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# A Theorem on the Closure of $\Omega$-Pure Subgroups of $C_\Omega$ Groups in the $\Omega$-Topology

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The closure of ascending unions of $\Omega$-pure subgroups of $C_\Omega$ groups is discussed. Applications are given to the torsion product, the balanced-projective dimension of $C_\Omega$ groups, and Kurepa's Hypothesis. 

In this note we will be concerned exclusively with abelian $p$-groups for some fixed prime $p$. If $\lambda \leqslant \Omega$ is a limit ordinal, a group $G$ is called a $C_\lambda$ if $G/G(\alpha)$ is a *dsc* (direct sum of countable groups) for all $\alpha < \lambda$. We call a $C_\Omega$ group $G$ *normal* if $G(\Omega) = 0$. In [K1] and [K2] the author considered the question of when $\operatorname{Tor}(A, B)$ is a dsc, which was originally posed by Nunke in [N]. The difficult case is where $A$ and $B$ are reduced groups of equal length. Considering the case where this length is $\Omega$, in [K1] it is shown that $\operatorname{Tor}(A, B)$ is a $C_\Omega$ group iff both $A$ and $B$ are $C_\Omega$ groups. Therefore, to determine when $\operatorname{Tor}(A, B)$ is a dsc of length $\Omega$ we must start with $A$ and $B$ being $C_\Omega$ groups. In [K1] it is also shown that if $A$ and $B$ are normal $C_\Omega$ groups whose *balanced-projective dimension* (b.p.d. for short; see [FH]) does not exceed 1, then $\operatorname{Tor}(A, B)$ is a dsc. We are therefore led to the question of whether every normal $C_\Omega$ group has b.p.d. at most 1. There is another way to view this question: it can be seen that if $X$ is a subgroup of the dsc $T$ and $K = T/X$, then $X$ is $\Omega$-pure iff its is balanced and $K$ is a $C_\Omega$ group (see Lemma 1 (b)). Therefore every closed $\Omega$-pure subgroup of a dsc is also a dsc iff every normal $C_\Omega$ group has b.p.d. at most 1. This theme is developed further in [K3].

A related issue concerns an invariant $\Gamma G$ defined in [K2] for $C_\Omega$ groups $G$, which is similar to constructions used in the study of $\kappa$-free groups and separable $p$-groups. In [K2] this invariant is used to give some conditions under which iterated torsion products are dsc's. As in the case of $\kappa$-free groups and separable groups it can be asked, what are the possible values that $\Gamma G$ can assume?



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All of these ideas are related to the closure properties of $\Omega$-pure subgroups of $C_\Omega$ groups in the $\Omega$-topology. When these subgroups are closed some authors call them *separable* (e.g., [FH]), but the use of this topological terminology seems more appropriate for our purposes. Our main result (Theorem 3) on the closure of these groups is proven in a manner similar to other results concerning the ordinal $\Omega$ (e.g., the fact that $\Omega$-high subgroups are not summable). We then apply our result to the above questions. Essentially this result allows us to concentrate on certain $\Omega$-pure subgroups of cardinality $\aleph_1$ which we can call $\kappa$-Kurepa subgroups, where $\kappa$ is a cardinal. As a simple consequence of this we will be able to show that the invariant $\Gamma G$ can take on only the values 0 and 1, and it will be nonzero only when there are $|G|$-Kurepa subgroups (Theorem 8). The main result of [K3] is that the existence of $\kappa$-Kurepa subgroups is equivalent to the existence of families of sets of cardinality $\kappa$ satisfying Kurepa's Hypothesis (KH). In this work we present some related results having to do with the iterated torsion product, $\mathrm{Tor}(G_1, ..., G_n)$, for normal $C_\Omega$ groups $G_1, ..., G_n$. The first (Theorem 15) says that for cardinals less than $\aleph_{n+1}$, to decide if this is a dsc we need only look for $\aleph_i$-Kurepa subgroups for $i = 1, ..., n$. We also establish the somewhat surprising results that this will always be a dsc iff there are no $\aleph_n$-Kurepa extensions (Theorem 17) and it will always have b.p.d. at most 1 iff there are no $\aleph_{n+1}$-Kurepa extensions (Theorem 19). Therefore, the possible size of a family of sets satisfying KH is to a large extent determined by the Tor functor. It is also clear from this that a total solution to Nunke's problem will intimately depend upon what set theoretic universe is being employed. We conclude by characterizing those normal $C_\Omega$ groups $G$ which have the property that $\mathrm{Tor}(X, G)$ is always a dsc for any normal $C_\Omega$ group $X$, at least in any set theoretic universe where $2^{\aleph_1} = \aleph_2$ and KH is valid.

We begin with some review. If $G$ is a group and $\lambda$ is an ordinal, then $G(\lambda) = \{g \in G : ht(g) \geq \lambda\}$. A subgroup $A$ of $G$ is called *$\lambda$-isotype* if $A(\alpha) = A \cap G(\alpha)$ for all $\alpha \leq \lambda$; it is called *$\lambda$-nice* if $(G/A)(\alpha) = \{G(\alpha) + A\}/A$ for all $\alpha < \lambda$; it is called *$\lambda$-balanced* if it is both $\lambda$-isotype and $\lambda$-nice. We assume familiarity with the standard results on $\lambda$-purity (see, for example [G]). Note that $\lambda$-purity is called $p^\lambda$-purity by some authors. The following gives several different characterizations of $\lambda$-purity in $C_\lambda$ groups.

LEMMA 1. *Suppose $\lambda \leq \Omega$ is a limit ordinal, $A$ is a subgroup of the $C_\lambda$ group $G$, and $K = G/A$. Then the following statements are equivalent:*

(a) *$A$ is a $\lambda$-pure subgroup,*

(b) *$A$ is a $\lambda$-balanced subgroup and $K$ is a $C_\lambda$ group,*

(c) *for every $\alpha < \lambda$, $0 \to A/A(\alpha) \to G/G(\alpha) \to K/K(\alpha) \to 0$ is a short exact sequence which splits,*

(d) *$A$ is a $\lambda$-isotype subgroup and for all $\alpha < \lambda$, $A/A(\alpha)$ is a summand of $G/G(\alpha)$,*

(e) *$A$ is a pure subgroup of $G$ and, under the natural embedding, $\mathrm{Tor}(A, H_\lambda)$ is a summand of $\mathrm{Tor}(G, H_\lambda)$, where $H_\lambda$ is the "generalized Prüfer group" of length $\lambda$.*

*Proof.* (a)$\Rightarrow$(b) By [G, Theorem 91] and [K2, Lemma 1].

(b)$\Rightarrow$(c) These sequences are clearly balanced exact. Since $K/K(\alpha)$ is a dsc, and hence a balanced-projective, they must split.

(c)$\Rightarrow$(d) Clear.

(d)$\Rightarrow$(a) By [G, Theorem 87].

(a)$\Leftrightarrow$(e) By [K2, Lemma 1].

If $\lambda$ is a limit ordinal, the $\lambda$-topology on a group $G$ is that defined using $\{G(\alpha)\}_{\alpha<\lambda}$ as a neighborhood base of 0. Because we will be very concerned with the closure of $\lambda$-pure subgroups in this topology, we include the next result for future reference.

LEMMA 2. *Suppose $\lambda$ is a limit ordinal, $G$ and $K$ are groups, and $\phi: G \to K$ is a surjection. Suppose further that $A = \ker \phi$ is a $\lambda$-nice subgroup of $G$ and $B$ is a subgroup of $G$ containing $A$. Then $\overline{\phi(B)} = \phi(\bar{B})$ (where the closures are in the $\lambda$-topology of $K$ and $G$ respectively). In particular, we have $\bar{A} = \phi^{-1}(K(\lambda))$.*

*Proof.* Since $A \subseteq G(\alpha) + B$ for every $\alpha < \lambda$, it is easily checked that

$$\bigcap_{\alpha<\lambda} \phi(G(\alpha)+B) = \phi\left(\bigcap_{\alpha<\lambda} \{G(\alpha)+B\}\right) = \phi(\bar{B}).$$

Therefore,

$$\begin{aligned}\overline{\phi(B)} &= \bigcap_{\alpha<\lambda} \{K(\alpha)+\phi(B)\} \\ &= \bigcap_{\alpha<\lambda} \{\phi(G(\alpha))+\phi(B)\} \\ &= \bigcap_{\alpha<\lambda} \phi(G(\alpha)+B) = \phi(\bar{B}).\end{aligned}$$

The second statement follows since $\phi(\bar{A}) = \overline{\phi(A)} = \overline{\{0\}} = K(\lambda)$.

From here on, all topological notions will be with respect to the $\Omega$-topology. Our main result, which we state next, essentially says that closure behaves continuously for smoothly ascending unions of $\Omega$-pure subgroups of $C_\Omega$ groups.

THEOREM 3. *Suppose $G$ is a $C_\Omega$ group, $\mu$ is a limit ordinal, and $\{A_i\}_{i<\mu}$ is a smoothly ascending chain of $\Omega$-pure subgroups of $G$. Then*

$$\overline{\bigcap_{i<\mu} A_i} = \bigcap_{i<\mu} \overline{A_i}.$$

*In particular, if for each $i<\mu$ $\overline{A_i} \subseteq A_{i+1}$, then the union is a closed subset of $G$.*

*Proof.* We may assume $A_0 = 0$. Let $A = \bigcup_{i<\mu} A_i$. Clearly

$$\bigcup_{i<\mu} \overline{A_i} \subseteq \bar{A}.$$

Conversely, suppose $\{x_\alpha\}_{\alpha<\Omega}$ is a net in $A$ which converges to $x \in \bar{A}$. We may assume that $x_\alpha - x_{\alpha'} \in G(\alpha)$ whenever $\alpha < \alpha' < \Omega$ (i.e., the net is *proper*). Suppose $x \notin \overline{A_i}$ for each $i < \mu$. In order to derive a contradiction, we inductively define ordinals $\alpha_n$ and $\beta_n$ for each non-negative integer $n$, such that

(1) $\Omega > \alpha_n > \alpha_{n-1}$, $\mu > \beta_n > \beta_{n-1}$,

(2) $x_{\alpha_n} \notin A_{\beta_{n-1}+1} + A(\alpha_n)$,

(3) $\beta_n = \max\{\beta : x_{\alpha_n} \notin A_\beta + A(\alpha_n)\}$.

To see how this can be accomplished, start with $\alpha_0 = \beta_0 = 0$, and assume that $\alpha_{n-1}$ and $\beta_{n-1}$ have been defined. Observe that if there is no $\alpha_n > \alpha_{n-1}$ satisfying (2), then $x \in \overline{A_{\beta_{n-1}+1}}$, which we have assumed to fail. Having then found $\alpha_n$, we can find $\beta_n$; since $\{A_\beta + A(\alpha_n) : \beta < \mu\}$ is a smoothly ascending chain with union $A$, there is a first ordinal $\delta$, such that $x_{\alpha_n} \in A_\delta + A(\alpha_n)$. Smoothness implies that $\delta$ is isolated, and (2) guarantees that $\beta_n = \delta - 1$ is strictly greater than $\beta_{n-1}$.

Observe that our construction yields

$$x_{\alpha_n} \in \{A_{\beta_n+1} + A(\alpha_n)\} - \{A_{\beta_n} + A(\alpha_n)\}$$

for each positive integer $n$. Let $\gamma < \Omega$ be greater than $\alpha_n$ for all $n$.

Since each $A_i$ is $\Omega$-pure in $G$ it is also $\Omega$-pure in $A_{i+1}$. So by Lemma 1 (c), for each $i$, $\{A_i + A(\gamma)\}/A(\gamma) \cong A_i/A_i(\gamma)$ is a summand of $\{A_{i+1} + A(\gamma)\}/A(\gamma) \cong A_{i+1}/A_{i+1}(\gamma)$. This means that there are subgroups $B_i$ for each $i$, such that

$$B_i + \{A_i + A(\gamma)\} = A_{i+1} + A(\gamma),$$

$$B_i \cap \{A_i + A(\gamma)\} = A(\gamma).$$

Therefore, for each $i$, the following two facts are valid:

$$A_i + A(\gamma) = \sum_{j<i} B_j$$

and

$$A/A(\gamma) \cong \bigoplus_{i<\mu} B_i/A(\gamma).$$

Since the composite of this last isomorphism with the natural map $A \to A/A(\gamma)$ preserve all heights strictly less than $\gamma$, it readily follows that whenever $\alpha < \gamma$ and $b_{i_j} \in B_{i_j}$ for $j = 1, ..., m$ we have

$$b_{i_1} + \cdots + b_{i_m} \equiv 0 \ (\text{mod } A(\alpha)) \Leftrightarrow \forall j = 1, ..., m, \ b_{i_j} \equiv 0 \ (\text{mod } A(\alpha)). \quad (*)$$

Suppose $x_\gamma = y_{i_1} + \cdots + y_{i_r}$, where each $y$ is in the corresponding $B$. We derive our contradiction by showing that the set $\{i_1, ..., i_r\}$ is infinite. Let $n$ be any positive integer. Since

$$A_{\beta_n+1} + A(\alpha_n) = \sum_{j<\beta_n+1} B_j + A(\alpha_n)$$

we know that

$$x_{\alpha_n} = z_{j_1} + \cdots + z_{j_s} + z_{\beta_n},$$

where each $z$ is in the corresponding $B$ and each $j$ is strictly less than $\beta_n$. Since $x_{\alpha_n} \notin A_{\beta_n} + A(\alpha_n)$, we can conclude that $z_{\beta_n} \not\equiv 0$ (mod $A(\alpha_n)$). Now,

$$y_{i_1} + \cdots + y_{i_r} = x_\gamma \equiv x_{\alpha_n} = z_{j_1} + \cdots z_{j_s} + z_{\beta_n} \qquad (\text{mod } A(\alpha_n))$$

and so

$$y_{i_1} + \cdots + y_{i_r} - z_{j_1} - \cdots - z_{j_s} - z_{\beta_n} \equiv 0 \qquad (\text{mod } A(\alpha_n)).$$

Therefore, by (*), we have $\beta_n \in \{i_1, ..., i_r\}$. Since this is true for every positive integer $n$ we have our contradiction.

Although it is not important for the applications contained in this work, it is perhaps of interest that in our last result we have not assumed that the union $A$ is also an $\Omega$-pure subgroup. The reader will note the sharp contrast between Theorem 3 and the situation for separable $p$-groups (i.e., $C_\omega$ groups). Clearly any basic subgroup of a group is the ascending union of direct summands of the group. These are ($\omega$-) pure subgroups which are closed in the $p$-adic topology. Their union, though, not only is not necessarily closed, it is in fact dense. Also, the reader familiar with [EH] will note the parellel between Theorem 3 and the *smooth continuity of closures* (SCC) property for separable $p$-groups defined therein.

We pause to review a few definitions. By an *$\Omega$-development* of the $C_\Omega$ group $G$ we mean a smoothly ascending chain of $\Omega$-pure subgroups $\{B_i\}$ such that $B_0 = 0$, $|B_i| < |G|$, and $\bigcup B_i = G$ (so an $\Omega$-development is a filtration consisting of $\Omega$-pure subgroups). We say the $\Omega$-development is *normal* if for each $i$, $(G/B_i)(\Omega)=0$ iff $(B_{i+1}/B_i)(\Omega)=0$. In other words, if $B_i$ is a closed subgroup of $B_{i+1}$ it is actually closed in $G$. Any $C_\Omega$ group of cardinality at least $\aleph_2$ has a normal $\Omega$-development (see [K2, Corollary 1]).

In our subsequent results we will be concerned with constructing $\Omega$-developments of $C_\Omega$ groups consisting entirely of groups closed in the $\Omega$-topology. To avoid repeating ourselves, we state the essential construction in the following somewhat awkward fashion.

LEMMA 4. *Suppose $G$ is a normal $C_\Omega$ of cardinality $\alpha$ and $\aleph$ is a regular cardinal with $\aleph_1<\aleph\leqslant\alpha$. Suppose whenever $R$ and $S$ are $\Omega$-pure subgroups of $G$ such that*

(1) $R\subseteq S\subseteq G$,

(2) *$R$ is closed,*

(3) $|R|<\alpha$,

(4) $|S/R|<\aleph$,

*we have that $|\bar{S}/R|<\aleph$. Then $G$ has an $\Omega$-development $\{B_i\}$ such that for each $i$, $B_i$ is a closed subgroup with $|B_{i+1}/B_i|<\aleph$.*

*Proof.* Suppose $\{g_i\}_{i<\alpha}$ is an enumeration of $G$. Define a smoothly ascending chain of subgroups $\{R_i\}_{i<\alpha}$ such that

(1) $R_0=0$,

(2) $g_i\in R_{i+1}$,

(3) $\overline{R_i}\subseteq R_{i+1}$,

(4) $R_i$ is pure in $G$,

(5) $\mathrm{Tor}(R_i, H_\Omega)$ is the sum of a subset of the terms in a fixed decomposition of $\mathrm{Tor}(G, H_\Omega)$ into countable groups,

(6) $|R_i|<\aleph$ when $i<\aleph$ and $|R_i|\leqslant|i|$ when $i\geqslant\aleph$,

(7) $|R_{i+1}/R_i|<\aleph$.

Suppose we have constructed our chain for each $i<j$ and we wish to construct $R_j$. Observe that (4) and (5), together with Lemma 1 (e), guarantee that each $R_i$ is an $\Omega$-pure subgroup. If $j$ is a limit ordinal, smoothness dictates that $R_j=\bigcup_{i<j}R_i$, and it is easy to check that (4)–(6) are satisfied. If $j=i+1$ is isolated, suppose $i=\beta+n$, where $\beta$ is 0 or a limit ordinal and $n<\omega$. By (3) and Theorem 3, $R_\beta$ is closed. By (7), $|R_i/R_\beta|<\aleph$, so by hypothesis, $|\overline{R_i}/R_\beta|<\aleph$. By an easy "back-and-forth" argument, there is an $R_{i+1}$ containing $\overline{R_i}$ and $g_i$ satisfying (4)–(7). This completes the construction.

We now restrict attention to those $R_i$ where $i$ is a limit (i.e., let $B_i=R_{\omega i}$).

These $B_i$ are closed $\Omega$-pure subgroups, and since the cofinality of $\aleph$ is greater than $\omega$, we have $|B_{i+1}/B_i| < \aleph$. This completes the proof.

Suppose $A$ is an $\Omega$-pure subgroup of the normal $C_\Omega$ group $G$. If $\kappa \geqslant \aleph_2$ is a cardinal, we say $A$ is a $\kappa$-Kurepa subgroup of $G$ (and $G$ is a $\kappa$-Kurepa extension of $A$) if $|A| = \aleph_1$ and $|\bar{A}| \geqslant \kappa$. A collection $\mathscr{F} \subseteq P(\Omega)$ is called a Kurepa family if for every countable $\beta$,

$$|\{X \cap \beta : X \in \mathscr{F}\}| \leqslant \aleph_0.$$

Kurepa's Hypothesis is the assertion that there are Kurepa families of cardinality greater than $\aleph_1$. This is independent of ZFC + GCH (see [Ku]). The similarity of terminology is explained by the following result of [K3]:

THEOREM 5. *If $\kappa \geqslant \aleph_2$ is a cardinal, then there exists a $\kappa$-Kurepa extension iff there exists a Kurepa family of cardinality $\kappa$.*

Since a Kurepa family can clearly have cardinality at most $2^{\aleph_1}$ this theorem implies that there are no $\kappa$-Kurepa extensions for $\kappa > 2^{\aleph_1}$. In the constructible universe there does exist a Kurepa family of cardinality $\aleph_2 = 2^{\aleph_1}$. We now have:

LEMMA 6. *If $\kappa \geqslant \aleph_2$ is a regular cardinal and $G$ is a normal $C_\Omega$ with no $\kappa$-Kurepa subgroups, then for every $\Omega$-pure subgroup $X$ of $G$, we have that $|X| < \kappa$ implies $|\bar{X}| < \kappa$.*

*Proof.* The result is true by hypothesis when $|X| \leqslant \aleph_1$. Suppose now that we have verified the result for all $\Omega$-pure subgroups of cardinality less than $\alpha$ and $|X| = \alpha$. Let $X_i$ for $i < \alpha$ be an $\Omega$-development of $X$. By the regularity of $\kappa$, we have

$$|\bar{X}| = \left| \bigcup_{i<\alpha} \overline{X_i} \right| = \sup_{i<\alpha} |\overline{X_i}| < \kappa.$$

THEOREM 7. *Suppose $\aleph > \aleph_1$ is a regular cardinal for which there are no $\aleph$-Kurepa extension. Then every normal $C_\Omega$ group $G$ has an $\Omega$-development $\{B_i\}$ such that for each $i$, $B_i$ is a closed subgroup with $|B_{i+1}/B_i| < \aleph$.*

*Proof.* In the notation of Lemma 4, by Lemmas 1 and 2, $G/R$ is a normal $C_\Omega$ group which by hypothesis has no $\aleph$-Kurepa subgroups. So by Lemmas 2 and 6 we have that

$$|S/R| < \aleph \Rightarrow |\bar{S}/R| = |\overline{S/R}| < \aleph,$$

and the result follows.

If $G$ is a $C_\Omega$ group of regular cardinality $\alpha \geqslant \aleph_2$ and $\{B_i\}_{i<\alpha}$ is a normal $\Omega$-development, let

$$\Gamma'\{B_i\} = \{i : (B_{i+1}/B_i)(\Omega) \neq 0\}$$

and $\Gamma G$ be the image of $\Gamma'\{B_i\}$ in $\mathscr{P}([0, \alpha))/I$, where $I$ is the boolean algebra ideal generated by the complements of the cub subsets of $[0, \alpha)$ (see for example [E]). This invariant measures how far $G$ is from having an $\Omega$-development consisting of closed $\Omega$-pure subgroups.

THEOREM 8. *Suppose $G$ is a normal $C_\Omega$ group of regular cardinality $\kappa \geqslant \aleph_2$. Then $\Gamma G \neq 0$ iff $G$ has a $\kappa$-Kurepa subgroup. In particular, $\Gamma G$ is always* 0 *or* 1.

*Proof.* If $A \subseteq G$ is a $\kappa$-Kurepa subgroup, then any $\Omega$-pure subgroup $B$ of $G$ containing $A$ of cardinality less than $\kappa$ cannot be closed. This clearly implies that $\Gamma G = 1$. Conversely, if $G$ has no $\kappa$-Kurepa subgroups, then Lemma 6 implies that we can apply Lemma 4 with $\aleph = \alpha = \kappa$ and conclude that $G$ has an $\Omega$-development consisting of closed subgroups. Therefore, $\Gamma G = 0$.

COROLLARY 9. *If $2^{\aleph_1} = \aleph_2$, then for any normal $C_\Omega$ group $G$ of regular cardinality at least $\aleph_3$ we have $\Gamma G = 0$.*

*Proof.* Immediate from the last result since there can be no $|G|$-Kurepa extensions.

We now wish to apply the above results to the Tor functor. We shall frequently have cause to refer to a smoothly ascending chain $\{B_i\}$ of $\Omega$-pure subgroups of a normal $C_\Omega$ group $G$. In this case we will always be assuming that the chain starts with $B_0 = 0$. If $B_i = G$ for all $i > 0$ we will call the chain *trivial*. Several of our subsequent constructions will use the following three facts:

THEOREM 10. [K2, Theorem 4]. *If $G_1, ..., G_n$ are normal $C_\Omega$ groups which are the ascending union of smooth chains $\{B_{1,j}\}, ..., \{B_{n,j}\}$ of $\Omega$-pure subgroups and for each i and j, the group*

$$X_{i,j} = \operatorname{Tor}(B_{1,j}, ..., B_{i-1,j}, B_{i,j+1}/B_{i,j}, B_{i+1,j+1}, ..., B_{n,j+1})$$

*is a dsc, then* $\operatorname{Tor}(G_1, ..., G_n)$ *is a dsc.*

THEOREM 11. *Suppose $X$ and $Y$ are normal $C_\Omega$ groups.*

(a) *If $Y$ is the union of a smoothly ascending chain of $\Omega$-pure subgroups, $\{B_i\}$, such that* $\operatorname{Tor}(X, B_{i+1}/B_i)$ *is a dsc for all i, then* $\operatorname{Tor}(X, Y)$ *is a dsc.*

(b) [K2, Theorem 3]. *If* $\aleph_1 \leqslant |X| < |Y|$ *and* $\operatorname{Tor}(X, Y)$ *is a dsc, then* $Y$ *has an* $\Omega$*-development* $\{B_i\}$ *such that for each* $i$, $|B_{i+1}/B_i| \leqslant |X|$ *and* $\operatorname{Tor}(X, B_{i+1}/B_i)$ *is a dsc.*

THEOREM 12. [K2, Theorem 5]. *If* $G_1, ..., G_n$ *are normal* $C_\Omega$ *groups of cardinality at most* $\aleph_{n-1}$ *then* $\operatorname{Tor}(G_1, ..., G_n)$ *is a dsc.*

Theorem 10 is actually slightly more general than [K2, Theorem 4], which assumes that the $B$'s are part of $\Omega$-developments, but the same proof clearly applies. Theorem 11 (a) is simply a particular case of Theorem 10 (using a trivial chain for $X$). We now mention the following simple fact:

LEMMA 13. *If* $X$ *is a balanced subgroup of a dsc* $G$ (*in particular, if it is a closed,* $\Omega$*-pure subgroup*) *and* $|X| \leqslant \aleph_1$, *then* $X$ *is a dsc.*

*Proof.* Clearly we may assume $|G| \leqslant \aleph_1$ and the result follows since the b.p.d. of any group of cardinality $\leqslant \aleph_1$ is at most 1 (see [FH]).

We wish to now extend the definition of $\kappa$-Kurepa extensions a bit. If $G$ is a normal $C_\Omega$ group and $A$ is an $\Omega$-pure subgroup of cardinality $\aleph_1$ we say $A$ is an $\aleph_1$-Kurepa subgroup iff it is either an $\aleph_2$-Kurepa subgroup or else a closed subgroup which fails to be a dsc.

PROPOSITION 14. *Suppose* $G$ *is a normal* $C_\Omega$ *group.*

(a) *If* $|G| < \aleph_2$, *then* $G$ *has no* $\aleph_1$*-Kurepa subgroups iff it is a dsc.*

(b) *If* $|G| = \aleph_2$, *then* $G$ *has no* $\aleph_1$*-Kurepa subgroups iff it has an* $\Omega$*-development consisting entirely of dsc's.*

*Proof.* In (a), ($\Leftarrow$) follows directly from Lemma 13. For ($\Rightarrow$), note that $G$ cannot be an $\aleph_1$-Kurepa subgroup of itself, so it must be a dsc. For (b), by Theorem 8, $G$ has an $\Omega$-development consisting of closed subgroups iff it has no $\aleph_2$-Kurepa subgroups and by Lemma 13 these will all be dsc's iff it has no $\aleph_1$-Kurepa subgroups.

The reason for the above terminology is contained in the following theorem, whose proof is similar to that of [K2, Theorem 10]:

THEOREM 15. *Suppose* $G_1, ..., G_n$ *are normal* $C_\Omega$ *groups of cardinality at most* $\aleph_n$. *Then* $\operatorname{Tor}(G_1, ..., G_n)$ *fails to be a dsc iff, possibly after reordering,* $G_i$ *has an* $\aleph_i$*-Kurepa subgroup for* $i = 1, ..., n$.

*Proof.* If we define $\operatorname{Tor}(G) = G$, the result for $n = 1$ is Proposition 14 (a). Assuming the result for $n - 1$ suppose first that each $G_i$ has an $\aleph_i$-Kurepa subgroup $A_i$, for $i = 1, ..., n$. If $\operatorname{Tor}(G_1, ..., G_n)$ is a dsc we can clearly express each $G_i$ as the union of a smoothly ascending chain, $\{B_{i,j}\}$, for $i = 1, ..., n$, such that

(1) $|B_{i,j}| \leqslant \aleph_{n-1}$,

(2) $B_{i,j}$ is, for each fixed $i$, either always or never closed,

(3) $\mathrm{Tor}(B_{1,j}, ..., B_{n,j})$ is a summand of $\mathrm{Tor}(G_1, ..., G_n)$.

To see this, let $B_{i,j}$ be the trivial chain if $|A_i| < \aleph_n$; otherwise, choose some $\Omega$-developments and find subdevelopments which satisfy (2) and (3). Observe that if $j$ is chosen large enough, then for each $i < n$, $A_i$ is an $\aleph_i$-Kurepa subgroup of $B_{i,j}$. For this $j$, $\mathrm{Tor}(B_{1,j}, ..., B_{n,j})$ is a summand of

$$\mathrm{Tor}(B_{1,j}, ..., B_{n-1,j}, B_{n,j+1})$$

and by (1) and Theorem 12 this last group is a dsc. Therefore the quotient

$$\mathrm{Tor}(B_{1,j}, ..., B_{n-1,j}, B_{n,j+1}/B_{n,j})$$

is also a dsc. Since $G_n$ has an $\aleph_n$-Kurepa subgroup, $(B_{n,j+1}/B_{n,j})(\Omega) \neq 0$, and by induction $\mathrm{Tor}(B_{1,j}, ..., B_{n-1,j})$ fails to be a dsc, so we have a contradiction (cf. [K2, Lemma 3]).

Conversely, if the stated condition fails, then after possibly reordering there exists a $k \leqslant n$ such that $G_1, ..., G_k$ all fail to have an $\aleph_k$-Kurepa subgroup. In particular, since $n \geqslant 2$, by Theorem 8, for $i = 1, ..., k$, $\Gamma G_i = 0$ whenever $|G_i| = \aleph_n$. For each $i = 1, ..., n$, let $\{B_{i,j}\}$ be $\Omega$-developments if $|G_i| = \aleph_n$ and be trivial chains otherwise. For $i \leqslant k$ we may assume each $B_{i,j}$ is closed. We claim that for each $i$ and $j$,

$$X_{i,j} = \mathrm{Tor}(B_{1,j}, ..., B_{i-1,j}, B_{i,j+1}/B_{i,j}, B_{i+1,j+1}, ..., B_{n,j+1})$$

is a dsc. For $i \leqslant k$ this follows from Theorem 12 since $(B_{i,j+1}/B_{i,j})(\Omega) = 0$. If $i > k$, then $B_{1,j}, ..., B_{k,j}$ clearly cannot contain $\aleph_k$-Kurepa subgroups, so by induction

$$\mathrm{Tor}(B_{1,j}, ..., B_{i-1,j}, B_{i+1,j+1}, ..., B_{n,j+1})$$

and hence $X_{i,j}$ is a dsc. We are now done by Theorem 10.

Recall that if $G$ is a normal $C_\Omega$ group of cardinality $\aleph_2$ then $\Gamma G = 0$ iff $G$ has b.p.d. at most 1 ([K2, Proposition 3 (b)]). It can also easily be checked that if $G$ is any group with $G(\Omega) = 0$ then the b.p.d. of $G$ is at most 2. We therefore have

COROLLARY 16. *Suppose $A$ and $B$ are normal $C_\Omega$ groups of cardinality $\aleph_2$. If both $A$ and $B$ have b.p.d. 2, then* $\mathrm{Tor}(A, B)$ *is not a dsc. In particular, the b.p.d. of $A$ is at most* 1 *iff* $\mathrm{Tor}(A, A)$ *is a dsc.*

*Proof.* This follows from the above observation, together with Theorems 8 and 15.

The following result is perhaps surprising because absolutely no restrictions are placed on the cardinality of the groups involved.

THEOREM 17. *If $n$ is a positive integer then the following statements are equivalent*:

(a) *There are no $\aleph_n$-Kurepa extensions,*

(b) $\mathrm{Tor}(G_1, ..., G_n)$ *is a dsc for every collection* $G_1, ..., G_n$ *of normal* $C_\Omega$ *groups.*

*Proof.* If there does exist an $\aleph_n$-Kurepa extension, then we can clearly find a normal $C_\Omega$ group $G$ of cardinality $\aleph_n$ which has an $\aleph_n$-Kurepa subgroup. So by Theorem 15, $\mathrm{Tor}(G, ..., G)$ is not a dsc.

Conversely, suppose there are no $\aleph_n$-Kurepa extensions. We prove the result by induction on the maximum of the cardinalities of the $G$'s, which we denote by $\alpha$. If $\alpha \leqslant \aleph_n$, the result follows from Theorem 15. Otherwise, by Theorem 7 we can express each $G_i$ as a smoothly ascending union of closed $\Omega$-pure subgroups $\{B_{i,j}\}$ of cardinality strictly less than $\alpha$ (if $|G_i| < \alpha$, let $B_{i,j}$ be the trivial chain). Since $(B_{i,j+1}/B_{i,j})(\Omega) = 0$, by induction we have that

$$X_{i,j} = \mathrm{Tor}(B_{1,j}, ..., B_{i-1,j}, B_{i,j+1}/B_{i,j}, B_{i+1,j+1}, ..., B_{n,j+1})$$

is a dsc and we are done by Theorem 10.

COROLLARY 18. *If $2^{\aleph_1} = \aleph_2$, then* $\mathrm{Tor}(A, B, C)$ *is always a dsc whenever $A$, $B$, and $C$ are normal $C_\Omega$ groups.*

Observe the sharp contrast between Theorem 17 and the situation for separable $p$-groups. As observed in [N], whether or not one assumes any special set theoretic axioms, it is easy to construct separable $p$-groups $G_1, ..., G_n$, for any positive $n$, such that $\mathrm{Tor}(G_1, ..., G_n)$ fails to be a direct sum of cyclic groups.

For the next result, we note the following fact from [K1]: if $M_\Omega$ is the standard elementary $S$-group of length $\Omega$ (so $M_\Omega$ is an $\Omega$-pure subgroup of the generalized Prüfer group, $H_\Omega$), then a $C_\Omega$ group $G$ has b.p.d. at most 1 iff $\mathrm{Tor}(G, M_\Omega)$ is a dsc.

THEOREM 19. *Suppose $n$ is a positive integer. The following are equivalent*:

(a) *There are no $\aleph_{n+1}$-Kurepa extensions;*

(b) *Whenever* $G_1, ..., G_n$ *are normal* $C_\Omega$ *groups,* $\mathrm{Tor}(G_1, ..., G_n)$ *has b.p.d. at most* 1;

(c) *Whenever* $G_1, ..., G_n$ *are normal* $C_\Omega$ *groups,* $\mathrm{Tor}(G_1, ..., G_n)$ *is a dsc, and* $C_1, ..., C_n$ *are closed* $\Omega$*-pure subgroups of* $G_1, ..., G_n$ *respectively, then* $\mathrm{Tor}(C_1, ..., C_n)$ *is also a dsc.*

*Proof.* To prove (a) implies (b), simply observe that Theorem 17 implies that $\mathrm{Tor}(M_\Omega, G_1, ..., G_n)$ is a dsc, so we are done by the above remark. For the converse, observe that if $G$ were a normal $C_\Omega$ group with an $\aleph_{n+1}$-Kurepa subgroup, then since $M_\Omega$ has an $\aleph_1$-Kurepa subgroup (namely itself), $\mathrm{Tor}(M_\Omega, G, ..., G)$ is not a dsc and hence $\mathrm{Tor}(G, ..., G)$ has b.p.d. 2. To prove (b) implies (c) note that there is an $\Omega$-pure exact sequence,

$$0 \to \mathrm{Tor}(C_1, G_2, ..., G_n) \to \mathrm{Tor}(G_1, ..., G_n) \to \mathrm{Tor}(G_1/C_1, G_2, ..., G_n) \to 0.$$

This sequence is clearly balanced, and since $C_1$ is closed, $(G_1/C_1)(\Omega) = 0$. Therefore $\mathrm{Tor}(G_1/C_1, G_2, ..., G_n)$ has b.p.d. at most 1 and so $\mathrm{Tor}(C_1, G_2, ..., G_n)$ is a dsc. Continuing in this way gives the result. For the converse, suppose $G_1, ..., G_n$ are normal $C_\Omega$ groups. Let

$$0 \to A \to T \to G_1 \to 0$$

be an $\Omega$-pure sequence with $T$ a dsc (for example let $T = \mathrm{Tor}(H_\Omega, G_1)$, $A = \mathrm{Tor}(M_\Omega, G_1)$). Then there is an $\Omega$-pure exact sequence

$$0 \to \mathrm{Tor}(A, G_2, ..., G_n) \to \mathrm{Tor}(T, G_2, ..., G_n) \to \mathrm{Tor}(G_1, G_2, ..., G_n) \to 0.$$

Clearly the middle group is a dsc and since $A$ is a closed $\Omega$-pure subgroup of $T$, the left group is also a dsc and so $\mathrm{Tor}(G_1, ..., G_n)$ has b.p.d. at most 1. This completes the proof.

As was pointed out in the proof of Theorem 15, if we define $\mathrm{Tor}(G) = G$, Theorems 15 and 17 are valid when $n = 1$.

We say a group is *close to being a dsc* if it is a normal $C_\Omega$ group of cardinality at most $\aleph_2$ which has no $\aleph_1$-Kurepa subgroups. If we assume the continuum hypothesis ($2^{\aleph_0} = \aleph_1$), we can exhibit a specific example of a group which is close to being a dsc, but fails to be a dsc as follows: By [RW] there is a balanced subgroup of a dsc group of cardinality $\aleph_2$ which fails to be a dsc. It is clear (using Lemma 13, for example) that this group is close to being a dsc. Similarly, if we assume KH, then there is clearly a normal $C_\Omega$ group $X$ of cardinality $\aleph_2$ and b.p.d. 2 (choose $X$ to have an $\aleph_2$-Kurepa subgroup). Then

$$0 \to \mathrm{Tor}(M_\Omega, X) \to \mathrm{Tor}(H_\Omega, X) \to X \to 0$$

is $\Omega$-pure and hence balanced. So since $\mathrm{Tor}(H_\Omega, X)$ is a dsc, once again by Lemma 13, $\mathrm{Tor}(M_\Omega, X)$ is close to being a dsc, but is not a dsc.

THEOREM 20. *The following statements are equivalent*:

(a) *There are no* $\aleph_3$-*Kurepa extensions*,

(b) *whenever* $G$ *is close to being a dsc*, $\mathrm{Tor}(G, Y)$ *is a dsc for every normal* $C_\Omega$ *group* $Y$.

*Proof*. For (a) implies (b), note that if $|Y| \leqslant \aleph_2$, the result follows directly form Theorem 15. In general, by Theorem 7, $Y$ has a closed $\Omega$-development $B_i$ such that $|B_{i+1}/B_i| \leqslant \aleph_2$ for each $i$. This part then follows from Theorem 11 (a). For (b) implies (a), note that if $X$ and $Y$ are normal $C_\Omega$ groups of cardinality $\aleph_2$ and $\aleph_3$ respectively, which contain $\aleph_2$- and $\aleph_3$-Kurepa subgroups, then by the above discussion, $G = \mathrm{Tor}(M_\Omega, X)$ is close to being a dsc and by Theorem 15, $\mathrm{Tor}(M_\Omega, X, Y) \cong \mathrm{Tor}(G, Y)$ fails to be a dsc.

COROLLARY 21. *Suppose* $2^{\aleph_1} = \aleph_2$ *and assume* $G$ *and* $H$ *are normal* $C_\Omega$ *groups of cardinality* $\aleph_3$. *If neither* $G$ *nor* $H$ *has an* $\aleph_1$-*Kurepa subgroup then* $\mathrm{Tor}(G, H)$ *is a dsc*.

*Proof*. Let $A_i$ and $B_i$ denote closed $\Omega$-developments of $G$ and $H$ respectively (these exist by Theorem 8). Clearly $A_i$ and $B_i$ cannot contain $\aleph_1$-Kurepa subgroups, so $\mathrm{Tor}(A_i, B_{i+1}/B_i)$ and $\mathrm{Tor}(A_{i+1}/A_i, B_{i+1})$ are dsc groups by the last theorem. The result then follows from Theorem 10.

We conclude with:

THEOREM 22. *Suppose* $2^{\aleph_1} = \aleph_2$ *and there exists a normal* $C_\Omega$ *group* $X$ *of cardinality* $\aleph_2$ *which has an* $\aleph_2$-*Kurepa subgroup. Then for a normal* $C_\Omega$ *group* $G$ *the following statements are equivalent*:

(a) $\mathrm{Tor}(X, G)$ *is a dsc*,

(b) $\mathrm{Tor}(Y, G)$ *is a dsc for any normal* $C_\Omega$ *group* $Y$,

(c) $G$ *is the smooth ascending union of* $\Omega$-*pure subgroups* $B_i$ *such that* $B_{i+1}/B_i$ *is close to being a dsc for each* $i$.

*Proof*. (b) implies (a) is trivial. (c) implies (b) follows from Theorem 20 and Theorem 11 (a). If $|G| \leqslant \aleph_2$, then (a) implies (c) follows from Theorem 15 (using the trivial chain), and if $|G| > \aleph_2$ we can use Theorem 11 (b). This completes the result.

Observe that the hypotheses of Theorem 22 are true in the constructible universe ($V = L$).

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