$$

Now, by the embedding $W_2^{2,1}(Q_T) \hookrightarrow L^{q_N}(Q_T)$ (see for instance [11, Lemma 3.2, p. 80]):

$$\|\zeta_\varepsilon\|^2_{L^{q_N}(Q_T)} \leqslant C_T \int_0^T\int_\omega e^{-r\alpha} w_\varepsilon^2 \, dx \, dt.$$

Going back to our control, we get using (33):

$$\|g_\varepsilon\|^2_{L^{q_N}(Q_T)} = \|\chi_\omega \zeta_\varepsilon\|^2_{L^{q_N}(Q_T)} \leqslant C \int_0^T\int_\omega e^{-r\alpha} w_\varepsilon^2 \, dx \, dt \leqslant C_T \|X_0\|^2_{L^2(\Omega)}. \tag{33}$$

From (33) and [11, Theorem 10.4, p. 621], it follows, at least for a subsequence, that for $\varepsilon \to 0$:

$$g_\varepsilon \rightharpoonup g \quad \text{weakly in } L^{q_N}(Q_T),$$
$$(u_\varepsilon, v_\varepsilon) \rightharpoonup (u, v) \quad \text{weakly in } L^2\big(0, T; H_0^1(\Omega)\big) \cap W^{2,1}_{q_N}(Q_T),$$

and $((u, v), g)$ satisfy (5) with $(u, v)(T) = 0$ and $\|\chi_\omega g\|^2_{L^{q_N}(Q_T)} \leqslant C_T \|X_0\|^2_{L^2(\Omega)}$. □

4. Local null controllability of (1)

Our main result is the following:

Theorem 6 *(Local controllability to the trajectories). Assume that $f_i \in C^2(\mathbb{R}^2, \mathbb{R})$ for $i = 1, 2$, and let $T > 0$, $1 \leqslant N < 6$. Assume also that there exists a global trajectory (ψ^*, w^*, g^*) of (1) such that*

$$\frac{\partial f_1}{\partial w}(\psi^*, w^*) \geqslant \mu > 0 \quad \text{a.e. on } \omega_b \times (0, T_0)$$

for some $0 < T_0 < T$ and $\omega_b \Subset \omega$. Then there is $\rho > 0$ such that if $\psi_0, \ w_0 \in H_0^1(\Omega) \cap W^{2(1-\frac{1}{q_N}),q_N}(\Omega)$ (q_N is defined in (32)) with $\|(\psi_0, \ w_0)\|_{L^\infty(\Omega)} \leqslant \rho$, one can find $g \in L^{q_N}(Q_T)$ such that there exists (ψ_g, w_g) solution of (3) with $\psi_g, \ w_g \ \in W^{2,1}_{q_N}(Q_T)$ and satisfying:

$$\psi_g(T) = 0, \qquad w_g(T) = 0.$$

Remark 3. It should be said that ω_b and T_0 are arbitrary in the assumption (34) and, so, it seems not to be a real restriction on the trajectory (ψ^*, w^*, g^*) of (1). For example, this hypothesis is satisfied by steady-state solutions of (1) if the nonlinearities f_1 and f_2 are sufficiently smooth.

Proof. For $R > 0$, set

$$K_R = \left\{(\psi, w) \in \big(L^\infty(Q_T)\big)^2; \ \big\|(\psi, w)\big\|_{L^\infty(Q_T)} < R\right\}$$

and consider the problem (5) with a fixed $(\psi, w) \in K_R$ in a, b, c and d. Since $f_i \in C^1(R^2, R)$ and thanks to (34), b will satisfy the assumptions of Theorem 3 for a sufficiently small $R > 0$.

For each $(\psi, w) \in K_R$, thanks to (34), we apply Lemma 5 and consider the set $\mathcal{F}(\psi, w) \subset L^2(Q_T)$ of all the solutions $u_g, v_g \in L^2(0, T; H_0^1(\Omega)) \cap W_{q_N}^{2,1}(Q_T)$ associated with any control $g \in L^{q_N}(Q_T)$ such that $(u_g, v_g)(T) = 0$ a.e. Ω and $\|\chi_\omega g\|^2_{L^{q_N}(Q_T)} \leqslant C_T \|X_0\|^2_{L^2(\Omega)}$. The set $\mathcal{F}(\psi, w)$ is a nonempty closed convex subset of $L^2(Q_T)$. On the other hand, $\mathcal{F}(K_R)$ is relatively compact in $L^2(Q_T)$ and exactly as in [5], $\mathcal{F}$ is semicontinuous using [11, Theorem 10.4]. To prove that $\mathcal{F}$ has a fixed point (clearly, a fixed point of $\mathcal{F}$ is a solution of (3)), it remains to show that there exists $R > 0$ such that $\mathcal{F}(K_R) \subset K_R$.

To do this, we first prove that

$$\left\|(u_g, v_g)\right\|^2_{L^\infty(Q_T)} \leqslant C_T \|X_0\|^2_{L^\infty(\Omega)}. \tag{34}$$

Exactly as in [2], we get

$$\begin{aligned}\left\|(u_g, v_g)(t)\right\|_{L^\infty(\Omega)} \leqslant C\Bigg(&\|X_0\|_{L^\infty(\Omega)} + T^{-\frac{N+2}{2q_N}+1}\|\chi_\omega g\|_{L^{q_N}(Q_T)} \\ &+ \left(1 + \left\|(a, b, c, d)\right\|_{L^\infty(Q_T)}\right)\int_0^t \left\|(u_g, v_g)_g(\tau)\right\|_{L^\infty(\Omega)}\, d\tau\Bigg)\end{aligned}$$

and from Gronwall's inequality:

$$\begin{aligned}\left\|(u_g, v_g)\right\|_{L^\infty(Q_T)} \leqslant{}& Ce^{C(1+\|(a,b,c,d)\|_{L^\infty(Q_T)})T} \\ &\times \left(\|X_0\|_{L^\infty(\Omega)} + T^{-\frac{N+2}{2q_N}+1}\|\chi_\omega g\|_{L^{q_N}(Q_T)}\right),\end{aligned} \tag{35}$$

and (34) follows from (35) and Lemma 5.

The local controllability follows from (34) by taking the initial data sufficiently small. □

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