Let $G$ denote a compact connected Lie group, with maximal torus $T$ (of dimension $d$, say). Let $\pi: G \to M_n \mathbb{C}$ be a unitary representation, and form the $n^\infty$ UHF $C^*$ algebra $A = \otimes M_n \mathbb{C}$ ($n$ is fixed); this is the infinite tensor product of copies of the $n \otimes n$ matrix ring. We have an action $\alpha: G \to \text{Aut}(A)$, by setting $\alpha(g) = \otimes \text{Ad} \pi(g)$. B. M. Baker suggested the name Xerox product type action, because the representation $\pi$ is duplicated over and over. We may form the crossed product $A \times \_ G$ and the fixed point algebra $A^G$ (or simply $A^G$ if there is little likelihood of ambiguity). By restriction of $\alpha$ to $T$, we also obtain a Xerox action of the $d$-torus $T$ (also called $\alpha^T$). Clearly $A^G$ is a unital subalgebra of $A^T$. The principal result asserts that every trace on $A^G$ extends to a trace on $A^T$. In the course of the proof, we also show that the Grothendieck group of $A^G$, $K_0(A^G)$, is finitely generated as a ring, and that $K_0(A^T)$ (which is also a ring) is finitely generated as a $K_0(A^G)$-module.

There are several consequences of these results. The space of faithful pure traces on $A^G$ is a dense open subset of the pure trace space of $A^G$, and moreover is homeomorphic to $(R^d)^+ / W$, the orbit space of the strictly positive $d$-tuples under the natural action of the Weyl group, provided that $\alpha$ is faithful. If $B = \otimes_{i=1}^\infty M_{n(i)} \mathbb{C}$ is a UHF algebra with corresponding product type action (not necessarily Xerox) $\beta = \otimes \text{Ad} \pi_i$, then the natural inclusion $A^G \alpha \to (A \otimes B)^G \alpha \beta$ induces a bijection on faithful pure traces (if $\alpha$ is faithful), and the set of the latter is dense in the pure trace space of $(A \otimes B)^{G, \alpha \beta}$.

Another result (not proved here) using the main result of the article asserts that a difference of characters of $G$ divides an actual character (within the representation ring of $G$) if and only if its restriction to $T$ divides a character of $T$.

* Supported in part by an operating grant from NSERC of Canada.
EXTENDING TRACES ON FIXED POINT C

The proof will be largely in the language of ordered $K_o$-theory. It is necessary that the reader be familiar with the theory of $AF$ $C^*$ algebras (all of the $C^*$ algebras occurring herein will be $AF$), and the translation via $K_o$ to dimension groups $[E, El, EHS, H, W]$. In particular, we assume that the reader understands the translation between traces on $AF$ algebras and states on dimension groups.

We also require some results from previous work on Xerox actions (usually expressed in terms of $K_o$), to be found in [H; Sects. I–IV] among other sources.

Section I discusses the situation when $G = U(n)$ is embedded in the standard manner in $M_n \mathbb{C}$. The results here are already well known, but they are used in Section II to show that in the general case, $K_o(A^n)$ is a finitely generated ring, and that $K_o(A^T)$ is finitely generated as a $K_o(A^n)$ module. These facts together with some standard invariant theory (for finite groups) are employed in Section III to prove the main result and some consequences. Section IV deals with a number of examples, where the traces are given by explicit density matrices and product states on $A$.

I would like to thank Geoffrey Price for pointing out a serious error in an earlier version of this article. The correct proof of the main result came out of a series of fascinating discussions and correspondence with Geoffrey Price and Bob Powers, together with a suggestion from Hyman Bass (which latter was absolutely crucial).

I. Standard Action of $U(n)$

Let $U$ denote $U(n)$, and form $\gamma(u) = \otimes \text{Ad} u$ in $\text{Aut}(A)$. Restricting $\gamma$ to the maximal torus $\bar{T}$ ($T^n$), we have a corresponding action of $\bar{T}$ on $A$. We wish to discuss $K_o$ of the inclusion $A^U \to A^T$. This material (with more details) can be found in [HR, VII; and W].

Let $e_i$ denote the $n \times n$ matrix with a 1 in the $i, i$ entry and zeros elsewhere, and set $E_i = e_i \otimes I \otimes \cdots$ in $A$. Clearly $E_i$ belongs to $A^T$, and thus represents an element of $K_o(A^T)$, $[E_i]$. By [HR, VII; or H, 1.4], $K_o(A^T)$ may be identified with the ring $\mathbb{Z}[x_i/\sum x_j]$. Set $X_i = x_i/\sum x_j$; then the identification is via $[E_i] \mapsto X_i$. Not only is $K_o(A^T)$ a ring and a partially ordered group additively (the latter since $A^T$ is an $AF$ algebra), but it is also a partially ordered ring. The positive cone is generated additively and multiplicatively by $\{X_i\}_{i=1}^n$.

The character of the restriction of the standard representation of $U(n)$, to $T$, its maximal torus, is $\sum x_i$. The action of the Weyl group on the variables $\{x_i\}$ extends to a similar permutation action on $\{X_i\}$; it is the full symmetric group on the subscripts.

Let $\sigma_1 = \sum x_i, \quad \sigma_2 = \sum_{i<j} x_i x_j, \ldots, \quad \sigma_n = x_1 x_2 \cdots x_n$ be the symmetric
functions on the letters \( \{x_i\} \). The representation ring of \( U \) is \( \mathbb{Z}[\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \sigma_n, \sigma_n^{-1}] \). By [HR, VII] (or see [W]) the map \( K_o(A^U) \to K_o(A^T) \) is an inclusion, and the image of \( K_o(A^U) \) is just \( \mathbb{Z}[Y_1, \ldots, Y_{n-1}] \), where \( Y_j = \sigma_{j+1}/\sigma_j^{n-1} \) (notice that the character of the standard representation of \( U \) is identified with \( \sigma_i \)) and the set \( \{Y_1, \ldots, Y_{n-1}\} \) is algebraically independent. We will see that the image of \( K_o(A^U) \) in \( K_o(A^T) \) is precisely the fixed point subring of \( K_o(A^T) \) under the action of \( S_n \) (the Weyl group of \( U \)). We write this

\[
K_o(A^U) = K_o(A^T)^{S_n}. \tag{1}
\]

The action of \( S_n \) here is given as a quotient of the permutation action on \( \mathbb{Z}[x_i] \) modulo the invariant ideal generated by \( \sum x_i - 1 \). By [J, 2.13, p. 133–134], \( \mathbb{Z}[x_i]^{S_n} \) is the ring of symmetric polynomials, so \( \mathbb{Z}[X_i]^{S_n} = \mathbb{Z}[Y_i] \) and thus (1) holds, and \( K_o(A^T) = \mathbb{Z}[X_i] \) is a finitely generated \( K_o(A^U) \)-module, each \( X_i \) is integral over \( \mathbb{Z}[Y_i] \), and \( K_o(A^U) \) is a finitely generated ring.

In this special case, we wish to show that the pure states on \( K_o(A^U) \) (normalized, so that the identity element is sent to 1) extend to pure states on \( K_o(A^T) \), or equivalently that pure traces on \( A^U \) extend to \( A^T \). The easiest way to see this is via a result in [Pl]. Every pure trace on \( A^U \) is given by \( \otimes \text{tr}(D) \), where \( D \) is a density matrix in \( M_n \mathbb{C} \) (that is, \( D \) is positive and \( \text{tr} D = 1 \)) and \( \text{tr} \) denotes the usual (unnormalized) trace on \( M_n \mathbb{C} \). Diagonalizing \( D \) (via an element of \( U \)) has no effect on the trace it determines, in the sense that \( \otimes \text{tr}(D) = \otimes \text{tr}(uDu^* \ldots) \) as positive linear functionals on \( A^U \), so that we may assume \( D \) is diagonal. We wish to show that the product linear functional \( t_D = \otimes \text{tr}(D) \) is a trace on \( A^T \). (To avoid confusion between states on \( K_o \) and states other than traces on \( C^* \) algebras, we refer to the latter as positive linear functionals or simply as linear functionals.)

To this end, we apply \( t_D \) to terms of the form \( e_{i(1)} \otimes e_{i(2)} \otimes \cdots \otimes e_{i(s)} \otimes I \otimes I \otimes \cdots \); the result is \( \prod_{k=1} t_{d_{i(k)}} \) (where \( D - \text{diag}(d_1, d_2, \ldots, d_n) \)). A simple computation reveals that \( D \otimes D \otimes \cdots \otimes D \) (\( s \) times) lies in the algebra generated by \( \bigotimes I \); \((\otimes I) \bigotimes I \otimes \cdots \). We thus deduce that \( t_D(a, a) = t_D(aa) \) for any \( a \) in \( A \) and \( a \) in \( A^T \) (\( (\otimes I) \otimes I \otimes \cdots \)). Thus \( t_D \) is a trace, and as its effect on \( K_o(A^T) \), \( [E_i] \mapsto d_i, [E_{i(s)}] \mapsto d_{i(s)} \), \( \prod_{k=1} ^{s} d_{i(k)} \), is multiplicative, \( t_D \) is a pure trace.

Thus we have shown (2) and (2'):

Every (pure) trace on \( A^U \) extends to a (pure) trace on \( A^T \). \tag{2}

Its translation to states on \( K_o \) is

Every (pure) state on \( K_o(A^U) \) extends to a (pure) state on \( K_o(A^T) \). \tag{2'}
II. Finite Generation of $K_0(A^G)$

Here $G$ is an arbitrary compact connected Lie group with maximal torus $T$ of dimension $d$, $\pi: G \to M_n(C)$ (same $n$ as in Sect. I) is a unitary representation, and $A$ is the $n^\infty$ UHF C$^*$ algebra, $\otimes M_n(C)$. The action of $G$ (and by restriction, of $T$) on $A$ is obtained by Xeroxing $\pi$ to obtain $\otimes \text{Ad } \pi: G \to \text{Aut}(A)$.

The fixed point algebra, $A^G$, is an $AF$ algebra and its ordered $K_0$-group is thus a dimension group, and in fact is given explicitly as follows. Form the representation ring of $G$, $\mathcal{R}(G)$, and invert $\chi$, the character of $G$ corresponding to $\pi$. Set $S^+ = \{\psi/\chi^k \mid \text{there exists } m \text{ such that } \psi \chi^m \text{ is a character}\}$). This is $K_0(A^G)$.

To recover $K_0(A^G)$, set $R_\chi$ to be the bounded subring of $S$, that is, $R_\chi = \{s \in S \mid \text{there exists } N \in \mathbb{N} \text{ with } -N \leq s \leq N\}$, and set $R_\chi^+ = S^+ \cap R_\chi$. The inclusion $R_\chi \to S$ describes the effect of embedding $A^G$ as a corner in $A \times_z G$, followed by taking $K_0([H, I; W])$.

If we do the same thing with $T$ replacing $G$, and set $P = \chi \mid T$, then $R_P$ has a very explicit form:

$$R_P = \mathbb{Z}[x^n/P; w \text{ is an exponent appearing in } P]$$

$$R_P^+ = \langle x^n/P \mid w \text{ is an exponent appearing in } P \rangle \text{ [H, I.4].}$$

Here of course $w = (w(1), \ldots, w(d))$ is a $d$-tuple of integers, and $x^n = x_1^{n(1)} \cdots x_d^{n(d)}$ is a monomial in the variables $x_1, \ldots, x_d$ (negative exponents are allowed)—$\mathcal{R}(T)$ is the Laurent polynomial ring in $d$ variables, $\mathbb{Z}[x_1, x_1^{-1}]$. The natural map $R_\chi \to R_P$ given by $\psi/\chi^k \mapsto (\psi \mid T)/P^k$ is order-preserving and a unital ring homomorphism, and this describes the map $K_0(A^G) \to K_0(A^T)$.

The generators, $x^n/P$, of $R_P$ correspond to projections in the first copy of $M_n(C)$, as follows. Write $P = \sum \lambda_w x^n$, and order the distinct exponents $w_1, w_2, \ldots, w_k$; abbreviate the corresponding multiplicities $\lambda_w$ by $\lambda^j_w$. For $z$ in $T = T^d$, say $z = (z_1, \ldots, z_d)$ and $w = (w(1), w(2), \ldots, w(d))$ in $\mathbb{Z}^d$, define $z^n = z_1^{n(1)} \cdots z_d^{n(d)}$. The representation $\pi \mid T$ with character $P$ is equivalent to

$$z \mapsto \text{diag}(z_1^{w_1}, \ldots, z_1^{w_k}, z_2^{w_1}, \ldots, z_2^{w_k}, \ldots, z_d^{w_1}, \ldots, z_d^{w_k}).$$

The centralizer of $\pi(T)$ is thus a direct product of $k$ matrix rings, of sizes $\lambda^1_w, \ldots, \lambda^k_w$, respectively. Let $e_j^j (j = 1, 2, \ldots, k)$ denote the diagonal idempotent $n \times n$ matrix with 1 in the $\sum_{s < j} \lambda^j_w + 1$ position (so $e^1 = e_1$) and zeroes elsewhere. Then $e_j$ is a minimal projection in the $j$th matrix ring in the centralizer of $\pi(T)$. Set $E_j = e_j \otimes I \otimes I \otimes \cdots$ in $A^T$. Then the computations in (e.g.) [HR, H, W] show that $[E_j]$ (the image in $K_0(A^T)$) is the element.
$x^d/P$ in $R_\rho$, and thus these generate $K_o(A^T)$ as a ring and $K_o(A^T)^+$ additively and multiplicatively.

Recall the notation of Section I. The generators of $K_o(A^T)$ (as a ring) are the $[E_i]$ (equivalence class computed with respect to $A^T$); these map, via the inclusion $A^T \to A^T$, to $[E_i]$ computed with respect to $A^T$. Each $E_i$ is equivalent to one of the $E'$, and as $K_o(A^T) \to K_o(A^T)$ is a ring homomorphism, and the latter is generated by $\{[E']\}$, it follows that the map is surjective.

Let $i: A^T \to A^T$ denote the inclusion. Then we have that $K_o(i)$ is an order-preserving surjective ring homomorphism. In fact, a little more is true. Since the positive cone is generated (additively and multiplicatively) by the $[E']$, it follows that $K_o(i)$ maps the positive cone of $K_o(A^T)$ onto the positive cone of $K_o(A^T)$. We summarize this in a proposition.

**Proposition II.1.** The inclusion $i: A^T \to A^T$ induces the map $K_o(i): K_o(A^T) \to K_o(A^T)$ which is a surjective order-preserving unital ring homomorphism, and moreover every element of $K_o(A^T)^+$ is the image of an element of $K_o(A^T)^+$.

With $U = U(n)$ as in Section I, we have the following commuting squares; the first consists of the obvious inclusions of $C^*$ algebras, and the second is $K_o$ of the first:

$$
\begin{array}{ccc}
A^U & \longrightarrow & A^G \\
\downarrow & & \downarrow \\
A^T & \longrightarrow & K_o(A^T) \\
\end{array}
\quad
\begin{array}{ccc}
K_o(A^T) & \longrightarrow & K_o(A^G) \\
\downarrow & & \downarrow \\
K_o(A^T) & \longrightarrow & K_o(A^T) \\
\end{array}
$$

The arrows on the right are order-preserving unital ring homomorphisms. The two vertical maps (on $K_o$) are described by restricting a quotient of elements of the representation ring to the maximal torus, and thus are both one to one. The natural actions of the Weyl groups (for $\bar{T}$ and for $T$) extends to $K_o(A^T)$ and $K_o(A^T)$, respectively, and we saw that $K_o(A^U)$ maps isomorphically to the fixed point subring of $K_o(A^T)$ under $S_n$ (the Weyl group of $U$). All we can say for the right vertical map is that the image is contained in $K_o(A^T)^W$, where $W$ denotes the Weyl group of $G$.

Now $K_o(A^T)$ has an explicit set of (ring) generators; we wish to conclude that $K_o(A^U)$ is also finitely generated. If we knew that $K_o(A^U) = K_o(A^T)^W$, then this would follow from a technique shown to me by Hyman Bass. However, we do know the result for $U$, so we can modify Bass's argument to deduce that $K_o(A^U)$ is finitely generated.

Let $C$ denote the image of $K_o(A^U)$ inside $K_o(A^G)$. As $K_o(A^U)$ is just a pure polynomial ring in the $Y$'s (see Sect. I), it is a noetherian ring. Moreover, $K_o(A^T)$ is a pure polynomial ring, and under the action of the
symmetric group, the fixed point subring of $K_o(A^T)$ is $K_o(A^U)$ (Eq. (1)). By
the remarks just after (1), $K_o(A^T)$ is a finitely generated $C$-module, thus so
is $K_o(A^T)$. As $C$ is noetherian, any $C$-submodule of a finitely generated
module is itself finitely generated. We deduce that the intermediate ring
$K_o(A^G)$ (between $K_o(A^U)$ and $K_o(A^T)$) is finitely generated as a $C$-module.

Since $C$ is also finitely generated as a ring and $K_o(A')$ is a ring and is
finitely generated as a ring. Moreover, $K_o(A^T)$ being the image of $K_o(A^T)$, is
finitely generated as a $C$-module, and thus is finitely generated as a
$K_o(A^G)$-module. We thus have the following results.

PROPOSITION II.2. The ring $K_o(A^G)$ is finitely generated and $K_o(A^T)$ is
finitely generated as a $K_o(A^G)$-module.

III. PROOF OF TRACE EXTENSION AND ITS CONSEQUENCES

Maintain the notation of Sections I and II. We are now in a position to
prove the main result of the paper, namely, that every pure trace on $A^G$
extends to a pure trace on $A^T$ (from which it follows that every trace on $A^G$
extends to a trace on $A^T$). We proceed via the translation to the $K_o$ groups.

Let $\tau$ be a pure state of $K_o(A^G)$ (which we assume to be normalized with
respect to the identity element), and consider the following diagram:

\[
\begin{array}{c}
K_o(A^G) \rightarrow K_o(A^T) \\
\downarrow \\
K_o(A^G) \rightarrow K_o(A^T)
\end{array}
\]

We are going to "tensor" this diagram with $C$, in order to use standard
results from invariant theory. We first observe that $K_o(A^T)$ is a subring of
the rational function field over $Q$, $Q(x_1, \ldots, x_d)$. Thus $R = K_o(A^T) \otimes Z C$
is a finitely generated $C$-algebra without zero-divisors. Moreover, by II.2, $R$ is a
finitely generated module over its subring $N = K_o(A^G) \otimes C$. There is a
natural homomorphism $\tau: N \rightarrow C$ obtained by tensoring $\tau$ with the identity
to get a map $N \rightarrow R \otimes Z C$, and following this by the multiplication map
$R \otimes C \rightarrow C$ ($r \otimes z \mapsto rz$).

LEMMA. [F, 5.51, 5.52(ii)]. Let $R$ be a finitely generated $C$-algebra
(which is a commutative domain), and let $W$ be a finite group of
automorphisms of $R$. Set $S = R^W$, the fixed point algebra, and let $N$ be any
subalgebra of $R$ such that $R$ is a finitely generated $N$-module.

(a) If $p: N \rightarrow C$ is an algebra homomorphism, then there is an algebra
homomorphism $\hat{p}: R \rightarrow C$ extending $p$, that is $\hat{p} \mid N = p$. 
(b) If \( q_1, q_2 : R \to C \) are two algebra homomorphisms such that \( q_1 | S = q_2 | S \), then there exists \( g \) in \( W \) such that \( g^* = q_2 \).

By part (a) of the lemma, \( \iota \) extends to an algebra homomorphism \( \iota : R \to C \). Let \( \iota \) denote the restriction of \( \iota \) to \( K_o(A^T) \), and set \( J \) to be the map \( K_o(A^U) \to K_o(A^G) \). By deleting the middle column of the preceding diagram, we have the following commuting square:

\[
\begin{array}{ccc}
K_o(A^U) & \xrightarrow{\iota} & R \\
\downarrow & & \downarrow \\
K_o(A^T) & \xrightarrow{\iota} & C
\end{array}
\]

The lower horizontal map is a ring homomorphism— we shall show that its range lies in \( R \), and it is positive (so is a pure state). Now \( \tau \circ J \) is a multiplicative state on \( K_o(A^U) \), so there exists an extension to a multiplicative state \( \tau : K_o(A^T) \to R \subset C \), (Sect. I, (2')). By complexifying \( K_o(A^T) \) (which is a polynomial ring over \( Z \)) and \( K_o(A^U) \) as we did \( K_o(A^T), K_o(A^G) \), we see that part (b) of the lemma is applicable to the complexified version of the homomorphisms \( \tau \circ K_o(i) \) and \( \tau \) (by Sect. I, (1)). Hence there exists an element \( g \) such that \( \tau^g = \tau \circ K_o(i) \). Thus the ordered set \( (\tau(X_j))_{j=1}^n \) is just a permutation of \( (\tau K_o(i)(X_j))_{j=1}^n \). Since the \( X_j \) generate the positive cone multiplicatively and additively of \( K_o(A^T) \), and since the \( \tau(X_j) \) are all nonnegative (as \( \tau \) is a state), it follows that \( \tau \circ K_o(i) \) is a state.

Given a positive element of \( K_o(A^T) \), \( r \), there exists by II.1 a positive element of \( K_o(A^U) \), \( s \), such that \( K_o(i)(s) = r \). Then \( \tau(r) = (\tau \circ K_o(i))(s) \geq 0 \) as \( \tau \circ K_o(i) \) is a state of \( K_o(A^T) \), thus \( \tau \) is a state of \( K_o(A^T) \). Being multiplicative, it is readily checked not to be a nontrivial convex combination of states, so is pure.

**Theorem.** Every pure state on \( K_o(A^G) \) extends to a pure state on \( K_o(A^T) \).

Translating back to \( C^* \) algebras, we obtain

**Theorem.** Let \( \pi : G \to M_n C \) be a unitary representation of the compact connected Lie group \( G \). Form the UHF algebra of type \( n^\infty \), \( A = \bigotimes M_n C \), and let \( \alpha \) be the Xerox product type action of \( G \) derived from \( \pi \). Let \( T \) be a maximal torus, and let \( T \) act on \( A \) by restriction. Then every pure trace on the fixed point algebra \( A^G \) extends to a pure trace on \( A^T \).

**Corollary.** The faithful pure traces on \( A^G \) form a dense open subset of the pure trace space. If the representation \( \pi \) is projectively faithful (i.e., \( \pi^{-1}(C.I) = \{ 1 \} \)) or simply if the projective kernel is finite, then the space of faithful pure traces is homeomorphic to \( (\mathbb{R}^d)^{**}/W \approx \mathbb{R}^d \), where \( d \) is the rank of \( G \).
EXTENDING TRACES ON FIXED POINT C*  

Proof. Given a pure trace, \( \tau \), of \( A^G \), there exists a pure trace \( \check{\tau} \) of \( A^T \) extending \( \tau \). By [H, III.3], there is a sequence of faithful pure traces \( \{ t_n \} \) on \( A^T \) converging weakly to \( \check{\tau} \). Clearly each \( \tau_n = t_n \mid A^G \) is a faithful trace; each is pure because of the multiplicativity of the state it induces on \( K_o(A^G) \); finally, \( \{ t_n \} \) converges weakly to \( \tau \). This establishes density.

If the representation is not already projectively faithful, we may make it so by a suitable telescoping, replacing \( G \) by a quotient group if necessary (as in [H, II]). Finiteness of the projective kernel ensures that the rank of the quotient group is that of \( G \). This does not affect \( A^G \) or \( A^T \). We may even assume (again, as in [H, II]), that \( \det(x(g)) = 1 \) for all \( g \) in \( G \), and thus by a further telescoping that the trivial character 1 appears in \( \chi \), the character of \( \pi \).

By [W], \( K_o(A \times T) = \mathbb{R}(G)[\chi^{-1}] \), and if \( P = \chi \mid T \), then \( K_o(A \times T) = \mathbb{R}(T)[P^{-1}] \), a pure state of \( K_o(A^G) \) (resp. \( K_o(A^T) \)) is faithful if and only if its value at \( \chi^{-1} \) (resp. \( P^{-1} \)) is not zero [H, I.3, II.5]. Hence the faithful pure states constitute a cozero set of a single element, so the set of them is open. The translation to traces on \( K_o \) of the fixed point algebras concludes the proof of the first sentence.

With \( \pi \) still assumed to be projectively faithful, the faithful pure traces are precisely the pure traces that extend to the crossed product (use the \( K_o \) formulation above—also see [H, II]). By [H, IV.6], the space of them is \( (\mathbb{R}^d)^+ / W \), which is homeomorphic to \( \mathbb{R}^d \). \( \Box \)

Corollary. Let \( \pi: G \to M_k(C) \) be a projectively faithful representation, and let \( \rho_i: G \to M_{k(i)}(C) \) be an infinite family of representations, indexed by \( N \). Set \( A = \otimes M_k(C) \), and \( B = A \otimes (\otimes_1^\infty M_{k(i)}(C)) \); let \( \alpha \) be the Xerox product type action on \( A \) obtained from \( \pi \), and let \( \beta: G \to \text{Aut}(B) \) be the product type (but not necessarily Xerox) action \( \alpha \otimes (\otimes \text{Ad} \rho_i) \). The inclusion \( A \to B \) induces an inclusion \( A^G \alpha \to B^G \beta \), which induces a one-to-one map on \( K_o \), and moreover an onto map on pure traces (of \( B^G \) onto those of \( A^G \)). Finally, the restriction map is a bijection on the pure faithful traces.

Proof. By [H, II.3] and a suitable telescoping, we may assume that the trivial character appears in \( \chi \).

We can pair off the characters of the \( \rho_i, \varphi_i \), with the infinitely many copies of the character of \( \pi, \chi \). By [HR, VII], we have a diagram for \( K_o(A \times T) \to K_o(\beta B \times \beta G) \),

\[
\begin{array}{cccccc}
\cdots \xrightarrow{\varphi_i} & \mathcal{A}(G) & \xrightarrow{\chi} & \mathcal{A}(G) & \xrightarrow{\chi} & \cdots \xrightarrow{\chi} & K_o(A \times T) \\
\cdots \xrightarrow{\varphi_i \cdot \varphi_j} & \mathcal{A}(G) & \xrightarrow{\chi \varphi_i \cdot \chi \varphi_j} & \mathcal{A}(G) & \xrightarrow{\chi \varphi_i \cdot \chi \varphi_j} & \cdots \xrightarrow{\chi \varphi_i \cdot \chi \varphi_j} & K_o(\beta B \times \beta G) \\
\end{array}
\]

Note that \( K_o(\beta B \times \beta G) \) is not generally a ring—but it is a rank one flat
module over $\mathcal{H}(G)$: A limit of rank one modules is rank one, and a limit of free modules is flat. We take the bounded submodules of both—in the case of $K_o(A \times \chi G)$, this is just what we called $R_\chi$. For $K_o(B \times \beta G)$, this is defined as $M = \{ s \in K_o(B \times \beta G) \mid \text{there exists } m \in \mathbb{N} \text{ such that } -m \cdot 1 \leq s \leq m \cdot 1 \}$, where $1$ is the trivial character. (We have assumed that $1$ appears in $\chi$.) Then $M = K_o(B^G)$, and the map on $K_o$ of the crossed products induces a map $R_\chi \to M$, which is just $K_o$ of the inclusion $A^G \to B^G$.

In particular, this imposes on $M$ the structure of an ordered $R_\chi$-module, of rank one—indeed, $M$ is embedded in the field of fractions of $R_\chi$ (because this is also the field of fractions of $K_o(A \times \chi G)$ and the module of fractions of $K_o(B \times \beta G)$). We now show that every pure state on $M$ is an $R_\chi$-module homomorphism (with suitable action on $R$).

Let $t : M \to R$ be a pure state, normalized so that $t(1) = 1$, where $1$ is the identity element of $R_\chi$. Because of our reduction to the case that $1$ appears in $\chi$, $\chi^{-1}$ belongs to $R_\chi$, and thus for all $h$ in $M$, $h \chi^{-1}$ is an element of $M$. For $r$ in $R^+$, define a new state via $t_r(h) = t(hr)/t(r)$, whenever $t(r) \neq 0$. Then as in the argument in [H, 1.2(c)], purity of $t$ ensures that $t_r(h) = t(h)$, so we deduce that $t(hr) = t(h) t(r)$ (for all $r$ in $R_\chi$), and that $t \mid R_\chi$ is multiplicative, hence pure as a state of $R_\chi$.

Let $t$ be a faithful pure state of $M$, so that its restriction to $R_\chi$ is also faithful and pure. Let $t'$ be any other pure state of $M$, and suppose that $t \mid R_\chi = t' \mid R_\chi$. It is easy to check that given $c$ in $M^+ \setminus \{0\}$, there exists $r$ in $R^+ \setminus \{0\}$ such that $cr \in R^+$. We thereby deduce that $t(c) t(r) = t(cr) = t'(c) t'(r) = t'(c) t(r)$. Since $t(r) \neq 0$, we deduce that $t(c) = t'(c)$, so $t = t'$. In fact, all that was needed for this to go through was that $t \mid R$ be faithful. Hence the map on pure faithful states of $M$ to those of $R_\chi$ is one to one. The same argument (that $c^{-1} R$ contains a positive element) also shows that if the image of a pure state is faithful, the original pure state is faithful as well.

Now we show that the range of the faithful pure states is all of the faithful pure states of $R_\chi$. We know that the latter are given by point evaluations in $(R^d)^{+*}$, say at $y = (r_1, ..., r_d)$. We can extend this to a state on $K_o(B \times \beta G)$ (and by restriction, to a state on $K_o(B^G)$) via the diagram,

\[ \mathcal{H}(G) \xrightarrow{x \varphi_1} \mathcal{H}(G) \xrightarrow{x \varphi_2} \mathcal{H}(G) \xrightarrow{x \varphi_3} \cdots \xrightarrow{x \varphi_d} K_o(B \times \beta G) \]

The first map to $R$ is evaluation at $y$; the second is evaluation at $y$ divided by the value of $\chi \cdot \varphi_1$ at $y$; the third is evaluation at $y$ divided by the value of $\chi^2 \varphi_1 \varphi_2$, and so on. This extends the original faithful state (defined on $K_o(A^G)$, and extended uniquely to $K_o(A \times \chi G)$). Moreover, its restriction
to $K_n(B^G) = M$ is pure because it has the module property, $t(cr) = t(c) t(r)$ for $c$ in $M$ and $r$ in $R_x$.

We thus have that the restriction map from pure states of $K_n(B^G)$ to those of $K_n(A^G)$ is a bijection on the faithful ones. Since the restriction of a pure state is multiplicative and therefore pure, the range of the pure state space is therefore a dense subset of the pure state space of $K_n(A^G)$. It is thus sufficient (to prove onto-ness) to show that the pure state space of $K_n(B^G)$ is compact. But this follows from the fact that a state of $K_n(B^G)$ is pure if and only if $t(cr) = t(c) t(r)$ for all $c$ in $M$, $r$ in $R_x$.

### IV. Examples

We shall give some examples of what the traces really look like, in terms of states on the $C^*$ algebra $A$. To begin with, we consider the case of the torus $T$ (of dimension $d$, say). In terms of pure states of $K_n(A^T)$, everything is known by the results in [H, III]. However, it is a bit tricky to write down all of the corresponding pure traces on $A^T$.

Recalling that $A = \otimes M_n(C)$, it has been shown [P1; P2, 4.1] that every pure trace on $A^T$ is given by the Xerox product state $\otimes \text{tr}(D - )$, where $\text{tr}$ is the usual trace on $M_n(C)$, $D$ is a density matrix in $M_n(C)$ (i.e., $D$ is positive of trace 1), and we may assume that $D$ lies in the algebra generated by $\pi(T)$ inside $M_n(C)$. Since all representations of tori can be diagonalized, we may assume $D$ is already diagonal. Unfortunately, it is not easy to decide which diagonal density matrices $D$ determine traces on $A^T$. We describe the complete collection of those that do.

Index the irreducible characters that appear in $P$ as $x^{w_1}, x^{w_2}, ..., x^{w_n}$ (some of the $w_i$'s will be the same in most cases of interest—notice that this differs from the notation in Sect. II), so that $P = \sum x^{w_i}$. The representation $\pi$ is then equivalent to

$$z \mapsto \text{diag}(z^{w_1}, z^{w_2}, ..., z^{w_n}),$$

and we assume that this is $\pi$. Form the following matrix, with entries from $R^d$:

$$\tilde{D} = P^{-1} \text{diag}(x^{w_1}, x^{w_2}, ..., x^{w_n}).$$

We may evaluate $\tilde{D}$ at any $r = (r_1, r_2, ..., r_d)$ in $(R^d)^{++}$, and obtain $\tilde{D}(r)$, an invertible density matrix. We will show that $\{\tilde{D}(r) \mid r \in (R^d)^{++}\}$ exhausts the faithful traces in the sense that every faithful pure trace is of the form $\otimes \text{tr}(\tilde{D}(r) - )$ for some $r$, and that every one of these yields a trace. Moreover, all pure traces are obtained by taking the closure of the
set of $\tilde{D}(r)$'s. If the action $\alpha = \otimes \Ad \pi$ is faithful (equivalently, $\pi$ is projectively faithful) distinct such density matrices will yield distinct pure traces.

Let $\tau$ be a faithful pure trace on $A^T$; it is of the form $\otimes \tr(\hat{D} - )$ for some diagonal density matrix $\hat{D}$; since $E_j$ (see Sect. I) lies in $A^T$, $De_j \neq 0$ (else $\tau$ would not be faithful) for $j = 1, \ldots, n$, and we deduce that $\hat{D}$ is invertible. Now the classes (in $K_\circ(A^T)$) of $E_j$, $[E_j]$ correspond to the elements $x^{v_j}/P$ in $R_p$. The pure faithful states of $R_p$ are given by a point evaluation, at some $r = (r_1, \ldots, r_d)$, a strictly positive $d$-tuple. Thus $[E_j]$ is sent to $x^{v_j}/P(r)$ (for some $r$) by $\tau$. But $D(r) e_j$ has trace exactly $x^{v_j}/P(r)$ and, in fact, the only nonzero entry of $De_j$ is the $j$th, and since the trace must be $x^{v_j}/P$ evaluated at $r$, it follows that $D = \tilde{D}(r)$.

On the other hand, given a point evaluation say at, $r$, there is a pure trace on $A^T$ to which it corresponds. Since there is thus a diagonal density matrix $D$ implementing it, the same argument shows that $\tilde{D}(r) - D$, providing that distinct points give distinct states on $K_\circ(A^T)$. This happens when $\pi$ is projectively faithful, which we may always assume [H, II]. Starting ab initio with a $\tilde{D}(r)$, it appears to be very difficult to prove directly that it implements a trace. Of course convergence of a sequence of $\tilde{D}(r)$'s is equivalent to weak convergence of the corresponding pure traces, so that by the density result of [H, III.3], the completion of the set $\{\tilde{D}(r)\}$ yields all the pure traces, and everything in the completion implements a pure trace.

Let $G$ be a compact connected Lie group with maximal torus $T$, let $\pi: G \to M_n \C$ be a representation, and form the Xerox action $\alpha = \otimes \Ad \pi$. Let $\chi$ be the character of $\pi$, and set $P = \chi | T$. We wish to determine the density matrices that will yield all the pure traces on $A^G$. Since they all extend to $A^T$, we can just take the density matrices that we have just computed for the toral action. However, any two density matrices in the same orbit (under $W$, the Weyl group of $G$) will yield the same trace on restriction to $A^G$ (the converse also holds—this is due to Price [P3]). So to obtain a complete set of traces (without redundance), just select a fundamental domain for the action of $W$ on $(\R^d)^{++}$, and take the $\tilde{D}(r)$ for $r$ in that domain only, together with paths to infinity which lie in the domain. We illustrate this with a number of examples.

**Example IV.1.** Let $G = SU(3)$, $n = 4$, and let $\pi$ be the standard representation direct sum the trivial one (for the standard representation alone, see [P1]). Then $\pi$ is a projectively faithful representation, so we require no modifications. The restriction to $T$ ($= T^2$) has character $P = x + y + x^{-1} y^{-1} + 1$, and the Weyl group, $S_3$, acts as the full permutation group on $\{x, y, x^{-1} y^{-1}\}$.

According to our earlier results, the traces on $A^T$ are parameterized by the density matrices $\tilde{D}(r, s) = \text{diag}(r, s, r^{-1} s^{-1}, 1)/(r + s + r^{-1} s^{-1} + 1)$ for
$r, s > 0$, and their limit points. These latter can be computed by means of the Newton diagram corresponding to the character, i.e., $cuv \log P$ (Fig. IV.1).

Following [H, III], the faces describe the limit points. The line joining $(0, 1)$ to $(1, 0)$ is described as the set of points $v$ in $cuv \log P$ such that $v(1, 1)^T = 1$; for $w$ in $cuv \log P$ but not on the line segment, $w(1, 1)^T < 1$. Then let $A_1, A_2$ be arbitrary positive real numbers, and set $X(t) = (A_1 t, A_2 t)$ (the $1$'s are the exponents of the $t$'s). As $A_1, A_2$ vary (in fact, we can keep one of them fixed), each of $\lim_{t \to \infty} X(t)$ is a density matrix and this exhausts those corresponding to this face. This yields some new density matrices: $D_1(a) = \{ \text{diag}(a, 1-a, 0, 0) \mid 0 < a < 1 \};$ here $a = A_1 / (A_1 + A_2)$.

By symmetry, we obtain two other 1-parameter families,

$$D_2(a) = \{ \text{diag}(0, a, 1-a, 0) \mid 0 < a < 1 \}$$

$$D_3(a) = \{ \text{diag}(1-a, 0, a, 0) \mid 0 < a < 1 \}.$$

Taking the limit points of these, we obtain the traces corresponding to the vertices, given by the density matrices

$$d_1 = \text{diag}(1, 0, 0, 0), \quad d_2 = \text{diag}(0, 1, 0, 0), \quad d_3 = \text{diag}(0, 0, 1, 0).$$

Of course the Weyl group permutes $d_1, d_2, \text{and} \quad d_3,$ and acts in the obvious way on the $D_1, D_2, D_3$ families. To obtain the (candidate) density matrices for $A^G$, we restrict to the fundamental domain. For the faithful traces, we obtain $\{ \tilde{D}(r, s) \mid r \geq s \geq r^{-1}s^{-1} \}$. The limit points of these include $D_1(a)$, but none of $D_2(a)$ or $D_3(a)$, and similarly $d_1$ but not $d_2$ or $d_3$. It is easily checked that these yield distinct pure traces on $A^G$, so we have a complete irredundant set of density matrices for $A^G$.

**Example IV.2.** Here $G = PSU(3)$ (in effect), $n = 8$, and $\pi: G \to M_8 C$ is the 8-dimensional irreducible representation of PSU(3). To write down the character, it is easier to realize this as a representation of $SU(3)$, by taking the standard representation, tensoring with its adjoint, and subtracting the trivial representation. So restricted to the maximal torus (of $SU(3)$), the
character is \((x + y + x^{-1}y^{-1})(x^{-1} + y^{-1} + xy) - 1 = P_0\), which we expand to obtain \(P_0 = 2 + xy^{-1} + x^2y + yx^{-1} + xy^2 + x^{-2}y^{-1} + x^{-1}y^{-2}\). This is of course not faithful (as a representation of \(T^2\)), but the kernel is the projective kernel, which is cyclic of order three. We eliminate this by rewriting \(P_0\) in terms of different monomials. The nonzero exponents corresponding to the monomials are \((1, -1), (2, 1), (-1, 1), (1, 2), (-2, -1), (-1, -2)\); the determinant of any pair of them is either 0, 3, or 6, so that \(\{(2, 1), (1, 2)\}\) is a \(\mathbb{Z}\)-basis for the subgroup they span. Rewriting in terms of this basis, we obtain a new character, \(P = 2 + xy^{-1} + x + x^{-1}y + y + x^{-1} + y^{-1}\). Its Newton polyhedron is a hexagon (Fig. IV.2).

Here there are six edges and six vertices. The faithful traces of \(A^T\) are given by \(\text{diag}(1, 1, rs^{-1}, r, r^{-1}s, s, r^{-1}, s^{-1})/P(r, s) = \tilde{D}(r, s)\) (assuming the representation is of the form \(z \mapsto \text{diag}(1, 1, z^{(1, 1)}, ...)\) (for \(r, s > 0\)). The limit traces affiliated with the path joining \((1, 0)\) to \((0, 1)\) are of the form \(D_i(a) = \text{diag}(0, 0, 0, 0, 1 - a, 0, 0, 0)\), and the five other \(1\)-parameter families are similarly obtained (notice that in each one, the first two entries are 0). Letting \(a\) tend to 1, we obtain \(d_i = \text{diag}(0, 0, 0, 1, 0, 0, 0, 0)\), and five others corresponding to the vertices. Then the pure traces of \(A^G\) are obtained from restricting the \(D(r, s)\) to \(r \geq s \geq r^{-1}s^{-1}\), together with \(D_i(a)\) and \(d_i\). It follows (again) that \(A^G\) has exactly one \(1\)-dimensional representation (corresponding to \(d_i\)) and no other finite dimensional ones. This phenomenon always occurs when the original representation is irreducible.

**Example IV.3.** We very quickly outline what happens when \(G = SU(3)\) and the representation is a direct sum of two irreducible ones with dominant weights \((2, 0)\) and \((2, 1)\), respectively. The former is 6-dimensional, and the latter is the 8-dimensional one discussed in the previous example. We have that \(\pi: G \to M_{14}\mathbb{C}\) is projectively faithful. This time there are two vertices of the Newton diagram in the fundamental domain, and so \(A^G\) has two \(1\)-dimensional representations whose traces correspond to the vertices. There are two edges of the Newton polyhedron in the fundamental domain, so there are two \(1\)-parameter families of traces (and the kernels of
the corresponding representations are all equal, and the quotient algebra is a fixed point algebra under $SU(2)$—cf. [H, VII.5]). The faithful traces are of the form $\tilde{D}(r, s)$ (inside $M_{14}C$) and present no new features.

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