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Martingale problem for superprocesses with non-classical branching functional

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Abstract

The martingale problem for superprocesses with parameters (ξ, Φ, k) is studied where $k(\mathrm{d}s)$ may not be absolutely continuous with respect to the Lebesgue measure. This requires a *generalization* of the concept of martingale problem: we show that for any process X which *partially* solves the martingale problem, an extended form of the liftings defined in [E.B. Dynkin, S.E. Kuznetsov, A.V. Skorohod, Branching measure-valued processes, Probab. Theory Related Fields 99 (1995) 55–96] exists; these liftings are part of the statement of the *full martingale problem*, which is hence not defined for processes X who fail to solve the *partial martingale problem*. The existence of a solution to the martingale problem follows essentially from Itô's formula. The proof of uniqueness requires that we find a sequence of (ξ, Φ, k^n)-superprocesses "approximating" the (ξ, Φ, k)-superprocess, where $k^n(\mathrm{d}s)$ has the form $\lambda^n(s, \xi_s)\,\mathrm{d}s$. Using an argument in [N. El Karoui, S. Roelly-Coppoletta, Propriété de martingales, explosion et représentation de Lévy–Khintchine d'une classe de processus de branchement à valeurs mesures, Stochastic Process. Appl. 38 (1991) 239–266], applied to the (ξ, Φ, k^n)-superprocesses, we prove, passing to the limit, that the *full martingale problem* has a unique solution. This result is applied to construct superprocesses with interactions via a Dawson–Girsanov transformation.

Keywords: Superprocesses; Martingale problem; Branching functional; Dawson–Girsanov transformation; Superprocess with interactions

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0. Introduction

0.1. Motivation

Let $(E, \mathcal{B})$ be a measurable space and let $\mathcal{M}_f$ denote the set of finite measures on $(E, \mathcal{B})$. We say that an $\mathcal{M}_f$-valued Markov process $X = (X_t, \Im, P_{r,\mu})$ is a superprocess if its transition probability $P_{r,\mu}$ satisfies the following formula:

$$v_{r,t} f(\mu) = \int_E \mu(\mathrm{d}x) v_{r,t} f(\delta_x)$$

where

$$v_{r,t} f(\mu) = -\log P_{r,\mu} \mathrm{e}^{-\langle X_t, f\rangle}, \quad f \in bp\mathcal{B}, \mu \in \mathcal{M}_f.$$

$v_{r,t} f$ is called the log-Laplace functional and is a semigroup,

$$v_{r,s}(v_{s,t}(f))(\mu) = v_{r,t} f(\mu), \quad \text{for } r < s < t.$$

Superprocesses can be characterized by *evolution equations* of the form

$$v_{r,t}(f)(x) = \pi_{r,x} f(\xi_t) - \pi_{r,x} \int_r^t \Phi(s, \xi_s, v_{s,t}(f)(\xi_s)) k(\mathrm{d}s)$$

where $\xi = (\xi_t, \Im, \pi_{r,x})$ is a Markov process, $k(\mathrm{d}s)$ is an additive functional of ξ and Φ is an operator, which admits only the log-Laplace $v_{r,t}(f)(x) := v_{r,t}(f)(\delta_x)$ as a solution. A detailed exposition of this approach can be found in [7].

The characterization of superprocesses by *evolution equations* has been achieved to a large extent. Indeed, under mild conditions on X, necessary conditions were found for (ξ, Φ, k) in [8]. Under slightly stronger conditions on X, the gap between the necessary and sufficient conditions was filled in [15]. On the other hand characterization of superprocesses in terms of martingale problems was stopped by difficulties arising when considering (ξ, Φ, k)-superprocess with k *non-classical*, i.e. with k non-absolutely continuous with respect to the Lebesgue's measure. In [16], Roelly-Coppoletta posed and solved the martingale problem for the $(\xi, (.)^2, \mathrm{d}s)$-superprocesses where ξ is a Feller process. El-Karoui and Roelly-Coppoletta [9] extended the result to a large class of $(\xi, \Phi, \mathrm{d}s)$-superprocesses where ξ is a Feller process. Fitzsimmons [11] obtained some results on the martingale problem for the $(\xi, \Phi, \mathrm{d}s)$-superprocesses (where ξ is a right process) and in particular he showed that interesting properties can be derived from a well posed martingale problem. Multitype superprocesses were characterized by martingale problems by Gorostiza and Lopez-Mimbela [13]. Fitzsimmons [12] also solved the martingale problem for the $(\xi, \Phi, \mathrm{d}s)$-superprocesses for ξ a right process and

$$\Phi(x, \lambda) = b(x)\lambda^2 + \int_0^\infty \left(\mathrm{e}^{-\lambda u} - 1 + \lambda u\right) n(x, \mathrm{d}u)$$

where $b(x)$ and the kernel $n(x, \mathrm{d}u)$ satisfy some properties. Dawson and Fleischmann showed in [3] that the one point catalytic super Brownian motion, that is the $(\xi, (.)^2, L^c)$-superprocesses (where L_t^c is the local time of the Brownian motion ξ at time t), solves a martingale problem related to the density of the occupation time process.

Difficulties are inherent even in the statement of a martingale problem for superprocesses with branching rates $k(\mathrm{d}s)$ which are not absolutely continuous with respect to the Lebesgue measure. The difficulties first come from the fact that it is not possible to get, in the case of a general k, the classical form of the $(\mathrm{A}, \mathcal{D}(\mathrm{A}))$-martingale problems, where A is an operator with domain $\mathcal{D}(\mathrm{A})$. The statement of the martingale problem itself is problematic. It requires (1) a *partial martingale problem* to identify (see Theorem 7) additive functionals K of X, corresponding to additive functionals k of the motion process ξ (the lifting K of k), and (2) for solutions to this *partial martingale problem*, a *full martingale problem* is needed to characterize the (ξ, Φ, k)-superprocess.

To illustrate this in the case of an absolutely continuous additive functional k, suppose that $X = (X_t, \Im, P_{r,\mu})$ is a process such that

$$t \mapsto \langle X_t, \varphi\rangle - \langle X_r, \varphi\rangle - \int_r^t \langle X_s, \mathrm{A}\varphi\rangle \,\mathrm{d}s$$

is a martingale for every φ in the domain $\mathcal{D}(\mathrm{A})$ of the infinitesimal generator A of ξ. This non-well posed *partial martingale problem* allows us to verify that for any measurable bounded nonnegative $\eta(s, x)$, if the additive functional $K^{\eta\,\mathrm{d}s}$ of X is given by

$$K^{\eta\,\mathrm{d}s}(\mathrm{d}s) = \int_E \eta(s, x) X_s(\mathrm{d}x)\,\mathrm{d}s,$$

then the process

$$s \mapsto K^{\eta\,\mathrm{d}s}(r, s] + \int_E \pi_{s,x}\left(\int_s^t \eta(u, \xi_u)\,\mathrm{d}u\right) X_s(\mathrm{d}x)$$

is a martingale. We call $K^{\eta\,\mathrm{d}s}(\mathrm{d}s)$ the *lifting* of $k(\mathrm{d}s) := \eta(s, \xi_s)\,\mathrm{d}s$. Now fix Φ and assume $k(\mathrm{d}s) = g(s, \xi_s)\,\mathrm{d}s$. Let

$$K^{\Phi(\varphi)\,\mathrm{d}k}(\mathrm{d}s) = \int_E \Phi(x, \varphi(x)) g(s, x) X_s(\mathrm{d}x)\,\mathrm{d}s$$

be the lifting of $\Phi(\xi_s, \varphi(\xi_s)) g(s, \xi_s)\,\mathrm{d}s$. The only solution to the *full martingale problem*

$$t \mapsto \exp(-\langle X_t, \varphi\rangle) + \int_r^t \exp(-\langle X_s, \varphi\rangle)\langle X_s, \mathrm{A}\varphi\rangle\,\mathrm{d}s - \int_r^t \exp(-\langle X_s, \varphi\rangle) K^{\Phi(\varphi)\,\mathrm{d}k}(\mathrm{d}s)$$

is the (ξ, Φ, k)-superprocess.

Recall from [9] that the martingale problem $(L, \mathcal{D})$ for the (ξ, Φ, k)-superprocesses with $k(\mathrm{d}s) = \mathrm{d}s$ is well posed where $\mathcal{D}$ is the class of functions on $\mathcal{M}_{\mathrm{f}}$ given by the formula $F(\mu) = f\langle \mu, \varphi\rangle$ for f infinitely differentiable with compact support on R, $\varphi \in \mathcal{D}(\mathrm{A})$ and $\varphi \geq 0$ and where, for every F in $\mathcal{D}$, $L(F)(\mu)$ is defined by

$$L(F)(\mu) = f'\langle \mu, \varphi\rangle\langle \mu, \mathrm{A}\varphi\rangle + f''\langle \mu, \varphi\rangle\langle b\varphi^2, \mu\rangle + \int_E \int_0^\infty G(F, x, u) n(x, \mathrm{d}u)\mu(\mathrm{d}x),$$

where $G(F, x, u)$ denotes the expression

$$G(F, x, u) = [f(\langle \mu, \varphi\rangle + u\varphi(x)) - f\langle \mu, \varphi\rangle - f'\langle \mu, \varphi\rangle u\varphi(x)].$$

In addition to the intrinsic interest of the martingale problem characterization of (ξ, Φ, k)-superprocesses, this can also be used (see Section 5) to construct superprocesses with interactions. Here the interaction is given by an additional term $\mathfrak{R}$, and the process is called the $(\xi, \Phi, k, \mathfrak{R})$-*superprocess with interactions*. It is characterized as the unique solution of a martingale problem obtained by a *Dawson–Girsanov transformation* of our martingale problem associated to the (ξ, Φ, k)-superprocess. The martingale problem formulation still holds the most promise for interacting models and developing a martingale problem in the general non-interacting case, as done in this paper, is a basic step.

0.2. Partial and full martingale problem

In general, a martingale problem can be formulated in the following way: first, to any (canonical càdlàg) process $X = (X_t, \Im, P_{r,\mu})$, a real valued process $t \mapsto (M^r_G)_t$, $t \geq r$, is defined up to $P_{r,\mu}$-indistinguishability, for every function G belonging to a certain set S. The (canonical càdlàg) process $X = (X_t, \Im, P_{r,\mu})$ or simply $P_{r,\mu}$ is said to be a *solution to the martingale problem* if the processes $t \mapsto (M^r_G)_t$ are $P_{r,\mu}$-martingales for every G in S. The martingale problem $((M^r_G), S)$ is said to be *well posed* if there exists one and only one solution to the martingale problem.

We see a well posed martingale problem as a "test" which characterizes a process. Pick a (canonical càdlàg) process $X = (X_t, \Im, P_{r,\mu})$. The test goes like this:

- For every $G \in S$, check if the process $t \mapsto (M^r_G)_t$ is a $P_{r,\mu}$-martingale.

If the test is a success, X is the only solution to the $((M^r_G), S)$ martingale problem. In the test, the order in which the processes $t \mapsto (M^r_G)_t$ (for $G \in S$) are tested has no importance. We introduce now a slight modification to this procedure. Let $S = S_1 \cup S_2$ where S_1 and S_2 are two disjoint sets. Our new "test" is the following:

- Test whether or not X is a solution to the $((M^r_G), S_1)$-martingale problem.
- If X is a solution to the $((M^r_G), S_1)$-martingale problem, test whether or not X is also a solution to the $((M^r_G), S_2)$-martingale problem.

The non-well-posed martingale problem $((M^r_G), S_1)$ is called the *partial martingale problem*. A solution to the partial martingale problem is called a solution to the *full martingale problem* if it is a solution to the $((M^r_G), S_2)$-martingale problem.

In this paper, partial martingale problems are used to determine certain additive functionals – in terms of the solutions of the partial martingale problem – which enter into the *statement* of the full martingale problem; hence the statement of the full martingale problem is simply not defined for processes $X = (X_t, \Im, P_{r,\mu})$ which are *not* solutions to the partial martingale problem.

0.3. Basic assumptions: Motion process ξ, branching mechanism Φ and branching functional k

Assumption 1. *Throughout this paper, the following assumptions are in* [2] *force:*

(a) (Phase space) (E, d) is a locally compact separable metric space. We denote by $\mathcal{B}$ the σ-algebra generated by d; given a family F of measurable functions, we denote by bF the bounded members of F and by pF the nonnegative $f \in F$. $C(E)$ denotes the set of continuous functions on E while $\hat{C}(E)$ denotes the set of members of $C(E)$ vanishing at infinity.

(b) (Measure space) $\mathcal{M}_{\mathrm{f}}$ (resp. $\mathcal{M}_1$) denotes the set of finite (resp. probability) measures on E, endowed with the topology of weak convergence.

(c) (Time interval) We consider stochastic processes on a fixed interval $I := [0, T]$, $T > 0$, or on subintervals of I.

(d) (Underlying particle's motion process ξ) Process $\xi = (\xi_t, \mathcal{F}, \pi_{r,x})$, is a (time homogeneous) Feller process living in a locally compact separable metric space (E, d). We denote by S_t the semigroup of ξ. We often make use of time inhomogeneous *notation* and in particular:

$$S_t^r(f)(x) := \pi_{r,x} f(\xi_t) := \pi_x f(\xi_{t-r}) = S_{t-r}(f)(x)$$
$$S_t(1) = 1.$$

$\mathcal{L} \supseteq \hat{C}(E)$ denotes an algebra of bounded measurable functions f such that $S_t(f)(x)$ is *strongly* continuous, that is

$$\|S_t(f)(.) - S_{t+h}(f)(.)\|_\infty \to 0 \quad \text{as } h \to 0.$$

Obviously, for ξ is Feller, a particular case is $\hat{C}(E) = \mathcal{L}$. We denote by $(\mathrm{A}, \mathcal{D}(\mathrm{A}))$ the infinitesimal (strong) generator of ξ.

(e) (Branching mechanism) $b(x)$ and $\ell(x, \mathrm{d}u)$ are respectively a measurable function and a kernel satisfying the conditions[1]:

$$0 \le b(x) \le 1, \quad 0 \le \int_0^\infty u^2 \ell(x, \mathrm{d}u) \le 1. \tag{1}$$

Throughout this paper we pose

$$\Phi(x, f(x)) = \frac{1}{2} b(x) f^2(x) + \int_0^\infty \mathcal{E}(uf)\, \ell(x, \mathrm{d}u)$$

where $\mathcal{E}(z) = \mathrm{e}^{-z} + z - 1$. We call Φ a *branching mechanism*. We use the notation $\Phi(x, f) := \Phi(x, f(x))$. In the same spirit as [9], we assume that for every $\varphi(x) \in \mathcal{D}(\mathrm{A})$, $\Phi(x, \varphi(x)) \in \mathcal{L}$. Moreover, we want that Φ be a *regular branching mechanism*, that is, $t \mapsto \Phi(w_t, \varphi_t(w_t))$ is càdlàg when $t \mapsto w_t$ and $t \mapsto \varphi_t(w_t)$ are càdlàg trajectories.

(f) (Branching functional) $k(\mathrm{d}s)$ is a continuous nonnegative additive functional of ξ satisfying the condition

$$h_t^r(x) := \pi_{r,x} k(r, t) \to 0 \quad \text{uniformly in } x \text{ as } t - r \to 0. \tag{2}$$

Note that, since we consider only our processes during the time interval $[0, T]$, this is equivalent to the "admissibility condition" in [7] according to [7, Lemma 3.3.1]. (Such additive functionals are called *admissible additive functionals*.) We assume that $h_t^r(.) \in \mathcal{L}$ for every r, t.

0.4. Partial martingale problem and liftings

In order only to be able to state the martingale problem for the (ξ, Φ, k)-superprocesses, we first need to extend the notion of lifting and projection introduced in [8] to the case where X may not be a Markov process. Given an $\mathcal{M}_\mathrm{f}$-valued Hunt process $X = (X_t, \Im, P_{r,\mu})$ and a E-valued Hunt process ξ, Dynkin, Kuznetsov and Skorohoddefined the lifting $A(\mathrm{d}s)$ of an

[1] Condition (1) means that for $t \in R_+$, $\mu \in \mathcal{M}_\mathrm{f}$, $\sup_{r \le t} P_{r,\mu} \langle X_t, 1\rangle^2 < \infty$. This is a basic assumption in [8] which insures the existence of liftings (see Theorem A.1). It is also assumed in [4] which guarantees the continuous dependence of (ξ, Φ, k)-superprocesses with respect to parameter k (see Theorem A.3).

additive functional $a(\mathrm{d}s)$ of ξ as an additive functional $A(\mathrm{d}s)$ of X such that for every $r, t \in R_+$, every $\mu \in \mathcal{M}_\mathrm{f}$ and every bounded nonnegative measurable $\varphi(.)$

$$P_{r,\mu} A(r,t] = \int_E \mu(\mathrm{d}x) P_{r,\delta_x} A(r,t] \tag{3}$$

and

$$P_{r,\mu} \int_r^\infty \varphi(s) A(\mathrm{d}s) = \pi_{r,\mu} \int_r^\infty \varphi(s) a(\mathrm{d}s) \tag{4}$$

where $\pi_{r,\mu}(.) := \int_E \mu(\mathrm{d}x)\pi_{r,x}(.)$. If $A(\mathrm{d}s)$ is a lifting of $a(\mathrm{d}s)$, then $a(\mathrm{d}s)$ is said to be the projection of $A(\mathrm{d}s)$. And in fact, given a *linear* additive functional $A(\mathrm{d}s)$ of X, that is an additive functional such that (3) is verified, one can find an additive functional $a(\mathrm{d}s)$ of ξ which is the projection of A. The authors proved that the lifting-projection relation establishes a one to one correspondence between the additive functionals of ξ and the linear additive functionals of X. Their proof makes use of the Markov property of X. For our purposes, it was necessary to reduce that condition to the assumption that a certain *partial martingale problem* is verified.

Definition 2 (*Partial Martingale Problem for ξ*). Let $r \in R_+$, $\mu \in \mathcal{M}_\mathrm{f}$ and let $X = (X_t, \Im, P_{r,\mu})$ satisfy the following conditions:

- X_t has its trajectories in $D_{[r,\infty)}(\mathcal{M}_\mathrm{f})$
- $P_{r,\mu}(X_r = \mu) = 1$.
- $\Im$ denotes the collection of filtrations $\{\Im_t^r\}_{t\in[r,\infty)}$ defined by

  $$\Im_t^r = \bigcap_{\varepsilon>0} \sigma(X_s : r \le s \le t+\varepsilon)^{P_{r,\mu}}$$

  where the superscript $^{P_{r,\mu}}$ denotes the completion with respect to $P_{r,\mu}$.
- $P_{r,\mu}^{\Im_t^r}$ denotes the conditional expectation with respect to $\Im_t^r$.

The process $X = (X_t, \Im, P_{r,\mu})$ will be said to be a solution to the (r,μ)-*partial martingale problem for ξ* if for every $\varphi \in \mathcal{D}(\mathrm{A})$

$$t \mapsto \langle X_t, \varphi\rangle - \langle X_r, \varphi\rangle - \int_r^t \langle X_s, \mathrm{A}\varphi\rangle\, \mathrm{d}s \tag{5}$$

is a $P_{r,\mu}$-martingale for $t \in [r, T]$.

The full martingale problem requires for its statement the notion of a lifting of an additive functional:

Definition 3 (*Extended Definition of Lifting*). Let $X = (X_t, \Im, P_{r,\mu})$ be a (canonical càdlàg $\mathcal{M}_\mathrm{f}$-valued) process and let $a(\mathrm{d}s)$ be an additive functional of ξ. A predictable right continuous additive functional $A(\mathrm{d}s)$ of X will be called a *lifting* of $a(\mathrm{d}s)$ if for every $t \ge r$, the process

$$s \mapsto A(r,s] + \int \pi_{s,x} a(s,t] X_s(\mathrm{d}x)$$

is a $P_{r,\mu}$-martingale for $s \in [r,t]$.

Note that this definition of liftings agrees with [8]. The following proposition (which will be proved in a further section) guaranties the existence and uniqueness of liftings for every solution $X = (X_t, \Im, P_{r,\mu})$ to the partial martingale problem.

Proposition 4 (*Liftings Existence and Uniqueness*). *Let the process $X = (X_t, \Im, P_{r,\mu})$ be a solution to the (r, μ)-partial martingale problem for ξ. Then for every additive functional $a(\mathrm{d}s)$ of ξ satisfying (2), there exists a unique lifting $A(\mathrm{d}s)$ of X. Moreover, it is a continuous additive functional.*

Notation 5 ($K^{\Phi(f)\,\mathrm{d}k}(\mathrm{d}s)$). Let f be a progressively measurable and bounded function, and let Φ be a branching mechanism. Then the additive functional $\Phi(\xi_s, f(s, \xi_s))k(\mathrm{d}s)$ satisfies (2), and we will denote by $K^{\Phi(f)\,\mathrm{d}k}(\mathrm{d}s)$ the lifting of $\Phi(\xi_s, f(s, \xi_s))k(\mathrm{d}s)$.

0.5. Full martingale problem

The "full martingale problem" characterization of superprocesses with parameters (ξ, Φ, k) is the main result of this paper.

Definition 6 (*Full Martingale Problem*). Let $r \in R_+$, $\mu \in \mathcal{M}_\mathrm{f}$. A solution $X = (X_t, \Im, P_{r,\mu})$ to the (r, μ)-partial martingale for ξ which is such that for every $\varphi \in \mathcal{D}(\mathrm{A})$ the process

$$t \mapsto \exp(-\langle X_t, \varphi\rangle) + \int_r^t \exp(-\langle X_s, \varphi\rangle)\langle X_s, \mathrm{A}\varphi\rangle\,\mathrm{d}s - \int_r^t \exp(-\langle X_s, \varphi\rangle) K^{\Phi(\varphi)\,\mathrm{d}k}(\mathrm{d}s) \tag{6}$$

is a $P_{r,\mu}$-martingale will be called a solution to the (r, μ)-*full martingale problem for* (ξ, Φ, k). We will say that X is a solution to the (ξ, Φ, k)-*full martingale problem* if it is a solution to the (r, μ)-*full martingale problem for* (ξ, Φ, k) for every $r \in R_+$, $\mu \in \mathcal{M}_\mathrm{f}$.

Theorem 7 (*Martingale Problem*). *Let $r \in R_+$, $\mu \in \mathcal{M}_\mathrm{f}$. Then $X = (X_t, \Im, P_{r,\mu})$ is a solution to the (r, μ)-full martingale problem for (ξ, Φ, k) if and only if $P_{r,\mu} = P_{r,\mu}^{(\xi,\Phi,k)}$, where $P_{r,\mu}^{(\xi,\Phi,k)}$ is the distribution of the (ξ, Φ, k)-superprocess.*

0.6. Outline

To show that the (ξ, Φ, k)-*full martingale problem* is well posed, i.e. to prove Theorem 7, we need to show that (a) the (ξ, Φ, k)-superprocess is a solution to the (ξ, Φ, k)-*full martingale problem* and (b) there is only one possible solution to the (ξ, Φ, k)-*full martingale problem.* Section 1 deals with the proof that the (ξ, Φ, k)-superprocess is a solution to the (ξ, Φ, k)-*full martingale problem*.

The proof that the solution is unique relies on a sequence of superprocesses that we construct to "approximate" (in a strong sense specified below) our given superprocess. The approximating superprocesses, $X^n = (\xi, \Phi, k^n)$, have the property that their branching additive functional rates $k^n(\mathrm{d}s)$ are absolutely continuous with respect to the Lebesgue measure: $k^n(\mathrm{d}s) = \lambda^n(s, \xi_s)\,\mathrm{d}s$. This is done in Section 2.

In Section 3, we study the connection between ξ and any solution X to the *partial martingale problem*. Firstly, additive functionals of ξ can be lifted (Proposition 4), but also the convergence of processes $s \mapsto F^n(s, \xi_s)$ to a process $s \mapsto F(s, \xi_s)$ can also be *"lifted"* to obtain the convergence of processes $s \mapsto \langle X_s, F^n(s, .)\rangle$ to the process $s \mapsto \langle X_s, F(s, .)\rangle$. Furthermore, the convergence (in some weak sense) of additive functionals $a^n(\mathrm{d}s)$ to their limit $a(\mathrm{d}s)$ implies the convergence of their liftings $K^{\mathrm{d}a^n}(\mathrm{d}s)$ to the lifting $K^{\mathrm{d}a}(\mathrm{d}s)$. For our purpose, these results are particularly interesting for $F^n(s, \xi_s) := v^n_{s,T}(\varphi)(\xi_s)$ and for $a^n(\mathrm{d}s) := k^n(\mathrm{d}s)$, where $v^n_{s,T}$ is the log-Laplace functional of the (ξ, Φ, k^n)-superprocess of Section 2.

Indeed, in Section 2, we first show that if $X = (X_t, \Im, P_{r,\mu})$ is a solution to the (r, μ)-*full martingale problem for* (ξ, Φ, k) then the processes

$$t \mapsto \exp(-\langle X_t, v^n_{t,T}(\varphi)\rangle) + \int_r^t \exp(-\langle X_s, v^n_{s,T}(\varphi)\rangle) \left\langle X_s, \mathrm{A}v^n_{s,T}(\varphi) + \frac{\partial}{\partial s} v^n_{s,T}(\varphi) \right\rangle \mathrm{d}s - \int_r^t \exp(-\langle X_s, v^n_{s,T}(\varphi)\rangle) K^{\Phi(v^n_{.,T}(\varphi))\,\mathrm{d}k}(\mathrm{d}s)$$

are martingales. But then, letting

$$K^n_1(\mathrm{d}s) := \langle X_s, \Phi(., v^n_{s,T}(\varphi))\lambda^n(s, .)\rangle \,\mathrm{d}s$$
$$K^n_2(\mathrm{d}s) := K^{\Phi(v^n_{.,T}(\varphi))\,\mathrm{d}k}(\mathrm{d}s)$$

and using the lifted convergence results of Section 3 we can prove, passing to the limit, that the processes

$$t \mapsto \exp(-\langle X_t, v_{t,T}(\varphi)\rangle)$$

are martingales, so

$$P_{r,\mu}(\exp(-\langle X_T, \varphi\rangle)) = \exp(-\langle \mu, v_{r,T}(\varphi)\rangle) = P^{(\xi,\Phi,k)}_{r,\mu}(\exp(-\langle X_T, \varphi\rangle))$$

and $P_{r,\mu} = P^{(\xi,\Phi,k)}_{r,\mu}$, which completes the argument.

Finally, in Section 5, the *full martingale problem for* (ξ, Ψ, k) is applied to construct superprocesses with interactions via a Dawson–Girsanov transformation for the binary branching $\Psi(s, x, \lambda) = \lambda^2$.

1. Proof of the existence of a solution to the martingale problem

In this section we prove the existence part of Theorem 7, that is, we show that the distribution $P^{(\xi,\Phi,k)}_{r,\mu}$ of the (ξ, Φ, k)-superprocess is a solution to the full martingale problem.

Clearly, the process (5) is a $P^{(\xi,\Phi,k)}_{r,\mu}$-martingale for every $\varphi \in \mathcal{D}(\mathrm{A})$. Existence and uniqueness of liftings is given from Theorem A.1. Let $C_t(\varphi)$ be the quadratic variation of the continuous martingale part of the semimartingale $\langle X_t, \varphi\rangle$. Then Itô's formula implies that

$$t \mapsto (\langle X_t, \varphi\rangle)^2 - (\langle X_r, \varphi\rangle)^2 - 2\int_r^t \langle X_s, \varphi\rangle\langle X_s, \mathrm{A}\varphi\rangle \,\mathrm{d}s - C_t(\varphi) + \sum_{r<s\le t} ((\langle X_{s-} + \Delta X_s, \varphi\rangle)^2 - (\langle X_{s-}, \varphi\rangle)^2 - 2\langle X_{s-}, \varphi\rangle\langle \Delta X_s, \varphi\rangle)$$

is a $P_{r,\mu}$-martingale for every $\varphi \in \mathcal{D}(\mathrm{A})$. Simplifying we obtain that

$$t \mapsto (\langle X_t, \varphi\rangle)^2 - (\langle X_r, \varphi\rangle)^2 - 2\int_r^t \langle X_s, \varphi\rangle\langle X_s, \mathrm{A}\varphi\rangle \,\mathrm{d}s - C_t(\varphi) + \sum_{r<s\le t} (\langle \Delta X_s, \varphi\rangle)^2$$

is a $P_{r,\mu}$-martingale for every $\varphi \in \mathcal{D}(\mathrm{A})$. By definition of the modified Lévy measure, this is the same thing as saying that

$$t \mapsto (\langle X_t, \varphi\rangle)^2 - (\langle X_r, \varphi\rangle)^2 - 2\int_r^t \langle X_s, \varphi\rangle\langle X_s, \mathrm{A}\varphi\rangle\, \mathrm{d}s$$
$$- C_t(\varphi) + \int_r^t \int_{\mathcal{M}_\mathrm{f}} \langle \mu, \varphi\rangle^2 L(\mathrm{d}s, \mathrm{d}\mu) \tag{7}$$

is a $P_{r,\mu}$-martingale for every $\varphi \in \mathcal{D}(\mathrm{A})$, where $\int_{\mathcal{M}_\mathrm{f}} \langle \mu, \varphi\rangle^2 L(\mathrm{d}s, \mathrm{d}\mu)$ is the lifting of $\int_0^\infty (u\varphi)^2(\xi_s)\ell(\xi_s, \mathrm{d}u)k(\mathrm{d}s)$. Note that (by definition of lifting)

$$P_{r,\mu} \int_r^t \int_{\mathcal{M}_\mathrm{f}} \langle \mu, \varphi\rangle^2 L(\mathrm{d}s, \mathrm{d}\mu) = \pi_{r,\mu} \int_r^t \int_0^\infty (u\varphi)^2(\xi_s)\ell(\xi_s, \mathrm{d}u)k(\mathrm{d}s). \tag{8}$$

Since the $P_{r,\mu}$-expectation of martingale (7) is zero, we can use (8) and the moment formulae of Theorem A.1 to calculate

$$P_{r,\mu}(C_t(\varphi)) = \pi_{r,\mu} \int_r^t b(\xi_s)\varphi^2(\xi_s)k(\mathrm{d}s).$$

Thus

$$P_{r,\mu}\left(\frac{1}{2}C_t(\varphi) - \int_r^t \hat{Q}(\varphi^2)(\mathrm{d}s)\right) = 0$$

where $\hat{Q}(\varphi^2)(\mathrm{d}s)$ is the lifting of $\frac{1}{2}b(\xi_s)\varphi^2(\xi_s)k(\mathrm{d}s)$. Therefore, since X_t is a Markov process, this implies that

$$t \mapsto \frac{1}{2}C_t(\varphi) - \int_r^t \hat{Q}(\varphi^2)(\mathrm{d}s)$$

is a martingale. But because $t \mapsto \frac{1}{2}C_t(\varphi) - \int_r^t \hat{Q}(\varphi^2)(\mathrm{d}s)$ is also a right continuous predictable process of integrable variation, we obtain that $\frac{1}{2}C_t(\varphi) \equiv \int_r^t \hat{Q}(\varphi^2)(\mathrm{d}s)$. We can apply Itô's formula which gives that

$$t \mapsto \exp(-\langle X_t, \varphi\rangle) + \int_r^t \exp(-\langle X_s, \varphi\rangle)\langle X_s, \mathrm{A}\varphi\rangle\, \mathrm{d}s$$
$$- \int_r^t \exp(-\langle X_s, \varphi\rangle)\hat{Q}(\varphi^2)(\mathrm{d}s)$$
$$- \int_r^t \exp(-\langle X_s, \varphi\rangle) \int_{\mathcal{M}_\mathrm{f}} \mathcal{E}(-\langle \mu, \varphi\rangle)L(\mathrm{d}s, \mathrm{d}\mu)$$

is a $P_{r,\mu}^{(\xi,\Phi,k)}$-martingale. But since $\hat{Q}(\varphi^2)(\mathrm{d}s)$ is the lifting of the additive functional $\frac{1}{2}b(\xi_s)\varphi^2(\xi_s)k(\mathrm{d}s)$ and $\int_{\mathcal{M}_\mathrm{f}} \mathcal{E}(-\langle \mu, \varphi\rangle)L(\mathrm{d}s, \mathrm{d}\mu)$ is the lifting of $\int_0^\infty \mathcal{E}(u\varphi(\xi_s))\ell(\xi_s, \mathrm{d}u)k(\mathrm{d}s)$, this can be rewritten to give that

$$t \mapsto \exp(-\langle X_t, \varphi\rangle) + \int_r^t \exp(-\langle X_s, \varphi\rangle)\langle X_s, \mathrm{A}\varphi\rangle\, \mathrm{d}s$$
$$- \int_r^t \exp(-\langle X_s, \varphi\rangle)K^{\Phi(\varphi)\,\mathrm{d}k}(\mathrm{d}s)$$

is a $P_{r,\mu}^{(\xi,\Phi,k)}$-martingale. □

2. Approximation of superprocesses

As explained in Section 0.6, in order to prove that the full martingale problem has only one solution, we need to *approximate* (in a rather strong sense specified below) superprocesses by other superprocesses with branching rate of the form $k^n(\mathrm{d}s) = \lambda^n(s, \xi_s)\,\mathrm{d}s$. This is done in Theorem 16 below which may have some independent interest. But before, some technical results are needed.

2.1. Some technical lemmas

Lemma 8. *Let $(\Omega, \Im, P)$ be a filtered probability space. and let $t \mapsto x_t^n$, $n \geq 1$ be right continuous processes such that $\sup_{\tau\in[r,t]} P(|x_\tau^n|) \to 0$, where the expression $\sup_{\tau\in[r,t]}$ denotes here the supremum over all stopping times τ such that $r \leq \tau \leq t$. Then*

$$\sup_{s\in[r,t]} \left|x_s^n\right| \to 0 \quad \text{in } P\text{-probability.}$$

Proof. Let $\eta > 0$. Let $\tau_\eta^n := \inf\{s \in [r,t] : |x_s^n| > \eta\}$, where $\inf\phi := t$. Then we have

$$\begin{aligned} P\left\{\sup_{s\in[r,t]} |x_s^n| > \eta\right\} &\leq P\{|x_{\tau_\eta^n}^n| \geq \eta\} \\ &\leq \frac{1}{\eta} P(|x_{\tau_\eta^n}^n|) \end{aligned}$$

and this converges to zero by hypothesis. $\square$

Lemma 9. *Let $(\Omega, \mathcal{G}, P)$ be a probability space and $a_n(\mathrm{d}s)$ be a sequence of random measures on $\mathbf{R}_+$ such that $P|a_n([0,t]) - a([0,t])| \to 0$ for every $t \geq 0$. Then there exists a subsequence a_{n_k} such that P-a.s. $a_{n_k} \Longrightarrow a$.*

Proof. With the use of Cantor's diagonalization method one finds a subsequence a_{n_k} such that

$$P\{|a_{n_k}[0,q] - a[0,q]| \to 0 \text{ for every rational } q \geq 0\} = 1.$$

But then, because the mappings $t \longmapsto a_n[0,t]$ are increasing, this implies that P-a.s. $a_{n_k} \Longrightarrow a$. $\square$

Lemma 10. *Let $k(\mathrm{d}s)$ be any additive functional of ξ. Let $S_t^r(f)(x) = S_{t-r}(f)(x)$ be the semigroup generated by ξ. For every $0 \leq r \leq s \leq t$ we have that $S_s^r(h_t^s)(x) \leq h_t^r(x)$ and*

$$|S_s^r(h_t^s)(x) - h_t^r(x)| = h_s^r(x)$$

where $h_t^s(x) = \pi_{s,x} k(s,t)$.

Proof.

$$\begin{aligned} S_s^r(h_t^s)(x) &= \pi_{r,x}(\pi_{s,\xi_s} k(s,t]) \\ &= \pi_{r,x}(k(s,t]) \\ &= \pi_{r,x}(k(r,t]) - \pi_{r,x}(k(r,s]) \\ &= h_t^r(x) - h_s^r(x). \quad \square \end{aligned}$$

2.2. A-*smooth approximation of superprocesses*

In this section we introduce the concept of A-*smooth approximation of superprocesses*. The key result here is Theorem 16 below, which states that, under Assumptions 0.3(a)–0.3(c), an A-smooth approximation exists.

Definition 11. A sequence $k^n(\mathrm{d}s)$ of additive functionals of ξ is said to be uniformly admissible if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $s, t \in [0, T]$, $|s - t| < \delta$ implies that $\sup_n \| {}^n h^s_t \|_\infty < \varepsilon$ where ${}^n h^s_t(x) = \pi_{s,x} k^n(s, t)$.

Definition 12. We say that a mapping $\psi : [0, T] \times E \to R_+$ is smooth for the strong generator $(\mathrm{A}, \mathcal{D}(\mathrm{A}))$ of ξ, or simply that ψ is smooth for A, if

(1) $\psi(s, .)$ belongs to $\mathcal{D}(\mathrm{A})$ for every s
(2) $\frac{\partial}{\partial s}\psi(s, x)$ exists for every s and

$$\left\| \frac{\psi(s+h, .) - \psi(s, .)}{h} - \frac{\partial}{\partial s}\psi(s, .) \right\|_\infty \to 0$$

(3) ψ, $\frac{\partial}{\partial s}\psi$ and $\mathrm{A}\psi$ are bounded and strongly continuous.

Definition 13. We say that (ξ, Φ, k^n)-superprocesses X^n form an A-smooth approximation for the (ξ, Φ, k)-superprocess X if:

- $k^n(\mathrm{d}s)$ has the form $\lambda^n(s, \xi_s)\,\mathrm{d}s$
- the log-Laplace functional v^n of X^n converges to the log-Laplace functional v of X.
- for every $f \in \mathcal{D}(\mathrm{A})$, the function $\psi^n(s, x) := v^n_{s,T}(f)(x)$ is smooth for A.

The proof of existence of an A-smooth approximation relies on the following lemma where we show that any admissible additive functional k can be approximated (in some rather strong sense) by additive functionals k^n of the form $k^n(\mathrm{d}s) = \lambda^n(s, \xi_s)\,\mathrm{d}s$. This is used in Theorem 16 to show that, then, the sequence of (ξ, Φ, k^n)-superprocesses form an A-smooth approximation for the (ξ, Φ, k)-superprocess.

Lemma 14. *Let $k(\mathrm{d}s)$ be a (continuous) admissible additive functional of a right process ξ. There exists a sequence of additive functionals $k^n(\mathrm{d}s)$ of the form*

$$k^n(\mathrm{d}s) = \lambda^n(s, \xi_s)\,\mathrm{d}s$$

such that

(i) $\sup_{0 \le s < t \le T} \sup_{x \in E} |{}^n h^s_t(x) - h^s_t(x)|$ *tends to zero as n tends to infinity, where* ${}^n h^s_t(x) = \pi_{s,x} k^n(s, t)$ *and* $h^s_t(x) = \pi_{s,x} k(s, t)$;
(ii) *the sequence $k^n(\mathrm{d}s)$ is uniformly admissible;*
(iii) *$k^n(r, \tau]$ converges to $k(r, \tau]$ in $L^1(\pi_{r,x})$ for every r-stopping time*[2] *τ (bounded by T) and every r, x;*
(iv) *for every r, x there exists a subsequence $\{k^{n_k}(\mathrm{d}s)\}_{k=1}^\infty$ converging weakly to $k(\mathrm{d}s)$.*

[2] By this we mean a stopping time $\tau \ge r$ with respect to the filtration $\{\mathcal{F}^r_t\}_{t \in [r, \infty)}$.

Proof. Let $t_i^n := \frac{i}{n}T$; choose $\frac{1}{n}T > \delta_n > 0$ such that for every $\alpha \le \beta$ such that $|\alpha - \beta| \le \delta_n$ we have

$$\|h_\beta^\alpha\|_\infty \le \frac{1}{n^2}.$$

Let us denote by pC_c^∞ the set of all infinitely differentiable nonnegative functions $f : R_+ \to R_+$ with a compact support. We denote by $\mathrm{supp}\{f\}$ the support of a function $f \in pC_c^\infty(R_+)$. Choose a function f_i^n in $pC_c^\infty(R_+)$ such that

(1) $\mathrm{supp}\{f_i^n\} \subset [t_i^n, t_i^n + \delta_n]$
(2) $\int f_i^n(s)\,\mathrm{d}s = 1$
(3) (for simplicity) $f_i^n(s)$ is a translation of $f_j^n(s)$.

Let

$$k^n(\mathrm{d}s) := \sum_{i=0}^{n-1} h_{t_{i+1}^n}^s(\xi_s) f_i^n(s)\,\mathrm{d}s$$

and

$${}^n h_t^s(x) := \pi_{s,x} k^n(s, t].$$

Note that

$${}^n h_T^{t_j^n}(x) = \sum_{i=j}^{n-1} \int_{t_i^n}^{t_i^n+\delta_n} f_i^n(s)\,\mathrm{d}s\, S_s^{t_j^n}(h_{t_{i+1}^n}^s)(x). \tag{9}$$

But for $s \in [t_i^n, t_i^n + \delta_n]$

$$\begin{aligned} S_s^{t_j^n}(h_{t_{i+1}^n}^s)(x) &= \pi_{t_j^n,x}(\pi_{s,\xi_s} k(s, t_{i+1}^n]) \\ &= \pi_{t_j^n,x}(k(s, t_{i+1}^n]) \\ &= \pi_{t_j^n,x}(k(t_i^n, t_{i+1}^n]) - \pi_{t_j^n,x}(k(t_i^n, s]) \\ &= \pi_{t_j^n,x}\left(\pi_{t_i^n,\xi_{t_i^n}} k(t_i^n, t_{i+1}^n]\right) - \pi_{t_j^n,x}\left(\pi_{t_i^n,\xi_{t_i^n}} k(t_i^n, s]\right) \\ &= \pi_{t_j^n,x}\left(h_{t_{i+1}^n}^{t_i^n}(\xi_{t_i^n})\right) - \pi_{t_j^n,x}\left(h_s^{t_i^n}(\xi_{t_i^n})\right). \end{aligned}$$

Thus

$$\left\| S_s^{t_j^n}(h_{t_{i+!}^n}^s)(x) - \pi_{t_j^n,x}\left(h_{t_{i+1}^n}^{t_i^n}(\xi_{t_i^n})\right)\right\|_\infty \le \max_{i=0,\ldots,n-1} \left\| h_{t_i^n+\delta_n}^{t_i^n}\right\|_\infty \le \frac{1}{n^2}.$$

Returning to Eq. (9) we get that

$$\max_{j=0,\ldots,n} \left\| {}^n h_T^{t_j^n}(x) - h_T^{t_j^n}(x)\right\|_\infty \le \frac{1}{n}.$$

Now if $s \in (t_{j-1}^n, t_j^n)$ we have that

$$\begin{aligned}\|{}^{n}h^{s}_{T}(x)-h^{s}_{T}(x)\|_{\infty} &= \left\|\pi_{s,x}\left(k^{n}(s,t^{n}_{j}]-k(s,t^{n}_{j}]\right)\right\|_{\infty}\\ &= +\left\|\pi_{s,x}\left({}^{n}h^{t^{n}_{j}}_{T}(\xi_{t^{n}_{j}})-h^{t^{n}_{j}}_{T}(\xi_{t^{n}_{j}})\right)\right\|_{\infty}\\ &\leq 2\sup_{s\in(t^{n}_{j-1},t^{n}_{j})}\left\|h^{s}_{t^{n}_{j}}\right\|_{\infty}+\frac{1}{n};\end{aligned}$$

and the last expression tends to zero as n tends to infinity. Moreover, since

$${}^{n}h^{s}_{t}(x)-h^{s}_{t}(x)=\pi_{s,x}(k^{n}(s,T]-k(s,T])-\pi_{s,x}(\pi_{t,\xi_t}k(t,T]-\pi_{t,\xi_t}k^{n}(t,T])$$

we easily derive that

$$\sup_{0\leq s,t\leq T,x\in E}|{}^{n}h^{s}_{t}(x)-h^{s}_{t}(x)|\to 0$$

as n tends to infinity. This establishes that $k^{n}(\mathrm{d}s)$ satisfies property (i). Property (ii) is an immediate consequence of (i).

It remains only to establish property (iii) and (iv). But property (i) implies that for every $r\geq 0$ and every $x\in E$, we have that

$$\sup_{\tau\in[r,T]}\pi_{r,x}(|{}^{n}h^{\tau}_{T}(\xi_{\tau})-h^{\tau}_{T}(\xi_{\tau})|)\to 0$$

where the supremum is taken over all r-stopping times τ such that $r\leq\tau\leq T$. Consequently, from Lemma 8, we obtain that $\sup_{s\in[r,T]}|^{n}\,h^{s}_{T}(\xi_s)-h^{s}_{T}(\xi_s)|\to 0$ in $\pi_{r,x}$-probability. One verifies easily that all the hypotheses of Theorem A.2 are verified, and this yields property (iii). Property (iv) is immediate from Lemma 9, and the proof is complete. □

Remark 15. In Lemma 14, the sequence of additive functional $k^{n}(\mathrm{d}s)$ can be chosen to have the form

$$k^{n}(\mathrm{d}s)=\sum_{i=0}^{n-1}h^{t^{n}_{i}}_{t^{n}_{i+1}}(\xi_s)f^{n}_{i}(s)\,\mathrm{d}s,$$

where $f^{n}_{i}\in pC^{\infty}_{c}(R_{+})$ for $n=1,2,\ldots;i=0,\ldots,n-1;$

Proof. Choose δ_n such that for every $r\geq 0$ and every $\alpha\leq\beta\leq\alpha+\delta_n$ we have

$$\max_{i=1,\ldots,n}\left\|S^{r}_{\alpha}(h^{t^{n}_{i}}_{t^{n}_{i+1}})-S^{r}_{\beta}(h^{t^{n}_{i}}_{t^{n}_{i+1}})\right\|_{\infty}+\left\|h^{\alpha}_{\beta}\right\|_{\infty}\leq\frac{1}{n^{2}}.$$

Proceed then exactly like in the proof of Lemma 14. Note that if $r\in\{t^{n}_{0},\ldots,t^{n}_{n}\}$ then

$${}^{n}h^{r}_{T}(x)=\sum_{t^{n}_{i}\geq r}\int_{t^{n}_{i}}^{t^{n}_{i}+\delta_n}f^{n}_{i}(s)\,\mathrm{d}s\,S^{r}_{s}(h^{t^{n}_{i}}_{t^{n}_{i+1}})(x).$$

But since for $s\in[t^{n}_{i},t^{n}_{i}+\delta_n]$ we have

$$\left\|S^{r}_{s}(h^{t^{n}_{i}}_{t^{n}_{i+1}})-S^{r}_{t^{n}_{i}}(h^{t^{n}_{i}}_{t^{n}_{i+1}})\right\|_{\infty}\leq\frac{1}{n^{2}}$$

and since

$$\sum_{t^{n}_{i}\geq r}\int_{t^{n}_{i}}^{t^{n}_{i}+\delta_n}f^{n}_{i}(s)\,\mathrm{d}s\,S^{r}_{t^{n}_{i}}(h^{t^{n}_{i}}_{t^{n}_{i+1}})(x)=h^{r}_{T}(x),$$

we get

$$\left\|{}^n h^r_T(x) - h^r_T(x)\right\| \le \frac{1}{n}.$$

The rest is similar to the proof of Lemma 14. □

Theorem 16. *There exists a uniformly admissible sequence of additive functionals $k^n(\mathrm{d}s)$ with $k^n(\mathrm{d}s) = \lambda^n(s, \xi_s)\,\mathrm{d}s$ which are such that the sequence of (ξ, Φ, k^n)-superprocesses form an* A*-smooth approximation for the (ξ, Φ, k)-superprocess. For every $(r, x) \in R_+ \times E$ and every r-stopping time $\tau \le T$, $k^n(r, \tau]$ converges in $L^1(\pi_{r,x})$ to $k(r, \tau]$.*

Proof. Let $\{k^n(\mathrm{d}s)\}$ be a collection of approximating additive functionals as in Remark 15. Let $v^n_{r,t}(f)(x)$ be the log-Laplace functional of the corresponding (ξ, Φ, k^n)-superprocess, for $n = 1, 2, \ldots$. According to Theorem A.3, $v^n_{r,t}(f)(x) \to v_{r,t}(f)(x)$ where $v_{r,t}(f)(x)$ is the log-Laplace functional of the (ξ, Φ, k)-superprocess. According to Theorem A.5, $v^n_{r,t}(f)(x)$ is smooth for A and $\mathrm{A}v^n_{s,T}(f)(x) + \frac{\partial}{\partial s} v^n_{s,T}(f)(x) = \Phi(x, v^n_{s,T}(f)(x))\lambda^n(s, x)$ for every $0 \le s \le T, x \in E$ and $f \in \mathcal{D}(\mathrm{A})$. □

3. The partial martingale problem

In this section we investigate some of the properties shared by all processes $X = (X_t, \Im, P_{r,\mu})$ which are solutions to the partial martingale problem. One of these properties is that, for such processes, liftings exist, and therefore, the full martingale problem *can be stated*.

We also prove that the convergence of processes $s \mapsto F^n(s, \xi_s)$ to a process $s \mapsto F(s, \xi_s)$ can be *"lifted"* to obtain the uniform convergence of processes $s \mapsto \langle X_s, F^n(s, .)\rangle$ to the process $s \mapsto \langle X_s, F(s, .)\rangle$. Also, the convergence (in some weak sense) of additive functionals $a^n(\mathrm{d}s)$ to their limit $a(\mathrm{d}s)$ implies the convergence of their liftings $K^{\mathrm{d}a^n}(\mathrm{d}s)$ to the lifting $K^{\mathrm{d}a}(\mathrm{d}s)$.

3.1. Connection between X and its particle motion ξ

The following result is due to Fitzsimmons [12, Corollary 2.8]. It establishes – via the partial martingale problem – a link between solutions X to the partial martingale problem and their projection ξ.

Lemma 17. *Let $X = (X_t, \Im, P_{r,\mu})$ be a solution to the partial martingale problem and let S_t be the semigroup of ξ. If τ is a bounded r-stopping time then for all $f \in b\mathcal{B}$*

$$P^{\Im^r_\tau}_{r,\mu}\langle X_{\tau+t}, f\rangle = \langle X_\tau, S_t f\rangle, \quad \textit{for every } t \ge 0$$

where $P^{\Im^r_\tau}_{r,\mu}$ denotes the conditional expectation with respect to $\Im^r_\tau$.

The following technical lemma will be used several times in this paper:

Lemma 18. *Let $X = (X_t, \Im, P_{r,\mu})$ be a solution to the partial martingale problem for ξ. Then for every $f \in b\mathcal{B}$, every $T > 0$ the process $t \mapsto \langle X_t, S_{T-t} f\rangle$ is a càdlàg martingale. In particular, for every r-stopping time τ bounded by T we have that*

$$P_{r,\mu}\langle X_\tau, S_{T-\tau} f\rangle = \langle \mu, S_{T-r} f\rangle.$$

Proof. From Lemma 17, we have that

$$P_{r,\mu}^{\Im_t^r}\langle X_{t+s}, S_{T-t-s}(f)\rangle = \langle X_t, S_{T-t}(f)\rangle$$

and hence the process $t \mapsto \langle X_t, S_{T-t}(f)\rangle$ is a martingale. Since it is dominated by $t \mapsto \|f\|_\infty \langle X_t, 1\rangle$, it belongs to class (D), according to [7, Lemma A.1.1]. If $f \in \mathcal{D}(\mathrm{A})$, then $S_t f \in \mathcal{D}(\mathrm{A})$ for every $t \geq 0$. Hence, for every t', the process $t \mapsto \langle X_t, S_{T-t'}(f)\rangle$ is a càdlàg process. Hence if Λ_n denotes a sequence of partitions $\{t_i^n\}_{i=0}^n$ of the interval $[r, T]$ with $\mathrm{mesh}\{\Lambda_n\} \to 0$, then the process x_t^n defined by

$$t \mapsto x_t^n := \sum_{i=0}^{n-1} 1_{[t_i^n, t_{i+1}^n)}(t)\langle X_t, S_{T-t_i^n} f\rangle$$

is a càdlàg process. Because $f \in \mathcal{D}(\mathrm{A})$, $1_{[t_i^n, t_{i+1}^n)}(t) S_{T-t_i^n} f(x)$ converges uniformly (in $x \in E$ and $t \in [r, T]$) to $S_{T-t}(f)(x)$. Therefore $t \mapsto x_t^n$ converges uniformly (in $t \in [r, T]$ for every $\omega \in \Omega$) to $t \mapsto \langle X_t, S_{T-t} f\rangle$. Consequently, $t \mapsto \langle X_t, S_{T-t} f\rangle$ is a càdlàg martingale. From the optional sampling theorem we get that for every r-stopping time τ bounded by T

$$P_{r,\mu}\langle X_\tau, S_{T-\tau} f\rangle = \langle \mu, S_{T-r} f\rangle. \tag{10}$$

The extension of equality (10) to arbitrary $f \in b\mathcal{B}$ follows from the fact that $\mathcal{D}(\mathrm{A})$ is dense, for the bounded pointwise convergence, in $b\mathcal{B}$. From lemma [7, A.1.1.D], we conclude from this equality that $t \mapsto \langle X_t, S_{T-t} f\rangle$ is a right continuous – and therefore càdlàg – martingale. □

Corollary 19. *Let $X = (X_t, \Im, P_{r,\mu})$ be a solution to the partial martingale problem for ξ. Let $\beta \in (r, T]$, $\alpha \in [r, T]$ and let $f(.) \in b\mathcal{B}$. Then the process*

$$t \mapsto x_t := 1_{[\alpha,\beta)}(t)\langle X_t, S_\beta^t f\rangle$$

is càdlàg, and moreover, for every $\delta > 0$ and every stopping time τ

$$P_{r,\mu} x_{\tau+\delta} = P_{r,\mu} 1_{[\alpha,\beta)}(\tau + \delta)\langle X_\tau, S_\beta^\tau f\rangle.$$

Proof. The process $t \mapsto \langle X_{t\wedge\beta}, S_\beta^{t\wedge\beta} f\rangle$ is a càdlàg martingale; so x_t is càdlàg. Let τ be a stopping time and $\delta > 0$. Note that without lost of generality, we can assume that $\tau + \delta \leq \beta$: this is due to the fact that for $\tau + \delta > \beta$, we have $x_{(\tau+\delta)\wedge\beta} = x_{\tau+\delta} = 0$. From the optional stopping time theorem, we get

$$P_{r,\mu}^{\Im_\tau^r}\langle X_{\tau+\delta}, S_\beta^{\tau+\delta} f\rangle = \langle X_\tau, S_\beta^\tau f\rangle$$

where $P_{r,\mu}^{\Im_\tau^r}(.)$ denotes the conditional expectation with respect to $\Im_\tau^r$. Because $1_{[\alpha,\beta)}(\tau+\delta) \in \Im_\tau^r$ this completes the proof. □

Corollary 20. *Let $X = (X_t, \Im, P_{r,\mu})$ be a solution to the partial martingale problem for ξ. Let $t_0 := r < t_1 < \cdots < t_n := T$ be a partition of $[r, T]$. Let $f^i(.) \in b\mathcal{B}$, that is a bounded $\mathcal{B}$-measurable function, for $i = 1, \ldots, n$. Then the process*

$$t \mapsto x_t := \sum_{i=0}^{n-1} 1_{[t_i, t_{i+1})}(t)\left\langle X_t, S_{t_{i+1}}^t f^{i+1}\right\rangle$$

is càdlàg, and for every stopping time τ and every $\delta > 0$ we have that

$$P_{r,\mu}(x_{\tau+\delta}) = P_{r,\mu} \sum_{i=0}^{n-1} 1_{[t_i,t_{i+1})}(\tau + \delta) \left\langle X_\tau, S^\tau_{t_{i+1}} f^{i+1} \right\rangle.$$

Proof. This is immediate from the above corollary. □

3.2. Liftings

Consider now the function $h^r_T(x) := \pi_{r,x} a(r, T]$ which is called the *characteristic* of the additive functional $a(\mathrm{d}s)$. Assume that h_T is bounded. Note that by Markov property, for every $0 \le r \le s \le T$ we have

$$S^r_s(h^s_T)(x) = \pi_{r,x}(\pi_{s,\xi_s} a(s, T]) = \pi_{r,x} a(s, T] \le h^r_T(x).$$

We use this in the following proof of the existence and uniqueness of liftings for solutions to the partial martingale problem.

Proof of Proposition 4. According to Lemma 17

$$P^{\Im^r_t}_{r,\mu} \langle X_{t+s}, f\rangle = \langle X_t, S_s f\rangle, \quad P_{r,\mu}\text{-almost surely for every } f \in b\mathcal{B}.$$

Consequently,

$$P^{\Im^r_t}_{r,\mu} \left\langle X_{t+s}, h^{t+s}_T \right\rangle = \left\langle X_t, S_s \left(h^{t+s}_T\right)\right\rangle \le \left\langle X_t, h^t_T \right\rangle,$$

and therefore process $t \mapsto x_t := \langle X_t, h^t_T\rangle$ is a supermartingale.

Let $\Lambda_n := r = t^n_0 < \cdots < t^n_n = T$ be a sequence of nested partitions of the interval $[r, T]$ with $\text{mesh}\{\Lambda_n\} \to 0$. According to Corollary 20, the processes

$$t \mapsto x^n_t := \sum_{i=0}^{n-1} 1_{[t^n_i,t^n_{i+1})}(t) \left\langle X_t, S^t_{t^n_{i+1}} h^{t^n_{i+1}}_T \right\rangle$$

are càdlàg. Since $a(\mathrm{d}s)$ is admissible, we have that, according to Lemma 10,

$$\max_{i=0,\dots,n-1} \sup_{t\in[t^n_i,t^n_{i+1})} \left\| S^t_{t^n_{i+1}} h^{t^n_{i+1}}_T - h^t_T \right\|_\infty \to 0.$$

Moreover, due to the fact that $t \mapsto \langle X_t(\omega), 1\rangle$ is càdlàg,

$$\sup_{s\in[r,T]} \langle X_s(\omega), 1\rangle < \infty.$$

We can thus conclude that

$$\lim_{n\to\infty} \sup_{t\in[r,T]} |x^n_t(\omega) - x_t(\omega)| = 0.$$

The uniform limit of a sequence of càdlàg functions being also càdlàg, we conclude that $t \mapsto x_t$ is càdlàg.

Thus, by Doob–Meyer decomposition theorem (cf. [7, Theorem A.1.1]), $t \mapsto x_t$ has a unique compensator $A(\mathrm{d}s)$ (which is, by definition of lifting, the unique lifting of $a(\mathrm{d}s)$) and

$$A(r, t] = \lim_{\Lambda} \sum_{i=1}^{n-1} P^{\Im^r_{t_i-}}_{r,\mu} \left\{\left\langle X_{t_i}, h^{t_i}_T\right\rangle - \left\langle X_{t_{i+1}}, h^{t_{i+1}}_T\right\rangle\right\}$$

$$= \lim_{\Lambda} \sum_{i=1}^{n-1} P_{r,\mu}^{\Im^r_{t_i-}} \left\{ \left\langle X_{t_i}, h_T^{t_i} \right\rangle - \left\langle X_{t_i}, S_{t_{i+1}}^{t_i} h_T^{t_{i+1}} \right\rangle \right\}$$

$$= \lim_{\Lambda} \sum_{i=1}^{n-1} P_{r,\mu}^{\Im^r_{t_i-}} \left\{ \left\langle X_{t_i}, h_{t_{i+1}}^{t_i} \right\rangle \right\} \tag{11}$$

weakly in $L^1(P_\mu)$ as Λ runs over a standard sequence of partitions $\Lambda = \{r = t_0 < t_1 < \cdots < t_n = t\}$ of the interval $[r, t]$. Moreover, the convergence in (11) is strong when A is continuous.

We now show that the lifting A of an admissible additive functional a is continuous. According to [7, Theorem A.1.1], A is continuous if and only if for every sequence of r-stopping times $\tau_n \nearrow \tau$, with $\tau_n < \tau$, we have $E_{r,\mu} x_{\tau_n} \searrow E_{r,\mu} x_\tau$.

Let the r-stopping times τ_n increase to τ. Clearly, since $x_t = 0$ for $t \geq T$, we can suppose that $\tau \leq T$. Choose ε and pick δ such that $|\alpha - \beta| \leq \delta$ implies $\|h^\alpha_\beta\|_\infty \leq \varepsilon$.

We have

$$P_{r,\mu} x_{\tau_n} \geq P_{r,\mu} x_\tau \geq P_{r,\mu} x_{\tau \vee (\tau_n + \delta)} = P_{r,\mu} 1_{\{\tau_n + \delta < \tau\}} x_\tau + P_{r,\mu} 1_{\{\tau_n + \delta \geq \tau\}} x_{\tau_n + \delta}.$$

But because x belongs to class (D), we have, for n big enough, that the right hand side of the above differs from $E_{r,\mu} x_{\tau_n + \delta}$ by a quantity which is less than or equal to ε. Therefore, for big n,

$$P_{r,\mu} x_{\tau_n + \delta} \leq P_{r,\mu} x_\tau + \varepsilon. \tag{12}$$

On the other hand we get from Lemma 10

$$\sum_{i=0}^{N-1} P_{r,\mu} 1_{\{\frac{i}{N} T \leq \tau + \delta < \frac{i+1}{N} T\}} \left\langle X_{\tau+\delta}, S^{\tau+\delta}_{\frac{i+1}{N} T} h_T^{\frac{i+1}{N} T} \right\rangle \to P_{r,\mu} \left\langle X_{\tau+\delta}, h_T^{\tau+\delta} \right\rangle \tag{13}$$

but by Corollary 20, the left hand side of (13) coincides with

$$\sum_{i=0}^{N-1} P_{r,\mu} 1_{\{\frac{i}{N} T \leq \tau + \delta < \frac{i+1}{N} T\}} \left\langle X_\tau, S^{\tau}_{\frac{i+1}{N} T} h_T^{\frac{i+1}{N} T} \right\rangle.$$

Another use of Lemma 10 gives

$$\sum_{i=0}^{N-1} P_{r,\mu} 1_{\{\frac{i}{N} T \leq \tau + \delta < \frac{i+1}{N} T\}} \left\langle X_\tau, S^{\tau}_{\frac{i+1}{N} T} h_T^{\frac{i+1}{N} T} \right\rangle \to P_{r,\mu} \left\langle X_\tau, S^\tau_{\tau+\delta} h_T^{\tau+\delta} \right\rangle,$$

and therefore

$$P_{r,\mu} \left\langle X_{\tau+\delta}, h_T^{\tau+\delta} \right\rangle = P_{r,\mu} \left\langle X_\tau, S^\tau_{\tau+\delta} h_T^{\tau+\delta} \right\rangle.$$

Thus, using (12), we have, for n big enough,

$$\begin{aligned} 0 \le P_{r,\mu} x_{\tau_n} - P_{r,\mu} x_\tau &\le \varepsilon + P_{r,\mu}(x_{\tau_n} - x_{\tau_n+\delta}) \\ &= \varepsilon + P_{r,\mu}\left(\left\langle X_{\tau_n}, h_T^{\tau_n}\right\rangle - \left\langle X_{\tau_n}, S_{\tau_n+\delta}^{\tau_n} h_T^{\tau_n+\delta}\right\rangle\right) \\ &= \varepsilon + P_{r,\mu}\left(\left\langle X_{\tau_n}, h_{\tau_n+\delta}^{\tau_n}\right\rangle\right) \\ &\le \varepsilon(1 + |\mu|). \end{aligned}$$

This shows that $P_{r,\mu} x_{\tau_n} \searrow P_{r,\mu} x_\tau$ and therefore, as pointed out earlier, the compensator A of x is continuous. □

3.3. Convergence for ξ versus convergence for X

Let $f^n(r,x)$ be a collection of nearly Borel functions, and consider the process $s \mapsto F_{s,T}^n(\xi_s)$, $s \in [0,T]$, where $F_{r,T}^n(x) := \pi_{r,x} f^n(T, \xi_T)$. To these processes correspond the "lifted" processes $s \mapsto \langle X_s, F_{s,T}^n \rangle$. In this subsection, we establish a criterion under which the pointwise convergence of $F_{r,T}^n(x)$ to $F_{r,T}(x)$ implies that the processes $s \mapsto \langle X_s, F_{s,T}^n \rangle$ converge *uniformly in* s to the process $s \mapsto \langle X_s, F_{s,T} \rangle$.

We are particularly interested in the processes $s \mapsto \langle X_s, v_{s,T}^n(.)\rangle$, where v^n is the log-Laplace functional of an A-smooth approximating sequence for the superprocess with parameters (ξ, Φ, k). We want to show that $s \mapsto \langle X_s, v_{s,T}^n(.)\rangle$ converges in probability uniformly in s to $s \mapsto \langle X_s, v_{s,T}(.)\rangle$.

We also establish a criterion under which the convergence of additive functionals $k^n(\mathrm{d}s)$ to an additive functional $k(\mathrm{d}s)$ implies the same convergence for their liftings $K^n(\mathrm{d}s)$ and $K(\mathrm{d}s)$. This is crucial for the proof of uniqueness to the martingale problem.

3.3.1. Uniform convergence of sequences of "lifted" processes

Notation 21. Let z_s be a function of $s \in [r,T]$. In the following, the expression z^* will denote

$$z^* := \sup_{t\in[r,T]} z_t.$$

Lemma 22. *Let $X = (X_t, \Im, P_{r,\mu})$ be a solution to the partial martingale problem for ξ. Let $f^n(t,x)$ be a sequence of uniformly bounded measurable functions satisfying the condition*

$$\sup_{0\le t\le T, x\in E, n\ge 1} |S_{t+\delta} f^n(t+\delta, x) - f^n(t,x)| \to 0 \quad \text{as } \delta \to 0. \tag{14}$$

Suppose that

$$f(t,x) = \lim_{n\to\infty} f^n(t,x). \tag{15}$$

Then the process

$$|x^n - x|^* := \sup_{t\in[r,T]} |\langle X_t, f^n(t,.)\rangle - \langle X_t, f(t,.)\rangle|$$

converges to zero in $P_{r,\mu}$-probability.

Proof. Let $t_i^m := r + \frac{i}{m}(T - r)$, for $i = 0, \ldots, m$. Define

$$
\begin{aligned}
x_t^n &:= \left\langle X_t, f^n(t,.)\right\rangle \\
x_t^\infty &:= \langle X_t, f(t,.)\rangle \\
x_t^{n,m} &:= \sum_{i=0}^{m-1} 1_{[t_i^m, t_{i+1}^m)}(t) \left\langle X_t, S^t_{t_{i+1}^m} f^n(t_{i+1}^m, .)\right\rangle \\
x_t^{\infty,m} &:= \sum_{i=0}^{m-1} 1_{[t_i^m, t_{i+1}^m)}(t) \left\langle X_t, S^t_{t_{i+1}^m} f(t_{i+1}^m, .)\right\rangle .
\end{aligned}
$$

Recall that for every ω, $\sup_{t\in[r,T]} \langle X_t(\omega), 1\rangle < \infty$. Thus, (14) implies that for every $\varepsilon > 0$ and for every m big enough, we have

$$
\sup_{n\in[1,\ldots,\infty]} |x^{n,m}(\omega) - x^n(\omega)|^* < \varepsilon.
$$

Therefore, it suffices to prove that for every $m > 0$, $|x^{n,m} - x^{\infty,m}|^*$ tends to zero in $P_{r,\mu}$-probability. This will clearly be verified if for every $c > 0$

$$
\sup_{t\in[r,c]} \left|\left\langle X_t, S_c^t f^n(c,.)\right\rangle - \left\langle X_t, S_c^t f(c,.)\right\rangle\right|
$$

converges to zero in $P_{r,\mu}$-probability. This is the case if

$$
\sup_{t\in[r,c]} \left\langle X_t, S_c^t \left|f^n(c,.) - f(c,.)\right|\right\rangle
$$

converges to zero in $P_{r,\mu}$-probability. To prove this, it suffices, according to Lemma 8, to check that

$$
\lim_{n\to\infty} \sup_{r\le\tau\le c} P_{r,\mu} \left\langle X_\tau, S_c^\tau \left|f^n(c,.) - f(c,.)\right|\right\rangle = 0.
$$

But for every $g \in \mathcal{B}$, the process $t \mapsto \left\langle X_t, S_c^t g\right\rangle$ is a càdlàg martingale. Hence, from the optional sampling theorem we get that, for every r-stopping time $\tau \le c$

$$
\begin{aligned}
P_{r,\mu} \left\langle X_\tau, S_c^\tau \left|f^n(c,.) - f(c,.)\right|\right\rangle &= P_{r,\mu} \left\langle X_r, S_c^r \left|f^n(c,.) - f(c,.)\right|\right\rangle \\
&= \left\langle \mu, S_c^r \left|f^n(c,.) - f(c,.)\right|\right\rangle .
\end{aligned}
$$

Because f^n converges to f, and because $\{f^n\}$ is uniformly bounded, the right hand side of the last equality tends to zero. □

Corollary 23. *Let $X = (X_t, \Im, P_{r,\mu})$ be a solution to the partial martingale problem for ξ. Let $k^n(\mathrm{d}s), k(\mathrm{d}s)$ be a collection of uniformly admissible additive functionals of ξ. Let v^n be the log-Laplace functional of the (ξ, Φ, k^n)-superprocess, and v the log-Laplace functional of the (ξ, Φ, k)-superprocess. Let $g \in \mathcal{L}$ and suppose that $v_{r,T}^n(g)(x)$ converges to $v_{r,T}(g)(x)$ for every r, x. Then $s \mapsto \langle X_s, v_{s,T}^n(g)\rangle$ converges uniformly (in s) to $s \mapsto \left\langle X_s, v_{s,T}(g)\right\rangle$ in $P_{r,\mu}$-probability.*

Proof. Let $g \in \mathcal{L}$. Note that $v_{r,T}^n(g)(x)$ and $\Phi(x, v_{r,T}^n(g)(x))$ are uniformly bounded. This implies that the family of additive functionals $\{k^{*n}\}$, defined by

$$
k^{*n}(\mathrm{d}s) := \Phi(\xi_s, v_{s,T}^n(f)(\xi_s))k^n(\mathrm{d}s),
$$

is uniformly admissible. It follows from Lemma 10 that (14) holds with $f^n(t,x) := \pi_{r,x} k^{*n}(r,T]$ and $f(t,x) := \pi_{r,x} k^*(r,T]$, where

$$k^*(\mathrm{d}s) := \Phi(\xi_s, v_{s,T}(f)(\xi_s))k(\mathrm{d}s).$$

This yields (14) with $f^n(t,x) := v^n_{r,T}(f)(x)$ and $f(t,x) := v_{r,T}(f)(x)$. The assumption that $v^n_{r,T}(f)(x)$ converges to $v_{r,T}(f)(x)$ is identical to (15). An appeal to Lemma 22 completes the proof. □

3.3.2. Convergence of additive functionals versus convergence of their liftings

Proposition 24. *Let $X = (X_t, \Im, P_{r,\mu})$ be a solution to the partial martingale problem for ξ. Let $k^n(\mathrm{d}s), k(\mathrm{d}s)$ be a collection of uniformly admissible additive functionals. Suppose that for every r, x we have that*

$$^n h^r_T(x) := \pi_{r,x} k^n(r,T] \to \pi_{r,x} k(r,T] =: h^r_T(x). \tag{16}$$

Then, for every r-stopping time $\tau \le T$, $K^n(r,\tau]$ converges to $K(r,\tau]$ in $L^1(P_{r,\mu})$, where $K^n(\mathrm{d}s)$ (resp. $K(\mathrm{d}s)$) is the lifting of $k^n(\mathrm{d}s)$ (resp. $k(\mathrm{d}s)$).

Proof. Because the additive functionals are uniformly admissible, we derive from Lemma 10 that condition (14) is verified with $f^n(t,x) = {}^n h^r_T(x)$ and $f(t,x) = h^t_T(x)$. Condition (16) is identical to condition (15) and therefore, according to Lemma 22,

$$\sup_{t\in[r,T]} \left|\left\langle X_t, {}^n h^t_T\right\rangle - \left\langle X_t, h^t_T\right\rangle\right| \to 0$$

in $P_{r,\mu}$-probability.

Clearly, for every r-stopping time τ bounded by T and every bounded random variable M

$$P_{r,\mu}\left(M\left\langle X_\tau, {}^n h^\tau_T\right\rangle\right) \to P_{r,\mu}\left(M\left\langle X_\tau, h^\tau_T\right\rangle\right).$$

We have already established, in Section 3.2, that processes

$$t \mapsto x^n_t := \left\langle X_t, {}^n h^t_T\right\rangle$$

are right continuous supermartingales of class (D) whose compensators are the liftings $K^n(\mathrm{d}s)$ of the additive functionals $k^n(\mathrm{d}s)$. In fact, since the additive functionals $k^n(\mathrm{d}s)$ are uniformly admissible, their characteristics ${}^n h^r_T(x)$ are uniformly bounded, so the processes $t \mapsto x^n_t$ belong uniformly to class (D).

It suffices only to appeal to Theorem A.2 to obtain the desired result. □

Corollary 25. *Under the hypotheses of Corollary 23, if $K^{\Phi(v^n)\,\mathrm{d}k^n}(\mathrm{d}s)$ is the lifting of $\Phi(s, \xi_s, v^n_{s,T}(g)(\xi_s))k^n(\mathrm{d}s)$, where $g \in \mathcal{L}$, then, for every r-stopping time $\tau \le T$, $K^{\Phi(v^n)\,\mathrm{d}k^n}(r,\tau]$ converges to $K^{\Phi(v)\,\mathrm{d}k}(r,\tau]$ in $L^1(P_{r,\mu})$.*

Proof. Clearly, $v^n_{r,T}(g)(x) - \pi_{r,x} g(\xi_t)$ converges to $v_{r,T}(g)(x) - \pi_{r,x} g(\xi_t)$ for every r, x. That is, if

$$^n\tilde{h}^r_T(x) := \pi_{r,x} \int_r^T \Phi(s, \xi_s, v^n_{s,\xi_s}(g)(\xi_s))k^n(\mathrm{d}s)$$

$$\tilde{h}^r_T(x) := \pi_{r,x} \int_r^T \Phi(s, \xi_s, v_{s,\xi_s}(g)(\xi_s))k(\mathrm{d}s)$$

then

$$ {}^n\tilde{h}^r_T(x) \to \tilde{h}^r_T(x) $$

for every $(r, x) \in [0, T] \times E$. Moreover, the additive functionals

$$ \Phi(s, \xi_s, v^n_{s,\xi_s}(g)(\xi_s))k^n(\mathrm{d}s) $$

are uniformly admissible. An appeal to Proposition 24 completes the proof. □

4. The full martingale problem: Uniqueness of the solution

We now prove the uniqueness of the solution to the full martingale problem. Assume for now on that $(X_t, \Im, P_{r,\mu})$ is a solution to the full martingale problem. Our first goal, in this section, is to "extend" the martingales (6) to the case where φ is a time dependent function.

4.1. Extension of the martingale problem to time dependent functions

Lemma 26. *Let $X = (X_t, \Im, P_{r,\mu})$ be a solution to the full martingale problem for (ξ, Φ, k) and let ψ be smooth for* A. *Then*

$$ t \mapsto \exp(-\langle X_t, \psi_t\rangle) + \int_r^t \exp(-\langle X_s, \psi_s\rangle) \left\langle X_s, \mathrm{A}\psi_s + \frac{\partial}{\partial s}\psi_s \right\rangle \mathrm{d}s $$
$$ - \int_r^t \exp(-\langle X_s, \psi_s\rangle) K^{\Phi(\psi)\,\mathrm{d}k}(\mathrm{d}s) \tag{17} $$

is a $P_{r,\mu}$-martingale, where $K^{\Phi(\psi)\,\mathrm{d}k}(\mathrm{d}s)$ is the lifting of $\Phi(\xi_s, \psi_s)k(\mathrm{d}s)$.

Proof. The proof is a generalization of Lemma 8 in [9] (see also [10, Lemma 4.3.4]). First, for a measurable function $f(s, x)$, let us define (when the expressions makes sense)

$$ u_f(s, X_t) := \exp(-\langle X_t, f(s, .)\rangle) $$
$$ v_f(s, X_t) := \exp(-\langle X_t, f(s, .)\rangle) \left\langle X_t, \frac{\partial}{\partial s} f(s, .) \right\rangle $$
$$ w_f(s, X_t) := \exp(-\langle X_t, f(s, .)\rangle)\langle X_t, \mathrm{A}f(s, .)\rangle. $$

Let ψ be smooth for A. Then we have

$$ u_\psi(t_2, X_{t_2}) - u_\psi(t_1, X_{t_2}) = -\int_{t_1}^{t_2} v_\psi(s, X_{t_2})\,\mathrm{d}s $$

and

$$ P^{\Im^r_{t_1}}_{r,\mu}[u_\psi(t_1, X_{t_2}) - u_\psi(t_1, X_{t_1})] = -P^{\Im^r_{t_1}}_{r,\mu}\left[\int_{t_1}^{t_2} w_\psi(t_1, X_s)\,\mathrm{d}s\right] $$
$$ - P^{\Im^r_{t_1}}_{r,\mu}\left[\int_{t_1}^{t_2} u_\psi(t_1, X_s) K^{\Phi(\psi_{t_1})\,\mathrm{d}k}(\mathrm{d}s)\right]. \tag{18} $$

Therefore, if Λ^n is a partition of $[t_1, t_2]$ with $\mathrm{mesh}\{\Lambda^n\} \to 0$ and ψ^n and X^n are defined by

$$ \psi^n(s, x) := \sum_{i=1}^{n} \psi(t^n_i, x) 1_{[t^n_i, t^n_{i+1})}(s) $$

$$X^n(s) := \sum_{i=1}^{n} X_{t_{i+1}^n} 1_{[t_i^n, t_{i+1}^n)}(s)$$

then, clearly,

$$\sum_{i=1}^{n-1} 1_{[t_i^n, t_{i+1}^n)}(s) K^{\Phi(\xi_s, \psi_{t_i^n}(\xi_s))\,\mathrm{d}k}(\mathrm{d}s) = K^{\Phi(\xi_s, \psi_s^n(\xi_s))\,\mathrm{d}k}(\mathrm{d}s)$$

(where $K^{\eta(s,\xi_s)\,\mathrm{d}k}(\mathrm{d}s)$ denotes here the lifting of $\eta(s,\xi_s)k(\mathrm{d}s)$) and we get (by summing the expressions in (18)) that

$$\begin{aligned} P_{r,\mu}^{\Im_{t_1}^r}[u_\psi(t_2, X_{t_2}) - u_\psi(t_1, X_{t_1})] &= -P_{r,\mu}^{\Im_{t_1}^r}\left[\int_{t_1}^{t_2} v_\psi(s, X_s^n)\,\mathrm{d}s\right] \\ &\quad - P_{r,\mu}^{\Im_{t_1}^r}\left[\int_{t_1}^{t_2} w_{\psi^n}(s, X_s)\,\mathrm{d}s\right] \\ &\quad + P_{r,\mu}^{\Im_{t_1}^r}\left[\int_{t_1}^{t_2} u_{\psi^n}(s, X_s) K^{\Phi(\psi^n)\,\mathrm{d}k}(\mathrm{d}s)\right]. \end{aligned} \tag{19}$$

We want to show that

$$\begin{aligned} &\lim_{n\to\infty} P_{r,\mu}^{\Im_{t_1}^r}\left[\int_{t_1}^{t_2} v_\psi(s, X_s^n)\,\mathrm{d}s\right] = P_{r,\mu}^{\Im_{t_1}^r}\left[\int_{t_1}^{t_2} v_\psi(s, X_s)\,\mathrm{d}s\right] \\ &\lim_{n\to\infty} P_{r,\mu}^{\Im_{t_1}^r}\left[\int_{t_1}^{t_2} w_{\psi^n}(s, X_s)\,\mathrm{d}s\right] = P_{r,\mu}^{\Im_{t_1}^r}\left[\int_{t_1}^{t_2} w_\psi(s, X_s)\,\mathrm{d}s\right] \\ &\lim_{n\to\infty} P_{r,\mu}^{\Im_{t_1}^r}\left[\int_{t_1}^{t_2} u_{\psi^n}(s, X_s) K^{\Phi(\psi^n)\,\mathrm{d}k}(\mathrm{d}s)\right] = P_{r,\mu}^{\Im_{t_1}^r}\left[\int_{t_1}^{t_2} u_\psi(s, X_s) K^{\Phi(\psi)\,\mathrm{d}k}(\mathrm{d}s)\right] \end{aligned} \tag{20}$$

which would complete the proof of the lemma.

(1°) Let us first show that the first two limits of (20) are verified. From Lebesgue's theorem, it suffices to prove that for a fixed $s \geq t_1$, we have that $P_{r,\mu}$-almost surely

$$\begin{aligned} &\langle X_s^n, \psi_s\rangle \to \langle X_s, \psi_s\rangle, \\ &\langle X_s, \psi_s^n\rangle \to \langle X_s, \psi_s\rangle, \\ &\langle X_s, \mathrm{A}\psi_s^n\rangle \to \langle X_s, \mathrm{A}\psi_s\rangle, \\ &\left\langle X_s^n, \frac{\partial}{\partial s}\psi_s\right\rangle \to \left\langle X_s, \frac{\partial}{\partial s}\psi_s\right\rangle. \end{aligned}$$

Only the last convergence is not straightforward. Let Q_+ denote the set of nonnegative rational numbers. Since $\frac{\psi_{s+h}-\psi_s}{h} \in \mathcal{D}(\mathrm{A})$, the processes

$$t \mapsto \left\langle X_t, \frac{\psi_{s+h} - \psi_s}{h}\right\rangle, \quad \text{where } s, h \in Q_+$$

are $P_{r,\mu}$-indistinguishable from right continuous processes. But since $\langle X_t, 1\rangle$ is a càdlàg process,

$$\sup_{r\leq t\leq T} \langle X_t(\omega), 1\rangle < \infty,$$

for every $\omega \in \Omega$. Using the facts that $\frac{\psi_{s+h}-\psi_s}{h} \to \frac{\partial}{\partial s}\psi_s$ uniformly in x, we obtain that $P_{r,\mu}$-almost surely, the mappings $\left\{t \mapsto \left\langle X_t, \frac{\partial}{\partial s}\psi_s\right\rangle\right\}_{s\in Q_+}$ are uniform limits of right continuous mappings. They are therefore also right continuous. Because the function $\frac{\partial}{\partial s}\psi_s$ is strongly continuous and bounded, it is easy to derive that $(t, s) \mapsto \left\langle X_t, \frac{\partial}{\partial s}\psi_s \right\rangle$ is jointly right continuous, $P_{r,\mu}$-almost surely. Hence, $\left\langle X_s^n, \frac{\partial}{\partial s}\psi_s \right\rangle \to \left\langle X_s, \frac{\partial}{\partial s}\psi_s \right\rangle$, $P_{r,\mu}$-almost surely, as wanted.

(2°) We now show that the third limit of (20) holds. Note that, for every ω, $s \mapsto \left\langle X_s, \psi_s^n \right\rangle(\omega)$ converges uniformly (in $s \in [r, T]$) to $s \mapsto \langle X_s, \psi_s \rangle(\omega)$. According to Proposition 24, $K^{\Phi(\psi^n)\,\mathrm{d}k}(r, \tau]$ converges to $K^{\Phi(\psi)\,\mathrm{d}k}(r, \tau]$ in $L^1(P_{r,\mu})$ for every r-stopping time $\tau \le T$. With Lemma 9, it is also possible to suppose (perhaps by taking a subsequence) that $K^{\Phi(\psi^n)\,\mathrm{d}k}(\mathrm{d}s)$ converges weakly to $K^{\Phi(\psi)\,\mathrm{d}k}(\mathrm{d}s)$. This yields (17) as wanted. □

4.2. *Proof of uniqueness for the full martingale problem*

We are now ready to show that the solution to the full martingale problem for (ξ, Φ, k) is unique, as stated in Theorem 7.

Step (1) According to Lemma 14, we can choose a uniformly admissible sequence of additive functionals $k^n(\mathrm{d}s) = \lambda^n(s, \xi_s)\,\mathrm{d}s$ such that if v^n (resp. v) is the log-Laplace of the (ξ, Φ, k^n)-superprocess (resp. (ξ, Φ, k)-superprocess) then for every nonnegative $\varphi \in \mathcal{D}(\mathrm{A})$

(i) $v_{r,T}^n(\varphi)(x)$ converges to $v_{r,T}(\varphi)(x)$ for every r, x;
(ii) $v_{r,T}^n(\varphi)(x)$ is smooth for A;
(iii) for every r, x and every r-stopping time $\tau \le T$, $k^n(r, \tau]$ converges in $L^1(\pi_{r,x})$ to $k(r, \tau]$.

Fix a nonnegative $\varphi \in \mathcal{D}(\mathrm{A})$. Let us define

$$K_1^n(\mathrm{d}s) := \left\langle X_s, \Phi(., v_{s,T}^n(\varphi))\lambda^n(s, .)\right\rangle \mathrm{d}s$$
$$K_2^n(\mathrm{d}s) := K^{\Phi(v_{.,T}^n(\varphi))\,\mathrm{d}k}(\mathrm{d}s).$$

Note that $K_1^n(\mathrm{d}s)$ is the lifting of $\Phi(\xi_s, v_{s,T}^n(\varphi))\lambda^n(s, \xi_s)\,\mathrm{d}s$. According to Proposition 24 and Corollary 25 we have that

(A) For every r-stopping time $\tau \le T$, both random variables $K_1^n(r, \tau]$ and $K_2^n(r, \tau]$ converge to $K^{\Phi(v_{.,T}(\varphi))\,\mathrm{d}k}(r, \tau]$ in $L^1(P_{r,\mu,})$.

Invoking Lemma 9 we are also allowed to assume (by mean of taking a subsequence) that a.s.

(B) $K_1^n(\mathrm{d}s)$ and $K_2^n(\mathrm{d}s)$ converges weakly to $K^{\Phi(v_{.,T}(\varphi))\,\mathrm{d}k}(\mathrm{d}s)$.

Moreover, from Corollary 23, it is also possible to suppose (by means of taking a subsequence) that

(C) $s \mapsto \langle X_s, v_{s,T}^n(\varphi)\rangle$ converges uniformly in $s \in [r, T]$ to $s \mapsto \langle X_s, v_{s,T}(\varphi)\rangle$.

Step (2) According to Lemma 26,

$$t \mapsto \exp\left(-\left\langle X_t, v_{t,T}^n(\varphi)\right\rangle\right) + \int_r^t \exp\left(-\left\langle X_s, v_{s,T}^n(\varphi)\right\rangle\right)\left\langle X_s, \mathrm{A}v_{s,T}^n(\varphi) + \frac{\partial}{\partial s}v_{s,T}^n(\varphi)\right\rangle \mathrm{d}s$$
$$- \int_r^t \exp\left(-\left\langle X_s, v_{s,T}^n(\varphi)\right\rangle\right) K^{\Phi(v_{.,T}^n(\varphi))\,\mathrm{d}k}(\mathrm{d}s)$$

is a martingale. Putting $x_t^n = \left\langle X_t, v_{t,T}^n(\varphi)\right\rangle$, the equality

$$\mathrm{A}v_{s,T}^n(\varphi) + \frac{\partial}{\partial s}v_{s,T}^n(\varphi) = \Phi(., v_{s,T}^n(\varphi))\lambda^n(s, .)$$

gives

$$e^{-x_t^n} = M_t^n(\varphi) + \int_r^t e^{-x_s^n} K_1^n(\mathrm{d}s) - \int_r^t e^{-x_s^n} K_2^n(\mathrm{d}s) \tag{21}$$

where $t \mapsto M_t^n(\varphi)$ is a $P_{r,\mu}$ martingale.

Clearly $e^{-x_t^n} \to e^{-x_t}$ pointwise and in $L^1(P_{r,\mu})$ where x_t is defined by $x_t := \langle X_t, v_{t,T}(\varphi)\rangle$ and v is the log-Laplace functional of the superprocess (ξ, Φ, k).

From (A), (B) and (C) we get that

$$\int_r^t e^{-x_s^n} K_1^n(\mathrm{d}s) - \int_r^t e^{-x_s^n} K_2^n(\mathrm{d}s) \to 0 \tag{22}$$

where the convergence holds in $L^1(P_{r,\mu})$.

That forces $M_t^n(\varphi)$ to converge in $L^1(P_{r,\mu})$ to a limit $M_t(\varphi)$ which has to be a martingale, and we get

$$P_{r,\mu}(e^{-x_T}) = P_{r,\mu}(e^{-x_r}),$$

which is precisely

$$P_{r,\mu}(\exp(-\langle X_T, \varphi\rangle)) = \exp(-\langle \mu, v_{r,T}(\varphi)\rangle). \tag{23}$$

Since T is arbitrary and since $p\mathcal{D}(\mathrm{A})$ is uniformly dense in the set of strictly positive members of $\hat{C}(E)$, it clearly follows from (23) that X is the superprocess with parameters (ξ, Φ, k). □

5. Application to superprocesses with interactions

We now introduce a Dawson–Girsanov transformation (cf. [1] and [2, Theorem 7.2.2]) for (ξ, Ψ, k)-superprocesses, where $\Psi(s, x, \lambda) = \lambda^2$.

It follows from [7] (see indeed [15, Remark 1.1]), that there exists a continuous version of the (ξ, Ψ, k)-superprocess.

Notation 27. Fix $r \geq 0$ and $\mu \in \mathcal{M}_\mathrm{f}$

(1) Let $X = (X_t, \Im, P_{r,\mu}^{(\xi,\Psi,k)})$ denote the canonical superprocess with parameter (ξ, Ψ, k) realized on $C_{[r,\infty)}(\mathcal{M}_\mathrm{f})$, the subspace of $D_{[r,\infty)}(\mathcal{M}_\mathrm{f})$ consisting of continuous trajectories.

(2) Let $\hat{Q}^{(\xi,\Psi,k)}(fg)(\mathrm{d}s)$ denote, for $f, g \in pb\mathcal{B}$, the lifting of the additive functional $\frac{1}{2}b^s(\xi_s)f(\xi_s)g(\xi_s)k(\mathrm{d}s)$. It follows from Itô's formula that, for every $\varphi \in \mathcal{D}(\mathrm{A})$, the $P_{r,\mu}^{(\xi,\Psi,k)}$-martingale

$$t \mapsto M_t(\varphi) := \langle X_t, \varphi\rangle - \langle X_r, \varphi\rangle - \int_r^t \langle X_s, \mathrm{A}\varphi\rangle\, \mathrm{d}s$$

is square integrable with quadratic variation

$$2\hat{Q}^{(\xi,\Psi,k)}(\varphi^2)(\mathrm{d}s).$$

One easily checks that

$$\langle M(f), M(g)\rangle(\mathrm{d}s) = 2\hat{Q}^{(\xi,\Psi,k)}(fg)(\mathrm{d}s). \tag{24}$$

(3) Let $M^{(\xi,\Psi,k)}$ denote the martingale measure extending the martingales $M_t(\varphi)$. It is an orthogonal martingale measure with intensity

$$\nu((r,t]\times A)=\int_r^t 2\hat{Q}^{(\xi,\Psi,k)}(1_A)(\mathrm{d}s).$$

We denote by $Q^{(\xi,\Psi,k)}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y)$ the covariance functional of $M^{(\xi,\Psi,k)}$. It is clear from (24) that

$$Q^{(\xi,\Psi,k)}(\mathrm{d}s,f,g)=2\hat{Q}^{(\xi,\Psi,k)}(fg)(\mathrm{d}s)$$

for every $f,g\in pb\mathcal{B}$.

(4) For any $g:\mathcal{M}_1\to C(E)$ bounded and measurable, and $M(\mathrm{d}s,\mathrm{d}y)$ an orthogonal martingale measure with covariance functional $Q(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y)$ we set

$$Z_g^M(t)=\int_r^t\int_E g(X_s,y)M(\mathrm{d}s,\mathrm{d}y)$$
$$Z_g^Q(t)=\frac{1}{2}\int_r^t\int_E\int_E g(X_s,x)g(X_s,y)Q(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y)$$
$$Z_g^{M,Q}(t)=\exp\left\{Z_g^M(t)-Z_g^Q(t)\right\}.$$

(5) Let ϱ denote a bounded and measurable mapping from $\mathcal{M}_1$ to $C(E)$.

(6) Define $\mathfrak{R}(\mathrm{d}s,\mathrm{d}x)=\int\varrho(X_s,y)Q^{(\xi,\Psi,k)}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y)$.

In addition to the Assumptions 1 and the notation above, we now require that[3]:

Assumption 28. Assume that for every $\theta>0$, and every $t\geq r$,

$$P_{r,\mu}^{(\xi,\Psi,k)}\left(\mathrm{e}^{\theta Q^{(\xi,\Psi,k)}((r,t],E,E)}\right)=P_{r,\mu}^{(\xi,\Psi,k)}\left(\mathrm{e}^{2\theta\hat{Q}^{(\xi,\Psi,k)}(1)(r,t]}\right)<\infty.$$

Theorem 29 (($\xi,(.)^2,k,\mathfrak{R}$)-*Superprocess with Interactions*). *There exists one and only one distribution* $P_{r,\mu}^{(\xi,\Psi,k,\mathfrak{R})}$ *on* $C_{[r,\infty)}(\mathcal{M}_\mathrm{f})$ *such that for every* $\varphi\in\mathcal{D}(\mathrm{A})$,

$$t\mapsto M_t^{\mathfrak{R}}(\varphi):=\langle X_t,\varphi\rangle-\langle X_r,\varphi\rangle-\int_r^t\langle X_s,\mathrm{A}\varphi\rangle\,\mathrm{d}s-\int_r^t\int\varphi(x)\mathfrak{R}(\mathrm{d}s,\mathrm{d}x)$$

is a continuous local martingale with increasing process

$$\int_r^t\int\int\varphi(y)\varphi(x)Q^{(\xi,\Psi,k)}(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y),$$

and such that $t\mapsto Z_{-\varrho}^{M^{\mathfrak{R}},Q^{(\xi,\Psi,k)}}(t)$ *is a martingale, where* $M^{\mathfrak{R}}$ *is the martingale measure coming from the martingales* $t\mapsto M_t^{\mathfrak{R}}(\varphi)$.

Proof. The proof is *identical* to the proof in [2, Theorem 7.2.2]. In Dawson's argument, one should only replace (1°) $Q(X_s,\mathrm{d}x,\mathrm{d}y)\,\mathrm{d}s$ by $Q(\mathrm{d}s,\mathrm{d}x,\mathrm{d}y)$; (2°) r by ϱ; (3°) $R(X_s,\mathrm{d}x)\,\mathrm{d}s$ by $\mathfrak{R}(\mathrm{d}s,\mathrm{d}x)$. □

[3] This condition can be compared to a condition in [6] which asserts that for every $t>0$ and every $\theta>0$, $\sup_{r<t}\sup_x\pi_{r,x}\mathrm{e}^{\theta k(r,t)}<\infty$.

Appendix A

Theorem A.1. *Let* $X = (X_t, \Im, P_{r,\mu})$ *be the superprocess with parameters* (ξ, Φ, k)*. Then* X *is a Hunt process, the lifting of every predictable additive functional with finite characteristic exists and the lifting of* $\ell(\xi_s, \mathrm{d}u)k(\mathrm{d}s)$ *is the modified Lévy measure* $L(\mathrm{d}s, \mathrm{d}\mu)$ *of* X*. In particular for every bounded measurable real valued function* f *we have*

$$P_{r,\mu} \int_r^t \int_{\mathcal{M}_\mathrm{f}} f(\langle \mu, \varphi \rangle) L(\mathrm{d}s, \mathrm{d}\mu) = \pi_{r,\mu} \int_r^t \int_0^\infty f(u\varphi(\xi_s))\ell(\xi_s, \mathrm{d}u)k(\mathrm{d}s). \tag{25}$$

The following moment formulae are satisfied:

(i) $P_{r,\mu}\langle X_t, f\rangle = \pi_{r,\mu} f(\xi_t)$

(ii)

$$P_{r,\mu}\langle X_t, f_1\rangle\langle X_t, f_2\rangle = \pi_{r,\mu} f_1(\xi_t)\pi_{r,\mu} f_2(\xi_t) + \pi_{r,\mu} \int_r^t \pi_{s,\xi_s} f_1(\xi_t)\pi_{s,\xi_s} f_2(\xi_t)k^{(2)}(\mathrm{d}s)$$

where $k^{(2)}(\mathrm{d}s) = \left(b(\xi_s) + \int_0^\infty u^2\ell(\xi_s, \mathrm{d}u)\right) k(\mathrm{d}s)$.

Proof. The E-valued process ξ is a Hunt process. Therefore the superprocess X is also a Hunt process (see [15, Theorem 6.32]). The existence and uniqueness of liftings is due to [8] (see [7, p. 83]). The fact that the modified Lévy measure $L(\mathrm{d}s, \mathrm{d}\mu)$ is the lifting of $\ell(\mathrm{d}s, \mathrm{d}\mu) := \ell(\xi_s, \mathrm{d}u)k(\mathrm{d}s)$ is also due to [8] (See [7, Theorem 6.1.1 and Section 6.8.1]). The formula (25) follows from the definition of the lifting of a measure valued additive functional, see [7, Equation 6.2.13a]. The moment formulae were established in [6, p. 1163]. □

Theorem A.2. *Let* x^n*,* $n = 1, \ldots, \infty$ *be right continuous supermartingales and* A^n*,* $n = 1, \ldots, \infty$ *their compensators. Assume the* x^n *belong uniformly to the class* (D)*. Assume also that for every stopping time* τ*,* x^n_τ *converges weakly in* L^1 *to* x^∞_τ *and that* $\sup_{0\le s\le T} |x^n_s - x_s|$ *converges to zero in probability. Then for every stopping time* τ*,* A^n_τ *converges to* A^∞_τ *in* L^1.

Proof. See [5, VII.19 and 20]. □

Theorem A.3. *Consider branching functionals* $k^1, \ldots, k^\infty = k$*, being uniformly of bounded characteristic. Suppose that for every starting point* $(r, x) \in [0, T] \times E$ *and every* r*-stopping time* $\sigma \le T$ *we know that* $k^n(r, \sigma]$ *converges to* $k(r, \sigma]$ *in* $L^1(\pi_{r,x})$ *as* $n \to \infty$*. Then the related log-Laplace functionals converge:*

$$v^n_{r,t}(f)(x) \underset{n}{\to} v_{r,t}(f)(x), \quad 0 \le r \le t \le T, x \in E, f \in pb\mathcal{B}.$$

Proof. See [4, Theorem 23]. □

Definition A.4. We say that a mapping $\Gamma(s, x, \lambda)$ is locally in λ strongly continuous if for every $s \ge 0$ and every $\Lambda \ge 0$

$$\lim_{t\to s} \sup_{x\in E, 0\le\lambda\le\Lambda} |\Gamma(s, x, \lambda) - \Gamma(t, x, \lambda)| = 0.$$

Theorem A.5. *Let $(\xi, \mathcal{F}, \pi_{r,x})$ be a time homogeneous right process with value in a metrizable Luzin space $(E, \mathcal{E})$. Let S_t denote the semigroup of ξ and let $\mathcal{L} \subseteq b\mathcal{E}$ denote the set of functions $f \in b\mathcal{E}$ such that $S_t(f)(x)$ is strongly continuous. Let $(\mathrm{A}, \mathcal{D}(\mathrm{A}))$ be the (strong) generator of S. Let $\Phi(s, x, \lambda)$ be a nonnegative mapping such that $\Phi(s, x, \varphi(x)) \in \mathcal{L}$ for every $\varphi \in \mathcal{D}(\mathrm{A})$ and such that for each $\Lambda, T \in R_+$,*

$$\|\Phi_s'\|_\infty \vee \|\Phi_s''\|_\infty \vee \|\Phi_\lambda'\|_\infty \vee \|\Phi_\lambda''\|_\infty =: M(\Lambda, T) =: M < \infty$$

where the supremum is taken over the triples (s, x, λ) such that $0 \le s \le T, x \in E, 0 \le \lambda \le \Lambda$. Assume that Φ and its derivatives are locally in λ strongly continuous. Then for each $\varphi \in \mathcal{D}(A)$, there exists a unique solution v to the equation

$$v_{t,T}(\varphi)(x) = S_{T-t}\varphi(x) - \int_t^T S_{r-t}[\Phi(r, v_{r,T}(\varphi))](x)\,\mathrm{d}r.$$

v satisfies the properties

(1) *$v_{t,T}(\varphi)(x)$ belongs to $\mathcal{D}(\mathrm{A})$ for every t;*
(2) *$\frac{\partial}{\partial t} v_{t,T}(\varphi)(x)$ exists and*

$$\left\| \frac{v_{t+h,T}(\varphi)(.) - v_{t,T}(\varphi)(.)}{h} - \frac{\partial}{\partial t} v_{t,T}(\varphi)(.) \right\|_\infty \to 0;$$

(3) *$v_{t,T}(\varphi)$, $\frac{\partial}{\partial t} v_{t,T}(\varphi)$ and $Av_{t,T}(\varphi)$ are bounded and strongly continuous.*

Moreover

$$\frac{\partial}{\partial t} v_{t,T}(\varphi)(x) + Av_{t,T}(\varphi)(x) = \Phi(t, x, v_{t,T}(\varphi)(x)).$$

Proof. See [14, Theorem 2]. □

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