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FINITE CONVERGENCE OF NONSMOOTH EQUATION BASED METHODS FOR AFFINE VARIATIONAL INEQUALITIES

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Abstract—This note shows that several nonsmooth equation based methods proposed recently for affine variational inequalities converge finitely under some standard assumptions.

Many algorithms for variational inequality related problems can be regarded as Newton-type algorithms applied to the nonsmooth equation formulation of the corresponding problems (e.g., [1,2]). See [3] for a brief survey and related literatures. Recently, following the pioneer work by Pang [4], a family of new algorithms, also based on the nonsmooth equation approach, has been proposed to solve the same class of problems [3,5,6]. Unlike traditional Newton-type methods for these problems (see [1,7]), the new algorithms not only converge fast locally [8], but also converge globally with the help of line search procedure [4,9]. In addition, they solve subproblems of lower dimension at each iteration and usually take very few iterations to converge. Computational studies [3,6,9,10] indicate that the new algorithms are very efficient. This note shows that under some standard assumptions these new algorithms converge finitely for affine variational inequality problems (VIP) which include linear complementarity problems (LCP) and quadratic programs (QP).

Let $\mathbf{f} : R^n \to R$ be a given affine function and S be a polyhedral set. The affine variational inequality is to find a vector $\mathbf{x} \in S$ such that for all $\mathbf{y} \in S$

$$(\mathbf{y} - \mathbf{x})^T \mathbf{f}(\mathbf{x}) \geq 0. \tag{1}$$

Suppose that S is defined by a system of equalities and inequalities:

$$S = \{\mathbf{x} \in R^n : \mathbf{g}(\mathbf{x}) \leq \mathbf{0}, \mathbf{h}(\mathbf{x}) = \mathbf{0}\},$$

where $\mathbf{g} : R^n \to R^m$ and $\mathbf{h} : R^n \to R^l$ are both affine functions. It is well known that the VIP (1) can be formulated as a system of nonsmooth equations. Two such equations will be considered in this note, both of which have been investigated in literature [5,6].

The first formulation, called Formulation 1, uses a min operator. Let $\mathbf{H}_1 : R^{n+m+l} \to R^{n+m+l}$ be a function defined by

$$\mathbf{H}_1(\mathbf{z}) = \begin{pmatrix} \mathbf{f}(\mathbf{x}) + \nabla \mathbf{g}^T \mathbf{u} + \nabla \mathbf{h}^T \mathbf{v} \\ \min\{-\mathbf{g}(\mathbf{x}), \mathbf{u}\} \\ -\mathbf{h}(\mathbf{x}) \end{pmatrix}, \tag{2}$$

where $\mathbf{z} = (\mathbf{x}, \mathbf{u}, \mathbf{v}) \in R^{n+m+l}$ and the min operator is taken component-wise. Then $\mathbf{x}$ solves the affine VIP if and only if it satisfies the equation $\mathbf{H}_1(\mathbf{z}) = \mathbf{0}$ for some $\mathbf{u} \geq \mathbf{0}$ and $\mathbf{v}$.

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The second formulation, called Formulation 2, uses the Minty-map. Denote

$$u_i^+ = \max\{u_i, 0\}, \quad u_i^- = \min\{u_i, 0\}, \qquad i = 1, \dots, m;$$

$$\mathbf{u}^+ = (u_1^+, \dots, u_m^+)^T, \quad \mathbf{u}^- = (u_1^-, \dots, u_m^-)^T.$$

Let $\mathbf{H}_2 : R^{n+m+l} \to R^{n+m+l}$ be a function defined by

$$\mathbf{H}_2(\mathbf{z}) = \begin{pmatrix} \mathbf{f}(\mathbf{x}) + \nabla \mathbf{g}^T \mathbf{u}^+ + \nabla \mathbf{h}^T \mathbf{v} \\ -\mathbf{g}(\mathbf{x}) + \mathbf{u}^- \\ -\mathbf{h}(\mathbf{x}) \end{pmatrix}. \tag{3}$$

Then $\mathbf{x}$ solves the affine VIP if and only if it solves the equation $\mathbf{H}_2(\mathbf{z}) = \mathbf{0}$ for some $\mathbf{u}$ and $\mathbf{v}$.

All the nonsmooth-equation based algorithms proposed so far differ either in the (nonsmooth) equation used or in the subproblems, which rely on certain index sets defined below. At a given vector $\mathbf{z} = (\mathbf{x}, \mathbf{u}, \mathbf{v})$, the index sets of Formulation 1 (for refined definition, see [5] are defined by:

$$\begin{aligned} \alpha(\mathbf{z}) &= \{i : -\mathbf{g}(\mathbf{x}) < u_i\}, \\ \beta(\mathbf{z}) &= \{i : -\mathbf{g}(\mathbf{x}) = u_i\}, \\ \gamma(\mathbf{z}) &= \{i : -\mathbf{g}(\mathbf{x}) > u_i\}, \end{aligned} \tag{4}$$

while those of Formulation 2 are defined by

$$\begin{aligned} \alpha(\mathbf{z}) &= \{i : u_i > 0\}, \\ \beta(\mathbf{z}) &= \{i : u_i = 0\}, \\ \gamma(\mathbf{z}) &= \{i : u_i < 0\}. \end{aligned} \tag{5}$$

For each iterate $\mathbf{z}$, a corresponding subproblem is constructed based on the above index sets. It could be a system of linear equations (see, for example, [8], a mixed linear complementarity problems [5,6], a quadratic program [3], or some combination of these problems [8]. We refer to the references for details of these constructions. The following general algorithm is designed to simplify the analysis in this note.

ALGORITHM 1.

Step 0. *(Initialization) Starting with some vector* $\mathbf{z}^0$ *(which may have to satisfy certain conditions).*

Step 1. *(Subproblem) Given* $\mathbf{z}^k$*, the subproblem generates* $\bar{\mathbf{z}}^k = \mathbf{G}(\mathbf{z}^k)$*, where* $\mathbf{G}(\cdot)$ *is a subproblem operator.*

Step 2. *(Line Search)* $\mathbf{z}^{k+1} = \mathbf{z}^k + \lambda^k(\bar{\mathbf{z}}^k - \mathbf{z}^k)$*, where* $0 < \lambda^k \le 1$ *is determined by a line search rule (assuming unit step size is adapted whenever possible).*

Step 3. *(Convergency Test) If* $\mathbf{z}^k$ *satisfies a prescribed tolerance, stop. Otherwise* $k = k + 1$*, go to Step 1.*

In Step 1, the subproblem operator $\mathbf{G}$ may differ from one algorithm to another algorithm. To simplify the notation, denote

$$s^* = s(\mathbf{z}^*), \quad s^k = s(\mathbf{z}^k), \quad s = \alpha, \beta, \gamma.$$

where $\mathbf{z}^*$ is a solution of the VIP. We now present the finite convergence result for Algorithm 1 assuming the operator $\mathbf{G}$ satisfies the following condition.

CONDITION 1. $\mathbf{z}^* = \mathbf{G}(\mathbf{z}^k)$ *provided that*

$$\alpha^* \subseteq \alpha^k \text{ and } \gamma^* \subseteq \gamma^k. \tag{6}$$

THEOREM 1. *Suppose the sequence* $\{\mathbf{z}^k\}$ *generated by Algorithm 1 converges to a solution* $\mathbf{z}^*$ *and the operator* $\mathbf{G}$ *satisfies Condition 1. Then there exists an integer* $K > 0$ *such that* $\mathbf{z}^K = \mathbf{z}^*$.

PROOF. Since $\{\mathbf{z}^k\}$ converges to $\mathbf{z}^*$ and the index sets α and β are defined by strict inequalities, there exists an integer $K > 0$ such that relation (6) holds for all $k \geq K-1$. Then $\bar{\mathbf{z}}^{K-1} = \mathbf{z}^*$ since the operator $\mathbf{G}$ satisfies Condition 1. By the assumption of the line search rule, $\lambda^{K-1} = 1$ and $\mathbf{z}^K = \mathbf{z}^*$. ∎

In the rest of the paper, we present several subproblems: some of them have been used in algorithms studied before and others are potentially useful for future algorithms. We then verify that the operator $\mathbf{G}$ defined by these subproblems satisfies Condition 1 and consequently establish the finite convergence for these algorithms. The following concept of b-regularity is important for our proof.

DEFINITION 1. *A vector $\mathbf{z}$ (with index sets α, β and γ given by either (4) or (5)) is said to be b-regular for the functions $\mathbf{H}_i$, $i = 1, 2$, if for every index set $\eta \subseteq \beta$ the following matrix is nonsingular:*

$$\begin{bmatrix} \nabla \mathbf{f} & \nabla \mathbf{g}_{\alpha\cup\eta}^T & \nabla \mathbf{h}^T \\ -\nabla \mathbf{g}_{\alpha\cup\eta} & \mathbf{0} & \mathbf{0} \\ -\nabla \mathbf{h} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{7}$$

Two functions are defined in order to describe the subproblems. Let η_1, η_2, η_3 be mutually exclusive index sets such that $\eta_1 \cup \eta_2 \cup \eta_3 = \{1, \ldots, m\}$. For Formulation 1, define a function $\tilde{\mathbf{H}}_1 : R^{n+m+l} \to R^{n+m+l}$ by

$$\tilde{\mathbf{H}}_1(\mathbf{z}, \eta_1, \eta_2, \eta_3) = \begin{pmatrix} \mathbf{f}(\mathbf{x}) + \nabla \mathbf{g}_{\eta_1}^T \mathbf{u}_{\eta_1} + \nabla \mathbf{g}_{\eta_2}^T \mathbf{u}_{\eta_2} + \nabla \mathbf{h}^T \mathbf{v} \\ -\mathbf{g}_{\eta_1}(\mathbf{x}) \\ \min\{-\mathbf{g}_{\eta_2}(\mathbf{x}), \mathbf{u}_{\eta_2}\} \\ \mathbf{u}_{\eta_3} \\ -\mathbf{h}(\mathbf{x}) \end{pmatrix}.$$

Similarly, let η_1, η_2, η_3 be mutually exclusive index sets such that $\eta_1 \cup \eta_2 \cup \eta_3 = \{1, \ldots, m\}$. For Formulation 2, define a function $\tilde{\mathbf{H}}_2 : R^{n+m+l} \to R^{n+m+l}$ by

$$\tilde{\mathbf{H}}_2(\mathbf{z}, \eta_1, \eta_2, \eta_3) = \begin{pmatrix} \mathbf{f}(\mathbf{x}) + \nabla \mathbf{g}_{\eta_1}^T \mathbf{u}_{\eta_1} + \nabla \mathbf{g}_{\eta_2}^T \max\{0, \mathbf{u}_{\eta_2}\} + \nabla \mathbf{h}^T \mathbf{v} \\ -\mathbf{g}_{\eta_1}(\mathbf{x}) \\ -\mathbf{g}_{\eta_2}(\mathbf{x}) + \min\{0, \mathbf{u}_{\eta_2}\} \\ -\mathbf{g}_{\eta_3}(\mathbf{x}) + \mathbf{u}_{\eta_3} \\ -\mathbf{h}(\mathbf{x}) \end{pmatrix}.$$

Notice that the functional form of $\tilde{\mathbf{H}}_i$, $i = 1, 2$, depends on the index sets.

We now describe the subproblems. The iteration index k of $\mathbf{z}, \alpha, \beta, \gamma$ is omitted for simplicity.

SUBPROBLEM 1. $\mathbf{G}(\mathbf{z}) = \{\tilde{\mathbf{z}} : \tilde{\mathbf{H}}_1(\tilde{\mathbf{z}}, \alpha \cup \beta_1, \beta_2, \beta_3 \cup \gamma) = \mathbf{0}\}$, *where $\beta_1, \beta_2, \beta_3$ are mutually exclusive subsets of β and $\beta_1 \cup \beta_2 \cup \beta_3 = \beta$.*

Subproblem 1 is designed for Formulation 1. It summarizes those algorithms that solve linear equations or a mixed linear complementarity problem at each iteration based on Formulation 1. If $\beta_1 = \beta_3 = \emptyset$, it reduces to a mixed linear complementarity problem, which is the subproblem described in [5]. If $\beta_2 = \emptyset$, it becomes a system of linear equations described in [8], it is also similar to the heuristic procedure described in [10] for solving linear complementarity problems.

SUBPROBLEM 2. $\mathbf{G}(\mathbf{z}) = \{\tilde{\mathbf{z}} : \tilde{\mathbf{H}}_2(\tilde{\mathbf{z}}, \alpha \cup \beta_1, \beta_2, \beta_3 \cup \gamma) = \mathbf{0}\}$, *where $\beta_1, \beta_2, \beta_3$ are mutually exclusive subsets of β and $\beta_1 \cup \beta_2 \cup \beta_3 = \beta$.*

Subproblem 2 parallels Subproblem 1 and is designed for Formulation 2. If $\beta_1 = \beta_3 = \emptyset$, it reduces to a mixed linear complementarity problem, which is the subproblem described in [6].

SUBPROBLEM 3. $\mathbf{G}(\mathbf{z}) = \arg\min\{\|\tilde{\mathbf{H}}_1(\tilde{\mathbf{z}}, \alpha \cup \beta_1, \beta_2, \beta_3 \cup \gamma)\|^2 \ (s.t.\ \mathbf{u}_{\alpha\cup\beta_1\cup\beta_3\cup\gamma} \geq \mathbf{0}, \mathbf{u}_{\beta_2} = \mathbf{0})\}$, *where $\beta_1, \beta_2, \beta_3$ are mutually exclusive subsets of β and $\beta_1 \cup \beta_2 \cup \beta_3 = \beta$.*

Subproblem 3 applies to Formulation 1 only and the constraints are optional. With the constraints, it is a generalized version of the NE/SQP algorithm for nonlinear complementarity problems described in [3]. Without the constraints, if $\beta_2 = \emptyset$, one can solve subproblem (6.3) of [8] by solving Subproblem 3 for all subsets β_1, β_3 of β.

SUBPROBLEM 4. $\mathbf{G}(\mathbf{z}) = \arg\min \{\|\tilde{\mathbf{H}}_2(\mathbf{z}, \alpha \cup \beta_1, \emptyset, \beta_2 \cup \gamma)\|^2 \quad (s.t.\ \mathbf{u}_{\beta_1} \geq \mathbf{0}, \mathbf{u}_{\beta_2} \leq \mathbf{0})\}$, *where* β_1, β_2 *are mutually exclusive subsets of* β *and* $\beta_1 \cup \beta_2 = \beta$.

Subproblem 4 applies to Formulation 2 only and the constraints are optional. It has not appeared in literature and could be used as a supplement of Subproblem 2 in case the latter fails to generate a new iterate that satisfies the line search rule.

The following result shows that the operator $\mathbf{G}$ defined by the above four subproblems satisfies Condition 1 under the b-regularity assumption.

PROPOSITION 1. *If* $\mathbf{z}^*$ *is a b-regular vector, the operator* $\mathbf{G}$ *defined by Subproblem* i, $i = 1, 2, 3, 4$ *satisfies Condition 1.*

PROOF. The proofs for the first two subproblems are similar. It suffices to show that if (6) holds, $\mathbf{z}^*$ is the unique solution of equations

$$\tilde{\mathbf{H}}_1(\mathbf{z}, \alpha \cup \beta_1, \beta_2, \beta_3 \cup \gamma) = \mathbf{0}, \tag{8}$$

$$\tilde{\mathbf{H}}_2(\mathbf{z}, \alpha \cup \beta_1, \beta_2, \beta_3 \cup \gamma) = \mathbf{0}. \tag{9}$$

By direct verification, one can see that $\mathbf{z}^*$ is a solution of equations (8) since relation (6) is true and by definition $\mathbf{u}^*_{\beta^*} = -\mathbf{g}_{\beta^*}(\mathbf{x}^*) = \mathbf{0}$. For Subproblem 1, suppose $\bar{\mathbf{z}}$ is another solution of equation (8). Without loss of generality, assume that

$$\bar{\mathbf{u}}_{\beta_4} > \mathbf{0}, \quad \bar{\mathbf{u}}_{\beta_5} = \mathbf{0}, \qquad \text{where } \beta_4 \cap \beta_5 = \emptyset, \quad \beta_4 \cup \beta_5 = \beta_2,$$

then $\mathbf{u}^*_{\beta_5} = \mathbf{0} = \bar{\mathbf{u}}_{\beta_5}$ since $\beta \subseteq \beta^*$. Substitute $\mathbf{z}^*$ and $\bar{\mathbf{z}}$ into equation (8) and subtract the resulting identities from each other. Taking into consideration the fact that $\mathbf{t}(\mathbf{x}) = \nabla \mathbf{t}\mathbf{x} + \mathbf{t}(\mathbf{0})$ for any affine function $\mathbf{t}$, we have

$$\begin{pmatrix} \nabla \mathbf{f} & \nabla \mathbf{g}^T_{\alpha\cup\beta_1\cup\beta_4} & \nabla \mathbf{h}^T \\ -\nabla \mathbf{g}_{\alpha\cup\beta_1\cup\beta_4} & \mathbf{0} & \mathbf{0} \\ -\nabla \mathbf{h} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \bar{\mathbf{x}} - \mathbf{x}^* \\ \bar{\mathbf{u}}_{\alpha\cup\beta_1\cup\beta_4} - \mathbf{u}^*_{\alpha\cup\beta_1\cup\beta_4} \\ \bar{\mathbf{v}} - \mathbf{v}^* \end{pmatrix} = \mathbf{0}.$$

Since $\alpha^* \subseteq \alpha \cup \beta_1 \cup \beta_4 \subseteq \alpha^* \cup \beta^*$ by relation (6), the matrix on the left hand side is nonsingular by the assumption of b-regularity. Therefore,

$$\bar{\mathbf{x}} = \mathbf{x}^*, \quad \bar{\mathbf{u}}_{\alpha\cup\beta_1\cup\beta_4} = \mathbf{u}^*_{\alpha\cup\beta_1\cup\beta_4}, \quad \bar{\mathbf{v}} = \mathbf{v}^*.$$

Consequently,

$$\bar{\mathbf{u}}_{\beta_3\cup\gamma} = \mathbf{g}_{\beta_3\cup\gamma}(\bar{\mathbf{x}}) = \mathbf{g}_{\beta_3\cup\gamma}(\mathbf{x}^*) = \mathbf{u}^*_{\beta_3\cup\gamma},$$

since $\beta_3 \cup \gamma \subseteq \beta^* \cup \gamma^*$. Summarizing the above argument, we obtain $\bar{\mathbf{z}} = \mathbf{z}^*$ for Subproblem 1. For Subproblem 2, suppose $\bar{\mathbf{z}}$ is another solution of equation (9), then $\bar{\mathbf{u}}_{\beta_2} = \mathbf{0} = \mathbf{u}^*_{\beta_2}$ by definition of the index set β, the definition of function $\tilde{\mathbf{H}}_2$ in (9), and the fact $\beta \subseteq \beta^*$. The rest of the proof is identical to that of Subproblem 1.

The proofs for the last two subproblems are essentially the same. Similar to the proof for Subproblem 1,2, one can verify that $\mathbf{z}^*$ is a solution of the quadratic programs in Subproblem 3,4 and the corresponding objective function values are zeros. Suppose $\bar{\mathbf{z}}$ is another solution, we must have

$$\tilde{\mathbf{H}}_1(\mathbf{z}^*, \alpha \cup \beta_1, \beta_2, \beta_3 \cup \gamma) = \tilde{\mathbf{H}}_1(\mathbf{z}, \alpha \cup \beta_1, \beta_2, \beta_3 \cup \gamma) = \mathbf{0},$$

$$\tilde{\mathbf{H}}_2(\mathbf{z}^*, \alpha \cup \beta_1, \emptyset, \beta_2 \cup \gamma) = \tilde{\mathbf{H}}_2(\mathbf{z}, \alpha \cup \beta_1, \emptyset, \beta_2 \cup \gamma) = \mathbf{0}.$$

Using the result for Subproblem 1,2, we have $\bar{\mathbf{z}} = \mathbf{z}^*$. ∎

Notice that if $\beta^* = \emptyset$ and the matrix (7) associated with index set α^* is nonsingular, then $\mathbf{z}^*$ is b-regular and the above finite convergence result holds.

The above finite convergence result can be extended easily to strictly convex QPs and LCPs. If $\mathbf{f}(\mathbf{x})$ is a gradient of some strictly convex quadratic function, the affine VIP reduces to a strictly convex QP and *b*-regularity becomes the familiar assumption that the binding constraints are linearly independent. Consequently, we have the following results.

COROLLARY 1. *Let* $\{\mathbf{z}^k\}$ *be a sequence generated by Algorithm 1 with its subproblem defined by Subproblem* i, $i = 1, 2, 3, 4$. *Suppose that* $\{\mathbf{z}^k\}$ *converges to the solution* $\mathbf{z}^*$ *of a strictly convex QP and that the rows of* $\nabla \mathbf{g}_{\alpha^* \cup \beta^*}, \nabla \mathbf{h}$ *are linearly independent. Then there exists an integer* K *such that* $\mathbf{z}^K = \mathbf{z}^*$.

When set S is restricted to a nonpositive orthant, the affine VIP reduces to an LCP: finding a vector $\mathbf{x}$ such that

$$\mathbf{f}(\mathbf{x}) \geq \mathbf{0}, \quad \mathbf{x} \geq \mathbf{0}, \quad \mathbf{x}^T \mathbf{f}(\mathbf{x}) = 0.$$

The corresponding nonsmooth equation formulations become

$$\mathbf{H}_1(\mathbf{x}) = \min \{\mathbf{f}(\mathbf{x}), \mathbf{x}\} = \mathbf{0} \quad \text{and} \quad \mathbf{H}_2(\mathbf{x}) = \mathbf{f}(\mathbf{x}^+) + \mathbf{x}^- = \mathbf{0},$$

respectively. The index sets are defined similar to (4) and (5) except that $\mathbf{u}$ is replaced by $\mathbf{x}$. The concept of *b*-regularity is simplified accordingly:

DEFINITION 2. *A vector* $\mathbf{x}$ *is said to be b-regular for function* $\mathbf{H}_i$, $i = 1, 2$, *if for every index set* $\alpha \subseteq \eta \subseteq \alpha \cup \beta$, *the matrix* $\nabla_\eta f_\eta$ *is nonsingular.*

By definition, any vector $\mathbf{x}$ associated with the LCP defined by a nonsingular matrix or a P-matrix is *b*-regular but the reverse is not necessarily true. The functions $\tilde{\mathbf{H}}_i : R^n \to R^n$, $i = 1, 2$, used in subproblems are simplified as follows:

$$\tilde{\mathbf{H}}_1(\mathbf{x}, \eta_1, \eta_2, \eta_3) = \begin{pmatrix} \mathbf{f}_{\eta_1}(\mathbf{x}) \\ \min \{\mathbf{f}_{\eta_2}(\mathbf{x}), \mathbf{x}_{\eta_2}\} \\ \mathbf{x}_{\eta_3} \end{pmatrix}, \quad \tilde{\mathbf{H}}_2(\mathbf{x}, \eta_1, \eta_2, \eta_3) = \begin{pmatrix} \mathbf{f}_{\eta_1}(\tilde{\mathbf{x}}) \\ \mathbf{f}_{\eta_2}(\tilde{\mathbf{x}}) + \min \{\mathbf{0}, \mathbf{x}_{\eta_2}\} \\ \mathbf{f}_{\eta_3}(\tilde{\mathbf{x}}) + \mathbf{x}_{\eta_3} \end{pmatrix},$$

where $\tilde{\mathbf{x}} = (\mathbf{x}_{\eta_1}, \max \{\mathbf{0}, \mathbf{x}_{\eta_2}\}, \mathbf{0}_{\eta_3}) \in R^n$. The following result can be shown in the same manner as for the affine VIP and the detailed proof is omitted.

COROLLARY 2. *Let* $\{\mathbf{x}^k\}$ *be a sequence generated by Algorithm 1 with its subproblem defined by Subproblem* i, $i = 1, 2, 3, 4$. *Suppose that* $\{\mathbf{x}^k\}$ *converges to a solution* $\mathbf{x}^*$ *of the LCP and that* $\mathbf{x}^*$ *is b-regular. Then there exists an integer* K *such that* $\mathbf{x}^K = \mathbf{x}^*$.

Obviously, if strict complementarity holds at the solution ($\beta^* = \emptyset$) and the matrix $\nabla_{\alpha^*} \mathbf{f}_{\alpha^*}$ is nonsingular, $\mathbf{x}^*$ is *b*-regular and the finite convergence result holds.

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