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Nonharmonic Fourier Series in the Control Theory of Distributed Parameter Systems*

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INTRODUCTION

This paper is part of a series dealing with the control theory of hyperbolic partial differential equations (see [1], [2]). In this series we have restricted ourselves to the study of the control of such processes with control parameters which are finite dimensional vector functions of time only. In actual applications the controls used to stabilize a vibrating body will of necessity be of this type. Moreover, it turns out that such controls are of considerable mathematical interest.

This paper is concerned with the *controllability* of such systems in the case where control is exercised by means of a force whose spatial distribution is fixed but whose sign and amplitude is variable with time. To avoid undue complexity the greater part of the paper treats the case of the nonuniform string. Nevertheless the methods used are also applicable to many other distributed parameter control problems as we shall show in the concluding section.

The work presented here was originally motivated by a very practical question which arises in control engineering. How can one control a finite number of modes of vibration in a distributed parameter system without adding undue amounts of energy to the neglected modes? In addition to results of more or less purely mathematical interest, we will provide at least a partial answer to this question.

As our method reduces essentially to the study of a moment problem, it would seem appropriate to note the differences between our work and that of A. G. Butkovskii [3]. Butkovskii makes an assumption, whose verification is not at all trivial, which in effect amounts to the assumption of controllability in finite time with a bounded control function and then proceeds to character-

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ize the time optimal control. Our goal is quite different, we pose an initial value problem and ask whether there is any control, bounded or not, which will reduce these initial conditions to zero in a given finite time interval. Thus our work is, logically, a necessary preparation for the study of Butkovskii's problem.

A second paper of Butkovskii [4] treats the case where $\rho(x)$ and $p(x)$ (see Section 1) are constant. This case is substantially less complicated than the problem which we shall consider in this paper.

1. Statement of the Control Problem and Principal Results

We consider the equation

$$\rho(x)\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) = g(x)f(t) \tag{1.1}$$

which describes forced motion of a string with density $\rho(x)$ and modulus of elasticity $p(x)$. The function $g(x)$ is an element of the space $L_2[0, 1]$, the interval $[0, 1]$ being the spatial extent of the string, and will be called the force distribution function. The functions $\rho(x)$ and $p(x)$ are positive and twice continuously differentiable on $[0, 1]$. A function $f(t)$ is an admissible control on the interval $[0, T]$ if it is an element of $L_2[0, T]$.

We will assume that $u(x, t)$ obeys the boundary conditions

$$A_0 u(0, t) + B_0 \frac{\partial u}{\partial x}(0, t) \equiv 0,$$

$$A_1 u(1, t) + B_1 \frac{\partial u}{\partial x}(1, t) \equiv 0, \tag{1.2}$$

where A_0, B_0, A_1, and B_1 are real constants with $A_0^2 + B_0^2 \neq 0$, $A_1^2 + B_1^2 \neq 0$.

The set of initial conditions which we will consider is the linear space I consisting of pairs of functions

$$u(x, 0) \equiv u_0(x), \qquad \frac{\partial u}{\partial t}(x, 0) \equiv v_0(x) \tag{1.3}$$

such that (1.2) is satisfied at time $t = 0$ while $d^2u_0(x)/dx^2$ and $dv_0(x)/dx$ are functions in $L_2[0, 1]$.

Problem A. *Let initial conditions (1.3) be given in the space I and let*

$T > 0$. *Does there exist an admissible control* $f(t)$ *on* $[0, T]$ *such that the solution of (1.1), (1.2) which satisfies these initial conditions also satisfies*

$$u(x, T) \equiv 0, \qquad \frac{\partial u}{\partial t}(x, T) \equiv 0, \qquad x \in [0, 1]\,? \tag{1.4}$$

If for fixed $T > 0$ the answer in Problem A is "yes" for *all* initial conditions in the space I, we will say that the system (1.1) (1.2) is *controllable in time* T.

It will be convenient to simplify the equation (1.1) by means of transformations of the variables. We set

$$u^* = \sqrt[4]{p(x)\,\rho(x)}\, u, \qquad x^* = \int_0^x \sqrt{\frac{\rho(\xi)}{p(\xi)}}\, d\xi, \tag{1.5}$$

and we obtain a new equation in u^* involving derivatives with respect to x^* and t. Reverting to the use of u and x rather than u^* and x^*, this equation is

$$\frac{\partial^2 u}{\partial t^2} - r(x)\, u - \frac{\partial^2 u}{\partial x^2} = \gamma(x) f(t), \tag{1.6}$$

where $r(x)$ is continuous on the interval

$$0 \leqslant x \leqslant \ell, \qquad \ell = \int_0^1 \sqrt{\frac{\rho(x)}{p(x)}}\, dx, \tag{1.7}$$

and $\gamma(x) \in L_2[0,]$. We have new boundary conditions

$$a_0 u(0, t) + b_0 \frac{\partial u}{\partial x}(0, t) \equiv a_1 u(\ell, t) + b_1 \frac{\partial u}{\partial x}(\ell, t) \equiv 0 \tag{1.8}$$

and again $a_0^2 = b_0^2 \neq 0$, $a_1^2 + b_1^2 \neq 0$. For (1.6) we pose the same control problem as we did for (1.1), replacing the boundary conditions (1.2) by (1.8) and $L_2[0, 1]$ by $L_2[0, \ell]$.

In the next section we will make a basic assumption on $\gamma(x)$. With this assumption our results will consist of three theorems whose proof occupies the greater part of the remainder of this paper.

THEOREM 1. *If* $T < 2\ell$ *the system (1.6) (equivalently the system (1.1)) is not controllable in time* T.

THEOREM 2. *If* $T > 2\ell$ *the system (1.6) (equivalently the system (1.1)) is controllable in time* T *and for each set of initial conditions in* I *the Problem* A *has infinitely many solutions* $f(t) \in L_2[0, T]$.

THEOREM 3. *If $T = 2\ell$, then*

(i) *when $b_0 = b_1 = 0$ the system (1.6) is controllable in time $T = 2\ell$ and for each set of initial conditions in I the solution set for Problem A is of the form $f(t) + E$, where $f(t)$ is a uniquely determined solution of Problem A and E is a fixed (for all initial conditions in I) one dimensional subspace of $L_2[0, 2\ell]$;*

(ii) *when exactly one of the numbers b_0, b_1 is different from zero, the system (1.6) is controllable in time $T = 2\ell$ and for each initial condition in I the Problem A has a unique solution;*

(iii) *when neither of the numbers b_0, b_1 is equal to zero, the system (1.6) is not controllable in time $T = 2\ell$ but becomes controllable if we replace I by a subspace $\hat{I} \subset I$ whose complement in I is one-dimensional.*

2. A MOMENT PROBLEM IN $L_2[0, T]$

Our approach to the above posed control problem uses the familiar method of separation of variables. We seek for solutions of the homogeneous equation corresponding to (1.6) of the forms $u(x, t) = \alpha(t)\phi(x)$. Then for some constant λ we must have

$$\frac{d^2\alpha(t)}{dt^2} + \lambda\alpha(t) = 0 \tag{2.1}$$

and

$$\frac{d^2\phi(x)}{dx^2} + r(x)\phi(x) + \lambda\phi(x) = 0. \tag{2.2}$$

We wish to find numbers λ_k such that for $\lambda = \lambda_k$ the Eq. (2.2) possesses a nontrivial solution which satisfies the same boundary conditions as imposed on $u(x, t)$ in (1.8). This is the Sturm-Liouville eigenvalue problem for which a vast literature exists. Briefly, it is known in the present case that there exists a strictly increasing sequence $\{\lambda_k\}$ of non-negative real numbers, $k = 0, 1, 2,...$, such that for $\lambda = \lambda_k$ the boundary value problem possesses a unique solution $\phi_k(x)$ with

$$\int_0^\ell \phi_k(x)\phi_\ell(x)\,dx = \delta_{k\ell}, \tag{2.3}$$

where $\delta_{k\ell}$ is Kronecker's delta. The $\phi_k(x)$ form an orthonormal basis for $L_2[0, \ell]$, i.e., given $\psi(x) \in L_2[0, \ell]$ we have

$$\psi(x) = \sum_{k=0}^{\infty} \psi_k\phi_k(x) = \sum_{k=0}^{\infty} \left(\int_0^\ell \psi(x)\phi(x)\,dx\right)\phi(x), \tag{2.4}$$

the convergence being with respect to the $L_2[0, \ell]$ norm, and

$$\sum_{k=0}^{\infty} \psi_k^2 < \infty. \tag{2.5}$$

Now let $u(x, t)$ be a solution of (1.6) which satisfies a set of initial conditions $u(x, 0) \equiv u_0(x)$, $(\partial u/\partial t)(x, 0) \equiv v_0(x)$, $0 \leqslant x \leqslant \ell$, in I and the boundary conditions (1.8). We expand $u(x, t)$, $u_0(x)$, $v_0(x)$ and $\gamma(x)$ as series in the $\phi_k(x)$:

$$u(x, t) = \sum_{k=0}^{\infty} \beta_k(t)\,\phi_k(x),$$

$$u_0(x) = \sum_{k=0}^{\infty} \mu_k \phi_k(x),$$

$$v_0(x) = \sum_{k=0}^{\infty} \nu_k \phi_k(x),$$

$$\gamma(x) = \sum_{k=0}^{\infty} \gamma_k \phi_k(x). \tag{2.6}$$

Then the $\beta_k(t)$ are solutions of

$$\frac{d^2\beta_k(t)}{dt^2} + \lambda_k \beta_k(t) = \gamma_k f(t) \tag{2.7}$$

for $k = 0, 1, 2,...$ and satisfy the initial conditions

$$\beta_k(0) = \mu_k, \qquad \frac{d\beta_k}{dt}(0) = \nu_k. \tag{2.8}$$

Problem A may then be replaced by

PROBLEM B. *Determine* $f(t) \in L_2[0, T]$ *such that for* $k = 0, 1, 2,...$ *the solutions* $\beta_k(t)$ *of (2.7), (2.8) also satisfy*

$$\beta_k(T) = \frac{d\beta_k}{dt}(T) = 0. \tag{2.9}$$

We now examine the numbers μ_k and ν_k more closely. We have

$$\mu_k = \int_0^{\ell} u_0(x)\,\phi_k(x)\,dx, \qquad \nu_k = \int_0^{\ell} v_0(x)\,\phi_k(x)\,dx. \tag{2.10}$$

Since $\phi_k(x)$ is an eigenfunction of the operator $L(\phi) = (d^2\phi/dx^2) + r(x)\phi$ corresponding to the eigenvalue λ_k, we have, for $\lambda_k \neq 0$,

$$\phi_k(x) = -\frac{1}{\lambda_k}\left[\frac{d^2\phi_k(x)}{dx^2} + r(x)\phi_k(x)\right]. \tag{2.11}$$

Substituting (2.11) into the first equation of (2.10) and integrating twice by parts, making use of the boundary conditions, we arrive at

$$\mu_k = -\frac{1}{\lambda_k}\int_0^{\ell}\left(\frac{d^2u_0(x)}{dx^2} + r(x)\,u_0(x)\right)\phi_k(x)\,dx. \tag{2.12}$$

Then making use of well known estimates on λ_k (see, e.g., [5] or [6]) and the definition of I, we have

$$\mu_k = \frac{1}{k^2}\hat{\mu}_k, \tag{2.13}$$

where

$$\sum_{k=0} \hat{\mu}_k^2 < \infty. \tag{2.14}$$

On the other hand, if we substitute (2.11) into the second equation of (2.10), integrate by parts once, use the asymptotic formulas for $d\phi_k(x)/dx$ found in [6] and the definition of I we see that

$$\nu_k = \frac{1}{k}\hat{\nu}_k, \tag{2.15}$$

where

$$\sum_{k=0}^{\infty} \hat{\nu}_k^2 < \infty. \tag{2.16}$$

To complete our discussion of the coefficients arising out of the equations (2.6) we make the following basic assumption.

ASSUMPTION. *The coefficients γ_k satisfy*

$$\liminf_{k\to\infty} k\,|\gamma_k| > 0, \tag{2.17}$$

and

$$\gamma_k \neq 0, \qquad k = 0, 1, 2,\ldots. \tag{2.18}$$

This assumption is satisfied, for example by $\gamma(x) \equiv x$ when $b_0 = b_1 = 0$.

Now let us set

$$\omega_k = \lambda_k^{1/2}, \qquad k = 0, 1, 2,\ldots \tag{2.19}$$

and use the variations of parameters formula to see that the solutions $\beta_k(t)$ of (2.7), (2.8) satisfy, for $\omega_k \neq 0$,

$$\beta_k(t) = \mu_k \cos(\omega_k t) + \frac{\nu_k}{\omega_k} \sin(\omega_k t) + \int_0^t \frac{\gamma_k}{\omega_k} \sin(\omega_k(t-s)) f(s)\, ds, \quad (2.20)$$

$$\frac{d\beta_k(t)}{dt} = -\mu_k \omega_k \sin(\omega_k t) + \nu_k \cos(\omega_k t) + \int_0^t \gamma_k \cos(\omega_k(t-s)) f(s)\, ds. \quad (2.21)$$

If in (1.2) we have $A_0 = A_1 = 0$ there is an eigenvalue $\lambda_0 = 0$. In this case

$$\beta_0(t) = \mu_0 + \nu_0 t + \int_0^t \gamma_0 (t-s) f(s)\, ds, \quad (2.22)$$

$$\frac{d\beta_0(t)}{dt} = \nu_0 + \int_0^t \gamma_0 f(s)\, ds. \quad (2.23)$$

If at time $t = T > 0$ the solutions $\beta_k(t)$ of (2.7), (2.8) are to satisfy the terminal condition (1.4), it is clear that we must have

$$\int_0^T \sin(\omega_k(T-s)) f(s)\, ds = -\frac{\mu_k \omega_k}{\gamma_k} \cos(\omega_k T) - \frac{\nu_k}{\gamma_k} \sin(\omega_k T), \quad (2.24)$$

$$\int_0^T \cos(\omega_k(T-s)) f(s)\, ds = \frac{\mu_k \omega_k}{\gamma_k} \sin(\omega_k T) - \frac{\nu_k}{\gamma_k} \cos(\omega_k T) \quad (2.25)$$

for $\omega_k \neq 0$, and in the case $\omega_0 = 0$

$$\int_0^T (T-s) f(s)\, ds = -\frac{\mu_0}{\gamma_0} - \frac{\nu_0}{\gamma_0} T, \quad (2.26)$$

$$\int_0^T f(s)\, ds = -\frac{\nu_0}{\gamma_0}. \quad (2.27)$$

Let us set $\tau = T - s$ and for each control function $f(t)$ let a corresponding function $h(\tau)$ be defined by

$$h(\tau) \equiv f(T-\tau) \equiv f(s). \quad (2.28)$$

Then, using (2.13)-(2.16) together with the well-known estimate

$$\omega_k = \mathcal{O}(k), \quad (2.29)$$

we see that Problems A and B finally reduce to the following.

MOMENT PROBLEM. *Let sequences $\{c_k\}$ and $\{d_k\}$, of real numbers be given satisfying*

$$\sum_{k=0}^{\infty} c_k^2 < \infty, \qquad \sum_{k=0}^{\infty} d_k^2 < \infty. \tag{2.30}$$

Let $T > 0$ be given. Does there exist a real-valued function $h(t) \in L_2[0, T]$ such that, for $\omega_k \neq 0$

$$\int_0^T \sin(\omega_k t)\, h(t)\, dt = c_k, \tag{2.31}$$

$$\int_0^T \cos(\omega_k t)\, h(t)\, dt = d_k, \tag{2.32}$$

and in the case $\omega_0 = 0$

$$\int_0^T th(t)\, dt = c_0, \tag{2.33}$$

$$\int_0^T h(t)\, dt = d_0? \tag{2.34}$$

We remark that the solution of this problem is easy when $r(x) = 0$ and either $a_0 = a_1 = 0$ or $b_0 = b_1 = 0$, for in those cases the ω_k are all multiples of a fixed positive number and familiar results from the theory of Fourier series may be used. The essence of the present paper is the solution of the above moment problem when the ω_k do not obey any such simple relationship. In order to accomplish this objective we shall first find it necessary to summarize the results available in the theory of so-called nonharmonic Fourier series.

2. NONHARMONIC FOURIER SERIES

Our task in this section is to examine the functions in the set

$$S = \{e^{i\omega_k t}, e^{-i\omega_k t} \mid k = 0, 1, 2, \ldots\} \tag{3.1}$$

or, in the case $\omega_0 = 0$,

$$S = \{1, t, e^{i\omega_k t}, e^{-i\omega_k t} \mid k = 1, 2, \ldots\} \tag{3.2}$$

as elements of the space $L_2[0, T]$, $T > 0$. The set S will be called *complete* if the smallest closed subspace of $L_2[0, T]$ which contains S is $L_2[0, T]$ itself;

excessive if a proper subset of S is complete in $L_2[0, T]$; *linearly independent* if S possesses a *biorthogonal set* in $L_2[0, T]$, i.e., if there exist two sequences $\{p_k(t)\}$, $\{q_k(t)\}$ in $L_2[0, T]$ with

$$\int_0^T e^{i\omega_k t}\, \bar{p}_\ell(t)\, dt = \delta_{k\ell} = \int_0^T e^{-i\omega_k t}\, \bar{q}_\ell(t)\, dt, \tag{3.3}$$

$$\int_0^T e^{i\omega_k t}\, \bar{q}_\ell(t)\, dt = 0 = \int_0^T e^{-i\omega_k t}\, \bar{p}_\ell(t)\, dt, \qquad k, \ell = 0, 1, 2, \ldots \tag{3.4}$$

and similar equations in the case where S is given by (3.2); *a basis for* $L_2[0, T]$ if S is both complete and linear independent; *deficient* if S is contained in a proper closed subspace H of $L_2[0, T]$.

The sequence of non-negative real numbers $\{\omega_k\}$ has *density* D if

$$\lim_{k\to\infty} \frac{k}{\omega_k} = D. \tag{3.5}$$

The same sequence possesses an *asymptotic gap* Γ if

$$\liminf_{k\to\infty} (\omega_{k+1} - \omega_k) = \Gamma. \tag{3.6}$$

We will proceed under the assumption that the sequence $\{\omega_k\}$ possesses a positive density D and that $\Gamma = 1/D$. This assumption is always satisfied by the ω_k introduced in the preceding section.

We will indicate how the properties listed above for S depend upon the relationship which the density D bears to the length T of the fundamental interval under consideration. The first important question is whether T is less than, greater than or equal to $2\pi D$.

The Case $T < 2\pi D$

It is known in this case (see, e.g., Levinson [7], p. 3) that S is excessive in $L_2[0, T]$ when given by either (3.1) or (3.2). As a result, the moment problem (2.31)-(2.34) has, in general, no solution.

The Case $T > 2\pi D$

The treatment of this case is not as straightforward as that of $T < 2\pi D$. In his paper [8] of 1942, L. Schwartz conjectured that the set S, as given by (3.1), could not be complete in $L_2[0, T]$ if $T > 2\pi D$. To the author's knowledge, this conjecture remains unproven unless somewhat stronger hypotheses are given.

In 1950 R. M. Redheffer, [9], proved that if

$$\limsup_{y\to\infty} \limsup_{x\to\infty} \frac{\Lambda(x+y) - \Lambda(x)}{y} < \frac{T}{2\pi} \tag{3.7}$$

the set S, as given by (3.1) is not complete in $L_2[0, T]$. Here $\Lambda(x)$ denotes the number of ω_k which are $< x$. He shows that there is a function

$$G(\omega) = H(\omega) \prod_{k=0}^{\infty} \left(1 - \frac{\omega^2}{{\omega_k}^2}\right) \tag{3.8}$$

which belongs to $L_2(-\infty, \infty)$, $H(0) \neq 0$, and a function $g(t) \not\equiv 0$ on $[0, T]$ such that $g(t) \in L_2[0, T]$ and

$$G(\omega) = \int_0^T g(t)\, e^{i\omega t}\, dt. \tag{3.9}$$

It is clear then that $g(t)$ is a nonzero element of $L_2[0, T]$ orthogonal to all of the functions in S and hence S is not complete in $L_2[0, T]$. If $\omega_0 = 0$, one employs the function

$$G(\omega) = \omega H(\omega) \prod_{k=1}^{\infty} \left(1 - \frac{\omega^2}{{\omega_k}^2}\right) \tag{3.10}$$

instead of (3.8). As a result we see that if t is removed from S as given by (3.2) then S is not complete in $L_2[0, T]$.

It is clear that one may adjoint to S, as given by (3.1), or to S as given by (3.2) without t, any finite number of functions of the form $e^{i\omega t}$ not already in the set and still fail to obtain a set complete in $L_2[0, T]$, for no finite number of ω's can alter (3.7). As long as S does not span $L_2[0, T]$ its members are linearly independent. (Schwartz, [8]). It follows that S as given by (3.2), including t, is also not complete in $L_2[0, T]$ since the deficiency of that set without t is infinite.

For S, as given by (3.1), the property of incompleteness in $L_2[0, T]$ implies, according to Schwartz, [8], the existence of a biorthogonal set for S in $L_2[0, T]$. What can we say about S as given by (3.2)? Let us note that for $\omega_0 > 0$

$$\lim_{\omega_0 \to 0+} \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i\omega_0} = t \tag{3.11}$$

uniformly for $t \in [0, T]$. It follows that

$$\lim_{\omega_0 \to 0+} \int_0^T g(t) \frac{e^{i\omega_0 t} - e^{i\omega_0 t}}{2i\omega_0}\, dt = \int_0^T g(t)\, t\, dt. \tag{3.12}$$

But then, using (3.9) with $G(\omega)$ given by (3.10),

$$\int_0^T g(t)\, dt = \lim_{\omega_0 \to 0+} \frac{G(\omega_0) - G(-\omega_0)}{2i\omega_0} = -i \frac{dG(\omega)}{d\omega}\Big|_{\omega=0} = -iH(0) \neq 0. \tag{3.13}$$

Thus $g(t)$, regarded as a linear functional on $L_2[0, T]$, vanishes on all elements of S, as given by (3.2), except the element t. Thus this set S is linearly independent and possesses a biorthogonal set. Of course, such a biorthogonal set is not, in general, uniquely determined whether S is given by (3.1) or (3.2). A unique biorthogonal set can be specified by requiring that its elements have norm as small as possible in $L_2[0, T]$, i.e., by requiring that the elements of the biorthogonal set lie in the closed subspace of $L_2[0, T]$ spanned by S. It is in this sense that we will refer to "the" biorthogonal set for S.

We are now ready to study the moment problem in this context. We do so, assuming in addition to (3.7) that

$$\liminf_{k\to\infty} (\omega_{k+1} - \omega_k) > \frac{T}{2\pi}. \tag{3.14}$$

Let square summable sequences $\{\tilde{c}_k \mid k = 0, 1,\ldots\}$, $\{\tilde{d}_k \mid k = 0, 1,\ldots\}$ of complex numbers be given. We wish to find an element $\tilde{h}(t)$ in $L_2[0, T]$ such that

$$\int_0^T \tilde{h}(t)\, e^{i\omega_k t}\, dt = \tilde{c}_k, \qquad k = 0, 1, 2,\ldots, \tag{3.15}$$

$$\int_0^T \tilde{h}(t)\, e^{-i\omega_k t}\, dt = \tilde{d}_k, \qquad k = 0, 1, 2,\ldots, \tag{3.16}$$

if none of the ω_k are zero. If $\omega_0 = 0$,

$$\int_0^T \tilde{h}(t)\, t\, dt = \tilde{c}_0, \tag{3.17}$$

$$\int_0^T \tilde{h}(t)\, dt = \tilde{d}_0. \tag{3.18}$$

First of all, it is clear that we can obtain at least a *formal* solution of this moment problem by setting

$$\tilde{h}(t) = \sum_{k=0}^{\infty} (\tilde{c}_k p_k(t) + \tilde{d}_k q_k(t)), \tag{3.19}$$

where the $p_k(t)$ and $q_k(t)$ are the elements of the biorthogonal set for S. We say *formal* because we do not know as yet that this series is convergent.

Now it is known (see R. P. Boas [10], S. Banach [11]) that (3.15), (3.16) possesses a solution $\tilde{h}(t) \in L_2[0, T]$ satisfying

$$\int_0^T |\tilde{h}(t)|^2\, dt \leqslant A \tag{3.20}$$

if and only if for every finite set $\{\zeta_k, \eta_k \mid k = 0, 1,..., K\}$ of complex numbers we have

$$\left| \sum_{k=0}^{K} \tilde{c}_k\zeta_k + \tilde{d}_k\eta_k \right|^2 \leqslant A \int_0^T \left| \sum_{k=0}^{K} (\zeta_k e^{i\omega_k t} + \eta_k e^{-i\omega_k t}) \right|^2 dt. \tag{3.21}$$

A result due to A. E. Ingham, [12] shows that if (3.14) is satisfied, there is a positive number M such that

$$\sum_{k=0}^{K} (|\zeta_k|^2 + |\eta_k|^2) \leqslant M \int_0^T \left| \sum_{k=0}^{K} (\zeta_k e^{i\omega_k t} + \eta_k e^{-i\omega_k t}) \right|^2 dt. \tag{3.22}$$

On the other hand,

$$\left| \sum_{k=0}^{K} \tilde{c}_k\zeta_k + \tilde{d}_k\eta_k \right|^2 \leqslant \left(\sum_{k=0}^{K} (|\tilde{c}_k|^2 + |\tilde{d}_k|^2) \right) \left(\sum_{k=0}^{K} (|\zeta_k|^2 + |\eta_k|^2) \right) \tag{3.23}$$

Setting $A = \sum_{k=0}^{\infty} (|\tilde{c}_k|^2 + |\tilde{d}_k|^2)\, M$ we have (3.21) and it follows that the moment problem (3.15), (3.16) has a solution $\tilde{h}(t)$.

When $\omega_0 = 0$ we can use the above techniques to show that there is an element $\hat{h}(t) \in L_2[0, T]$ satisfying (3.15), (3.16) for $k \geqslant 1$ and (3.18). Then letting $g(t)$ be the function defined in (3.9) we set

$$\tilde{h}(t) = \hat{h}(t) + i \frac{g(t)}{H(0)} \left[\tilde{c}_0 - \int_0^T \hat{h}(t)\, t\, dt \right] \tag{3.22}$$

to obtain a solution of the complete moment problem.

Once we know that there is at least one solution of the moment problem it is easy to show that there is a unique solution $h^*(t)$ of least $L_2[0, T]$ norm. This is also the unique solution of the moment problem which lies in the closed subspace of $L_2[0, T]$ spanned by S. We will show that the series (3.19) converges and that it converges to the particular solution $h^*(t)$ of the moment problem which we have just described. Keeping in mind that the $p_k(t)$, $q_k(t)$ of (3.19) lie in the closed subspace spanned by S, we define $h_p(t)$ by

$$\tilde{h}_p(t) = \sum_{k=0}^{p} (\tilde{c}_k p_k(t) + \tilde{d}_k q_k(t)), \tag{3.23}$$

and $h_p^*(t)$ is taken to be the solution of least $L_2[0, T]$ norm for the moment problem (3.15), (3.16) with $\tilde{c}_k$, $\tilde{d}_k$ replaced by 0 for $k \leqslant p$. Then $\tilde{h}_p(t) + h^*(t)$ is a solution of the moment problem (3.15), (3.16) which lies in the closed subspace spanned by S. Hence we conclude that

$$\tilde{h}_p(t) + h_p^*(t) = h^*(t).$$

But

$$\| h_p^* \|^2 \leqslant \sum_{k=p+1}^{\infty} (| \tilde{c}_k |^2 + | \tilde{d}_k |^2) M \tag{3.24}$$

and the square summability of the $\{\tilde{c}_k\}$ and $\{\tilde{d}_k\}$ then shows that

$$\lim_{p\to\infty} \| h_p^* \| = 0, \tag{3.25}$$

hence

$$\lim_{p\to\infty} \| \tilde{h}_p(t) - h^*(t) \| = 0. \tag{3.26}$$

Therefore we have

$$h^*(t) = \sum_{k=0}^{\infty} (\tilde{c}_k p_k(t) + \tilde{d}_k q_k(t)) \tag{3.27}$$

and the series converges in the $L_2[0, T]$ norm.

The above argument is strictly valid only for the case $\omega_0 \neq 0$ but it is easily extended to the other case.

The Case $T = 2\pi D$

It is clear from what has already been said that if we wish the moment problem to possess exactly one solution in $L_2[0, T]$ we must have

$$T = 2\pi D. \tag{3.28}$$

It is equally clear that this alone is not enough to guarantee exactly one solution. For we may adjoin to or take away from S any finite number of functions of the form $e^{i\omega t}$ without changing the density. To get the result which we want it is necessary to assume that the real numbers ω_k are close to k in some sense. Considerable work has been done in this context: see Paley and Wiener [13]; Duffin and Eachus, [14]; Riesz and Sz.-Nagy, [15]; V. D. Golovin, [16]; N. Levinson, [7]. We will use in this paper the recent work of M. I. Kadec, [17].

Let us consider a set S:

$$S = \{e^{i\omega_k t} \mid k = 0, \pm 1, \pm 2,...\}. \tag{3.29}$$

S is called a Riesz basis for $L_2[0, T]$ if for each function

$$\theta(t) = \sum_{k=-\infty}^{\infty} \zeta_k e^{i\omega_k t} \tag{3.30}$$

we have, for $0 < A \leqslant B < \infty$,

$$A\left(\sum_{k=-\infty}^{\infty} | a_k |^2\right) \leqslant \| \theta \|^2 \leqslant B\left(\sum_{k=-\infty}^{\infty} | a_k |^2\right). \tag{3.31}$$

We have noted the case $T > 2\pi D$ that the first of these inequalities implies the existence of a solution of the moment problem. When S is a basis for $L_2[0, T]$ the solution is, of course, unique.

According to Kadec, [17], S is a Riesz basis for $L_2[0, 2\pi D]$ if

$$\sup_k \left| \omega_k - \frac{k}{D} \right| < \frac{1}{4D}. \tag{3.32}$$

The constant on the right is known, from the work of Levinson, [7], to be the best possible.

Now, let us suppose that all of the ω_k are distinct and for some $\epsilon > 0$

$$\left| \omega_k - \frac{k}{D} \right| < \frac{1}{4D} - \epsilon, \qquad | k | > K, \tag{3.33}$$

where K is a positive integer. Then there arises the question as to whether or not S remains a Riesz basis for $L_2[0, 2\pi D]$. It is clear that the set

$$S = \{e^{i(k/D)t} \mid | k | \leqslant K\} \cup \{e^{i\omega_k t} \mid | k | > K\} \tag{3.34}$$

is such a basis. Let us define a compact (i.e., completely continuous) linear transformation T_0 on $L_2[0, 2\pi D]$ by

$$\begin{aligned} T_0(e^{i(k/D)t}) &= e^{i\omega_k t}, \qquad | k | \leqslant K \\ T_0(e^{i\omega_k t}) &= 0 \qquad\qquad | k | > K. \end{aligned} \tag{3.35}$$

Then the bounded linear transformation $I + T_0$ is defined on $L_2[0, 2\pi D]$ and carries the Riesz basis $\hat{S}$ into S.

It is well known that $I + T_0$ possesses a bounded inverse if and only if $(I + T_0)(f) = 0 \Rightarrow f = 0$, $f \in L_2[0, 2\pi D]$. This is part of the statement of the Fredholm alternative. Now let f be expressed as a linear combination of the elements of $\hat{S}$ and assume $f \neq 0$. If $(I + T_0)(f) = 0$, there is an integer k, $| k | \leqslant K$, such that $e^{i\omega_k t}$ can be written as a linear combination of the remaining elements of S. But then, using the fact that all but finitely many of the elements of S are elements of $\hat{S}$, it is easy to see that $\hat{S}$ spans only a proper closed subspace of $L_2[0, 2\pi D]$. But the result of Schwartz referred to earlier states that when this is the case the elements of $\hat{S}$ are linearly independent, none can be written as linear combinations of the others. Thus we have a contradiction at $I + T_0$ must be invertible. But then it is clear that S is also a Riesz basis for $L_2[0, 2\pi D]$.

Thus, returning to S as given by (3.1), we see that if (3.33) is satisfied for $k > K$ we can solve the moment problem consisting of all but one, which is arbitrary, of the equations (3.15), (3.16). For S as given by (3.2) we can solve all equations but (3.17) in the moment problem.

We conclude this section with two remarks. First of all, if S as given by (3.1) is replaced by a set of functions $e^{i(\omega_k+\delta)t}$, where δ is any fixed real number, all of the results of this section remain true, for the transformation $f(t) \to e^{i\delta t} f(t)$ is unitary. Secondly, it is clear that a solution of the moment problems (3.15)-(3.18) implies a solution of (2.31)-(2.34) and the solution of (2.31)-(2.34) which has least $L_2[0, T]$ norm must be a real-valued function. Naturally in the case $T = 2\pi D$ when one of the Eqs. (3.15)-(3.18) cannot be satisfied, a corresponding equation must be eliminated from (2.31)-(2.34).

4. Proof of Theorems 1, 2, 3

Obviously most of the work involved in proving these theorems has already been done in Sections 2 and 3. All we need to do here is to indicate that the real numbers ω_k introduced in (2.19) satisfy conditions which correspond to the assumptions made on the ω_k in Section 3. The material which establishes this is readily available in the literature. We will describe the results briefly and give references.

First of all it is well known, see, e.g., Tricomi [6], that the solutions λ_k of the eigenvalue problem (2.2), for any set of boundary conditions (1.8), are such that $| \omega_k - (\pi k/l) |$ (recall $\omega_k^2 = \lambda_k$) is uniformly bounded. As a consequence we conclude that the density of the ω_k is given by

$$D = \frac{\ell}{\pi}. \tag{4.1}$$

Thus the critical interval length is

$$2\pi D = 2l. \tag{4.2}$$

Theorem 1 then follows immediately from the result cited under the case $T < 2\pi D$ in Section 3.

It is also shown in [6], and we will expand upon this in connection with Theorem 3, that the asymptotic gap for the ω_k is given by

$$\Gamma = \lim_{k\to\infty} (\omega_{k+1} - \omega_k) = \frac{\pi}{\ell} = \frac{1}{D}. \tag{4.3}$$

Thus the results of Redheffer and Ingham cited in Section 3 apply and Theorem 2 follows from the work done in Section 3 on the moment problem for $T > 2\pi D$.

We turn now to Theorem 3. The eigenvalue problem (2.2) has, for each of the boundary conditions (i), (ii), and (iii) listed in the statement of Theorem 3, a "prototype problem" in the form of the eigenvalue problem

$$\frac{d^2\phi(x)}{dx^2} + \lambda\phi(x) = 0 \tag{4.4}$$

with boundary conditions

(i) $\phi(0) = \phi(\ell) = 0;$

(ii) $\frac{d\phi}{dx}(0) = 0, \quad \phi(\ell) = 0 \quad \text{or} \quad \phi(0) = 0, \quad \frac{d\phi}{dx}(\ell) = 0;$

(iii) $\frac{d\phi}{dx}(0) = 0, \quad \frac{d\phi}{dx}(\ell) = 0.$ (4.5)

These correspond to a uniform string with (i) both ends fixed; (ii) one end fixed, the other free to slide; (iii) both ends free to slide. The eigenvalues $\tilde{\lambda}_k$ of the prototype problem are such that $\tilde{\omega}_k = \tilde{\lambda}_k^{1/2}$ is given by, for $k = 0, 1, 2,...$

(i) $\tilde{\omega}_k = \frac{(k+1)\pi}{\ell};$

(ii) $\tilde{\omega}_k = \frac{(k+\frac{1}{2})\pi}{\ell};$

(iii) $\tilde{\omega}_k = \frac{k\pi}{\ell};$ (4.6)

and it is shown, e.g., in Tricomi's book that the ω_k for the original problem satisfy

$$\tilde{\omega}_k = \tilde{\omega}_k + \mathcal{O}\left(\frac{1}{k}\right) \tag{4.7}$$

in all three cases. From Section 3 we see, therefore, that the question of solvability of moment problems is entirely the same for the $e^{\pm i\omega_k t}$ as for the $e^{\pm i\tilde{\omega}_k t}$.

The work of Section 3 on the case $T = 2\pi D$ shows that the functions $e^{\pm i\omega_k t}$ arising in case (i) together with the function 1 form a Riesz basis for $L_2[0, 2\pi D]$ when (4.7) holds. Thus a solution of the moment problem (2.31), (2.32) always exists and is not unique for we may specify

$$\int_0^{2\ell} h(t)\,dt = c \tag{4.8}$$

for any choice whatsoever of the real number c. In the case of the uniform string with fixed endpoints this is illustrated in the fact that a force constant over an interval of length equal to the period of the fundamental mode merely returns the string to its original state over that length of time. The interpretation is somewhat more complicated for the nonuniform string and is, we believe, best left in the form of the statement (4.8).

In case (ii) the functions $e^{\pm i\omega_k t}$ form a Riesz basis for $L_2[0, 2\pi D] = L_2[0, 2\ell]$. For the functions $e^{\pm i\tilde{\omega}_k t}$ are obtained from the familiar Fourier series functions $e^{\pm i(k\pi/\ell)t}$, $k = 0, 1, 2,...$ by applying the unitary transformation $f(t) \to f(t)\, e^{i(\pi/2\ell)t}$. Thus in this case the moment problem has exactly one solution. We merely multiply each of the $e^{\pm i\omega_k t}$ by $e^{-i(\pi/2\ell)t}$ and use the results of Section 3.

The functions we are concerned with in the prototype problems (iii) are 1, t, $e^{\pm i(k\pi/\ell)t}$, $k = 1, 2,...$, the Fourier series functions for the interval $[0, 2\ell]$ plus t, which is excessive. The analysis given in Section 3 together with (4.7) shows that in case (iii) the functions $e^{\pm i\omega_k t}$, or 1, t, $e^{\pm i\omega_k t}$, $k = 1, 2,...$, if $\omega_0 = 0$, form a Riesz basis for $L_2[0, 2\pi D] = L_2[0, 2\ell]$ if one element is removed. When $\omega_0 \neq 0$ this can be any element. Whether or not this is the case for $\omega_0 = 0$ is not so clear but one certainly obtains a Riesz basis if t is removed from the set. In any event the truth of Theorem 3, part (iii) is evident. In the case of the uniform string with free endpoints, the prototype problem for (iii), one can bring the string to equilibrium in time $T = 2\pi D$ but the actual location of the motionless string at the end of the time interval is fixed by the initial conditions. In any longer interval we can stop all motion and place the string where we will.

5. Concluding Remarks

We will conclude this paper with a remark concerning the applicability of our work in actual control engineering problems and an indication of what further developments may be expected along similar lines.

An important problem in control engineering is that of controlling a finite number of modes of vibration of a distributed parameter system without accidentally adding excessive energy to any of the neglected modes. The analysis presented in the previous sections provides at least a mathematical solution for this problem. Let us suppose that we wish to stop vibration in the first $p + 1$ modes in a time interval of length T, $T \geqslant 2\pi D$. To do so we need only to solve a moment problem (2.31)-(2.34) wherein $c_k = d_k = 0$, $k > p$. The solutions, in terms of a biorthogonal set $\{p_k(t), q_k(t)\}$ for the functions $\sin \omega_k t$, $\cos \omega_k t$ (or t, 1, $\sin \omega_k t$, $\cos \omega_k t$, as the case may be) is just

$$h(t) = \sum_{k=0}^{p} (c_k p_k(t) + d_k q_k(t)). \tag{5.1}$$

The corresponding control function $f(t)$, given by (2.28), will indeed stop all motion in the first p modes, as desired. As for the other modes, it is immediately obvious from (2.20) and (2.21) that their state at the end of the time interval will be the same as if no control at all (i.e., $f(t) \equiv 0$) had been applied during the interval. Thus, if this procedure is used over successive intervals of length $T \geqslant 2\pi D$ in order to counteract the effects of continuing disturbances, the controls themselves will have no cumulative effect on the higher order modes, i.e., $h > p$.

If the functions $p_k(t)$, $q_k(t)$ of the biorthogonal set have once been obtained for $k \leqslant p$, the calculation of the control function is completely trivial. Compare (2.24)-(2.27) with (2.31)-(2.34). We hope to devote a later paper to the question of numerical approximation of the functions $p_k(t)$ and $q_k(t)$.

Thus far in this paper we have dealt exclusively with the Eq. (1.1) which describes the motion of a nonuniform string. The method has much wider application, however. As an example, consider the simple beam equation which, after suitable normalization, is

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = g(x) f(t). \tag{5.2}$$

The eigenvalues λ_0, λ_1,... for

$$\frac{d^4\phi(x)}{dx^4} - \lambda\phi(x) = 0 \tag{5.3}$$

are such that $\omega_k = (\lambda_k)^{1/2}$ satisfies

$$\omega_k = (n + q)\,\pi + o(1), \tag{5.4}$$

where q is a rational number which depends upon the boundary conditions imposed. In the case where one end of the beam is clamped while the other end of the beam is free, $q = \frac{1}{2}$ and a unique control function $f(t)$ exists if $T = 2$. No control exists if $T < 2$ and the control exists but is not unique if $T > 2$. This should be compared with case (ii) in Theorem 3. If both ends are free we have two zero eigenvalues $\omega_0 = \omega_1 = 0$ and thereafter (5.4) is obeyed with $q = -\frac{1}{2}$. As a result one can stop all vibration in time $T = 2$ and bring either, but not both, of the modes corresponding to the eigenfunctions $\phi_0(x) \equiv 1$, $\phi_1(x) \equiv x$ to any state desired. Comparable results obtain for other boundary conditions. Naturally, appropriate assumptions on $g(x)$ and the smoothness of the initial conditions are necessary.

Indeed, the methods presented here should find application in all vibration problems involving one-dimensional continua, for in all such cases the ω_k will possess a finite positive density. One can expect difficulties to arise from multiple eigenvalues and failure to satisfy an asymptotic gap condition.

For higher dimensional continua, such as membranes, plates, cubes of elastic material, etc., the density of the ω_k is always infinite and controllability in our sense, is never possible in finite time. It is interesting to ask what happens for $T = \infty$, however.

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