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Parametrized spaces model locally constant homotopy sheaves

Michael A. Shulman

Department of Mathematics, University of Chicago, 5734 S. University Ave, Chicago, IL 60615, USA

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Abstract

We prove that the homotopy theory of parametrized spaces embeds fully and faithfully in the homotopy theory of simplicial presheaves, and that its essential image consists of the locally homotopically constant objects. This gives a homotopy-theoretic version of the classical identification of covering spaces with locally constant sheaves. We also prove a new version of the classical result that spaces parametrized over X are equivalent to spaces with an action of ΩX. This gives a homotopy-theoretic version of the correspondence between covering spaces and π_1-sets. We then use these two equivalences to study base change functors for parametrized spaces.

Keywords: Parametrized spaces; Ex-spaces; Homotopy sheaves; Simplicial presheaves; Model category; Right Quillen embedding; Fibration; Base change

1. Introduction

Recently there has been growing interest in doing homotopy theory 'relative' to a base topological space. One motivation for this is to find a framework which includes both local cohomology and generalized cohomology theories, since clearly such a generalization requires a notion of 'spectrum relative to a base space', or at least of 'space relative to a base space'. In this paper we focus on spaces for simplicity; we hope to deal with spectra in a later paper.

There are two general approaches to such a relative theory in the literature: one involving 'sheaves of spaces on B', or *homotopy sheaves* (also known as *stacks*), such as that of [14,16], and one involving 'spaces over B', or *parametrized spaces*, such as that of [22]. Formal comparisons of the two, however, are difficult to find in the literature. In this paper, we state and prove such a comparison; our slogan is that *parametrized spaces are equivalent to locally constant homotopy sheaves*.

Our inspiration comes from the well-known equivalence between the following three categories.

(i) Locally constant sheaves of sets on B.
(ii) Covering spaces over B (which are fibrations with discrete fibers).
(iii) Sets with an action of $\pi_1(B)$. If B is not path-connected, we use instead the fundamental groupoid $\Pi_1(B)$.

E-mail address: shulman@math.uchicago.edu.

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Our goal is to prove a 'homotopical' version of this. Specifically, we will show that the following three homotopy theories are equivalent.

(a) Homotopy sheaves on B which are 'locally constant'.
(b) Fibrations over B.
(c) (If B is path-connected) spaces with an action of ΩB. We regard ΩB as representing the automorphisms of the base point of '$\Pi_\infty(B)$', the 'fundamental ∞-groupoid' of B.

Often, of course, we use a larger category of models. We find the homotopy sheaves as the fibrant objects in a model structure on the category of simplicial presheaves, and the fibrations over B as the fibrant objects in a model structure on the category of all spaces over B. We also refer to this latter as the homotopy theory of *parametrized spaces*.

Our method of proof is also similar to the '0-dimensional' version. One way to prove the equivalence between (i) and (ii) is to first prove that the category of all sheaves of sets on B is equivalent to the category of local homeomorphisms (or 'etale spaces') over B, and then identify the covering spaces as the local homeomorphisms which are 'locally constant'. Analogously, we will prove the equivalence between (a) and (b) by using a different model structure on spaces over B, due to [13], whose homotopy theory is equivalent to that of homotopy sheaves and in which all objects are fibrant. We will show that a model structure for spaces parametrized over B embeds into this model structure, and that its image consists of the 'locally constant' homotopy sheaves.

Likewise, the equivalence between (ii) and (iii) goes by taking the fiber of a covering space, with action induced by path-lifting around loops in B. We prove the equivalence between (b) and (c) using a homotopical version of this.

Our motivating analogy also suggests other aspects of the relationship between homotopy sheaves and parametrized spaces. For example, since covering spaces over B are equivalent to $\pi_1(B)$-sets, they depend only on homotopy-theoretic information about B, while the category of all sheaves of sets on B determines B essentially up to homeomorphism. Analogously, the homotopy theory of parametrized spaces is invariant under weak equivalences of the base space, while that of homotopy sheaves is not. This is not a problem with either approach, merely a difference in emphasis: homotopy theorists are only interested in spaces as homotopy types, while sheaf theorists are interested in spaces, such as spectra of rings, which carry more information than their ordinary weak homotopy type can support.

Another important difference has to do with base change functors and homology and cohomology. Under the correspondence between (i) and (ii), the sheaf cohomology of a locally constant sheaf of groups on B is identified with the local cohomology of B with coefficients in the corresponding local system. However, while it is easy to also define *homology* with local coefficients, it is quite difficult to define 'sheaf homology' in general, and this difference carries over to the homotopical version.

The analogues of homology and cohomology in relative homotopy theory are, respectively, derived left and right adjoints $f_!$ and f_* to the pullback functor f^* for a map f of base spaces; when f is the projection $r : B \to *$, we expect to recover homology from $r_!$ and cohomology from r_*. For homotopy sheaves, the adjunction $f^* \dashv f_*$ is well-behaved, but in general f^* has no left adjoint. For parametrized spaces, on the other hand, the adjunction $f_! \dashv f^*$ is well-behaved, while the right adjoint f_* is harder to get a handle on. A right adjoint f_* on the homotopy-category level was shown to exist in [22] only by using Brown representability, and only on connected spaces.

One motivation for our comparison result is the hope to shed some light on the right adjoint f_* for parametrized spaces. We will show that the derived functor f^* for parametrized spaces agrees with the derived functor f^* for the corresponding locally constant homotopy sheaves; in particular, the f^* for homotopy sheaves preserves locally constant objects. The functors f_*, on the other hand, agree whenever f is a fibration between locally compact CW complexes, but in general the f_* for homotopy sheaves need not preserve locally constant objects.

It follows that for such fibrations, the f_* for parametrized spaces can be computed by passing through homotopy sheaves. This is not a huge gain in generality, since f_* can be computed by existing methods when f is a bundle of cell complexes, but we give some motivation for believing that it is almost best possible. We also give examples in which the f_* for homotopy sheaves is very different from the f_* for parametrized spaces.

The equivalence between (b) and (c) is more promising for the construction of f_*, at least at a formal level. We will show that this equivalence preserves all the base change functors, and in particular that a derived right adjoint f_* can always be constructed for parametrized spaces by passing through ΩB-spaces. This requires no assumptions on the map f and no connectivity assumptions on the spaces involved. However, this equivalence involves a chain of two adjunctions in different directions, so to actually compute f_* in this way may be impractical.

The plan of this paper is as follows. In Sections 2 and 3 we recall some facts about point-set topology and model structures for parametrized spaces. Sections 4–7 are then devoted to the equivalence between (a) and (b). In Section 4 we define a model structure for homotopy sheaves using simplicial presheaves and compare it to the model structure from [13] which uses actual spaces over a base space. In Section 5, we prove that the homotopy theory of parametrized spaces embeds into that of homotopy sheaves, and in Section 6 we prove that the image consists of the 'locally constant' homotopy sheaves. Then in Section 7 we compare the base change functors in the two situations.

Sections 8 and 9 deal with the equivalence between (b) and (c). In Section 8 we prove that when G is a grouplike topological monoid, such as a Moore loop space ΩA, the homotopy theories of G-spaces (with the underlying weak equivalences, not the weak equivalences usually used in equivariant homotopy theory) and of spaces parametrized over BG are equivalent. Since any connected space A is weakly equivalent to $B(\Omega A)$, and parametrized spaces are invariant under weak equivalence of the base space, this shows that spaces over A are equivalent to ΩA-spaces. Finally, in Section 9 we show that this equivalence preserves all the base change functors.

An important technical tool in our work is a new model structure for topological spaces discovered by Cole [2], obtained by *mixing* the 'standard' model structure constructed by Quillen [24] with the 'classical' model structure constructed by Strøm [28]. In Cole's model structure the weak equivalences are the *weak* homotopy equivalences, while the fibrations are the *Hurewicz* fibrations, and the cofibrant objects are the spaces of the homotopy type of a CW complex. This model structure is arguably closer to classical homotopy theory than is the standard model structure, and its cofibrant and fibrant objects are also better behaved and preserved by more constructions.

Results analogous to ours can be found in [29], which works with simplicial sets and Kan fibrations rather than topological spaces and Hurewicz fibrations. This provides another formalization of the slogan that parametrized spaces model locally constant homotopy sheaves. However, a topological approach is of independent interest for many reasons.

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2. Point-set topology

In the parametrized world there are always some point-set topological issues that must be dealt with. It is by now generally accepted that a good category of topological spaces for homotopy theory must be cartesian closed, and the most common choice is the category of *compactly generated* spaces; that is, weak Hausdorff k-spaces (see [21, Ch. 5]). However, in the parametrized case one wants the category of spaces *over* every base space B to also be cartesian closed. Standard categorical arguments show that this is equivalent to the existence, for any map $f : A \to B$ of base spaces, of a right adjoint f_* to the pullback functor f^*.

However, this extra desideratum is false for compactly generated spaces. Various remedies are possible. One is to restrict the structure maps $X \to B$ of spaces over B, and the transition maps $f : A \to B$ of base spaces, to be open maps, as is done in [13]. However, in some cases this is too restrictive; for example, it disallows diagonal maps $\Delta : B \to B \times B$. Another solution is to use a topological quasitopos, such as pseudotopological spaces (see [30]) or subsequential spaces (see [15]).

We adopt instead the solution used in [22]: we require base spaces to be compactly generated, but allow total spaces to be arbitrary k-spaces, not necessarily weak Hausdorff. The references given in [22, §1.3] show that if B is compactly generated, the category $\mathscr{K}/B$ of k-spaces over B is cartesian closed, and if $f : A \to B$ is a continuous map between compactly generated spaces, the pullback functor $f^* : \mathscr{K}/B \to \mathscr{K}/A$ has not only a left adjoint $f_!$ but a right adjoint f_*. The same is true if we consider the categories $\mathscr{K}_B$ of sectioned spaces over B.

The left adjoint $f_! : \mathscr{K}/A \to \mathscr{K}/B$ is simply given by composition with f. In the sectioned case, $f_! : \mathscr{K}_A \to \mathscr{K}_B$ is defined by a pushout, which in the case of the map $r : B \to *$ simply quotients out the section.

We think of the right adjoint f_* as a 'space of relative sections'. When f is the map $A \to *$, the space $f_* X$ is simply the space of global sections of $X \xrightarrow{p} A$; that is, the subspace of $\mathscr{K}(A, X)$ consisting of the maps $A \xrightarrow{s} X$ such that $ps = 1_A$.

From now on, when we speak of *a space over B* it is to be understood that B is compactly generated and the total space is a k-space. Although our point-set conventions are different than those of [13], it is readily seen that all the proofs in [13] carry over without difficulty to our setting, so this will be our last comment on the difference.

One fundamental result we will need is the following. Recall (for example, from [9, §1.5]) that the following conditions on a CW complex X are equivalent.

(i) X is *locally finite*, meaning that each point has a neighborhood which intersects only finitely many cells.
(ii) X is locally compact.
(iii) X is metrizable.
(iv) X is first countable.

Theorem 2.1. *(See [13, 6.6].) If X is a CW complex with the above properties, then any open subspace of X has the homotopy type of a CW complex.*

Proof. Given a class $\mathscr{C}$ of spaces, in [12] a space is defined to be an ANR($\mathscr{C}$) (absolute neighborhood retract) if it is a neighborhood retract of every space in $\mathscr{C}$ that contains it as a closed subset. If $\mathscr{C}$ is the class of metric spaces, an ANR($\mathscr{C}$) is called a *metric ANR* or just an *ANR*. By [12, 11.4], every CW complex is an ANR(M), where M is the class of 'M-spaces' defined there. The remarks before [12, 10.4] show that an ANR(M) is a metric ANR just when it is metrizable.

Thus, since our CW complex X is assumed metrizable, it is an ANR. But by [9, A.6.4], any open subset of an ANR is an ANR, and by [9, 5.2.1] spaces of the homotopy type of CW complexes coincide with spaces of the homotopy type of ANRs. □

This is important because in order to compare spaces over X to sheaves on X, we need to consider *sections* over open subsets $U \subset X$. Of course, sections over U are particular maps out of U, and we know that only spaces of the homotopy type of CW complexes are 'homotopically good' for mapping out of.

3. Model structures for parametrized spaces

There are several model structures on the category $\mathscr{K}/B$ of spaces over B. Any model structure on $\mathscr{K}$ gives rise, by standard arguments, to a model structure on $\mathscr{K}/B$. The most well-known model structures on $\mathscr{K}$ are the following.

(i) The *standard* or q-model structure, in which the weak equivalences are the weak homotopy equivalences, the fibrations are the Serre fibrations, and the cofibrations are the retracts of relative cell complexes. This is the model structure originally constructed by Quillen in [24].
(ii) The *classical* or h-model structure, in which the weak equivalences are the homotopy equivalences, the fibrations are the Hurewicz fibrations (or 'h-fibrations'), and the cofibrations are the closed Hurewicz cofibrations. This model structure was constructed in [28].

However, as mentioned in Section 1, there is also a *mixed* model structure, which was discovered by Cole.

Theorem 3.1. *(See [2].) Suppose that a category $\mathscr{C}$ has two model structures, called the q-model structure and the h-model structure, such that*

- *Every h-equivalence is a q-equivalence, and*
- *Every h-fibration is a q-fibration.*

Then $\mathscr{C}$ also has a mixed *or m-*model structure *in which*

- *The weak equivalences are the q-equivalences,*
- *The fibrations are the h-fibrations,*
- *The cofibrations are the h-cofibrations which factor as a q-cofibration followed by an h-equivalence, and the m-cofibrant objects are the h-cofibrant objects which have the h-homotopy type of a q-cofibrant object.*

If the q-model structure is left or right proper, so is the m-model structure. If the q- and h-model structures are both monoidal, so is the m-model structure.

As is evident, we prefix model-theoretic words like 'equivalence', 'fibration', 'cofibration', and 'cofibrant' with a letter to indicate which model structure we are referring to. However, we continue to refer to q-equivalences and h-fibrations rather than m-equivalences and m-fibrations. If we want to make clear which model structures are being mixed, we may refer to the mixing of the q- and h-model structures as the $m_{q,h}$-model structure.

In the case of $\mathscr{K}$, the standard and classical model structures mix to give a model structure in which the weak equivalences are the *weak* homotopy equivalences and the fibrations are the Hurewicz fibrations. A map $f : A \to X$ is an m-cofibration if and only if it is a Hurewicz cofibration that is cofiber homotopy equivalent under A to a relative CW complex. In particular, the m-cofibrant objects are the spaces of the homotopy type of a CW complex; thus we can rephrase Theorem 2.1 by saying that a locally compact CW complex is *hereditarily m-cofibrant*. Since the q- and h-model structures on $\mathscr{K}$ are monoidal, so is the m-model structure.

The mixed model structure packages classical information in an abstract way. For example, it is true in the generality of Theorem 3.1 that a q-equivalence between m-cofibrant objects is an h-equivalence. Note that the identity functor is a Quillen equivalence between the m- and q-model structures, and that unlike the q-model structure, neither the h- nor the m-model structure on $\mathscr{K}$ is cofibrantly generated.

We denote the model structures on $\mathscr{K}/B$ obtained from the h, q, and m-model structures on $\mathscr{K}$ by the same letters. Note that the m-model structure on $\mathscr{K}/B$ induced by the m-model structure on $\mathscr{K}$ is the same as the model structure obtained by mixing the q- and h-model structures on $\mathscr{K}/B$.

We also have a *fiberwise* or f-model structure on $\mathscr{K}/B$, whose weak equivalences are the *fiberwise* homotopy equivalences (hereafter *f-equivalences*); see [22, §5.1] for more details. However, the q-model structure does not mix with the f-model structure, since not every f-fibration is a q-fibration.

The homotopy theory on $\mathscr{K}/B$ we are interested in is that modeled by the Quillen equivalent q- and m-model structures. It was observed in [22] that the q-model structure on $\mathscr{K}/B$ is not good enough for some purposes because it has too many cofibrations, and of course the m-structure has even more. Thus, a main technical result of [22] was the construction of a Quillen equivalent 'qf-model structure' with better formal properties. The qf-structure will not play any role for us, however, since we will be more interested in controlling the fibrations than the cofibrations. For this, the best choice is the m-model structure, in which the fibrant objects are the Hurewicz fibrations over B.

By standard arguments, each model structure on $\mathscr{K}/B$ gives rise to a corresponding model structure on the category $\mathscr{K}_B$ of *sectioned* spaces over B, since the latter is just the category of pointed objects (that is, objects under the terminal object) in $\mathscr{K}/B$. In $\mathscr{K}_B$ one may also consider 'fiberwise pointed' homotopy equivalences, fibrations, and so on; these form an 'fp-model structure' in the category $\mathscr{U}_B$ of compactly generated spaces over B, but it is unknown whether they do so in $\mathscr{K}_B$; see [22, 5.2.9].

Recall that for any map $f : A \to B$, we have a string of adjunctions $f_! \dashv f^* \dashv f_*$ at the point-set level.

Proposition 3.2. *(See [22, §7.3].) The adjunction $f_! \dashv f^*$ is Quillen for the q- and m-model structures. If f is a q-equivalence, then $f_! \dashv f^*$ is a Quillen equivalence. If f is a bundle whose fibers are cell complexes, then $f^* \dashv f_*$ is Quillen for the q-model structures.*

The results in [22] are only stated for the sectioned case of $\mathscr{K}_B$, but the proofs remain valid in the unsectioned case of $\mathscr{K}/B$.

This implies that we always have derived adjunctions $\mathbf{L}_q f_! \dashv \mathbf{R}_q f^*$ at the level of homotopy categories, and that when f is a bundle of cell complexes, we also have a derived adjunction $\mathbf{L}_q f^* \dashv \mathbf{R}_q f_*$. Because in the latter case f^* is left and right Quillen for the same model structure, its left and right derived functors agree. We decorate $\mathbf{L}$ and $\mathbf{R}$ with a q to remind us that these are derived functors with respect to the q-equivalences; since left and right derived functors are determined by the weak equivalences of a model structure, the derived functors are the same whether we use the q- or the m-model structures.

For maps f other than bundles of cell complexes, the functor f_* is difficult to get a handle on homotopically. It is proven in [22, 9.3.2], using Brown representability, that in the sectioned case, for any map f the functor $\mathbf{R}_q f^*$ has a partial right adjoint defined on connected objects. However, in general no relationship between this functor and the point-set level functor f_* is known. In Sections 7 and 9 we will see that this problem can be partially remedied by passing across one or the other of our equivalences.

4. Model structures for homotopy sheaves

There are several ways to make the notion of 'homotopy sheaves' precise. Probably the most common approach is the following. Let B be a space and let $\mathscr{B}$ denote the poset of open sets in B. We write $\mathcal{S}$ for the category of simplicial sets, equipped with its usual model structure. The category $\mathcal{S}^{\mathscr{B}^{op}}$ of simplicial presheaves on $\mathscr{B}$ then has a *projective* model structure in which the weak equivalences and fibrations are objectwise. We now localize this structure at a suitable set of maps to obtain a new model structure whose fibrant objects may be called 'homotopy sheaves'.

First we introduce some notation. We write $y : \mathscr{B} \hookrightarrow Set^{\mathscr{B}^{op}}$ for the Yoneda embedding, so that yU is the presheaf represented by an open subset $U \subset B$, i.e.

$$yU(V) = \begin{cases} \{*\} & \text{if } V \subset U, \\ \emptyset & \text{otherwise.} \end{cases}$$

If $U = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ is an open cover of an open subset $U \subset B$, we write $y\mathcal{A}$ for the following subfunctor of yU:

$$y\mathcal{A}(V) = \begin{cases} \{*\} & \text{if } V \subset U_\alpha \text{ for some } \alpha \in \mathcal{A}, \\ \emptyset & \text{otherwise.} \end{cases}$$

We write $\mathscr{I}_B$ for the set of all inclusions

$$y\mathcal{A} \hookrightarrow yU \tag{4.1}$$

ranging over all open covers $U = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ of open subsets $U \subset B$.

A presheaf of sets is a sheaf, in the usual sense, just when it sees all the maps in $\mathscr{I}_B$ as isomorphisms. Thus, it makes sense to localize $\mathcal{S}^{\mathscr{B}^{op}}$ at $\mathscr{I}_B$ (considered as a set of maps between discrete simplicial presheaves) and call the resulting model structure the *homotopy sheaf model structure*. A simplicial presheaf is fibrant in this model structure when it is objectwise fibrant and moreover sees all the maps $\mathscr{I}_B$ as weak equivalences; we call such an object a *homotopy sheaf*. We denote the homotopy category of this model structure by $\mathrm{HoSh}(B)$.

Remark 4.2. We could also, if we wished, use the category $\mathscr{K}^{\mathscr{B}^{op}}$ of presheaves of topological spaces. The standard Quillen equivalence

$$|-| : \mathcal{S} \rightleftarrows \mathscr{K} : S$$

lifts to a Quillen equivalence between the projective model structures on $\mathcal{S}^{\mathscr{B}^{op}}$ and $\mathscr{K}^{\mathscr{B}^{op}}$, and thence to a Quillen equivalence between homotopy sheaf model structures. In this paper we will use the simplicial version, because it is easier to write down explicit projective-cofibrant replacements. However, in an equivariant context the discrete category $\mathscr{B}$ may need to be replaced by a topologically enriched category, in which case the use of spaces rather than simplicial sets would become important.

Remark 4.3. The above construction is the same idea followed in [16], although there the localization is done using quasi-categories rather than model categories. However, the elements of the model-categorical approach can be found in [16, §7.1]. This model structure for homotopy sheaves is not equivalent to that of [14], which is constructed by localizing with respect to the larger class of *hypercoverings* (see [7]). Several arguments for using coverings rather than hypercoverings can be found in [16], in particular the result we quote below as Theorem 4.7.

As we mentioned in Section 1, however, there is also a model structure on the category $\mathscr{K}/B$ which is Quillen equivalent to the above simplicial model for homotopy sheaves. This model structure was called the *fine* model structure in [13] where it was first defined; an essentially identical model structure was also constructed in [16, §7.1.2]. We will call it the *ij-model structure*.

If $X \to B$ is a space over B and $U \subset B$ is an open set, we denote by $\underline{\Gamma}(U, X)$ the *space of sections of* X *over* U. It can be defined as the mapping space $\mathrm{Map}_B(U, X)$ in the topologically enriched category $\mathscr{K}/B$, or more abstractly as $r_* j^* X$ where $j : U \hookrightarrow X$ is the inclusion and $r : U \to *$ is the projection. The underlying sets $\Gamma(U, X)$ of the spaces $\underline{\Gamma}(U, X)$ form the ordinary sheaf of sections of X, but the spaces of sections carry more information about the topology of X. This enables us, for instance, to distinguish between X and the local homeomorphism (or 'etale space') corresponding to its ordinary sheaf of sections.

We now define the following classes of maps.

- The ij-equivalences are the maps f over B such that $\underline{\Gamma}(U, f)$ is a q-equivalence for all open sets $U \subset B$.
- The ij-fibrations are the maps f over B such that $\underline{\Gamma}(U, f)$ is a q-fibration for all open $U \subset B$. In particular, every space over B is ij-fibrant.
- Of course, the ij-cofibrations are the maps over B having the left lifting property with respect to the ij-trivial ij-fibrations.

It is proven in [13] that the above classes of maps define a topological model structure on $\mathscr{K}/B$. It is clearly cofibrantly generated; a set of generating cofibrations is

$$\{U \times S^{n-1} \hookrightarrow U \times D^n\colon\ n \in \mathbb{N}, U \subset B \text{ open}\}$$

and a set of generating trivial cofibrations is

$$\{U \times D^{n-1} \hookrightarrow U \times D^n\colon\ n \in \mathbb{N}, U \subset B \text{ open}\}.$$

Since the generating cofibrations are f-cofibrations and the generating trivial cofibrations are f-trivial f-cofibrations, the identity functor of $\mathscr{K}/B$ is left Quillen from the ij-model structure to the f-model structure. Moreover, we have the following fact.

Lemma 4.4. *Any f-equivalence is an ij-equivalence, and a map between ij-cofibrant objects is an ij-equivalence if and only if it is an f-equivalence.*

Proof. This follows from the fact that the model structure is topological (that is, it is a $\mathscr{K}$-model category), using the topological version of [10, 9.5.16]. Alternately, for the first statement one may observe that r_* and j^* both preserve homotopies. Therefore, if g is an f-equivalence, $\Gamma(U, g) = r_* j^* g$ is an h-equivalence and hence a q-equivalence for all open $U \subset X$, and thus g is an ij-equivalence. The second statement follows from this and the fact that the identity functor is left Quillen from the ij-model structure to the f-model structure. □

This implies that unlike the q-model structure, the ij-model structure on $\mathscr{K}/B$ *does* mix with the f-model structure to give a mixed $m_{ij,f}$-model structure. We will make no essential use of this model structure, but its existence is interesting.

Analogously, in the induced ij-model structure on $\mathscr{K}_B$, any fp-equivalence (in fact, any f-equivalence) is an ij-equivalence, and a map between ij-cofibrant objects is an ij-equivalence if and only if it is an fp-equivalence. Recall, though, that an 'fp-model structure' is not known to exist on $\mathscr{K}_B$, so we do not have any '$m_{ij,fp}$-model structure'.

We now describe the equivalence between the ij-model structure and the homotopy sheaf model structure. There is a canonical adjoint pair

$$|-|_B : \mathcal{S}^{\mathscr{B}^{op}} \rightleftarrows \mathscr{K}/B : S^B. \tag{4.5}$$

The right adjoint, called the *relative singular complex*, is defined by

$$S^B(X)(U) = S\big(\underline{\Gamma}(U, X)\big), \tag{4.6}$$

where S is the usual total singular complex of a space. The left adjoint $|-|_B$ is called the *relative geometric realization*; it takes a simplicial presheaf F to the tensor product of functors $i \otimes_{\mathscr{B}} |F|$, where $|F|$ denotes the objectwise geometric realization of F and $i : \mathscr{B} \to \mathscr{K}/B$ sends each open set $U \subset B$ to itself, considered as a space over B.

We say that a topological space is *hereditarily paracompact* if all its open subsets are paracompact. This is true, for example, if the space is metrizable. Moreover, all CW complexes are hereditarily paracompact (see [9, §1.3]). The version of the following result in [16] applies more generally, but we will only be interested in the hereditarily paracompact case.

Theorem 4.7. *(See [16, 7.1.4.5].) If B is Hausdorff and hereditarily paracompact, then the adjunction* (4.5) *defines a Quillen equivalence between the homotopy sheaf model structure and the ij-model structure. In particular, we have* $\mathrm{HoSh}(B) \simeq \mathrm{Ho}_{ij}(\mathscr{K}/B)$.

Idea of proof. We will not give the whole proof, but we give enough of it to explain the need for the hypotheses on B. By definition of ij-equivalences and ij-fibrations, the adjunction is Quillen for the projective model structure and the ij-model structure. Thus, to show that it is Quillen for the homotopy sheaf model structure, it suffices to show that the left derived functor of $|-|_B$ (with respect to the projective model structure) takes the maps $\mathscr{I}_B$ to ij-equivalences.

To calculate $\mathbf{L}_{proj}|-|_B$, we must replace objects by projective-cofibrant ones. The presheaf yU is already projective-cofibrant, but $y\mathcal{A}$ is not. We can give an explicit description of a cofibrant replacement for $y\mathcal{A}$ as follows: choose a total ordering of $\mathcal{A}$, and define $\widetilde{y\mathcal{A}}$ to be the geometric realization of the following simplicial object in $\mathcal{S}^{\mathscr{B}^{op}}$.

$$\cdots \coprod_{\alpha\leqslant\beta\leqslant\gamma} y(U_\alpha \cap U_\beta \cap U_\gamma) \rightrightarrows \coprod_{\alpha\leqslant\beta} y(U_\alpha \cap U_\beta) \rightrightarrows \coprod_{\alpha} yU_\alpha \tag{4.8}$$

Since $y\mathcal{A}$ is the coequalizer of the last two face maps, it admits a map from $\widetilde{y\mathcal{A}}$, which is a projective-cofibrant replacement.

Now, the relative realization of yU is just the space U over B. The relative realization of $\widetilde{y\mathcal{A}}$ can be described as a subset of $B \times [0,1]^{\mathcal{A}}$ by using barycentric coordinates in each simplex. The points of $|\widetilde{y\mathcal{A}}|_B$ are those pairs (b,ϕ), where $b \in B$ and $\phi : \mathcal{A} \to [0,1]$, such that

- $\phi(\alpha) > 0$ for only finitely many α,
- if $\phi(\alpha) > 0$, then $b \in U_\alpha$, and
- $\sum_\alpha \phi(\alpha) = 1$.

The topology of $\widetilde{y\mathcal{A}}$ is generally finer than that induced from $B \times [0,1]^{\mathcal{A}}$, but this is largely irrelevant since the identity map is a homotopy equivalence between the two topologies; see [9, 3.3.7].

The map $|\widetilde{y\mathcal{A}}|_B \to |yU|_B = U$ is the obvious projection. A section of this projection over B is precisely a partition of unity subordinate to the cover (U_α). Since by assumption, U is Hausdorff and paracompact, such a section exists, and a linear homotopy shows that it is actually the inclusion of a fiberwise deformation retract. Since f-equivalences are ij-equivalences, we see that $\mathbf{L}_{proj}|-|_B$ takes the maps in $\mathscr{I}_B$ to ij-equivalences, and hence the adjunction is Quillen for the homotopy sheaf model structure.

Finally, the functor S^B reflects weak equivalences by definition of the ij-equivalences. Thus, by [11, 1.3.16], to obtain a Quillen equivalence it suffices to show that for any projective-cofibrant simplicial presheaf X, the map $X \to S^B|X|_B$ is an $\mathscr{I}_B$-localization. This is proven in [16, §7.1.4] using another, more complicated, partition-of-unity argument. □

Remark 4.9. The preceding proof breaks down if we localize $\mathcal{S}^{\mathscr{B}^{op}}$ at all hypercovers instead: the relative realization of a hypercover is not necessarily an ij-equivalence.

We now show that the base change functors in the two cases also agree. Suppose that $f : A \to B$ is a continuous map, where A and B are Hausdorff and hereditarily paracompact. Then, as observed in [13, 5.9], the adjunction

$$f^* : \mathscr{K}/B \rightleftarrows \mathscr{K}/A : f_* \tag{4.10}$$

is Quillen for the ij-structures, since f^* preserves the generating cofibrations and trivial cofibrations. It thus gives rise to a derived adjunction which we denote $\mathbf{L}_{ij} f^* \dashv \mathbf{R}_{ij} f_*$.

On the other hand, the functor $f^{-1} : \mathscr{B} \to \mathscr{A}$ induces, by precomposition, a functor $f_* : \mathcal{S}^{\mathscr{A}^{op}} \to \mathcal{S}^{\mathscr{B}^{op}}$, which has a left adjoint f^* given by Kan extension.

Proposition 4.11. *The adjunction*

$$f^* : \mathcal{S}^{\mathscr{B}^{op}} \rightleftarrows \mathcal{S}^{\mathscr{A}^{op}} : f_* \tag{4.12}$$

is Quillen for the homotopy sheaf model structures.

Proof. Since f_* preserves objectwise fibrations and weak equivalences, the adjunction is Quillen for the projective model structures. Thus, by definition of Bousfield localization, it suffices to show that $\mathbf{L}_{proj} f^*$ takes the maps in $\mathscr{I}_B$ to $\mathscr{I}_A$-local equivalences. However, since f^* takes the representable functor yU to $y(f^{-1}(U))$, for any cover $U = \bigcup U_\alpha$ in B it takes the diagram (4.8) to the corresponding diagram for the cover $f^{-1}(U) = \bigcup f^{-1}(U_\alpha)$ in A. Since it also preserves colimits, it takes the resulting cofibrant replacement for a map in $\mathscr{I}_B$ to the corresponding replacement for the corresponding map in $\mathscr{I}_A$, which is clearly an $\mathscr{I}_A$-local equivalence. □

Thus, we also have a derived adjunction $\mathbf{L}_{sh} f^* \dashv \mathbf{R}_{sh} f_*$.

Theorem 4.13. *The derived adjunctions of $f^* \dashv f_*$ for the ij-model structure and the homotopy sheaf model structure agree under the Quillen equivalence* (4.5). *More precisely, we have isomorphisms*

$$\mathbf{R}_{sh} f_* \circ \mathbf{R} S^A \cong \mathbf{R} S^B \circ \mathbf{R}_{ij} f_*$$

and

$$\mathbf{L}|-|_B \circ \mathbf{L}_{sh} f^* \cong \mathbf{L}_{ij} f^* \circ \mathbf{L}|-|_A.$$

Proof. Since deriving Quillen adjunctions is functorial, it suffices to check that the point-set level adjunctions agree. But if $X \in \mathscr{K}/A$ and $U \in \mathscr{B}$, we have $\underline{\Gamma}(U, f_* X) \cong \underline{\Gamma}(f^{-1}U, X)$, from which we see that $f_* \circ S^A \cong S^B \circ f_*$ as desired. The other isomorphism follows formally. □

Remark 4.14. Of course, the functor $f^*: \mathscr{K}/B \to \mathscr{K}/A$ also has a left adjoint $f_!$. It is observed in [13, 5.9] that when f is an embedding, the adjunction $f_! \dashv f^*$ is also Quillen for the ij-structures. On the other hand, in general the functor $f^*: \mathcal{S}^{\mathscr{B}^{op}} \to \mathcal{S}^{\mathscr{A}^{op}}$ will not have a left adjoint at all.

By standard model-category arguments, the ij-structure and the homotopy sheaf structure give rise to model structures on the corresponding pointed categories $\mathscr{K}_B$ and $\mathcal{S}_*^{\mathscr{B}^{op}}$. The following fact implies that Theorem 4.7 descends to the pointed case as well.

Proposition 4.15. *(See [11, 1.3.5 and 1.3.17].) If $F: \mathscr{C} \rightleftarrows \mathscr{D}: G$ is a Quillen adjunction, there is a corresponding Quillen adjunction $F_*: \mathscr{C}_* \to \mathscr{D}_*: G_*$ between the corresponding pointed model categories. If in addition $F \dashv G$ is a Quillen equivalence and the terminal object of $\mathscr{C}$ is cofibrant and preserved by F, then $F_* \dashv G_*$ is also a Quillen equivalence.*

Corollary 4.16. *The sectioned adjunction*

$$|-|_B : \mathcal{S}_*^{\mathscr{B}^{op}} \rightleftarrows \mathscr{K}_B : S^B \tag{4.17}$$

defines a Quillen equivalence between the model category of pointed homotopy sheaves and the sectioned ij-model structure. For a map $f: A \to B$, the sectioned adjunctions $f^* \dashv f_*$ are again Quillen in both cases and their derived functors agree under the equivalence (4.17).

Remark 4.18. The adjunction (4.5) actually factors through the category $\mathscr{K}^{\mathscr{B}^{op}}$ of topological homotopy sheaves:

$$\mathcal{S}^{\mathscr{B}^{op}} \rightleftarrows \mathscr{K}^{\mathscr{B}^{op}} \rightleftarrows \mathscr{K}/B,$$

where the first adjunction is the Quillen equivalence from Remark 4.2. By the 2-out-of-3 property for Quillen equivalences, it follows that the adjunction

$$\mathscr{K}^{\mathscr{B}^{op}} \rightleftarrows \mathscr{K}/B$$

is a Quillen equivalence between the topological homotopy sheaf model structure and the ij-model structure. Analogous remarks apply to the base change functors and the pointed variants.

5. Parametrized spaces embed in homotopy sheaves

We now want to show that the homotopy theory of parametrized spaces embeds in that of homotopy sheaves. First we introduce some terminology.

Definition 5.1. We say that a Quillen adjunction $F:\mathscr{C} \rightleftarrows \mathscr{D}:G$ is a *right Quillen embedding* from $\mathscr{D}$ to $\mathscr{C}$ if, for any fibrant $Y \in \mathscr{D}$, the canonical map

$$FQGY \to FGY \to Y \tag{5.2}$$

is a weak equivalence, where Q denotes cofibrant replacement in $\mathscr{C}$.

We regard a right Quillen embedding as exhibiting the homotopy theory of $\mathscr{D}$ as a 'sub-homotopy-theory' of the homotopy theory of $\mathscr{C}$. Of course, there is a dual notion of left Quillen embedding. For example, the identity functor of $\mathscr{K}$ is a left Quillen embedding from the q- or m-model structure to the h-model structure.

It is well known that a Quillen adjunction is a Quillen equivalence just when it induces an equivalence of homotopy categories. There is an analogue for Quillen embeddings.

Proposition 5.3. *A Quillen adjunction $F \dashv G$ is a right Quillen embedding if and only if the right derived functor* $\mathbf{R}G$ *is full and faithful on homotopy categories.*

Proof. $\mathbf{R}G$ is full and faithful just when the counit

$$\mathbf{L}F \circ \mathbf{R}G \to \mathrm{Id}_{\mathrm{Ho}\,\mathscr{D}} \tag{5.4}$$

is an isomorphism, but (5.4) is represented on the point-set level by (5.2), so the former is an isomorphism just when the latter is a weak equivalence. □

Remark 5.5. By [11, 1.3.16], a right Quillen embedding is a Quillen equivalence if and only if the left adjoint F reflects weak equivalences between cofibrant objects. This is a homotopical version of the fact that a full and faithful right adjoint is an equivalence if and only if its left adjoint reflects isomorphisms.

We are now working towards showing that the identity adjunction of $\mathscr{K}/B$ is a right Quillen embedding from the m-model structure (that is, the $m_{q,h}$-model structure) to the ij-model structure. We begin with a sequence of lemmas.

Lemma 5.6. *If B is hereditarily m-cofibrant, then the identity functor of $\mathscr{K}/B$ is a left Quillen functor from the ij-model structure to the m-model structure.*

Proof. We must show that the generating cofibrations and trivial cofibrations for the ij-structure are m-cofibrations and m-trivial m-cofibrations. However, the generating trivial cofibrations for the ij-structure are f-equivalences, and therefore q-equivalences, so it suffices to show that the generating ij-cofibrations are m-cofibrations. Since the generating cofibrations have the form $U \times S^{n-1} \hookrightarrow U \times D^n$ for some open set $U \subset B$, and U is m-cofibrant by assumption, this follows from the fact that the m-model structure is monoidal. □

In fact, as pointed out by the referee, the identity functor is also left Quillen from the mixed $m_{ij,f}$-model structure to the m-model structure. This is because the m-fibrations are the h-fibrations, which are also f-fibrations, and by Lemma 5.6 all m-trivial m-fibrations are ij-equivalences. Thus we have the following diagram of identity functors which are all left Quillen functors. Both horizontal arrows on the left are Quillen equivalences, and both horizontal arrows on the right are left Quillen embeddings.

$$\begin{array}{ccccc}
(\mathscr{K}/B, ij) & \longrightarrow & (\mathscr{K}/B, m_{ij,f}) & \longrightarrow & (\mathscr{K}/B, f) \\
 & & \downarrow & & \downarrow \\
(\mathscr{K}/B, q) & \longrightarrow & (\mathscr{K}/B, m_{q,h}) & \longrightarrow & (\mathscr{K}/B, h)
\end{array}$$

Lemma 5.7. *Let B be hereditarily m-cofibrant and let $QF \to F$ be a q-cofibrant replacement of $F \in \mathscr{K}$. Then*

$$B \times QF \to B \times F \tag{5.8}$$

is an ij-cofibrant replacement of the product projection $B \times F \to B$.

Proof. It is clear that $B \times QF$ is ij-cofibrant, since any decomposition of QF into cells $S^{n-1} \hookrightarrow D^n$ gives a corresponding decomposition of $B \times QF$ into cells $B \times S^{n-1} \to B \times D^n$, which are generating ij-cofibrations. It remains to show that (5.8) is an ij-equivalence. For any product projection $B \times C \to B$, we have a homeomorphism

$$\underline{\Gamma}(U, B \times C) \cong \operatorname{Map}(U, C),$$

so applying $\underline{\Gamma}(U, -)$ to (5.8) yields the map

$$\operatorname{Map}(U, QF) \to \operatorname{Map}(U, F). \tag{5.9}$$

Since $QF \to F$ is a q-equivalence and U has the homotopy type of a CW complex, (5.9) is also a q-equivalence. This is true for all open $U \subset B$, so the map (5.8) is an ij-equivalence, as desired, and thus an ij-cofibrant replacement of $B \times F$. □

The following lemma is stronger than what we need in this section, but we will use it again in Section 7.

Lemma 5.10. *Let $f : A \to B$ be a map between hereditarily m-cofibrant spaces, where B is contractible. Let X be an h-fibrant object of $\mathscr{K}/B$ and let $QX \to X$ be an ij-cofibrant replacement. Then its pullback $f^*QX \to f^*X$ is both a q-equivalence and an ij-equivalence.*

In particular, when f is the identity of B, this says that $QX \to X$ itself is also a q-equivalence.

Proof. Since f-equivalences are both q-equivalences and ij-equivalences, and are preserved under pullback, we can work up to f-equivalence. For example, since any two ij-cofibrant replacements for X are f-equivalent, it suffices to show the result for *some* ij-cofibrant replacement. And since B is contractible, any h-fibrant $X \to B$ is f-equivalent to a product projection, so we may as well assume that X itself is a product projection $B \times F \to B$. In this case, we can use as our ij-cofibrant replacement the map $B \times QF \to B \times F$ from Lemma 5.7, for some q-cofibrant replacement $QF \to F$. But the pullback of this map along any $f : A \to B$ is just $A \times QF \to A \times F$, which is both a q-equivalence and an ij-equivalence (the latter by Lemma 5.7). □

Theorem 5.11. *If B is a locally compact CW complex, the identity adjunction of $\mathscr{K}/B$ is a right Quillen embedding from the m-model structure to the ij-model structure.*

Proof. We have shown in Lemma 5.6 that the identity adjunction is Quillen, so it remains to show that if $X \to B$ is h-fibrant, then $QX \to X$ is a q-equivalence (here Q denotes ij-cofibrant replacement).

Since B is a CW complex, it is locally contractible, so it has a cover (U_α) by contractible open sets with inclusions $j_\alpha : U_\alpha \hookrightarrow B$. For any α, the functor $j_\alpha^* : \mathscr{K}/B \to \mathscr{K}/U_\alpha$ preserves ij-equivalences and ij-cofibrations, so $j_\alpha^* QX \to j_\alpha^* X$ is again an ij-cofibrant replacement of a Hurewicz fibration. But since U_α is contractible, Lemma 5.10 tells us that $j_\alpha^* QX \to j_\alpha^* X$ is a q-equivalence preserved under pullbacks. In particular, if $j : U \hookrightarrow U_\alpha$ is any open subset, the further restriction $j^* j_\alpha^* QX \to j^* j_\alpha^* X$ is also a q-equivalence.

It follows that $QX \to X$ restricts to a q-equivalence over all open sets in the cover of B consisting of all finite intersections of the sets U_α. Since this cover is closed under finite intersections by construction, it follows from [19, 1.4] that $QX \to X$ is also a q-equivalence, as desired. □

Since the identity is a Quillen equivalence between the ij-model structure and the $m_{ij,f}$-model structure, it follows that the identity is also a right Quillen embedding from the m-model structure to the $m_{ij,f}$-model structure.

Corollary 5.12. *If B is a locally compact CW complex, then the relative realization-singular complex adjunction*

$$|-|_B : \mathcal{S}^{\mathscr{B}^{op}} \rightleftarrows \mathscr{K}/B : S^B$$

is a right Quillen embedding from the m-model structure to the homotopy sheaf model structure.

Proof. All CW complexes are Hausdorff and hereditarily paracompact, so we can compose the right Quillen embedding from Theorem 5.11 with the Quillen equivalence from Theorem 4.7. □

This result shows that parametrized spaces do, in fact, embed 'homotopically fully and faithfully' into homotopy sheaves. In particular, at the level of homotopy categories we have an adjunction

$$\iota^\star : \mathrm{HoSh}(B) \simeq \mathrm{Ho}_{ij}(\mathscr{K}/B) \rightleftarrows \mathrm{Ho}_q(\mathscr{K}/B) : \iota_\star$$

in which the right adjoint is full and faithful. The existence of $\iota_\star$, though not its full-and-faithfulness, was observed in [13].

Remark 5.13. Unlike Theorem 4.7, Theorem 5.12 remains true if we localize $\mathcal{S}^{\mathscr{B}^{op}}$ at all *hypercovers*, because the realization of any hypercover *is* a q-equivalence, though not an ij-equivalence—this follows from the proof of [1, Thm. 12.1]. In other words, *all locally constant homotopy sheaves are hypercomplete*. Thus, [29] was able to prove a simplicial version of Theorem 5.12 using a localization at all hypercovers.

The only property of a locally compact CW complex used in Theorem 5.11, aside from hereditary m-cofibrancy, is that it is locally contractible. For Theorem 5.12, we also need it to be Hausdorff and hereditarily paracompact. Thus, we can abstract the necessary properties of B as follows.

Definition 5.14. We say that a space is a *good ancestor* if it is

(i) compactly generated,
(ii) Hausdorff and hereditarily paracompact,
(iii) hereditarily m-cofibrant, and
(iv) locally contractible.

Any locally compact CW complex is a good ancestor. Moreover, any open subspace of a good ancestor is a good ancestor; that is, the property of being a good ancestor is itself hereditary. This will be important in Section 7.

Finally, most of the results of this section have corresponding versions for the sectioned theory, by the following lemma.

Lemma 5.15. *If $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ is a right Quillen embedding and the terminal object of $\mathscr{C}$ is cofibrant and preserved by F, then $F_* : \mathscr{C}_* \rightleftarrows \mathscr{D}_* : G_*$ is also a right Quillen embedding.*

Proof. Since the terminal object of $\mathscr{C}$ is cofibrant, any cofibrant object of $\mathscr{C}_*$ is also cofibrant in $\mathscr{C}$. The fact that F preserves the terminal object implies that the pointed adjunction $F_* \dashv G_*$ is defined simply by applying F and G to underlying objects. Thus, if Y is fibrant in $\mathscr{D}_*$, the map $F_*Q_*G_*Y \to Y$ is just $FQGY \to Y$, which is a weak equivalence since Y is also fibrant in $\mathscr{D}$. □

Corollary 5.16. *If B is a good ancestor, the identity functor of $\mathscr{K}_B$ is a right Quillen embedding from the m-model structure to the ij-model structure, and the pointed adjunction*

$$|-|_B : \mathcal{S}_*^{\mathscr{B}^{op}} \rightleftarrows \mathscr{K}_B : S^B$$

is a right Quillen embedding from the m-model structure to the homotopy sheaf model structure.

Proof. The terminal object is cofibrant in all model structures under consideration, the identity functor clearly preserves it, and it is easy to see that so does the relative geometric realization. Thus we can apply Lemma 5.15. □

Of course, since there is no known 'fp-model structure', the statements about the $m_{ij,f}$-model structure have no sectioned analogue.

6. The essential image

We would now like to identify the image of the right Quillen embedding from Theorem 5.11. As explained in Section 1, our intuition is that it consists of the locally constant homotopy sheaves. Of course, we need to make precise what we mean by 'locally constant' in a homotopical sense. From now on, we will take the ij-structure as our model for homotopy sheaves.

Definition 6.1. We say that an object $X \to B$ of $\mathscr{K}/B$ is *constant* if it is isomorphic in $\mathrm{Ho}_{ij}(\mathscr{K}/B)$ to one of the form $B \times F \to B$. We say that it is *locally constant* if B admits an open cover (U_α), with inclusions $j_\alpha : U_\alpha \hookrightarrow B$, such that $j_\alpha^* X$ is constant for all α.

We have the following trivial observation.

Lemma 6.2. *If B is locally contractible, then any Hurewicz fibration $X \to B$ is locally constant.*

Proof. Take a cover by contractible opens; then $j_\alpha^* X$ is a fibration over a contractible space, hence f-equivalent to a product projection. □

We observe that the essential image of the embedding $\iota_\star : \mathrm{Ho}_q(\mathscr{K}/B) \hookrightarrow \mathrm{Ho}_{ij}(\mathscr{K}/B)$ consists of the objects of $\mathrm{Ho}_{ij}(\mathscr{K}/B)$ isomorphic to Hurewicz fibrations, since the latter are the fibrant objects in the m-model structure. Thus, this image is contained in the locally constant objects. We now intend to show that conversely, any locally constant object of $\mathrm{Ho}_{ij}(\mathscr{K}/B)$ is isomorphic to a Hurewicz fibration. We begin with the following lemma which clarifies the structure of locally constant objects.

Lemma 6.3. *Let B be a good ancestor and $X \to B$ be ij-cofibrant and locally constant. Then X is locally f-equivalent to a product projection. In particular, it is a quasifibration.*

Proof. Since X is locally constant, we have a cover (U_α) such that $j_\alpha^* X$ is isomorphic in $\mathrm{Ho}_{ij}(\mathscr{K}/U_\alpha)$ to a product projection $U_\alpha \times F_\alpha$. Let $QF_\alpha \to F_\alpha$ be a q-cofibrant replacement; then by Lemma 5.7, $U_\alpha \times QF_\alpha$ is an ij-cofibrant replacement for $U_\alpha \times F_\alpha$. Therefore, since $j_\alpha^* X$ is also cofibrant, the composite isomorphism $j_\alpha^* X \cong U_\alpha \times QF_\alpha$ in $\mathrm{Ho}_{ij}(\mathscr{K}/B)$ is realized by an ij-equivalence in $\mathscr{K}/U_\alpha$. And since this is an ij-equivalence between ij-cofibrant objects, it is actually an f-equivalence. Thus, X is locally f-equivalent to a product projection.

Now, since $j_\alpha^* X$ is f-equivalent to an h-fibration, it is a 'halb-fibration' (see [3,4]), and in particular a quasifibration. Since f-equivalences and h-fibrations are preserved by restricting to open subspaces, this is also true of $j^* X$ for any open set $j : U \hookrightarrow X$ where $U \subset U_\alpha$ for some α, and in particular for finite intersections of the U_α. Thus, B has an open cover which is closed under finite intersections and over which X is a quasifibration. Standard criteria (e.g., [5]) then imply that X itself is a quasifibration. □

Theorem 6.4. *If B is a good ancestor, then any locally constant object of $\mathrm{Ho}_{ij}(\mathscr{K}/B)$ is isomorphic in $\mathrm{Ho}_{ij}(\mathscr{K}/B)$ to a Hurewicz fibration. Therefore, the essential image of $\mathrm{Ho}_q(\mathscr{K}/B)$ in $\mathrm{Ho}_{ij}(\mathscr{K}/B)$ consists precisely of the locally constant objects.*

Proof. Let X be locally constant. Since every object of $\mathrm{Ho}_{ij}(\mathscr{K}/B)$ is isomorphic to an ij-cofibrant one, we may assume that X is ij-cofibrant. Thus, by Lemma 6.3, there is a cover (U_α) and f-equivalences

$$U_\alpha \times F_\alpha \to j_\alpha^* X. \tag{6.5}$$

Moreover, since B is paracompact, we may assume by refinement that the cover (U_α) is numerable.

Now, let $X \to RX$ be an h-fibrant replacement; we want to show that it is actually an ij-equivalence. By [13, 6.1], since the cover (U_α) is numerable, it suffices to show that the induced map

$$j_\alpha^* X \to j_\alpha^* RX \tag{6.6}$$

is an ij-equivalence for each α. By definition of a quasifibration, the map $X \to RX$ induces a q-equivalence on all fibers, and therefore so does $j_\alpha^* X \to j_\alpha^* RX$. Moreover, since (6.5) is an f-equivalence, it induces an h-equivalence on fibers, and thus the composite

$$U_\alpha \times F_\alpha \to j_\alpha^* X \to j_\alpha^* RX \tag{6.7}$$

induces a q-equivalence on all fibers. But both $U_\alpha \times F_\alpha$ and $j_\alpha^* RX$ are h-fibrant, so by the five lemma, (6.7) is itself a q-equivalence. Again since both are h-fibrant, [13, 6.5] (due to Lewis) implies that (6.7) is an ij-equivalence.

Finally, since (6.5) is also an ij-equivalence, the 2-out-of-3 property implies that (6.6) is too. This shows that $X \to RX$ is an ij-equivalence, and thus X is isomorphic in $\mathrm{Ho}_{ij}(\mathscr{K}/B)$ to the Hurewicz fibration RX. □

7. Base change and homotopy sheaves

We now consider the relationship between the base change functors for parametrized spaces and for homotopy sheaves. This is nontrivial because f^* has a *left* derived functor $\mathbf{L}_{sh} f^* \cong \mathbf{L}_{ij} f^*$ for homotopy sheaves but a *right* derived functor $\mathbf{R}_q f^*$ for parametrized spaces. However, we will prove that the two agree up to homotopy. Recall that we write $\iota^\star : \mathrm{Ho}_{ij}(\mathscr{K}/B) \rightleftarrows \mathrm{Ho}_q(\mathscr{K}/B) : \iota_\star$ for the right Quillen embedding from Section 5.

Theorem 7.1. *For any map $f : A \to B$ between good ancestors, we have a natural isomorphism*

$$\mathbf{L}_{ij} f^* \circ \iota_\star \cong \iota_\star \circ \mathbf{R}_q f^* \tag{7.2}$$

in both the sectioned and unsectioned cases.

Proof. We prove the unsectioned case first. Since $\iota_\star$ and $\mathbf{R}_q f^*$ are both right derived functors for the same model structures, their composition is just given by their point-set composite applied to a fibrant object; in other words, $f^* X$ where $X \to B$ is an h-fibration. On the other hand, $\iota_\star X$ is again X (when X is h-fibrant), but to compute $\mathbf{L}_{ij} f^*(\iota_\star X)$ we must replace X by an ij-cofibrant object QX. Since this comes with an ij-equivalence $QX \xrightarrow{\sim} X$, we have a canonical map

$$f^* QX \to f^* X \tag{7.3}$$

which represents a map

$$\mathbf{L}_{ij} f^*(\iota_\star X) \to \iota_\star(\mathbf{R}_q f^* X). \tag{7.4}$$

In the terminology of [26], this is the 'derived natural transformation' of the point-set level equality $f^* \circ \mathrm{Id} = \mathrm{Id} \circ f^*$.

We claim that (7.4) is an isomorphism, or equivalently that (7.3) is an ij-equivalence. Let (U_α) be a numerable cover of B by contractible opens. Then

$$j_\alpha^* QX \to j_\alpha^* X \tag{7.5}$$

is again an ij-cofibrant replacement in $\mathscr{K}/U_\alpha$. By Lemma 5.10, since U_α is contractible and X is h-fibrant, any pullback of (7.5) to another good ancestor is an ij-equivalence.

In particular, this applies to the pullback along the restriction $f_\alpha : f^{-1}(U_\alpha) \to U_\alpha$ of f; thus the map

$$f_\alpha^* j_\alpha^* QX \to f_\alpha^* j_\alpha^* X$$

is an ij-equivalence. But if we write $i_\alpha : f^{-1}(U_\alpha) \hookrightarrow A$ for the inclusion, then we have $f i_\alpha = j_\alpha f_\alpha$ and hence $f_\alpha^* j_\alpha^* \cong i_\alpha^* f^*$, so the map

$$i_\alpha^* f^* QX \to i_\alpha^* f^* X$$

is also an ij-equivalence over $f^{-1}(U_\alpha)$ for all α. Since the cover $(f^{-1}(U_\alpha))$ of A is also numerable, it follows from [13, 6.1] that $f^* QX \to f^* X$ is an ij-equivalence over A, as desired.

In the sectioned case, we again have a map

$$\mathbf{L}_{ij} f^*(\iota_\star X) \to \iota_\star(\mathbf{R}_q f^* X)$$

represented by the map

$$f^* QX \to f^* X, \tag{7.6}$$

where X is h-fibrant in $\mathscr{K}_B$ and now Q denotes ij-cofibrant replacement in $\mathscr{K}_B$. But if we forget the sections, we see that X is also h-fibrant in $\mathscr{K}/B$, and since the terminal object of $\mathscr{K}/B$ is ij-cofibrant, QX is also an ij-cofibrant replacement in $\mathscr{K}/B$. Thus, applying the result for the unsectioned case, we see that (7.6) is an ij-equivalence in $\mathscr{K}/A$, hence also in $\mathscr{K}_A$. □

Since $\iota_\star$ has a left adjoint $\iota^\star$, the isomorphism (7.2) has a 'mate'

$$\iota^\star \circ \mathbf{L}_{ij} f^* \to \mathbf{R}_q f^* \circ \iota^\star. \tag{7.7}$$

Similarly, since $\mathbf{L}_{ij} f^*$ has a right adjoint $\mathbf{R}_{ij} f^*$, and, in the sectioned case, $\mathbf{R}_q f^*$ has a partial right adjoint $\mathbf{M} f_*$ defined on connected spaces (obtained using Brown representability), (7.2) has another 'partial mate'

$$\iota_\star \circ \mathbf{M} f_* \to \mathbf{R}_{ij} f_* \circ \iota_\star \tag{7.8}$$

defined on subcategories of connected spaces. (The 'M' may stand either for 'middle' or 'mysterious'.)

If f is a bundle of cell complexes, then f^* is also left Quillen for the q-structures, so $\mathbf{R}_q f^* = \mathbf{L}_q f^*$ also has a totally defined right adjoint $\mathbf{R}_q f_*$. In this case we have an analogous transformation

$$\iota_\star \circ \mathbf{R}_q f_* \to \mathbf{R}_{ij} f_* \circ \iota_\star \tag{7.9}$$

which is defined everywhere.

Standard categorical arguments show that (7.8) or (7.9) is an isomorphism if and only if (7.7) is. Thus, since $\mathbf{M} f_*$ is difficult to get a handle on, it is natural to focus our efforts on (7.7) instead. The main result is the following. This is a special case of the results of [26] regarding mates of derived natural transformations.

Proposition 7.10. *The transformation* (7.7) *at an ij-cofibrant space X over B is isomorphic to the map*

$$f^* X \to f^* RX, \tag{7.11}$$

where $X \to RX$ is an h-fibrant replacement.

Proof. The map (7.7) is defined to be the composite

$$\iota^\star \circ \mathbf{L}_{ij} f^* \to \iota^\star \circ \mathbf{L}_{ij} f^* \circ \iota_\star \circ \iota^\star \xrightarrow{\cong} \iota^\star \circ \iota_\star \circ \mathbf{R}_q f^* \circ \iota^\star \to \mathbf{R}_q f^* \circ \iota^\star, \tag{7.12}$$

where the first map is the unit, and the last the counit, of the adjunction $\iota^\star \dashv \iota_\star$. We now trace this through on the point-set level.

We start with an ij-cofibrant object $X \to B$, so that $\iota^\star(\mathbf{L}_{ij} f^* X)$ is given simply by $f^* X$. The first map is the unit of $\iota^\star \dashv \iota_\star$ at X, which is just the map $X \to RX$. Since we must apply $\mathbf{L}_{ij} f^*$ and then $\iota^\star$ to this, the first map is actually represented on the point-set level by

$$Qf^* QX \to Qf^* QRX \tag{7.13}$$

where Q denotes ij-cofibrant replacement. We have a diagram

$$\begin{array}{ccccc} Qf^* QX & \xrightarrow{\sim} & f^* QX & \xrightarrow{\sim} & f^* X \\ \downarrow & & \downarrow & & \downarrow \\ Qf^* QRX & \xrightarrow[\sim]{} & f^* QRX & \longrightarrow & f^* RX \end{array} \tag{7.14}$$

where the left-hand vertical arrow is (7.13), and in which the marked arrows are ij-equivalences, the top-right one since X is already ij-cofibrant. The diagram commutes by the naturality of Q and R.

We must then compose this with the isomorphism (7.2) at RX, which is obtained by applying f^* to the ij-cofibrant replacement map $QRX \to RX$. In our case, we must then apply $\iota^\star$ to this, which involves another ij-cofibrant replacement; thus the middle isomorphism in (7.12) is represented on the point-set level by

$$Qf^* QRX \to Qf^* RX. \tag{7.15}$$

We can add this map to (7.14), together with another square which commutes by naturality, to obtain the following.

$$\begin{array}{ccccc} Qf^*QX & \xrightarrow{\sim} & f^*QX & \xrightarrow{\sim} & f^*X \\ \downarrow & & \downarrow & & \downarrow \\ Qf^*QRX & \xrightarrow[\sim]{} & f^*QRX & \longrightarrow & f^*RX \\ & \searrow & & \nearrow & \\ & & Qf^*RX & & \end{array} \tag{7.16}$$

Finally, we must compose with the counit of $\iota^\star \dashv \iota_\star$ at Qf^*RX, which is simply the map $Qf^*RX \to f^*RX$ at the bottom right of (7.16). Hence, by the commutativity of (7.16), the composite of all three is equal to the composite

$$Qf^*QX \xrightarrow{\sim} f^*QX \xrightarrow{\sim} f^*X \to f^*RX.$$

The first two maps are ij-equivalences between ij-cofibrant objects, hence f-equivalences and so also q-equivalences. Thus, modulo these isomorphisms, (7.7) is equal to (7.11). □

This enables us to show easily that (7.7) is an isomorphism in some cases.

Theorem 7.17. *The transformations* (7.7) *and* (7.8) *are isomorphisms whenever f is a q-fibration between good ancestors, as is* (7.9) *when f is a bundle of cell complexes between good ancestors.*

Proof. As observed in [22, 7.3.4], f^* preserves all q-equivalences when f is a q-fibration, and $X \to RX$ is certainly a q-equivalence. □

However, we can also use Proposition 7.10 to construct counterexamples in which (7.7), and hence (7.8), is not an isomorphism. This phenomenon is closely related to [22, 0.0.1].

Counterexample 7.18. Let $f : A \to B$ be a map between good ancestors, where B is path connected, and let $U \subset B$ be an open set disjoint from the image $f(A)$. Then $U \to B$ is an ij-cofibrant object of $\mathscr{K}/B$ and $f^*U = \emptyset$, hence $\iota^\star(\mathbf{L}_{ij} f^*U) = \emptyset$ as well. However, since B is path-connected, there are paths connecting points in $f(A)$ to points in U, so f^*RU will not be empty; thus $\mathbf{R}_q f^*(\iota^\star U)$ is not empty and (7.7) is not an equivalence.

This very general example makes us suspect that (7.7) will not be an isomorphism for 'most' maps f. In fact, any map $f : A \to B$ for which (7.7) is an isomorphism must be 'almost a fibration' in the following sense. Any open $U \subset B$ is ij-cofibrant as an object of $\mathscr{K}/B$, so if (7.7) is an isomorphism at U, the map

$$f^*U \to f^*RU$$

must be a q-equivalence. But f^*U is just $f^{-1}(U)$, so this says that the preimage of U is equivalent to its 'homotopy preimage'. The analogous statement for *points*, rather than open sets, is what characterizes a quasifibration. We conjecture that (7.7) being an isomorphism implies that f is actually a quasifibration, but we have so far been unable to prove this.

Remark 7.19. We noted in Section 4 that when f is an embedding, the adjunction $f_! \dashv f^*$ is also Quillen for the ij-model structures. Therefore, in this case the left derived functor $\mathbf{L}_{ij} f^*$ is isomorphic to the right derived functor $\mathbf{R}_{ij} f^*$, so the isomorphism (7.2) follows formally because all functors involved are Quillen right adjoints and they commute on the point-set level. It follows that we also have an isomorphism

$$\iota^\star \circ \mathbf{L}_{ij} f_! \cong \mathbf{L}_q f_! \circ \iota^\star. \tag{7.20}$$

Moreover, in this case we have a canonical transformation

$$\mathbf{L}_{ij} f_! \circ \iota_\star \to \iota_\star \circ \mathbf{L}_q f_! \tag{7.21}$$

which is represented on the point-set level by the composite

$$f_!(Q_{ij}X) \to f_!X \to R(f_!X).$$

Here Q_{ij} denotes ij-cofibrant replacement, R denotes h-fibrant replacement, and X is assumed m-cofibrant and h-fibrant over A (which is a subspace of B). Since $f_!(Q_{ij}X)$ is supported only on A, while $R(f_!X)$ is supported on all path-components of B which intersect A, this can only be an ij-equivalence if A is a union of path components of B.

We end this section with some remarks about the potential utility of Theorem 7.17 for computing the mysterious functor $\mathbf{M}f_*$. The fact that $f^* \dashv f_*$ is Quillen for the q-model structures whenever f is a bundle of cell complexes implies that in this case, $\mathbf{M}f_*$ is isomorphic to $\mathbf{R}_q f_*$ and thus may be computed by first applying q-fibrant replacement and then the point-set level functor f_*. In particular, this applies when f is the projection $r: A \to *$ for a cell complex A, giving a way to compute 'fiberwise generalized cohomology'.

By comparison, Theorem 7.17 tells us that if f is any q-fibration between good ancestors, then $\mathbf{M}f_*$ may be computed by first applying an h-fibrant replacement and then the point-set level f_*. This is slightly better since it applies to q-fibrations which are not necessarily bundles. However, since our spaces must essentially be open subspaces of locally compact CW complexes, it doesn't give a way to compute fiberwise generalized cohomology for many new base spaces.

8. *G*-spaces and *BG*-spaces

We now consider the homotopy-theoretic version of the equivalence between locally constant sheaves and π_1-sets. Our intuition is that spaces parametrized over A should be equivalent to spaces with an action of the 'fundamental ∞-groupoid' $\Pi_\infty(A)$. Topologically speaking, at least if A is connected, $\Pi_\infty(A)$ can be represented by the loop space ΩA (where we choose a base point arbitrarily). We can choose a topological model for ΩA, such as the Moore loop space or the realization of the Kan loop group, which is a grouplike topological monoid; then A can be reconstructed, up to q-equivalence, as the classifying space of ΩA.

Moreover, if A is m-cofibrant, then so is ΩA by [23]. Since the homotopy theory of parametrized spaces is invariant under q-equivalences of the base space, it is harmless to assume that A is m-cofibrant. Thus, for the rest of this section we make the following assumption.

Assumption 8.1. *G is a compactly generated m-cofibrant grouplike topological monoid whose identity is a nondegenerate basepoint* (*that is, $* \to G$ is an h-cofibration*).

Of course, we are thinking of $G = \Omega A$ for a connected m-cofibrant space A which admits a nondegenerate basepoint. We intend to compare the homotopy theory of spaces with a G-action to the homotopy theory of spaces parametrized over BG. The results in this section are basically folklore. A bijection between equivalence classes can be found in the survey article [27], and a full equivalence of homotopy theories using simplicial fibrations can be found in [6,8]; our use of the m-model structure on $\mathscr{K}/BG$ will allow us to prove the strong result while using only topological spaces.

We will also need a model structure on $G\mathscr{K}$, the category of (left) G-spaces and G-equivariant maps. If G is a topological group and $\mathcal{H}$ is a set of closed subgroups of G, there is a cofibrantly generated model structure on $G\mathscr{K}$ in which the weak equivalences are the G-maps which induce q-equivalences on H-fixed point spaces for all $H \in \mathcal{H}$; we may call this the $q\mathcal{H}$-model structure. This is most frequently used in equivariant homotopy theory when $\mathcal{H}$ is the set of all closed subgroups of G; see, for example, [20]. However, we will be interested instead in the case when $\mathcal{H}$ consists only of the trivial subgroup $\{e\}$. We call this the *qe-model structure* and refer to its weak equivalences as *e-equivalences*. This model structure exists for any topological monoid G.

We now construct a Quillen equivalence between the qe-model structure on $G\mathscr{K}$ and the m-model structure on $\mathscr{K}/BG$. There is an obvious functor from $G\mathscr{K}$ to $\mathscr{K}/BG$ given by the Borel construction; a G-space X is mapped to $EG \times_G X = B(*, G, X)$, equipped with its projection to $BG = B(*, G, *)$. This functor has a right adjoint, which takes a space $Y \to BG$ over BG to the space $\mathrm{Map}_{BG}(EG, Y)$ of maps from EG to Y over BG, equipped with the left G-action induced from the right action of G on EG. Thus we have an adjoint pair

$$B(*, G, -): G\mathscr{K} \rightleftarrows \mathscr{K}/BG : \mathrm{Map}_{BG}(EG, -). \tag{8.2}$$

Since EG is contractible, we can think of $\mathrm{Map}_{BG}(EG, Y)$ as a 'homotopy fiber' of Y which is chosen in a clever way so as to inherit a strict G-action. Our first observation is that this intuition is precise when Y is fibrant.

Lemma 8.3. *Under Assumption* 8.1*, if $Y \to BG$ is an h-fibration, the map*

$$\mathrm{Map}_{BG}(EG, Y) \to \mathrm{Map}_{BG}(*, Y) = \mathrm{fib}(Y), \tag{8.4}$$

induced by the inclusion of the basepoint $* \to EG$, is an h-trivial h-fibration.

Proof. The map $* \to EG$ is an h-equivalence, and Assumption 8.1 ensures that it is also an h-cofibration. Thus, since the h-model structure on $\mathscr{K}$ is monoidal, the induced pullback corner map

$$\mathrm{Map}(EG, Y) \to \mathrm{Map}(*, Y) \times_{\mathrm{Map}(*, BG)} \mathrm{Map}(EG, BG) = Y \times_{BG} \mathrm{Map}(EG, BG)$$

is an h-trivial h-fibration. Since (8.4) is the pullback of this map along

$$i \times q : \mathrm{fib}(Y) \to Y \times_{BG} \mathrm{Map}(EG, BG),$$

where $i : \mathrm{fib}(Y) \hookrightarrow Y$ is the inclusion and $q : * \to \mathrm{Map}(EG, BG)$ picks out the canonical map $EG \to BG$, (8.4) is also an h-trivial h-fibration. $\square$

Theorem 8.5. *Under Assumption* 8.1*, the adjunction* (8.2) *is a Quillen equivalence between the qe-model structure and the m-model structure.*

Proof. The qe-model structure is cofibrantly generated, so to show that (8.2) is Quillen, it suffices to show that the left adjoint takes the generating qe-cofibrations and trivial cofibrations to m-cofibrations and trivial cofibrations. The generating qe-cofibrations are the maps

$$G \times S^{n-1} \to G \times D^n,$$

which are taken by the Borel construction to

$$EG \times S^{n-1} \to EG \times D^n.$$

By [17, A.6], EG is m-cofibrant because G is. Thus, since the m-structure is monoidal, these maps are m-cofibrations. The case of the generating trivial qe-cofibrations is analogous.

We now show that the adjunction is a Quillen equivalence. Let X be a qe-cofibrant G-space and let Y be an h-fibrant space over BG; we must show that a map $f : B(*, G, X) \to Y$ is a q-equivalence if and only if its adjunct $\hat{f} : X \to \mathrm{Map}_{BG}(EG, Y)$ is a q-equivalence. Actually, we will show that this is true for *any* G-space X and any h-fibrant Y over BG.

By [18, 7.6], since G is grouplike, $B(*, G, X) \to BG$ is a quasifibration. Therefore, a map $f : B(*, G, X) \to Y$ is a q-equivalence if and only if it induces a q-equivalence on fibers. But the fiber of $B(*, G, X)$ (over the base point) is just X, so this is true if and only if $X \to \mathrm{fib}(Y)$ is a q-equivalence. We now have a commutative triangle

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{fib}(Y) \\ & {\scriptstyle \hat{f}} \searrow \quad \nearrow & \\ & \mathrm{Map}_{BG}(EG, Y) & \end{array}$$

We have just argued that the horizontal map is a q-equivalence precisely when f is. Since the right-hand diagonal map is an h-equivalence by Lemma 8.3, the desired result follows from the 2-out-of-3 property. $\square$

It follows that for any connected nondegenerately based m-cofibrant space A, we have a chain of equivalences of homotopy categories

$$\mathrm{Ho}_q(\mathscr{K}/A) \simeq \mathrm{Ho}_q(\mathscr{K}/B\Omega A) \simeq \mathrm{Ho}_e\big((\Omega A)\mathscr{K}\big).$$

If A is not m-cofibrant, we can first replace it by a CW complex $\widetilde{A}$ and obtain a longer chain of equivalences.

Remark 8.6. There is also an *h-model structure* on $G\mathscr{K}$ in which the weak equivalences, fibrations, and cofibrations are the equivariant homotopy equivalences (where the homotopy inverse and homotopies must also be equivariant), equivariant Hurewicz fibrations, and equivariant Hurewicz cofibrations. Any h-equivalence is an e-equivalence and any h-fibration is a qe-fibration, so there is a mixed *me-model structure*, and the adjunction (8.2) can be shown to also be a Quillen equivalence between the me-model structure and the m-model structure.

Finally, by Proposition 4.15, we have a corresponding result in the sectioned and pointed cases.

Corollary 8.7. *Under Assumption* 8.1*, the pointed/sectioned version of* (8.2)*,*

$$B(*, G, -) : G\mathscr{K}_* \rightleftarrows \mathscr{K}_{BG} : \mathrm{Map}_{*,BG}(EG, -), \tag{8.8}$$

is also a Quillen equivalence.

9. Base change and *G*-spaces

We now compare the base change functors for parametrized spaces with those for G-spaces. If $f : G \to H$ is a map of topological monoids, it induces a restriction functor $f^* : H\mathscr{K} \to G\mathscr{K}$, which has both adjoints $f_!$ and f_* given by left and right Kan extension. It is easy to see that the adjunction $f_! \dashv f^*$ is Quillen for the qe-model structures, since f^* preserves fibrations and weak equivalences. We denote the resulting derived adjunction by $\mathbf{L}_e f_! \dashv \mathbf{R}_e f^*$.

The map f also induces a map $Bf : BG \to BH$ and thus the usual string of adjunctions $(Bf)_! \dashv (Bf)^* \dashv (Bf)_*$ between $\mathscr{K}/BG$ and $\mathscr{K}/BH$. As always, the adjunction $(Bf)_! \dashv (Bf)^*$ is Quillen for the q- and m-model structures. We write

$$\mathbf{L}B(*, G, -) : \mathrm{Ho}_e(G\mathscr{K}) \rightleftarrows \mathrm{Ho}_q(\mathscr{K}/BG) : \mathbf{R}\mathrm{Map}_{BG}(EG, -)$$

for the derived equivalence of the Quillen equivalence from Theorem 8.5.

Theorem 9.1. *If $f : G \to H$ is a map between topological monoids satisfying Assumption* 8.1*, then we have a natural isomorphism*

$$\mathbf{R}\mathrm{Map}_{BG}(EG, -) \circ \mathbf{R}_q(Bf)^* \cong \mathbf{R}_e f^* \circ \mathbf{R}\mathrm{Map}_{BH}(EH, -). \tag{9.2}$$

Proof. By the composability of Quillen adjunctions, we have isomorphisms

$$\mathbf{R}\mathrm{Map}_{BG}(EG, -) \circ \mathbf{R}_q(Bf)^* \cong \mathbf{R}\mathrm{Map}_{BG}\big(EG, (Bf)^*-\big),$$
$$\mathbf{R}_e f^* \circ \mathbf{R}\mathrm{Map}_{BH}(EH, -) \cong \mathbf{R}\big(f^*\mathrm{Map}_{BH}(EH, -)\big).$$

Now, for any space Y over BH, there is a canonical morphism

$$f^*\mathrm{Map}_{BH}(EH, Y) \to \mathrm{Map}_{BG}\big(EG, (Bf)^*Y\big) \tag{9.3}$$

induced by the map $Ef : EG \to EH$ over Bf. Moreover, the following triangle commutes.

$$\begin{array}{ccc} f^*\mathrm{Map}_{BH}(EH, Y) & \longrightarrow & \mathrm{Map}_{BG}(EG, (Bf)^*Y) \\ & \searrow \quad \swarrow & \\ & \mathrm{Map}_{BH}(*, Y) = \mathrm{Map}_{BG}(*, (Bf)^*Y) = \mathrm{fib}(Y) & \end{array}$$

By Lemma 8.3, the diagonal maps are q-equivalences when Y is h-fibrant, hence in this case (9.3) is also a q-equivalence. Thus it represents an isomorphism (9.2) of derived functors, as desired. □

Corollary 9.4. *We also have a natural isomorphism*

$$\mathbf{L}_q(Bf)_! \circ \mathbf{L}B(*, G, -) \cong \mathbf{L}B(*, H, -) \circ \mathbf{L}_e f_!. \tag{9.5}$$

This means that under the identification of $\mathrm{Ho}_e(G\mathscr{K})$ with $\mathrm{Ho}_q(\mathscr{K}/BG)$, the derived adjunctions of $f_! \dashv f^*$ and $(Bf)_! \dashv (Bf)^*$ agree. It is easy to check that these results remain true in the pointed/sectioned case.

It follows from general results about diagram categories in [25, §22] that the adjunction

$$f^* : H\mathscr{K} \rightleftarrows G\mathscr{K} : f_*,$$

while not in general a Quillen adjunction, does have a derived adjunction. The right derived functor $\mathbf{R}_e f_*$ can be computed explicitly as a cobar construction:

$$\mathbf{R}_e f_*(X) = C(H, G, X).$$

Moreover, since f^* preserves all e-equivalences, its left and right derived functors agree, so we obtain a chain of derived adjunctions

$$\mathbf{L}_e f_! \dashv \mathbf{R}_e f^* \cong \mathbf{L}_e f^* \dashv \mathbf{R}_e f_*.$$

In particular, $\mathbf{R}_e f^*$ has a right adjoint $\mathbf{R}_e f_*$. Since $\mathbf{R}_e f^*$ is isomorphic to $\mathbf{R}_q(Bf)^*$, it follows that the latter also has a totally defined right adjoint, without the need to appeal to Brown representability. The same is true in the pointed/sectioned case.

We can use this, in theory, to compute $\mathbf{R}_q g_*$ for an arbitrary map $g : A \to D$ between connected base spaces, by passing along the chain of Quillen equivalences and computing $\mathbf{R}_e(\Omega g)_*$. This procedure may be too complicated to be useful in practice, however.

Remark 9.6. Of course, the restriction to connected base spaces is innocuous in the case considered here: since $\mathscr{K}/(A \sqcup B) \simeq \mathscr{K}/A \times \mathscr{K}/B$, we can deal with non-connected base spaces by splitting them up into their path components. However, in an equivariant context, this restriction becomes more problematic because 'connectedness' is a subtler notion. This does not necessarily mean that our intuition that spaces over B are equivalent to $\Pi_\infty(B)$-spaces is wrong equivariantly, just that our naive approach using loop spaces fails.

The correct equivariant notion of 'homotopy sheaf' is likewise somewhat unclear. If G is a topological group and B is a G-space, an equivariant ij-model structure on $G\mathscr{K}/B$ is constructed in [13]. The weak equivalences (resp. fibrations) are the maps inducing weak equivalences (resp. fibrations) on spaces of H-equivariant sections over U, whenever $H \leqslant G$ is a closed subgroup and $U \subset B$ is an H-invariant open set. If we let $\mathscr{O}_G(B)$ denote the full topological subcategory of $G\mathscr{K}/B$ spanned by the objects $G \times_H U$ for such pairs (H, U), then these weak equivalences and fibrations are created by the functor

$$\begin{aligned} G\mathscr{K}/B &\to \mathscr{K}^{\mathscr{O}_G(B)^{op}} \\ X &\mapsto \mathrm{Map}_B(-, X), \end{aligned} \tag{9.7}$$

where $\mathscr{K}^{\mathscr{O}_G(B)^{op}}$ is the category of topological presheaves on the topologically enriched category $\mathscr{O}_G(B)$. Thus, (9.7) is right Quillen from the ij-structure on $G\mathscr{K}/B$ to the projective model structure on $\mathscr{K}^{\mathscr{O}_G(B)^{op}}$. We may hope to localize the projective model structure to make this adjunction into a Quillen equivalence, but the correct covers to use are not obvious.

The theory of parametrized spaces works just as well equivariantly, as is evident in [22], but it is also unclear whether it embeds in the theory of equivariant homotopy sheaves sketched above.

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