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The 2-stage Euclidean algorithm and the restricted Nagata's pairwise algorithm

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ABSTRACT

As with Euclidean rings and rings admitting a restricted Nagata's pairwise algorithm, we will give an internal characterization of 2-stage Euclidean rings. Applying this characterization we are capable of providing infinitely many integral domains which are ω-stage Euclidean but not 2-stage Euclidean. Our examples solve finally a fundamental question related to the notion of k-stage Euclidean rings raised by G.E. Cooke [G.E. Cooke, A weakening of the Euclidean property for integral domains and applications to algebraic number theory I, J. Reine Angew. Math. 282 (1976) 133–156]. The question was stated as follows: "I do not know of an example of an ω-stage euclidean ring which is not 2-stage euclidean."

Also, in this article we will give a method to construct the smallest restricted Nagata's pairwise algorithm θ on a unique factorization domain which admits a restricted Nagata's pairwise algorithm. It is of interest to point out that in a Euclidean domain the shortest length $d(a,b)$ of all terminating division chains starting from a pair (a,b) and the value $\theta(a,b)$ with g.c.d.$(a,b) \neq 1$ can be determined by each other.

1. Introduction

As different generalizations of the Euclidean algorithm, the concept of the 2-stage Euclidean algorithm was introduced by Cooke [6] in 1976, the concept of Nagata's pairwise algorithm was

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introduced by Nagata [10] in 1988, and the concept of the restricted Nagata's pairwise algorithm (RNPA) was pointed out by Leu [8] in 2008. It is quite surprising that, in appearance, the 2-stage Euclidean algorithm is a much more natural generalization of the Euclidean algorithm than the restricted Nagata's pairwise algorithm is, but, from Motzkin and Samuel's point of view, the restricted Nagata's pairwise algorithm has more close genetic relation to the Euclidean algorithm than the 2-stage Euclidean algorithm has (see Sections 2 and 4 of [8] and Section 4 here for details). Another evidence to support this point of view is that every ring admitting a RNPA is a principal ideal ring (see [10]), on the other hand, the ring $\mathbb{A}$ of all algebraic integers is a non-principal ideal domain which admits a 2-stage Euclidean algorithm (see [6]).

In Section 2 we will reinforce an internal characterization of unique factorization domains which admit a restricted Nagata's pairwise algorithm, and, as a crucial corollary to this characterization, we will give a method to construct the smallest restricted Nagata's pairwise algorithm θ on a unique factorization domain E provided that E admits a RNPA. We will also point out that in a Euclidean domain the shortest length $d(a,b)$ of all terminating division chains starting from a pair (a,b) and the value $\theta(a,b)$ with g.c.d.$(a,b) \neq 1$ can be determined by each other (see Corollary 2.4). In Section 4 we will derive an internal characterization of 2-stage Euclidean rings which is an analog of Proposition 2.1 of [8] on Euclidean rings and of Corollary 4.9 of [8] on rings admitting a restricted Nagata's pairwise algorithm. Using our characterization of 2-stage Euclidean rings we obtain successfully infinitely many integral domains E which are ω-stage Euclidean but not 2-stage Euclidean (see Example 4 of Section 5). Our examples solve finally a fundamental question related to the notion of k-stage Euclidean rings raised by Cooke on page 137 of [6] which was stated as follows:

"I do not know of an example of an ω-stage euclidean ring which is not 2-stage euclidean."

In this article a ring means a commutative ring with identity $1 \neq 0$.

2. The restricted Nagata's pairwise algorithm

We start with the definition of a restricted Nagata's pairwise algorithm.

Definition 1. Let E be a ring and W a well-ordered set. We say that a mapping ρ from $E \times E$ into W gives E a restricted Nagata's pairwise algorithm (RNPA) if and only if ρ satisfies the following conditions:

(1) If $a, b \in E$ and $u, v \in E^*$, then $\rho(au, bv) = \rho(a,b)$, where E^* is the unit group of E.
(2) If $b \in aE$ and $b \notin aE^* = \{ae \mid e \in E^*\}$, then $\rho(a,a) < \rho(b,b)$.
(3) If $b - c \in aE$, then $\rho(a,b) = \rho(a,c)$.
(4) For each pair (a,b) in $E \times E$, there are $q, r \in E$ so that $b = qa + r$ with either $r = a$ or $\rho(r,a) < \rho(a,b)$.
(5) For b coprime to a, $\rho(a,b) = \rho(a, 1_E)$.

Remark 1. A mapping ρ satisfying the conditions (1)–(4) above is now called a Nagata's pairwise algorithm which is equivalent to the pairwise algorithm given by M. Nagata (cf. [10] and [4, Proposition 2]). To define the smallest restricted Nagata's pairwise algorithm on a ring E with a restricted Nagata's pairwise algorithm, we may assume that W is an ordinal, with elements customarily denoted by $0, 1, 2, 3, \dots, \omega, \omega+1, \dots, 2\omega, \dots$, and $\mathrm{card}(E \times E) < \mathrm{card}(W)$ (cf. Section 3 or [8, Section 4]).

Proposition 2.1. *Let E be a ring admitting a Nagata's pairwise algorithm. Then starting from any pair (a,b) in $E \times E$, there exists a terminating k-stage division chain for some rational integer k.* (*See Section* 3 *for definition of a terminating k-stage division chain.*)

Proof. Let $\rho : E \times E \longrightarrow W$ be a Nagata's pairwise algorithm on E, where W is an ordinal such that $\mathrm{card}(E \times E) < \mathrm{card}(W)$. For each pair (a,b) in $E \times E$, there are $q, r \in E$ so that $b = qa + r$ with either

$r = a$ or $\rho(r, a) < \rho(a, b)$. Since W is a well-ordered set, there exist $k \in \mathbb{N}$, the set of positive integers, and $q_i, r_i \in E$ for $i = 1, 2, \ldots, k$ such that $\rho(r_{k-1}, r_{k-2}) < \cdots < \rho(r_1, a) < \rho(a, b)$ (set $r_0 = a$) and

$$\begin{aligned} b &= q_1 a + r_1, \\ a &= q_2 r_1 + r_2, \\ &\vdots \\ r_{k-2} &= q_k r_{k-1} + r_k \quad \text{with } r_k = r_{k-1}. \end{aligned}$$

The case $k = 1$ corresponds to the equation $b = qa + a$.

This completes the proof. □

In [8], Leu provided an internal characterization of unique factorization domains admitting a restricted Nagata's pairwise algorithm. It involves the following transfinite construction.

The transfinite construction of type RNPA. Let E be a ring and W an ordinal such that $\operatorname{card}(E \times E) < \operatorname{card}(W)$. We set $\tilde{E}_{-1} = \{0\}$ and $-1 < \alpha$ for every α in W. For α in W, we define $\tilde{E}_\alpha$ by transfinite induction as follows: the set $\tilde{E}'_\alpha = \bigcup_{\beta < \alpha} \tilde{E}_\beta$ (where $\beta \in \{-1\} \cup W$) is already defined and $\tilde{E}_\alpha$ is the union of $\{0\}$ and the set of all $a \in E$ such that $(E/aE)^\circ \subseteq \pi_a(\tilde{E}'_\alpha)$, where $\pi_a : \tilde{E}'_\alpha \longrightarrow E/aE$ is the canonical map and $(E/aE)^\circ$ is the set of all distinct cosets $b + aE$ with b coprime to a. In particular, $\tilde{E}_0 = \{0\} \cup E^*$.

The following proposition is Theorem 4.5 of [8] which we record here for completeness.

Proposition 2.2. *Let E be a unique factorization domain (UFD). Then E admits a restricted Nagata's pairwise algorithm if and only if the sequence $(\tilde{E}_\alpha)_{\alpha \in W}$ exhausts E, where W is an ordinal such that* $\operatorname{card}(E \times E) < \operatorname{card}(W)$.

Remark 2. Let $\mathbb{A}$ be the ring of all algebraic integers. It is known that $\mathbb{A}$ is not a principal ideal domain, hence $\mathbb{A}$ does not admit a RNPA by [10]. In Section 5 we will prove that $\tilde{\mathbb{A}}_1 = \mathbb{A}$. By Proposition 2.2, $\mathbb{A}$ is not a UFD. The existence of a ring like $\mathbb{A}$ shows the hypothesis that E is a unique factorization domain in Proposition 2.2 is essential.

Remark 3. The restricted Nagata's pairwise algorithm defined in the proof of Proposition 2.2 is not the smallest restricted Nagata's pairwise algorithm on E (see [8]). The following theorem is a crucial corollary to Proposition 2.2. The algorithm defined in it is indeed the smallest restricted Nagata's pairwise algorithm on E.

Theorem 2.3. *Let E be a unique factorization domain and the sequence $(\tilde{E}_\alpha)_{\alpha \in W}$ exhaust E. Then the mapping $\rho : E \times E \longrightarrow W$ defined in the following is the smallest restricted Nagata's pairwise algorithm on E, where W is an ordinal such that* $\operatorname{card}(E \times E) < \operatorname{card}(W)$:

(i) *For α in W, if $a \in \tilde{E}_\alpha \setminus \tilde{E}'_\alpha$ and $b \in E$, which is coprime to a, we define $\rho(a, b) = \alpha$.*
(ii) *For nonzero nonunit element a in E and $b \in aE$, write $a = p_1 p_2 \cdots p_t$, where $p_1, p_2, \ldots, p_t$ are irreducibles in E. We define $\rho(a, b) = t$.*
(iii) *We define*

$$\rho(0, b) = \begin{cases} \omega, & \text{if } b = 0; \\ t + 1, & \text{if } b = u q_1 q_2 \cdots q_t, \end{cases}$$

where ω denotes the first transfinite ordinal, $u \in E^*$, and $q_1, q_2, \ldots, q_t$ irreducible elements of E. If $b \in E^*$, then $\rho(0, b) = 1$.

(iv) *For nonzero nonunit elements a, b in E, which have a greatest common divisor $s \notin E^*$, we define $\rho(a, b) = \rho(s, s) + d(a', b') - 1$, where $a', b' \in E$ such that $a = a's$ and $b = b's$, and $d(a', b') = \min\{k \in \mathbb{N} \mid$ there exists a terminating k-stage division chain starting from the pair $(a', b')\}$. (Propositions 2.1 and 2.2 ensure that $d(a', b')$ exists, hence $d(a, b) = d(a', b')$. For α, not the last, in W, $\alpha + 1$ is the immediate successor of α.)*

Proof. We first show that ρ is a restricted Nagata's pairwise algorithm: It is easy to verify that ρ satisfies the conditions (1), (2), (3) and (5) of Definition 1. To verify that ρ satisfies the condition (4) of Definition 1, we divide the arguments into four cases.

For each pair (a, b) in $E \times E$:

Case 1. If $b \in aE$, then $b = qa + a$ for some $q \in E$.

Case 2. If $a = 0$ and $b \neq 0$, then we have $b = 0 + b$ with $\rho(b, 0) < \rho(0, b)$.

Case 3. If a is a nonzero nonunit element, b coprime to a, and $a \in \tilde{E}_\alpha \setminus \tilde{E}'_\alpha$, then there exist $q, r \in E$ such that $b = qa + r$ with nonzero $r \in \tilde{E}'_\alpha$, whence we have $\rho(r, a) < \alpha = \rho(a, b)$.

Case 4. If a is a nonzero nonunit element, and $s \notin aE^* \cup E^*$, a greatest common divisor of $\{a, b\}$, then $a = a's$ and $b = b's$ for some a', b' in E, which are relatively prime. We have $d(a', b') = n \geqslant 2$. Thus there exist $q_i, r_i \in E$ for $i = 1, 2, \ldots, n$ such that

$$\begin{aligned} b' &= q_1 a' + r_1, \\ a' &= q_2 r_1 + r_2, \\ &\vdots \\ r_{n-2} &= q_n r_{n-1} + r_n \quad \text{with } r_n = r_{n-1}. \end{aligned}$$

This gives $b = q_1 a + r_1 s$. We now show that $\rho(r_1 s, a) < \rho(a, b)$.

Subcase 4.1. $r_1 \in E^*$. Since $a = a's = a' r_1^{-1} r_1 s$, therefore $a \in r_1 sE$. By (ii) and (iv), $\rho(r_1 s, a) = \rho(s, s) < \rho(s, s) + d(a', b') - 1 = \rho(a, b)$.

Subcase 4.2. $r_1 \notin E^*$. Since $\text{g.c.d.}(r_1, a') = 1$, we have $\text{g.c.d.}(r_1 s, a) = s$. By (iv), $\rho(r_1 s, a) = \rho(s, s) + d(r_1, a') - 1 < \rho(s, s) + d(a', b') - 1 = \rho(a, b)$.

This proves that ρ is a restricted Nagata's pairwise algorithm on E. We next show that ρ is the smallest restricted Nagata's pairwise algorithm.

Let θ be the smallest restricted Nagata's pairwise algorithm on E (see [8, Proposition 4.2] for the existence of θ). We want to prove that $\rho = \theta$:

(i) For α in W, by Theorem 4.3 of [8], we know that $\tilde{E}_\alpha = \hat{E}_\alpha$ and $\tilde{E}'_\alpha = \hat{E}'_\alpha$, where $\hat{E}_\alpha = \{0\} \cup \{a \in E \setminus \{0\} \mid \theta(a, 1) \leqslant \alpha\}$ and $\hat{E}'_\alpha = \{0\} \cup \{a \in E \setminus \{0\} \mid \theta(a, 1) < \alpha\}$. If $a \in \tilde{E}_\alpha \setminus \tilde{E}'_\alpha$ and $b \in E$, which is coprime to a, we have $\theta(a, b) = \theta(a, 1) = \alpha = \rho(a, b)$.

(ii) For nonzero nonunit element a in E and $b \in aE$, write $a = p_1 p_2 \cdots p_t$, where $p_1, p_2, \ldots, p_t$ are irreducibles in E. By (2) of Definition 1, we have

$$0 = \theta(1,1) < \theta(p_1, p_1) < \theta(p_1 p_2, p_1 p_2) < \cdots < \theta(p_1 p_2 \cdots p_t, p_1 p_2 \cdots p_t)$$
$$= \theta(a,a) \leqslant \rho(a,a) = t.$$

This implies that $\theta(a,a) = t = \rho(a,a)$. Applying (3) of Definition 1, we know that $\theta(a,b) = \theta(a,a) = t = \rho(a,b)$.

(iii) Case $a = 0$, $b = 0$. Let p be an irreducible element in E and $i \in \mathbb{N}$. By (ii) above and (2) of Definition 1, we have that $i = \theta(p^i, p^i) < \theta(0,0) \leqslant \rho(0,0) = \omega$. Thus $\theta(0,0) = \omega = \rho(0,0)$.
Case $a = 0$, $b = uq_1 q_2 \cdots q_t$ for $u \in E^*$ and $q_1, q_2, \ldots, q_t$ irreducibles in E. For pair $(0,b)$, by (4) of Definition 1, there exist q, r in E such that $b = q \cdot 0 + r$ with $\theta(r,0) = \theta(b,0) < \theta(0,b)$. Applying (i) and (ii) above, we have that $t = \theta(b,0) < \theta(0,b) \leqslant \rho(0,b) = t+1$. Thus $\theta(0,b) = t+1 = \rho(0,b)$.

(iv) Let a and b be nonzero nonunit elements in E with g.c.d.$(a,b) = s \notin aE^* \cup E^*$. For such pair (a,b), by the proof of Proposition 2.1, there exists a terminating k-stage division chain with q_i, r_i in E for $i = 1, 2, \ldots, k$ such that $\theta(r_{k-1}, r_{k-2}) < \theta(r_{k-2}, r_{k-3}) < \cdots < \theta(r_1, a) < \theta(a,b)$ and

$$b = q_1 a + r_1,$$
$$a = q_2 r_1 + r_2,$$
$$\vdots$$
$$r_{k-2} = q_k r_{k-1} + r_k \quad \text{with } r_k = r_{k-1}.$$

Thus $r_{k-1} = su$ for some unit u in E^*. Write $a = a's$, $b = b's$ and $r_i = r_i's$ for $a', b', r_i' \in E$, $i = 1, 2, \ldots, k$. We then have

$$b' = q_1 a' + r_1',$$
$$a' = q_2 r_1' + r_2',$$
$$\vdots$$
$$r_{k-2}' = q_k r_{k-1}' + r_k' \quad \text{with } r_k' = r_{k-1}'.$$

This implies that $d(a', b') \leqslant k$. By (ii) above and Definition 1, we have

$$\rho(s,s) = \rho(r_{k-1}, r_{k-1}) = \theta(r_{k-1}, r_{k-1}) = \theta(r_{k-1}, r_{k-2})$$
$$< \theta(r_{k-2}, r_{k-3}) < \cdots < \theta(r_1, a) < \theta(a,b) \leqslant \rho(a,b)$$
$$= \rho(s,s) + d(a', b') - 1 \leqslant \rho(s,s) + k - 1.$$

This ascending chain gives that $\rho(s,s) + k - 1 \leqslant \theta(a,b) \leqslant \rho(s,s) + k - 1$. Hence $\theta(a,b) = \rho(a,b) = \rho(s,s) + k - 1$ and $d(a', b') = k$.

The theorem is proved. □

Remark 4. By Theorem 2.3 above and Remark 6 of [8], we know that the restricted Nagata's pairwise algorithm $\psi(a,b) = \min\{\phi(au) \mid u \in E^*\}$, induced by the smallest Euclidean algorithm ϕ on a principal ideal domain E, is not the smallest restricted Nagata's pairwise algorithm on E.

We end this section with a byproduct of Theorem 4.5 of [8] and Theorem 2.3.

Corollary 2.4. *Let a and b be nonzero elements in a domain E admitting a restricted Nagata's pairwise algorithm. Then the shortest length $d(a,b)$ of all terminating division chains starting from the pair (a,b) and the value $\theta(a,b)$ with* g.c.d.$(a,b) \neq 1$ *can be determined by each other, where θ is the smallest RNPA on E. If* g.c.d.$(a,b) = 1$, *then $d(a,b) \leqslant \theta(a,b)+1$.*

Proof. Let $s = \text{g.c.d.}(a,b) \neq 1$. Then $a = a's$ and $b = b's$ for some $a', b' \in E$. Since $d(a,b) = d(a',b')$ and $\theta(a,b) = \theta(s,s) + d(a',b') - 1 = \theta(s,s) + d(a,b) - 1$ by Theorem 2.3(iv), it follows that the values $d(a,b)$ and $\theta(a,b)$ can be determined by each other.

For the case g.c.d.$(a,b) = 1$, let p be an irreducible element in E and ρ_0 the restricted Nagata's pairwise algorithm defined in the proof of Theorem 4.5 of [8]. We then have

$$\theta(ap,bp) = \theta(p,p) + d(a,b) - 1 \leqslant \rho_0(ap,bp) = \rho_0(p,p) + \rho_0(a,b)$$
$$= \theta(p,p) + \theta(a,b).$$

It gives that $d(a,b) \leqslant \theta(a,b) + 1$.

The corollary is proved. □

3. The *k*-stage Euclidean algorithm

The following definitions of a k-stage Euclidean algorithm and an ω-stage Euclidean algorithm respectively are generalizations of the ones introduced by Cooke [6].

Definition 2. Let E be a ring. A sequence of equations (with $a, b, q_i, r_i \in E$)

$$\begin{aligned} b &= q_1 a + r_1, \\ a &= q_2 r_1 + r_2, \\ &\vdots \\ r_{k-2} &= q_k r_{k-1} + r_k \end{aligned} \tag{3.1}$$

is called a k-stage division chain starting from the pair (a,b). Such a division chain is called a terminating k-stage division chain or a terminating division chain of length k if the last remainder r_k is equal to either 0 or r_{k-1}.

Definition 3. We say that a ring E is a k-stage Euclidean ring with respect to ψ if we can find a mapping $\psi : E \longrightarrow W$, a well-ordered set, with the properties

(1) $\psi(a) = 0 \Leftrightarrow a = 0$,
(2) there is a $k \in \mathbb{N}$ such that for every pair a, b in E with $a \neq 0$, there exists an n-stage division chain starting from (a,b) for some $n \leqslant k$ (cf. (3.1)) with

$$\psi(r_n) < \psi(a). \tag{3.2}$$

Such a mapping ψ is called a k-stage Euclidean algorithm (k-SEA) on E. A ring E is called an ω-stage Euclidean ring with respect to ψ if ψ satisfies (1) and (2)$'$ for every pair (a,b) with $a \neq 0$, there exists a k-stage division chain (3.1) for some k such that the last remainder r_k satisfies (3.2).

Remark 5. Let a, b, q, and r be elements in a ring E with $a \neq 0$ and $b = qa + r$. We can extend the 1-stage division chain $b = qa + r$ to a 2-stage division chain $b = (q+1)a + (-a+r)$ and $a = (-1)(-a+r) + r$ with the same last remainder r.

Proposition 3.1. *Let E, A, and B be rings such that $E = A \times B$. Then E is a k-stage Euclidean ring if and only if A and B are k-stage Euclidean rings.*

Proof. Let ψ be a k-stage Euclidean algorithm from E into a well-ordered set W. For $x \in A$, set $\psi_1(x) = \psi(x, 0)$. Since ψ is a k-SEA on $A \times B$, so for each pair a, b in A with $a \neq 0$, there exist $q_i, r_i \in A$ and $p_i, s_i \in B$ for $i = 1, 2, \ldots, k$ such that

$$(b, 0) = (q_1, p_1)(a, 0) + (r_1, s_1),$$

$$(a, 0) = (q_2, p_2)(r_1, s_1) + (r_2, s_2),$$

$$\vdots$$

$$(r_{k-2}, s_{k-2}) = (q_k, p_k)(r_{k-1}, s_{k-1}) + (r_k, s_k)$$

with $\psi(r_k, s_k) < \psi(a, 0)$ (cf. Remark 5). It is clear that $s_1 = s_2 = \cdots = s_k = 0$. Therefore we have

$$b = q_1 a + r_1,$$

$$a = q_2 r_1 + r_2,$$

$$\vdots$$

$$r_{k-2} = q_k r_{k-1} + r_k$$

with $\psi_1(r_k) = \psi(r_k, 0) < \psi(a, 0) = \psi_1(a)$. Thus A is a k-stage Euclidean ring. Similarly, B is a k-stage Euclidean ring too.

Conversely, if $\psi_1 : A \longrightarrow W_1$ and $\psi_2 : B \longrightarrow W_2$ give A and B a k-stage Euclidean algorithm, respectively, where W_1 and W_2 are well-ordered sets. By applying Remark 5 and the arguments used in the proof of Proposition 6 of [11] on finite product of Euclidean rings, we can define a k-SEA on E by means of ψ_1 and ψ_2 in a natural way. $\square$

Similarly we also have the following.

Corollary 3.2. *Let E, A, and B be rings such that $E = A \times B$. Then E is an ω-stage Euclidean ring if and only if A and B are ω-stage Euclidean rings.*

Two algorithms (either both k-SEAs or both ω-SEAs) $\psi : E \longrightarrow W$, $\psi' : E \longrightarrow W'$ on a ring E are said to be isomorphic if there exists an order-isomorphism $h : \psi(E) \longrightarrow \psi'(E)$ such that $\psi' = h \circ \psi$. It is easy to see that isomorphic algorithms have the same properties. Thus, since all well-ordered sets with cardinal $\leqslant \mathrm{card}(E)$ are order isomorphic to proper initial segments of any well-ordered set W such that $\mathrm{card}(W) > \mathrm{card}(E)$, all the algorithms on the ring E may be constructed to take their values in the fixed well-ordered set W. For precision sake, we may assume that W is an ordinal, with elements customarily denoted by $0, 1, 2, 3, \ldots, \omega, \omega + 1, \ldots, 2\omega, \ldots$, and $\mathrm{card}(E) < \mathrm{card}(W)$.

Proposition 3.3. *If $\psi_\beta : E \longrightarrow W$ is any nonempty family of k-stage Euclidean algorithms (resp. ω-SEAs) on a ring E, then $\psi = \inf_\beta \psi_\beta$ is also a k-stage Euclidean algorithm (resp. ω-SEA) on E.*

Proof. Clearly $\psi(0) = 0$. For $a \in E$, if $\psi(a) = 0$, then there exists an index β such that $0 = \psi(a) = \psi_\beta(a)$, whence $a = 0$. Thus $\psi(a) = 0$ if and only if $a = 0$.

For every pair a, b in E with $a \neq 0$, there exist an index β and $q_i, r_i \in E$ for $i = 1, 2, \ldots, k$ such that $\psi(a) = \psi_\beta(a)$ and

$$b = q_1 a + r_1,$$
$$a = q_2 r_1 + r_2,$$
$$\vdots$$
$$r_{k-2} = q_k r_{k-1} + r_k$$

with $\psi_\beta(r_k) < \psi_\beta(a)$ which implies that $\psi(r_k) \leqslant \psi_\beta(r_k) < \psi_\beta(a) = \psi(a)$. This proves that ψ is a k-SEA (resp. ω-SEA) on E. $\square$

Remark 6. Proposition 3.3 shows that if a ring E is k-stage Euclidean (resp. ω-stage Euclidean), then E admits the smallest k-stage Euclidean algorithm (resp. the smallest ω-SEA).

4. The 2-stage Euclidean algorithm

Theorem 4.1. *Let $\tau : E \longrightarrow W$ be the smallest 2-stage Euclidean algorithm on a 2-stage Euclidean ring E. For $\alpha \in W$ set $\overline{E}_\alpha = \{b \in E \mid \tau(b) \leqslant \alpha\}$, $\overline{E}'_\alpha = \{b \in E \mid \tau(b) < \alpha\}$, and ${}_2E_0 = \{0\}$. For nonzero $\alpha \in W$ set ${}_2E_\alpha$ be the set of all $b \in E$ such that $E/bE = \pi_b({}_2E'_\alpha \cup {}_2E^b_\alpha)$, where ${}_2E'_\alpha = \bigcup_{\beta<\alpha} {}_2E_\beta$, ${}_2E^b_\alpha = \{d \in E \mid d|(b - e)$ for some $e \in {}_2E'_\alpha\}$, and $\pi_b : {}_2E'_\alpha \cup {}_2E^b_\alpha \longrightarrow E/bE$ is the canonical map. Then $\overline{E}_\alpha = {}_2E_\alpha$ for all $\alpha \in W$.*

Proof. Clearly we have that $\overline{E}_0 = \{0\} = {}_2E_0$. For $\alpha \neq 0$ in W, assuming $\overline{E}_\beta = {}_2E_\beta$ for all $\beta < \alpha$ in W, we want to prove that $\overline{E}_\alpha = {}_2E_\alpha$.

For nonzero $b \in \overline{E}_\alpha$, if $a + bE$ is any coset, then there exist q_1, q_2, d, e in E such that

$$a = q_1 b + d,$$
$$b = q_2 d + e$$

with $\tau(e) < \tau(b) \leqslant \alpha$. This implies that $e \in \overline{E}'_\alpha = \bigcup_{\beta<\alpha} \overline{E}_\beta = \bigcup_{\beta<\alpha} {}_2E_\beta = {}_2E'_\alpha$ and hence $d \in {}_2E^b_\alpha$, which shows that $b \in {}_2E_\alpha$. Thus $\overline{E}_\alpha \subseteq {}_2E_\alpha$.

Conversely, consider nonzero $b \in {}_2E_\alpha$ and suppose that $\tau(b) > \alpha$. Now define $\tau_1 : E \longrightarrow W$ by

$$\tau_1(x) = \begin{cases} \alpha, & \text{if } x = b; \\ \tau(x), & \text{otherwise.} \end{cases}$$

We claim that τ_1 is a 2-stage Euclidean algorithm: It is obvious that $\tau_1(x) = 0$ iff $x = 0$. As for the condition (2) of Definition 3, we divide the arguments into three cases. For every pair (c, a) in E with $c \neq 0$:

Case 1. $a = 0$. Clearly $0 = 0 \cdot c + 0$ with $\tau_1(0) = 0 < \tau_1(c)$.

Case 2. $a \neq 0$, $c = b$. Since $b \in {}_2E_\alpha$, there exist $q_1, d \in E$ such that $a = q_1 b + d$ with either $d \in {}_2E'_\alpha = \overline{E}'_\alpha$ or $d \in {}_2E^b_\alpha$. That is either $\tau_1(d) = \tau(d) < \alpha = \tau_1(b)$ or there existing $e \in {}_2E'_\alpha = \overline{E}'_\alpha$, and $q_2 \in E$ such that $b = q_2 d + e$ with $\tau_1(e) = \tau(e) < \alpha = \tau_1(b)$.

Case 3. $a \neq 0$, $c \neq b$. Since τ is a 2-SEA, there exist $q_1, q_2, r_1, r_2 \in E$ such that $a = q_1 c + r_1$, $c = q_2 r_1 + r_2$ with $\tau_1(r_2) \leqslant \tau(r_2) < \tau(c) = \tau_1(c)$.

It follows that τ_1 is indeed a 2-SEA on E. This contradicts the fact that τ is the smallest 2-SEA. Therefore we have $\tau(b) \leqslant \alpha$, that is $b \in \overline{E}_\alpha$.

We conclude that $\overline{E}_\alpha = {}_2E_\alpha$ for all $\alpha \in W$. $\square$

Theorem 4.1 on 2-SEA is an analog of Proposition 10 of [11] on Euclidean algorithm and of Theorem 4.3 of [8] on RNPA. As in [8], [9] and [11], the transfinite construction described in Theorem 4.1 may be performed in any ring.

The transfinite construction of type 2-SEA. Let E be a ring and W an ordinal such that $\mathrm{card}(E) < \mathrm{card}(W)$. We set ${}_2E_0 = \{0\}$. For $\alpha > 0$ in W, we define ${}_2E_\alpha$ by transfinite induction as follows: the set ${}_2E'_\alpha = \bigcup_{\beta<\alpha} {}_2E_\beta$ is already defined and ${}_2E_\alpha$ is the set of all $b \in E$ such that $E/bE = \pi_b({}_2E'_\alpha \cup {}_2E^b_\alpha)$, where ${}_2E^b_\alpha = \{d \in E \mid d|(b-e)$ for some $e \in {}_2E'_\alpha\}$ and $\pi_b : {}_2E'_\alpha \cup {}_2E^b_\alpha \longrightarrow E/bE$ is the canonical map.

Remark 7. It is clear that the sequence $({}_2E_\alpha)_{\alpha\in W}$ is increasing. Since $r \in {}_2E'_\alpha$ implies that $\pm(b-r) \in {}_2E^b_\alpha$, and $r + bE = -(b-r) + bE$, we have $\pi_b({}_2E'_\alpha \cup {}_2E^b_\alpha) = \pi_b({}_2E^b_\alpha)$. The reason we adopt $\pi_b({}_2E'_\alpha \cup {}_2E^b_\alpha)$ is because on a ring E, to investigate the relation between 2-stage Euclidean algorithms and restricted Nagata's pairwise algorithms, the expression $\pi_b({}_2E'_\alpha \cup {}_2E^b_\alpha)$ is more suggestive than $\pi_b({}_2E^b_\alpha)$.

Theorem 4.2. *Let E be a ring and W an ordinal such that* $\mathrm{card}(E) < \mathrm{card}(W)$. *Then $E = \bigcup_{\alpha\in W} {}_2E_\alpha$ if and only if E admits a 2-stage Euclidean algorithm.*

Proof. If $E = \bigcup_{\alpha\in W} {}_2E_\alpha$, then we define a map $\tau : E \longrightarrow W$ by $\tau(x) = \alpha$ if and only if $x \in {}_2E_\alpha \setminus {}_2E'_\alpha$. For any pair b, c in E with $b \neq 0$, if $b \in {}_2E_\alpha \setminus {}_2E'_\alpha$, then $\tau(b) = \alpha$ and $\pi_b({}_2E^b_\alpha) = \pi_b({}_2E'_\alpha \cup {}_2E^b_\alpha) = E/bE$. Thus there exist $q_1 \in E$ and $d \in {}_2E^b_\alpha$ such that $c = q_1b + d$. Since $d \in {}_2E^b_\alpha$, there exist $e \in {}_2E'_\alpha$ and $q_2 \in E$ such that $b = q_2d + e$ with $\tau(e) < \alpha = \tau(b)$. This proves that τ is a 2-SEA on E.

Conversely, if E admits a 2-SEA, then, by Theorem 4.1, $E = \bigcup_{\alpha\in W} \overline{E}_\alpha = \bigcup_{\alpha\in W} {}_2E_\alpha$. □

If a ring $E = \bigcup_{\alpha\in W} {}_2E_\alpha$ with W an ordinal such that $\mathrm{card}(E) < \mathrm{card}(W)$, then, by Theorem 4.1, the 2-SEA τ defined in the proof of Theorem 4.2 is the smallest 2-stage Euclidean algorithm on E.

Remark 8. Theorem 4.2 plays a key role in searching for integral domains to answer an open question raised by Cooke [6]. In Section 5 we will give integral domains which are ω-stage Euclidean but not 2-stage Euclidean.

We close this section with a proposition concerning finite-valued.

Proposition 4.3. *Let E be a ring such that E/bE is a finite ring for every $b \neq 0$ in E. If E admits a 2-stage Euclidean algorithm, then the smallest 2-stage Euclidean algorithm τ is finite-valued on E.*

Proof. By Theorem 4.2, we have $E = \bigcup_{\alpha\in W} {}_2E_\alpha$, where W is an ordinal such that $\mathrm{card}(E) < \mathrm{card}(W)$. If τ is not finite-valued on E, then, by Theorem 4.1, there is an element $b \in {}_2E_\omega \setminus {}_2E'_\omega$, where ω denotes the first transfinite ordinal. We have $\tau(b) = \omega$. Since $E/bE = \pi_b({}_2E'_\omega \cup {}_2E^b_\omega) = \pi_b({}_2E^b_\omega)$, every coset $c + bE$ admits a representative r with $r \in {}_2E^b_n$ for some finite value n. By the hypothesis E/bE is finite, whence $m = 1 + \sup(n)$ is an ordinary integer. By the transfinite construction of type 2-SEA on E, we have $b \in {}_2E_m$, a contradiction. Hence τ is finite-valued on E. □

5. Examples

We start with an example which plays multiple crucial roles in studying the Euclid-type algorithms.

Example 1. Let $\mathbb{A}$ be the ring of all algebraic integers. It is well known that $\mathbb{A}$ is not a principal ideal domain. Cooke [6] proved that $\mathbb{A}$ is a 2-stage Euclidean domain. Let a be a nonzero nonunit element in $\mathbb{A}$. For every $b + a\mathbb{A} \in (\mathbb{A}/a\mathbb{A})^\circ$, according to H.W. Lenstra, Jr. [7] and [1], there exists $q \in \mathbb{A}$ such that $b - aq = u$, a unit in $\mathbb{A}$. This implies that $b + a\mathbb{A} = u + a\mathbb{A} \in \pi_a(\tilde{\mathbb{A}}_0)$, hence $a \in \tilde{\mathbb{A}}_1$. It follows that $\mathbb{A} = \tilde{\mathbb{A}}_1$. Since $\mathbb{A}$ is not a principal ideal domain, $\mathbb{A}$ cannot admit a restricted Nagata's pairwise algorithm (see [10]). By Proposition 2.2, $\mathbb{A}$ is not a UFD. The existence of ring $\mathbb{A}$ shows that the hypothesis in Proposition 2.2 requiring a ring E being a UFD is essential. Since $\mathbb{A}$ is a 2-stage Euclidean domain but not a UFD, by Proposition 2 of [6], there exists a nonzero nonunit element γ in $\mathbb{A}$ which cannot factor as a finite product of irreducible elements. Note that we also have $\mathbb{A} = {}_2\mathbb{A}_1$.

The following proposition is a general version of Proposition 1 of [6]. The proof of it is the same as that of [6].

Proposition 5.1. *A ring E is ω-stage Euclidean if and only if every pair (a, b) in E with $a \neq 0$ has a terminating k-stage division chain for some k.*

Remark 9. By Propositions 2.1 and 5.1, we know that a ring E admitting a Nagata's pairwise algorithm is an ω-stage Euclidean ring.

Example 2. The results of Cohn [5] and Cooke [6] show that the principal ideal rings of integers of $\mathbb{Q}(\sqrt{-d})$ for $d = 19, 43, 67, 163$ are not ω-stage Euclidean. Thus, in these complex quadratic rings, there always exists a pair (a, b) which does not have a terminating division chain.

Example 3. The notion of quasi-Euclidean rings appeared in Bougaut [3] is equivalent to the notion of ω-stage Euclidean rings. In [3] Bougaut proved that every absolutely flat ring is a quasi-Euclidean ring. Here we shall prove that every absolutely flat ring E is a 2-stage Euclidean ring with $E = \tilde{E}_1$.

Instead of giving the definition of an absolutely flat ring, we adopt to give the following lemma which can be found in [2, Chapter 2, Exercise 27].

Lemma 5.2. *Let E be a ring. Then the following are equivalent:*

(1) *E is absolutely flat.*
(2) *Every principal ideal of E is idempotent.*
(3) *Every principal ideal of E is generated by an idempotent element.*

A Boolean ring is absolutely flat. More generally, a ring E in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x) is absolutely flat (see [2, Chapter 1, Exercise 7]). As an example, let $E = F_2 \times F_2 \times F_2 \times \cdots$ be the direct product of copies of F_2, a field of 2 elements, indexed by the positive integers. It is an easy check to verify that E is an absolutely flat ring but not a principal ideal ring.

Proposition 5.3. *Let E be an absolutely flat ring. Then*

(1) *E is a 2-stage Euclidean ring.*
(2) *$E = \tilde{E}_1$.*

Proof. (1) For nonzero $a \in E$, let $b + aE$ be any coset in E/aE. Since E is an absolutely flat ring, there exists an idempotent element a_0 in E such that $aE = a_0E$. Write $d = a_0 + b - a_0b$. Then $d + aE = b + aE$ and $a_0d = a_0^2 + a_0b - a_0^2b = a_0$. Since $a_0 | a$, we have $d | a$. It implies that $d \in {}_2E_1^a$. This shows that $a \in {}_2E_1$. It follows that $E = {}_2E_1$. Thus, by Theorem 4.2, we prove that E is a 2-stage Euclidean ring.

(2) Let a, b and d be elements in E as described in (1). Suppose b is coprime to a. Then $1 = \text{g.c.d.}(b, a) = \text{g.c.d.}(a, d)$ and $d | a$ imply that $d \in E^*$. This shows that $a \in \tilde{E}_1$. It follows that $E = \tilde{E}_1$. □

Remark 10. Let E, not a field, be an absolutely flat ring. From the proof of Proposition 5.3, we observe that ${}_2E_0 = \{0\} = \tilde{E}_{-1}$, ${}_2E_1 = E \supsetneqq \{0\} \cup E^* = \tilde{E}_0$, and $\tilde{E}_1 = E$. It shows that the sequence ${}_2E_\alpha$ is one step earlier than the sequence $\tilde{E}_\alpha$ to exhaust the ring E.

Example 4. Let $E = \mathbb{Z} + x\mathbb{Q}[x]$ be a subring of the polynomial ring $\mathbb{Q}[x]$. Every element in E is a polynomial in the indeterminate x with rational coefficients whose constant term is an integer. Then E is an ω-stage Euclidean domain, but it is not a 2-stage Euclidean domain.

More generally, we have the following theorem.

Theorem 5.4. *Let R be an integral domain and let F be the field of fractions of R. Let $E = R + xF[x] \subset F[x]$ be the subring of polynomials in the indeterminate x over F with constant term from R.*

(1) *R is a field if and only if E is a unique factorization domain.*
(2) *R is an ω-stage Euclidean domain if and only if E is an ω-stage Euclidean domain.*
(3) *Suppose R, not a field, has only finitely many units and there exists a map $\varphi : R \longrightarrow W$ with the properties*
  (i) *$\varphi(a) \leqslant \varphi(ab)$ for $a, b \in R$ with $b \neq 0$;*
  (ii) *for every a in R there exists an irreducible element p_a in R such that $\varphi(p_a) > \varphi(a)$,*
  where W is an ordinal such that $\operatorname{card}(E) < \operatorname{card}(W)$. Then E is not a 2-stage Euclidean domain.

Proof. (1) Suppose R is a field. Then $E = R[x]$ is indeed a UFD. Conversely, suppose E is a UFD. Assume that R is not a field. Then the reducible polynomial x can be written as a finite product of irreducibles of E. Applying it one can reach a contradiction. Hence R is a field.

(2) Suppose R is an ω-stage Euclidean domain. To prove E being an ω-stage Euclidean domain, by Proposition 5.1, it is equivalent to prove that for every pair $f(x)$, $g(x)$ in E with $f(x) \neq 0$ there exists a terminating k-stage division chain starting from the pair $(f(x), g(x))$ for some k.

We proceed by induction on $n = \operatorname{degree} f(x)$ with $f(x) \neq 0$. If $\operatorname{degree} f(x) = 0$, i.e., $f(x) \in R$ a nonzero constant polynomial, then for every $g(x)$ in E we can write $g(x) = q_1 f(x) + r_1$, where r_1 is the constant term of $g(x)$ and $q_1 = (g(x) - r_1)/f(x)$. Since $f(x), r_1 \in R$, if $r_1 \neq 0$ then by hypothesis the pair $(r_1, f(x))$ has a terminating division chain in R. It then follows that the pair $(f(x), g(x))$ has a terminating division chain in E. For positive integer n, suppose that every pair $(f(x), g(x))$ in E with $f(x) \neq 0$ and $\operatorname{degree} f(x) < n$ has a terminating division chain in E. We want to prove that every pair $(f(x), g(x))$ in E with $\operatorname{degree} f(x) = n$ has a terminating division chain in E.

Case 1. $\operatorname{degree} g(x) < n$. We first write $g(x) = 0 \cdot f(x) + g(x)$. Next by induction hypothesis the pair $(g(x), f(x))$ has a terminating division chain in E. Putting them together the pair $(f(x), g(x))$ has a terminating division chain in E.

Case 2. $\operatorname{degree} g(x) = \operatorname{degree} f(x) = n$. In this case we first find an element m in R such that $mf(x) = ax^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $mg(x) = bx^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$ with $a, b \in R$ and $a_i, b_i \in F$ for $i = 0, 1, \ldots, n-1$. Since R is an ω-stage Euclidean domain, there exist $s \in \mathbb{N}$ and $q_i, r_i \in R$ for $i = 1, 2, \ldots, s$ such that

$$\begin{aligned} b &= q_1 a + r_1, \\ a &= q_2 r_1 + r_2, \\ &\vdots \\ r_{s-2} &= q_s r_{s-1} + 0, \end{aligned}$$

a terminating s-stage division chain starting from the pair (a, b). Applying this division chain we have

$$\begin{aligned} mg(x) &= q_1\bigl(mf(x)\bigr) + \bigl(r_1 x^n + h_1(x)\bigr), \\ mf(x) &= q_2\bigl(r_1 x^n + h_1(x)\bigr) + \bigl(r_2 x^n + h_2(x)\bigr), \\ &\vdots \\ r_{s-2}x^n + h_{s-2}(x) &= q_s\bigl(r_{s-1}x^n + h_{s-1}(x)\bigr) + h_s(x), \end{aligned} \tag{5.1}$$

where degree $h_i(x) \leqslant n-1$ and the constant term of $h_i(x)$ being a multiple of m in R for $i = 1, 2, \ldots, s$. The division chain (5.1) gives a division chain for the pair $(f(x), g(x))$ with the last remainder $\frac{1}{m}h_s(x)$. Since degree $\frac{1}{m}h_s(x) \leqslant n-1$, by induction hypothesis the pair $(\frac{1}{m}h_s(x), \frac{1}{m}(r_{s-1}x^n + h_{s-1}(x)))$ has a terminating division chain in E. It then follows that the pair $(f(x), g(x))$ has a terminating division chain in E.

Case 3. degree $g(x) >$ degree $f(x) = n$. Since $F[x]$ is a Euclidean domain, there exist $q(x)$, $r(x)$ in $F[x]$ such that $g(x) = q(x)f(x) + r(x)$ with $r(x) = 0$ or degree $r(x) <$ degree $f(x)$. Write $q(x) = xq_1(x) + c$ with $q_1(x) \in F[x]$ and $c \in F$. Then $g(x) = xq_1(x)f(x) + cf(x) + r(x)$ with degree$(cf(x) + r(x)) \leqslant$ degree $f(x)$. Since the constant terms of $cf(x) + r(x)$ and $g(x)$ are the same, we have $cf(x) + r(x) \in E$. By induction hypothesis and Case 2, the pair $(cf(x) + r(x), f(x))$ has a terminating division chain in E. It then follows that the pair $(f(x), g(x))$ has a terminating division chain in E.

We conclude that E is an ω-stage Euclidean domain.

Conversely, suppose E is an ω-stage Euclidean domain. For every pair a, b in $R(\subset E)$ with $a \neq 0$, there exist $s \in \mathbb{N}$, $f_i(x), g_i(x) \in E$ for $i = 1, 2, \ldots, s$ such that

$$\begin{aligned} b &= f_1(x)a + g_1(x), \\ a &= f_2(x)g_1(x) + g_2(x), \\ &\vdots \\ g_{s-2}(x) &= f_s(x)g_{s-1}(x) + 0. \end{aligned}$$

Let q_i, r_i be the constant terms of f_i and g_i respectively. Then we have $q_i, r_i \in R$ for $i = 1, 2, \ldots, s$ and

$$\begin{aligned} b &= q_1a + r_1, \\ a &= q_2r_1 + r_2, \\ &\vdots \\ r_{s-2} &= q_sr_{s-1} + 0. \end{aligned}$$

This proves that R is an ω-stage Euclidean domain.

(3) We shall prove that $\bigcup_{\alpha \in W} {}_2E_\alpha \neq E$. First, for $\alpha = 0$ in W, we have ${}_2E_0 = \{0\} \subset R$. For $\alpha \neq 0$ in W, assuming ${}_2E_\beta \subset R$ for all $\beta < \alpha$ in W, we want to prove that ${}_2E_\alpha \subset R$. Suppose now there exists an element $f = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in {}_2E_\alpha \setminus R$ with $a_n \neq 0$. Write $a_n = b/a$ with nonunits $a, b \in R$. By (ii) there exists an irreducible element p in R such that $\varphi(p) > \max\{\varphi(a), \varphi(u_1 - a), \ldots, \varphi(u_m - a)\}$, where $\{u_1, u_2, \ldots, u_m\}$ is the set of all units of R. Since $f \in {}_2E_\alpha$, the nonzero coset $\frac{b}{p}x^n + fE$ admits a representative g with $g \in {}_2E_\alpha^f$. Thus $g|(f - e)$ for some $e \in {}_2E'_\alpha = \bigcup_{\beta<\alpha} {}_2E_\beta \subset R$. This implies that degree $g \leqslant$ degree$(f - e) = n$, whence $g = \frac{b}{p}x^n + fc$ for some $c \in R$. By (i) and $\varphi(p) > \varphi(a)$, we have $\frac{b}{p} + \frac{cb}{a} \neq 0$, which is the leading coefficient of g. Hence degree $g = n =$ degree$(f - e)$. Therefore there exists an element d in R such that $f - e = dg$. The coefficients of x^n of $f - e$ and dg are equal which induces $p = (a + cp)d$. By (i) and $\varphi(p) > \varphi(u_i - a)$ for $i = 1, 2, \ldots, m$, we have that $a + cp$ is not a unit in R. Since p is irreducible, we have $d \in R^*$. This implies that $\varphi(a) = \varphi(p(d^{-1} - c)) \geqslant \varphi(p)$, a contradiction. We conclude that ${}_2E_\alpha \subset R$ for all $\alpha \in W$, i.e., $\bigcup_{\alpha \in W} {}_2E_\alpha \neq E$.

By Theorem 4.2, we prove that E is not a 2-stage Euclidean domain. □

By applying Theorem 5.4 we can find infinitely many integral domains E, including $E = \mathbb{Z} + x\mathbb{Q}[x]$, which are ω-stage Euclidean domains but not 2-stage Euclidean domains. For example, let p be a prime, $n \geqslant 1$ an integer, K the finite field with p^n elements, $R = K[y]$ the ring of polynomials in the indeterminate y over K, and F the field of fractions of R. Let $E = R + xF[x]$ and the map $\varphi(b) =$ the degree of b in y for every $b \in R$. By Theorem 5.4(3) E is not a 2-stage Euclidean domain. On the other hand, since $R = K[y]$ is a Euclidean domain, hence R is an ω-stage Euclidean domain. By Theorem 5.4(2) E is an ω-stage Euclidean domain. Of course more examples of E which are ω-stage Euclidean but not 2-stage Euclidean can be found.

We have answered the following fundamental question raised by Professor Cooke [6]:

"I do not know of an example of an ω-stage euclidean ring which is not 2-stage euclidean."

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