Large Time Behaviour of Solutions of the
Porous Medium Equation in Bounded Domains*

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INTRODUCTION

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, with boundary $\partial \Omega$, and suppose that $u = u(x, t)$ is a solution of the initial-boundary value problem

\begin{align*}
  u_t &= \Delta(u^m), \quad \text{in } \Omega \times \mathbb{R}^+, \\
  u(x, t) &= 0, \quad \text{in } \partial \Omega \times \mathbb{R}^+, \\
  u(x, 0) &= u_0(x), \quad \text{in } \Omega.
\end{align*}

Here $m \geq 1$ is constant, $u_0(x)$ is a given nonnegative function, and

\begin{align*}
  \mathbb{R}^+ &= \{ t : 0 < t < +\infty \}.
\end{align*}

If $m = 1$ the partial differential equation in Problem (I) is just the classical equation of heat conduction and it is well known that under appropriate conditions on $u_0$ and $\Omega$, $u(\cdot, t) \to 0$ as $t \to +\infty$ (see, for example, [9]). More precisely, let $0 < \lambda_0 < \lambda_1 < \cdots$ denote the eigenvalues of the Laplace operator on $\Omega$ with Dirichlet boundary conditions, and let $\phi_0(x)$ denote the eigenfunction corresponding to the smallest eigenvalue $\lambda_0$. We take $\phi_0$ to be positive and normalize so that $\|\phi_0\|_2 = 1$. Then one can show that

\begin{align}
  \|u(\cdot, t)\|_2 &\leq K_1 e^{-\lambda_0 t} \quad \text{for } t \geq 0, \\
  \|e^{\lambda_0 t}u(\cdot, t) - (\phi_0, u_0) \phi_0\|_2 &\leq K_2 e^{-(\lambda_1 - \lambda_0)t} \quad \text{for } t \geq 0.
\end{align}

Here $\| \cdot \|_2$ denotes the $L^2(\Omega)$-norm and $(\cdot, \cdot)$ denotes the usual inner product on $L^2(\Omega)$. The constants $K_1$ and $K_2$ depend only on $u_0$ and $\Omega$.

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The object of this paper is to study the asymptotic behaviour as $t \to +\infty$ of nonnegative solutions of Problem (I) with $m > 1$, and to derive estimates which are analogous to (1) and (2). In case $m > 1$ the equation $u_t = \Delta (u^m)$ is often called the porous medium equation since it first arose in the study of gas flows in homogeneous porous media [17]. It also arises in a variety of other applications; for example, in modeling the diffusion of an electron–ion plasma [15] and in describing the dynamics of biological populations whose mobility is density dependent [13].

In order to discuss the asymptotic behaviour of nonnegative solutions of Problem (I) we must, of course, be sure that such solutions exist. In general, Problems (I) is not solvable in the classical sense and it is necessary to introduce a suitable class of weak (or generalized) solutions (cf. [18]). This is done in Section 3, and in the Appendix we prove that, under the assumptions on $u_0$ and $\Omega$ given in Section 1, Problem (I) possesses a unique solution in the class under consideration. An important by-product of the existence theory is a Comparison Principle for weak solutions of Problem (I) which is the basic tool in this study.

Our main result is the estimate

$$
\|(1 + t)^{\gamma} u(\cdot, t) - f\| \leq K (1 + t)^{-1}
$$

for $t > 0$, (3)

which holds for solutions of Problem (I). Here $\| \cdot \|$ denotes the $L^\infty(\Omega)$-norm; $\gamma = 1/(m - 1)$; $K$ is a positive constant which depends only on $m$, $n$, $u_0$, and $\Omega$; and the function $f(x)$ is the nontrivial solution of the problem

$$
\Delta (f^m) + \gamma f = 0 \quad \text{in} \quad \Omega,
$$

$$
f = 0 \quad \text{on} \quad \partial \Omega.
$$

(11)

The existence and uniqueness of $f$ is proved in Section 2. A different proof was given by Alikakos in Ref. [1]. In unpublished work, M. G. Crandall and L. C. Evans have also proved the convergence of $(1 + t)^{\gamma} u$ to $f$, but have not estimated the rate of convergence. As will be shown later, estimate (3) is sharp.

The first step in proving (3) is to show that there exists a constant $\tau_1 > 0$ depending only on $m$, $n$, $u_0$, and $\Omega$ such that

$$
0 \leq u(x, t) \leq (\tau_1 + t)^{-\gamma} f(x) \quad \text{in} \quad \bar{\Omega} \times \mathbb{R}^+.
$$

(4)

This is done in Section 4 using the Comparison Principle and one of the comparison functions introduced in Section 3. A version of estimate (4) is given by Evans in Ref. [8] and a more detailed estimate is derived by Alikakos in Ref. [1] using a modification of the DeGiorgi–Moser iteration.
Observe that if $f$ is the solution of Problem (II) then for any constant $\tau > 0$ the function

$$u^*(x, t) \equiv (\tau + t)^{-\gamma}f(x)$$

is a solution of the porous medium equation which vanishes on $\partial \Omega \times \mathbb{R}^+$. That is, $u^*(x, t)$ is the solution of Problem (I) with $u_0 = f_{\tau^{-\gamma}}$. Thus estimate (4) is the best possible.

The next step in proving (3) is to show that even if $u_0$ has compact support in $\Omega$ there exists a finite time $T$ such that the solution of Problem (I), with initial function $u_0$, is positive throughout $\Omega$ for all times $t > T$. This is done in Section 5. In Section 6 we extend this positivity result and prove that there are constants $T^* \geq 0$ and $\tau_0 > 0$ which depend only on $m, n, u_0$ and $\Omega$ such that

$$u(x, t) \geq (\tau_0 + t)^{-\gamma}f(x) \quad \text{in} \quad \Omega \times [T^*, +\infty). \quad (5)$$

The proofs of positivity and the estimate (5) rely on the Comparison Principle and the comparison functions introduced in Section 3. As we show in Section 6, (3) is a consequence of (4) and (5).

In Section 7 we apply our results to derive the analog of estimate (3) for solutions of the problem

$$u_t = \Delta(u^m) + \mu u \quad \text{in} \quad \Omega \times \mathbb{R}^+, \quad \text{(III)}$$

$$u(x, t) = 0 \quad \text{in} \quad \partial \Omega \times \mathbb{R}^+, \quad u(x, 0) = u_0(x) \quad \text{in} \quad \Omega$$

with $\mu > 0$. Problems of this sort arise in the Gurtin–MacCamy theory of density dependent diffusion of biological populations [13]. MacCamy [16] has obtained the basic convergence result without an estimate of the rate of convergence in the one-dimensional case by methods which are quite different from ours.

The equation $u_t = \Delta(u^m)$ also occurs in recent work in plasma physics for values of $m \in (0, 1)$. This is the so-called "fast diffusion" case, which is characterized by the existence of a finite extinction time; that is, there exists a $T \in \mathbb{R}^+$ which depends only on the data such that for the solution $u$ of Problem (I), $u(\cdot, t) = 0$ in $\Omega$ for all $t \geq T$. Assuming the existence and some regularity for the solution of Problem (I), Berryman and Holland [6] have recently established the stability of separable solutions in the fast diffusion case.
1. Preliminaries

Throughout this paper we shall always use the following assumptions about the domain \( \Omega \) and the initial function \( u_0 \):

(H1) \( \Omega \) is a bounded arcwise connected open set with compact closure whose boundary, \( \partial \Omega \), is compact and of class \( C^3 \).

(H2) \( u_0(x) \) is a nonnegative continuous function defined on \( \bar{\Omega} \) such that \( u_0 = 0 \) on \( \partial \Omega \) and \( u_0^m \in C^1(\bar{\Omega}) \).

We shall refer to these assumptions collectively as (H). The assumption that \( \partial \Omega \) is of class \( C^3 \) is stronger than necessary. We have adopted it for the sake of the convenience offered by the lemma given below and since it is close enough to the condition that \( \partial \Omega \) be of class \( C^{2+1/m} \) which is actually used in Section 2. We have made no particular effort to find the weakest possible hypothesis.

In our estimates we seek constants which depend only on \( m, n, u_0, \) and \( \Omega \). We shall adopt the convention that such constants are characterized as "depending only on the data."

Let \( d: \Omega \to [0, +\infty) \) be given by

\[
d(x) = \min\{|x - z|: z \in \partial \Omega\},
\]

that is, \( d(x) \) is the distance from \( x \) to \( \partial \Omega \). In terms of \( d(x) \) we define the sets

\[
\Omega_s = \{x \in \bar{\Omega}: 0 \leq d(x) < s\}
\]

and

\[
\Sigma_s = \bar{\Omega} \setminus \Omega_s = \{x \in \Omega: d(x) \geq s\}.
\]

In several places in the sequel we shall use the following properties of \( d(x) \).

Lemma 1. Let \( \Omega \) satisfy (H1). There is a constant \( \sigma \in \mathbb{R}^+ \) such that for every \( x \in \Omega_s \) there is a unique \( z(x) \in \partial \Omega \) which satisfies

\[
d(x) = |x - z(x)|.
\]

Moreover, \( d(x) \in C^2(\Omega_s) \).

A proof of Lemma 1 can be found in Ref. [22] and we omit it.\(^1\)

Let \( M_r: \partial \Sigma_r \to \partial \Omega \) be the mapping given by \( x \to z(x) \). It is a simple consequence of Lemma 1 that \( M_r \) is a homeomorphism for all \( r \in [0, \sigma) \). Note that this implies that \( \Sigma_r \) is arcwise connected for each \( r \in [0, \sigma) \).

\(^1\) There is a minor inaccuracy in Ref. [22] in the assertion that \( \sigma = 1/K \), where \( K \) is the bound for the principal curvatures of \( \partial \Omega \). Actually all that one can say is \( 0 < \sigma \leq 1/K \).
In this section we consider the nonlinear eigenvalue problem

\[-\Delta u = \lambda u^\delta \quad \text{in} \quad \Omega,\]
\[u = 0 \quad \text{on} \quad \partial \Omega,\]  

(II')

where \(\delta \in (0, 1)\). Clearly this problem reduces to Problem (II) if we set \(u = f^m, \lambda = \gamma, \) and \(\delta = 1/m\). To ensure that the right-hand side of the equation has a meaning we shall consider only nonnegative solutions of Problem (II').

If \(n = 1\), Problems (II) and (II') can be solved easily by means of first integrals and one finds that for each positive \(\lambda\) there exists a unique classical solution which is strictly positive in \(\Omega\). Here we shall establish the corresponding result for general \(n\) using a result of Amann [2].

Let \(u \in C^2(\Omega) \cap C(\overline{\Omega})\) be a solution of Problem (II') for \(\lambda = \tilde{\lambda}\). We say that \(u\) is an eigenfunction of Problem (II') corresponding to \(\tilde{\lambda}\) if \(u > 0\) and \(u \neq 0\) in \(\Omega\). If \(u\) is an eigenfunction of Problem (II') corresponding to \(\tilde{\lambda}\), then the function

\[u = \left(\frac{\lambda}{\tilde{\lambda}}\right)^{1/(1-\delta)} \tilde{u}\]

is an eigenfunction of Problem (II') corresponding to \(\lambda\). Thus, in particular, it suffices to prove the existence of eigenfunctions for a single value of \(\lambda\), say, \(\lambda = 1\).

The main result of this section is the existence and uniqueness of the eigenfunction for Problem (II') corresponding to any value of \(\lambda \in \mathbb{R}^+\).

**Proposition 1.** If \(\Omega\) satisfies (H1) then for each \(\lambda \in \mathbb{R}^+\) Problem (II') has a unique eigenfunction \(u\). Moreover, \(u \in C^{2+\delta}(\Omega), u(x) > 0\) for \(x \in \Omega,\) and

\[\frac{\partial u}{\partial v}(x) \equiv \nabla u(x) \cdot v(x) < 0 \quad \text{for} \quad x \in \partial \Omega.\]

Here \(v(x)\) denotes the outward directed unit normal vector to \(\partial \Omega\) at the point \(x\).

We shall prove Proposition 1 in several steps. To begin with, observe that if \(u\) is an eigenfunction of Problem (II') corresponding to \(\lambda \in \mathbb{R}^+\) then \(u = 0\) on \(\partial \Omega, u \geq 0\) in \(\Omega\) and \(u \neq 0\). Moreover, \(-\Delta u = \lambda u^\delta \geq 0\) in \(\Omega\). Therefore if
$u \in C^1(\Omega)$, it follows from the Strong Maximum Principle and the Boundary Point Lemma [20] that

\[ u > 0 \quad \text{in } \Omega \quad \text{and} \quad \partial u / \partial n < 0 \quad \text{on } \partial \Omega. \]  

(2.1)

Next we show that Problem (II') possesses an eigenfunction $u \in C^{2+\delta}(\Omega)$ corresponding to $\lambda = 1$. To prove this write $g(u) = u^\delta$. Then $g \in C^6(\mathbb{R}^+)$ and $g(0) = 0$. Hence, by Theorem 3 of [2] it suffices to establish the existence of a function $v \in C^{2+\delta}(\Omega)$ such that

\[ -\Delta v \geq v^\delta \quad \text{in } \Omega, \]

\[ v \geq 0 \quad \text{on } \partial \Omega, \]

and a function $w$ such that $0 < w \leq v$ in $\Omega$ and

\[ -\Delta w \leq w^\delta \quad \text{in } \Omega, \]

\[ w = 0 \quad \text{on } \partial \Omega. \]

To construct the function $v$, let $\Omega'$ be a smoothly bounded domain such that $\Omega \subset \Omega'$ and consider the eigenvalue problem

\[ -\Delta z = \mu z \quad \text{in } \Omega', \]

\[ z = 0 \quad \text{on } \partial \Omega'. \]

It is well known [20] that the principal eigenvalue $\mu_0$ is positive, and that the corresponding eigenfunction $\phi_0$ can be chosen so that $\phi_0 > 0$ in $\Omega'$ and $\max \{\phi_0(x) : x \in \Omega'\} = 1$. By the continuity of $\phi_0$

\[ \beta \equiv \inf \{\phi_0(x) : x \in \Omega\} > 0. \]

We now set $v = k\phi_0$, where $k \in \mathbb{R}^+$ is a constant to be chosen. Then

\[ -\Delta v - v^\delta = \mu_0 k \phi_0 - k^\delta \phi_0^\delta - k^\delta = \mu_0 k k^\delta (k^{1-\delta} - 1/\beta) \]

in $\Omega$. Thus

\[ -\Delta v \geq v^\delta \quad \text{in } \Omega \]

if

\[ k = (\beta \mu_0)^{-1/(1-\delta)}. \]

Next we construct the function $w$. For this purpose consider the eigenvalue problem

\[ -\Delta z = \rho z \quad \text{in } \Omega, \]

\[ z = 0 \quad \text{on } \partial \Omega \]
with principal eigenvalue $\rho_0$ and corresponding eigenfunction $\psi_0$. Again $\rho_0 > 0$, $\psi_0 > 0$ in $\Omega$, and we choose $\max \{\psi_0(x) : x \in \Omega\} = 1$. Put $w = c\psi_0$, where $c \in \mathbb{R}^+$ is a constant to be chosen. Then

$$-\Delta w - w^\delta = \rho_0 c\psi_0 - c^\delta \psi_0^\delta$$

$$= \rho_0 (c\psi_0)^\delta \left( (c\psi_0)^{1-\delta} - \frac{1}{\rho_0} \right) \leq \rho_0 (c\psi_0)^\delta \left( c^{1-\delta} - \frac{1}{\rho_0} \right).$$

Hence

$$-\Delta w \leq w^\delta \quad \text{in} \quad \Omega$$

if $c \leq \rho_0^{-\frac{1}{(1-\delta)}}$. Finally, because $v \geq k\beta$, we can choose $c$ so small that $v \geq w$ in $\overline{\Omega}$. This completes the proof of existence.

The eigenfunction whose existence we have just established is unique in the class $C^2(\Omega) \cap C^1(\overline{\Omega})$. To prove this suppose that there exist two distinct eigenfunctions $u$ and $v$ in class $C^2(\Omega) \cap C^1(\overline{\Omega})$ corresponding to $\lambda = 1$. Without loss of generality, we may assume that $u \not\geq v$, that is, the inequality $u < v$ holds for at least some points of $\Omega$. Define

$$r_0 = \sup \{ r > 0 : 5 \leq 3 u \text{ in } \overline{\Omega} \}.$$ 

By continuity, $u \not\geq v$ implies that $r_0 > 1$. On the other hand, for all sufficiently large $r \in \mathbb{R}^+$ we have $tu \geq v$ in $\overline{\Omega}$. This follows since $u$ satisfies (2.1) and $v \in C^1(\overline{\Omega})$. Thus $r_0 \in (1, +\infty)$.

Set

$$z(x) = r_0 u(x) - v(x).$$

Then $z(x) \geq 0$ in $\Omega$ and $z(x) = 0$ on $\partial \Omega$. Moreover,

$$-\Delta z = (r_0 - r_0^\delta) u^\delta + (r_0^\delta v^\delta - v^\delta > 0 \quad \text{in} \quad \Omega,$$

since $r_0 > 1$ and both $u$ and $v$ are positive in $\Omega$. Thus, by the Strong Maximum Principle and the Boundary Point Lemma,

$$z(x) > 0 \quad \text{in} \quad \Omega \quad (2.2)$$

and

$$\frac{\partial z}{\partial v}(x) < 0 \quad \text{for} \quad x \in \partial \Omega. \quad (2.3)$$

Let $\tau_n = r_0 - 1/n$ for integers $n \geq 1$. In view of the definition of $r_0$, there exists a point $x_n \in \Omega$ such that $\tau_n u(x_n) - v(x_n) < 0$ for each $n$. Since $\overline{\Omega}$ is compact, there exists a subsequence, which we again call $\{x_n\}$, such that
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\( x_n \in \Omega \) and \( x_n \to x_0 \in \widetilde{\Omega} \). If \( x_0 \in \Omega \) then \( z(x_0) = 0 \) in contradiction to (2.2). Thus \( x_0 \in \partial \Omega \). Therefore, if \( n \) is sufficiently large we have \( d(x_n) < \sigma \) and it follows from Lemma 1 that there is a unique \( z_n \equiv z(x_n) \in \partial \Omega \) such that

\[
d(x_n) = |x_n - z_n|.
\]

By the Theorem of the Mean,

\[
0 < -\frac{\tau_n u(x_n) - v(x_n)}{d(x_n)} = \nabla \{\tau_n u(x_n) - v(x_n)\} \cdot v(z_n),
\]

where \( \bar{x}_n \) is a point on the line joining \( x_n \) and \( z_n \). Now let \( n \to +\infty \). Since \( z_n \to x_0 \) it follows that

\[
\frac{\partial z}{\partial v}(x_0) \geq 0.
\]

However, this contradicts (2.3) so that the uniqueness of the eigenfunction is proved.

3. TRANSIENT SOLUTION: PROBLEM (I)

In this section we discuss some general results concerning solutions of Problem (I) and introduce some special solutions of the porous medium equation which will be used later on as comparison functions.

It is well known that because of the degeneracy of the porous medium equation,

\[
u_t = \Delta(u^m),
\]

our initial-boundary value problem, Problem (I), may not possess a solution in the classical sense [18]. Thus it is necessary to introduce a suitable class of weak (or generalized) solutions. Our definition of weak solution is closely related to the definitions given by Oleinik et al. [18] for the case \( n = 1 \) and by Sabinina [21] for the Cauchy problem with \( n \geq 1 \).

Let \( Q_T = \Omega \times (0, T] \) and \( S_T = \partial \Omega \times (0, T] \) for any \( T \in \mathbb{R}^+ \). A function \( u: \bar{Q}_T \to [0, +\infty) \) is said to be a weak solution of Problem (I) in \( Q_T \) if it satisfies the following conditions:

(i) For each \( (y, t) \in S_T \),

\[
\lim_{Q_T \ni (x, t) \to (y, t)} u(x, t) = 0.
\]

(ii) \( \nabla(u^m) \) exists in the sense of distributions in \( Q_T \) and

\[
\int_{Q_T} \{ u^2 + |\nabla(u^m)|^2 \} \, dx \, dt < +\infty.
\]
(iii) \( u \) satisfies
\[
\int_{Q_T} \{ \nabla \phi \cdot \nabla (u^m) - \phi u \} \, dx \, dt = \int_{\Omega} \phi(x, 0) u_0(x) \, dx
\]
for all \( \phi \in C^1(\bar{Q}_T) \) which vanish on \( S_T \) and \( \Omega \times \{ T \} \).

In the Appendix we prove the following basic results concerning weak solutions of Problem (I).

**Proposition 2.** If (H) holds then Problem (I) possesses a unique weak solution in \( Q_T \) for any \( T \subseteq \mathbb{R}^+ \).

**Proposition 3 (The Comparison Principle).** Suppose that \( \Omega \) satisfies (H1), and that \( u_0 \) and \( u_0^* \) both satisfy (H2). Let \( u \) and \( u^* \) denote the weak solutions of Problem (I) in \( Q_T \) with initial functions \( u_0 \) and \( u_0^* \), respectively. Then \( u_0 \geq u_0^* \) in \( \Omega \) implies \( u \geq u^* \) in \( Q_T \).

Note that in the definition of weak solutions of Problem (I) we do not require that \( u \in C(\bar{Q}_T) \). In the neighbourhood of any point \( (x, t) \in Q_T \) such that \( u(x, t) > 0 \), the weak solutions \( u \) is, in fact, a \( C^\infty \)-function. The proof of this is the same as the proof of the corresponding fact in the one-dimensional case given in [4]. For \( T' \in \mathbb{R}^+ \) such that \( \text{supp } u(\cdot, t) \subseteq \Omega \) for all \( t \in [0, T'] \), \( u \) is the weak solution of the Cauchy problem with initial function \( u_0 \) and the continuity of \( u \) in \( \bar{Q}_T \), follows from recent results of Caffarelli and Friedman [7]. On the other hand, as we show in Section 5, there exists a \( T \geq 0 \) such that \( u > 0 \) in \( \Omega \times [T, +\infty) \). Then in view of (i) in the definition of weak solution, \( u \) is continuous in \( \bar{Q} \times [T, +\infty) \). Thus the continuity of \( u \) is in doubt only in the case in which \( \text{supp } u_0 \subseteq \Omega \) and in that case only on the interval between the last time for which \( \{ \text{supp } u(\cdot, t) \} \cap \partial \Omega = \emptyset \) and the first time for which \( \text{supp } u(\cdot, t) = \bar{\Omega} \). In a recent paper [12], Gilding and Peletier have been able to adapt the results of Caffarelli and Friedman to prove that \( u \in C(\bar{Q}_T) \).

The comparison principle stated in Proposition 3 is the main tool in this paper and we shall derive our estimates by comparing the given solution of Problem (I) with suitably chosen functions. Below we shall give two families of these functions. Both of them belong to a class of similarity solutions of the form
\[
\bar{u}(x, t) = (t + \tau)^{-\alpha} g(\eta)
\]
with \( \eta = |x| (t + \tau)^{-\beta} \), where \( \alpha \) and \( \beta \) are positive numbers, related by the equation
\[
(m - 1) \alpha + 2\beta = 1.
\]
and \( \tau \) is an arbitrary positive number. Moreover, the function \( g \) satisfies the ordinary differential equation

\[
(g^m)'' + \frac{n-1}{\eta} (g^m)' + \beta \eta g' + \alpha g = 0, \quad \eta > 0.
\]

For \( n = 1 \), this class of similarity solutions was studied in some detail by Gilding and Peletier [10, 11].

A. For the first family of similarity solutions we choose \( \alpha = n\beta \). Then Eq. (3.1) can be solved explicitly, yielding

\[
g(\eta) = c(n, m)(a^2 - \eta^2)^{\frac{\gamma}{\nu}}, \quad 0 < \eta \leq a,
\]

in which \( c(n, m) = [2m(n + 2\gamma)]^{-\gamma} \) with \( \gamma = 1/(m - 1) \), and \( a \) is an arbitrary positive number. Thus we obtain the two-parameter family of weak solutions of the Cauchy problem

\[
u(x, t; a, \tau) = c(t + \tau)^{-\alpha} \{ [a^2 - |x|^2 (t + \tau)^{-2\beta}]_+ \}^{\gamma},
\]

where \( \{ \phi \}_+ = \max(0, \phi) \) and \( \alpha = n\beta = n\{(m - 1)n + 2\}^{-1} \). This family of solutions had been obtained earlier by Barenblatt [5] and Pattle [19].

B. For the second family of comparison functions we consider the initial value problem

\[
(g^m)'' + \frac{n-1}{\eta} (g^m)' + \beta \eta g' + \alpha g = 0, \quad \eta > 0,
\]

\[
g(0) = c, \quad g'(0) = 0,
\]

where \( c \) is an arbitrary positive number. Problem (IV) is degenerate at \( \eta = 0 \) if \( m > 1 \). However, it can be shown by means of an argument based on the contraction mapping theorem that there exists a unique solution \( g \in C^2([0, \eta_0]) \) for some small \( \eta_0 \). Moreover, this solution can be continued as long as it is positive. If we multiply the equation by \( \eta^{n-1} \) and integrate from 0 to \( \eta \) we obtain

\[
(g^m)'(\eta) = -\frac{\alpha - n\beta}{\eta^{n-1}} \int_0^\eta \zeta^{n-1} g(\zeta) \, d\zeta - \beta \eta g(\eta) \leq -\beta \eta g(\eta)
\]

provided that \( \alpha \geq n\beta \). This implies that

\[
g(\eta) \leq \left( c^{m-1} \frac{m-1}{2m} \beta \eta^2 \right)^{\gamma},
\]
and hence
\[ a = \sup \{ \eta > 0 : g > 0 \text{ on } [0, \eta) \} < \infty. \]

Moreover, it follows from (3.2) that

\[ K \equiv (g^m)'(a) = -\frac{a - \eta}{a^{n-1}} \int_0^a \zeta^{n-1} g(\zeta) \, d\zeta, \quad (3.3) \]

In particular, if we set \( a = 2n\beta \), then \( g^m \) is a decreasing function and hence, in view of (3.3), \(-\beta ca \leq \kappa < 0\).

Let us denote the solution of Problem (IV) by \( g(\eta; c) \) and the corresponding values of \( a \) and \( \kappa \) by \( a(c) \) and \( \kappa(c) \). Then it can readily be verified that

\[ g(\eta; c) = cg(c^{-m-1/2} \eta; 1) \]

and

\[ a(c) = c^{m-1/2} a(1), \quad \kappa(c) = c^{m+1/2} \kappa(1). \quad (3.4) \]

Then the second two-parameter family of comparison functions will be

\[ u(x, t; c, \tau) = (t + \tau)^{-\alpha} \left[ g(\eta; c) \right]_+, \]

where it should be recalled that

\[ \text{supp } u(x, t; c, \tau) = \{(x, t) : |x| \leq a(c)(t + \tau)^{\beta}, \, t > 0\} \]

and \( \alpha = 2n\beta = n((m - 1)n + 1)^{-1} \).

Observe that while the functions \( u(\cdot, \cdot; a, \tau) \) are weak solutions of the Cauchy problem for the porous medium equation, the functions \( v(\cdot, \cdot; c, \tau) \) are not. The reason is that the functions \( v \) do not have the right behaviour at the interface, that is, at the boundary of their support. Specifically, the analogue of our condition (iii) in the standard definition of weak solution of the Cauchy problem would require that \( (g^m)'(a; c) = 0 \) and, as we have seen, this does not occur. However, in the expanding domain \( |x| < a(c)(t + \tau)^{\beta} \), \( v \) is, of course, a classical solution of the porous medium equation with zero data on the lateral boundary \( |x| = c(t + \tau)^{\beta} \). It is precisely these properties which makes the family \( u(\cdot, \cdot; c, \tau) \) useful.

The families of functions \( u \) and \( v \) defined above will both be used as lower bounds for the given solution of Problem I. The function \( u \) gives the dispersal of gas due to a point source at \( x = 0 \) and \( t = -\tau \) \([5, 19]\). It will be used to show that, after a finite time, the gas fills the entire volume \( \Omega \). Then, when the gas has reached the boundary, and the density is kept at zero there, we shall employ the function \( v \) to show that the normal derivative of \( u^m \) becomes negative at the boundary.
4. AN UPPER BOUND

We are now ready to prove the first estimate given in the Introduction. Let $f = f(x)$ denote the nontrivial solution of Problem (II). By Proposition 1, this solution exists, and it is unique. Let $Q$ denote the cylinder

$$\{(x, t): x \in \Omega, t > 0\}.$$

**Theorem 1.** Suppose $\Omega$ and $u_0$ satisfy hypothesis (H), and let $u$ be the corresponding solution of Problem (I). Then there exists a constant $\tau_1 > 0$, which only depends on the data, such that

$$u(x, t) \leq (\tau_1 + t)^{-\gamma} f(x) \quad \text{in} \quad \bar{Q}.$$

**Proof:** Consider as comparison function the solution $v$ of the porous medium equation given by

$$v(x, t) = (\tau + t)^{-\gamma} f(x),$$

with $\tau$ a positive constant. Suppose there exists a constant $\tau, > 0$ such that

$$0 \leq v(x, t) < (1/n) u_0(x) \quad \text{in} \quad \bar{Q}.$$

(4.1)

If $\tau = \tau_1$, then $v(x, 0) \geq u_0(x)$ in $\bar{Q}$ and hence, by the Comparison Principle, the result follows.

To prove that (4.1) holds we suppose to the contrary that there exists a sequence $\{x_n\} \subset \bar{Q}$ such that

$$0 \leq f(x_n) < (1/n) u_0(x_n)$$

or, equivalently,

$$0 \leq f^m(x_n) < (1/n^m) u_0^m(x_n).$$

(4.2)

By the compactness of $\bar{Q}$, we may assume that $\{x_n\}$ converges to $\tilde{x} \in \bar{Q}$ as $n \to \infty$. Moreover the boundedness of $u_0$ and the positivity of $f$ imply that $\tilde{x} \in \partial \Omega$. If we now divide both sides of (4.2) by $d(x_n)$ we find, by the argument used at the end of the proof of Proposition 1, that

$$\frac{\partial}{\partial y} |f(\tilde{x})|^m = 0.$$

Since this contradicts Proposition 1, we conclude that (4.1) holds for some $\tau_1 \in \mathbb{R}^+$ so that the theorem is proved.
In order to complete the program outlined in the Introduction we shall need a lower bound for the weak solution \( u \) of Problem (I). The required bound will be derived in Section 6, but first we must show that there is a \( T \in [0, +\infty) \) such that \( u(\cdot, t) > 0 \) in \( \Omega \) for all \( t \geq T \). If \( u_0 > 0 \) in \( \Omega \) then, by the Comparison Principle, \( u(\cdot, t) > 0 \) in \( \Omega \) for all \( t \geq 0 \). In that case we can take \( T = 0 \) and proceed directly to Section 6. In the present section we shall be mainly concerned with the case in which the support of \( u_0 \) is a proper subset of \( \Omega \) and we shall show that, nevertheless, \( u \) is eventually positive throughout \( \Omega \).

**Proposition 4.** Assume that \( \Omega \) and \( u_0 \) satisfy (H) and let \( u \) denote the weak solution of Problem (I). If \( u_0 \not\equiv 0 \) in \( \Omega \) then there exists a \( T \in [0, +\infty) \), which depends only on the data, such that
\[
\text{u(x, t) > 0 in } \Omega \times [T, +\infty).
\]

The set valued function \( \text{supp } u(\cdot, t) \) is nondecreasing in \( t \). The proof of this assertion is the same as the proof for the case \( n = 1 \) given by Kalashnikov in Ref. [14]. Thus, to prove Proposition 4 it suffices to to show the existence of a \( T \in [0, +\infty) \) such that \( u(\cdot, T) > 0 \) in \( \Omega \). In Ref. [14] Kalashnikov also proves the one-dimensional version of Proposition 4.

The main tool in the proof of the proposition is the following technical lemma in which we estimate how long it takes to transmit positivity from a ball \( B_\rho(x_0) \subset \Omega \) to a neighbouring ball \( B_\rho(x^*) \). In this lemma we shall use the similarity solutions of Type (A) introduced in Section 3, that is, solutions of the porous medium equation of the form
\[
u(x, t; a, \gamma) = c(t + \tau)^{-\alpha} \left\{ |a^2 - \eta^2| + \gamma \right\},
\]
where
\[
\eta = |x - y| (t + \tau)^{-\beta}
\]
for arbitrary \( y \in \mathbb{R}^n \). Here \( c, \alpha, \beta, \) and \( \gamma \) are constants depending only on \( m \) and \( n \), while \( a \) and \( \tau \) are positive parameters. Moreover,
\[
\text{supp } u(x - y, 0; a, \tau) = \{ x \in \mathbb{R}^n : |x - y| \leq a\tau^\beta \}.
\]

**Lemma 2.** Suppose that
\[
u(x, 0) \geq \nu(x - y, 0; a, \tau)
\]
(5.1)
for some $y \in \Omega$ and some $\theta \in [0, +\infty)$, where $\rho \equiv a \tau^\theta < d(y)/2$. Then for any $a^* \in (0, a/2]$ and $x^*$ such that $|x^* - y| = a^* \tau^\theta$.

$$u(x, t^* + \theta) \geq u(x - x^*, 0; a - a^*, \tau^*),$$

where

$$\tau^* = \left(\frac{a}{a - a^*}\right)^{1/\theta} \tau \quad \text{and} \quad t^* - \tau^* = \tau.$$

In particular,

$$t^* \leq (2^{1/\theta} - 1) \tau.$$  (5.2)

**Proof.** To simplify notation we set

$$\tilde{u}(x, t) = u(x - x^*, t; a - a^*, \tau).$$

Observe that

$$\text{supp } \tilde{u}(x, 0) = \{x \in \mathbb{R}^n : |x - x^*| \leq (a - a^*) \tau^\theta\}.$$

Thus, if $x \in \text{supp } \tilde{u}(x, 0)$ then

$$|x - y| \leq |x - x^*| + |x^* - y| \leq (a - a^*) \tau^\theta + a^* \tau^\theta = a \tau^\theta,$$

that is,

$$\text{supp } \tilde{u}(x, 0) \subset B_{\rho}(y) \subset \Omega.$$  (5.3)

Moreover, we claim that

$$\tilde{u}(x, 0) \leq u(x - y, 0; a, \tau) \quad \text{in } \Omega.$$  (5.4)

In view of (5.3) it suffices to verify (5.4) on $\text{supp } \tilde{u}(x, 0)$. On that set, (5.4) is equivalent to

$$|x - y|^2 - |x - x^*|^2 \leq (a^2 - (a - a^*)^2) \tau^{2\theta}. $$  (5.5)

In view of (5.1), (5.3), and (5.4) we can apply the Comparison Principle and conclude that

$$\tilde{u}(x, t) \leq u(x, \theta + t)$$

as long as $\text{supp } \tilde{u}(:, t) \subset \Omega$. In view of the definitions of $t^*$ and $\tau^*$, $(a - a^*)(\tau^* + \tau)^\theta = a \tau^\theta = \rho$ so that $\text{supp } \tilde{u}(\cdot, t^*) = B_{\rho}(a^*)$. On the other
hand, \( \bar{B}_\rho(x^*) \subset B_{2\rho}(y) \). Hence \( 2\rho < d(y) \) implies that \( \operatorname{supp} \bar{u}(\cdot, t^*) \subset \Omega \) and we conclude that

\[
\bar{u}(\cdot, t^*) \leq u(\cdot, \theta + t^*) \quad \text{in} \quad \Omega.
\]

Finally we note that \( t + \tau = t - t^* + \tau^* \) implies

\[
u(x - x^*, t; a - a^*, \tau) = u(x - x^*, t - t^*; a - a^*, \tau^*)
\]

so that, in particular,

\[
\bar{u}(x, t^*) = u(x - x^*, 0; a - a^*, \tau^*).
\]

**Proof of Proposition 4.** If \( u_0 > 0 \) in \( \Omega \) there is nothing to prove. Therefore assume that \( u_0 \neq 0 \) and has compact support in \( \Omega \). Hence, there exists a point \( x_0 \in \Omega \) such that \( u = u_0(x_0) > 0 \). Fix \( s_1 \in (0, \sigma) \). In view of the continuity of \( u_0 \) there exists a \( \delta \) such that \( 0 < \delta < d(x_0) \) and \( u_0 \geq \mu/2 \) in \( B_{\delta/2}(x_0) \). Set \( s = \min(\delta, s_1) \). Then \( 0 < s \leq s_1 < \sigma, s < d(x_0) \) and \( u_0 \geq \mu/2 \) in \( B_{s/2}(x_0) \). The domain \( \Sigma_s = \{x \in \Omega: d(x) \geq s\} \) is compact and

\[
\Sigma_s \subset \bigcup_{x \in \Sigma_s} B_{s/2}(x) \subset \Omega.
\]

By the Heine–Borel–Lebesgue theorem, a finite number, \( N \), of points \( x_1, x_2, \ldots, x_N \) can be found in \( \Sigma_s \) such that

\[
\Sigma_s \subset \bigcup_{j=1}^{N} B_j \subset \Omega,
\]

where

\[
B_j = B_{s/2}(x_j) \quad (j = 1, 2, \ldots, N).
\]

Clearly, we can add the ball \( B_0 = B_{s/2}(x_0) \) to the finite covering set without altering any properties of that set. We claim that there exists a \( T' \in \mathbb{R}^+ \) depending only on \( m, n, u_0 \), and \( \Omega \) such that

\[
u(x, T') > 0 \quad \text{for} \quad x \in \mathcal{B} \equiv \bigcup_{j=0}^{N} B_j.
\] (5.6)

Fix \( k \in \{1, 2, \ldots, N\} \). Since \( \Sigma_s \) is connected (cf. Section 1) there exists a polygonal path \( \gamma_k \) from \( x_0 \) to \( x_k \), where the segments of \( \gamma_k \) join the centers of various balls \( B_j \). Observe that \( \gamma_k \) has at most \( N \) segments and that each segment has length at most \( s \). Moreover, \( z \in \gamma_k \) implies that \( B_{s/2}(z) \subset \Omega \). To prove (5.6) we shall use Lemma 2 to transmit the positivity from \( B_0 \) to \( B_k \) by means of a finite sequence of balls of radius \( s/2 \) with centers on \( \gamma_k \). To start
this process we must first show that \(a\) and \(\tau\) can be chosen in such a way that Lemma 2 can be applied to \(u\) in \(B_0\).

Recall that \(u_0 \geq \mu/2 > 0\) in \(B_0 = B_{\sqrt[s]{2}}(x_0)\). Thus

\[
u_0(x) \geq u(x - x_0, 0; a, \tau)
\]

if we choose

\[
supp u(x - x_0, 0; a, \tau) = \{x \in \mathbb{R}^n : |x - x_0| \leq a\tau^a = B_{\sqrt[s]{2}}(x_0)
\]

and

\[
max u(x - x_0, 0; a, \tau) = u(0, 0; a, \tau) = c\tau^{-a}a^{2\tau} = \mu/2.
\]

That is, (5.7) holds if

\[
a = s \left( \frac{\sqrt[s]{2}^{-1}a}{cs^{2\tau}} \right)^{\beta/(2\beta + \alpha)} \quad \text{and} \quad \tau = \left( \frac{cs^{2\tau}}{2\sqrt[s]{2}^{-1}a} \right)^{1/(2\beta + \alpha)}.
\]

The exact values of \(a\) and \(\tau\) are not important; what matters is that they depend only on \(m, n, s, \) and \(\mu\).

We can now use Lemma 2 inductively to transmit the positivity of \(u_0\) in \(B_0\) along the successive segments of \(\gamma_k\) to \(B_k\). Specifically, we conclude that there exist positive numbers \(a_k, T_k,\) and \(\tau_k\) such that \(a_k\tau_k^a = s/2\) and

\[
u(x, T_k) \geq u(x - x_k, 0; a_k, \tau_k).
\]

The numbers \(a_k, T_k,\) and \(\tau_k\) can be expressed in terms of \(a\) and \(\tau\) as given in (5.8) and so depend only on \(m, n, s, \) and \(\mu\). Let \(\bar{y}\) denote a generic segment in \(\gamma_k\). Since \(|\bar{y}| < s\) no more than four applications of Lemma 2 are required to transmit positivity from \(B_{\sqrt[s]{2}}(y)\) to \(B_{\sqrt[s]{2}}(z)\). Thus at most \(4N\) applications of Lemma 2 are required to reach \(B_k\) from \(B_0\) and it follows from (5.2) that

\[
T_k \leq (2^{4N/\beta} - 1) \tau.
\]

Therefore (5.6) holds if we take \(T' = (2^{4N/\beta} - 1) \tau\). Since \(s, \mu,\) and \(N\) depend on \(u_0\) and \(\Omega\), we conclude that \(T'\) depends only on \(m, n, u_0,\) and \(\Omega\).

To complete the proof of the proposition we must show that \(u\) eventually becomes positive in \(\Omega \setminus \mathcal{B}\). Fix \(s_2 \in (s_1, \sigma)\). Since the mapping \(M_r\), defined in Section 1 is a homeomorphism for \(r \in (0, \sigma)\), it is easily verified that

\[
\Omega_{s_2} \subset \bigcup_{y \in \partial \Sigma_{s_2}} B_{s_2}(y) = \Omega,
\]
where \( \Omega_{s_2} \) denotes the interior of \( \Omega_{s_2} \). Thus it suffices to show the existence of a \( T'' \in \mathbb{R}^+ \) which depends only on \( m, n, u_s, \) and \( \Omega \) such that
\[
u(x, T' + T'') > 0 \quad \text{in} \quad B_{s_2}(y)
\]
for every \( y \in \partial\Omega_{s_2} \). Once this has been done, Proposition 4 holds with \( T = T' + T'' \).

\( \Sigma_s \) is a compact subset of \( \mathcal{B} \) and \( u(\cdot, T') > 0 \) on \( \mathcal{B} \). Therefore \( u(\cdot, T') \) is continuous on \( \mathcal{B} \) and uniformly continuous on \( \Sigma_s \). Let
\[
u \equiv \min\{u(y, T'): y \in \partial\Omega_{s_2}\}.
\]
Since \( \partial\Omega_{s_2} \subset \Sigma_s \), we have \( \nu > 0 \). Moreover, there exists a \( \rho \) depending only on \( v \) and \( s \) such that \( B_\rho(y) \subset \Sigma_s \) and
\[
u(x, T') > u(y, T') - \frac{v}{2} \geq \frac{v}{2}
\]
for all \( x \in B_\rho(y) \).

Fix \( y \in \partial\Omega_{s_2} \). Then
\[
u(x, T') \geq u(x, y, 0; a, \tau)
\]
provided that \( a \tau^\beta = \rho \) and \( c \tau^{-a} a^{2\gamma} = v/2 \). By the Comparison Principle
\[
u(x, T' + t) \geq u(x - y, t; a, \tau)
\]
for all \( t \) such that \( \text{supp} \ u(x - y, t; a, \tau) \subset \Omega \), that is, for all \( t \leq T'' \), where \( T'' \) is defined by
\[
\text{supp} \ u(x - y, T''; a, \tau) = \overline{B}_{s_2}(y).
\]
(5.9)

By a straightforward computation, (5.9) implies that
\[
T'' = \frac{s_2^{\gamma/3} - \rho^{\gamma/3}}{a^{1/3}}.
\]
Finally observe that \( T'' \) depends on \( m, n, s_2, \rho, \) and \( v \), and therefore ultimately only on the data.

6. The Asymptotic Form of \( u \)

In this section we shall derive a lower bound for the weak solution \( u \) of Problem (I). This bound, together with the upper bound obtained in Section 4, will enable us to determine the asymptotic form \( u \). However, to begin with, we shall need a rather technical result.
Let $e: \bar{\Omega} \to \mathbb{R}$ be defined by

$$-\Delta e = 1 \quad \text{in} \quad \Omega,$$

$$e = 0 \quad \text{on} \quad \partial\Omega.$$ 

Since $\Omega$ satisfies (H1), $e \in C^{2+\alpha}(\bar{\Omega})$. By the Maximum Principle and the Boundary Point Lemma for elliptic equations, $e > 0$ in $\Omega$ and $\partial e/\partial v < 0$ on $\partial\Omega$ [20]. In terms of $e$, we define the cone

$$K \equiv \{ \phi \in C^1(\bar{\Omega}) : \phi \geq ke \text{ in } \bar{\Omega} \text{ for some constant } k \in \mathbb{R}^+ \}.$$ 

It is well known that if $f$ denotes the solution of Problem (II) then $f^m \in K$ [3]. Since we expect $(1 + t)^m u$ to behave asymptotically like $f$, it is reasonable to expect that $u^m(\cdot, t) \in K$ for all sufficiently large $t$. To prove this we will need the similarity solutions of Type (B) introduced in Section 3. Recall that these are solutions of the porous medium equation of the form

$$v(x - y, t; c, \tau) = (t + \tau)^{-\alpha} \left[ g(\eta; c) \right]_+,$$

where

$$\eta = |x - y| (t + \tau)^{-\beta}$$

and $g(\eta; c)$ is a decreasing function of $\eta$, with $g(0; c) = c$, $g'(0, c) = 0$ and

$$\text{supp } g(\eta; c) = \{ \eta : 0 \leq \eta \leq a \}.$$ 

Here $a = a(c) = c^{(m-1/2)}a(1)$, and $\alpha$ and $\beta$ are constants which depend only on $m$ and $n$.

**Proposition 5.** Assume that $\Omega$ and $u_0$ satisfy (H), and let $u$ denote the solution of Problem (I). There exists a $T^* \in [0, +\infty)$ which depends only on the data such that $u^m(\cdot, T^*) \in K$.

**Proof.** By Proposition 4, there exists a $T \in [0, +\infty)$ such that $u(\cdot, T) > 0$ in $\Omega$. Without loss of generality we can assume that $T = 0$. Otherwise we simply write $u(\cdot, t) = u(\cdot, T + \tau') \equiv \tilde{u}(\cdot, \tau')$, prove the existence of $T'$ such that $\tilde{u}^m(\cdot, T') \in K$, and set $T^* = T + T'$.

Fix $s \in (0, \sigma)$, $y \in \partial\Sigma$, and set

$$\mu = \min\{u_n(x) : x \in \bar{\Sigma}_{s, y}\}.$$ 

In view of our assumption that $T = 0$ it is clear that $\mu > 0$. Let

$$\tilde{v}(x, t) \equiv v(x - y, t; c, \tau),$$
where the parameters \( c \) and \( \tau \) are chosen such that \( \text{supp} \\bar{v}(x, 0) = \overline{B}_y(y) \) and \( \bar{v}(x, 0) \leq \bar{v}(y, 0) = \mu \). After some calculation (with reference to the formulae given in Section 3) one finds that

\[
c = \mu \tau^a \quad \text{and} \quad \tau = \left( \frac{s}{2a(1)\mu^{(m-1)/2}} \right)^2.
\]

As in Section 5, the exact values of \( c \) and \( \tau \) are unimportant; what matters is that they depend only on the data and, in particular, are independent of \( y \). The support of \( \bar{v}(x, t) \) expands as \( t \) increases and first contacts \( \partial \Omega \) when

\[
t = T^* \equiv (2^{1/\beta} - 1) \tau.
\]

Observe that \( T^* \) depends on the data, but not on \( y \).

We claim that

\[
2^*(x, t) > \bar{v}^m(x, t) \quad \text{in} \quad C^* \equiv \overline{\Delta} \times [0, T^*]. \tag{6.1}
\]

Let \( A = \{ \text{supp} \\bar{v} \} \cap C^* \). Then (6.1) holds on the closure of \( C^* \setminus A \) since on that set \( u^m > 0 = \bar{v}^m \). For each integer \( p \geq 1 \) let

\[
A_p = \{(x, t) \in A: \bar{v}^m(x, t) > 1/p \}.
\]

Note that \( A_p \neq \emptyset \) for all sufficiently large \( p \). As is shown in the Appendix, \( u \) is the limit of a decreasing sequence \( \{u_p\} \) of \( C^{2,1}(C^*) \) functions with \( u_p^m \geq 1/p \). Thus \( u_p^m \geq \bar{v}^m \) on \( A \setminus A_p \). Moreover, in view of our choice of \( c \) and \( \tau \),

\[
u_p^m(x, 0) \geq u^m(x, 0) \geq \mu^m \geq \bar{v}^m(x, 0).
\]

Since both \( u_p^m \) and \( \bar{v}^m \) are positive and belong to the class \( C^{2,1}(A_p) \) it follows from the standard Maximum Principle for parabolic equations that \( u_p^m \geq \bar{v}^m \) in \( A_p \) and, therefore, in all of \( A \). Assertion (6.1) is obtained by letting \( p \to +\infty \).

Since \( y \in \partial \Sigma_s \) and \( s \in (0, \sigma) \), there exists a unique \( z(y) \in \partial \Omega \) such that \( s = d(y) = |y - z(y)| \). Let \( v \) denote the inward-pointing unit normal to \( \partial \Omega \) at \( z(y) \) and define \( x: [0, s] \to \mathbb{R}^n \) by

\[
x(\zeta) = y - \zeta v.
\]

Then \( x(0) = y, x(s) = z(y) \), and

\[
(d \circ x)(\zeta) = |x(\zeta) - z(y)| = s - \zeta.
\]
By the Theorem of the Mean

\[ \bar{v}^m(x(\zeta), T^*) = (T^* + \tau)^{-\alpha_m} \left\{ g^m(s(T^* + \tau)^{-\beta}; c) + \frac{\zeta - s}{(T^* + \tau)^\beta} (g^m)' (\theta(T^* + \tau)^{-\beta}; c) \right\} \]

\[ = (T^* + \tau)^{-(\alpha_m + \beta)} |x(\zeta) - z(y)| (-g^m)' (\theta(T^* + \tau)^{-\beta}; c) \]  

(6.2)

for some \( \theta = \theta(\zeta) \in (\zeta, s) \). Recall that \( (g^m)' (a; c) = \kappa < 0 \). In view of the continuity of \( (g^m)' \), there exists an \( s_0 \in (0, s) \) such that

\[ (g^m)' (\zeta(T^* + \tau)^{-\beta}; c) < \kappa/2 < 0 \quad \text{for all } \zeta \in [s_0, s]. \]

Thus it follows from (6.1) and (6.2) that

\[ u^m(x(\zeta), T^*) \geq \frac{\kappa}{2} (T^* + \tau)^{-(\alpha_m + \beta)} (d \circ x)(\zeta) \]

\[ = \omega(d \circ x)(\zeta) \quad \text{for } \zeta \in [s_0, s], \]  

(6.3)

where \( \omega \in \mathbb{R}^+ \) is a constant which depends on the data, but not on \( y \) or \( \zeta \). Observe that (6.3) holds for arbitrary \( y \in \Sigma \), and for all \( x \) on the inward-directed normal through \( y(x) \) provided that \( d(x) < s - s_0 \). As we remarked in Section 1, the normal map \( M_r \) is a homeomorphism for \( r \in [0, \sigma) \). Therefore, it follows from (6.3) that

\[ u^m(x, T^*) \geq \omega d(x) \quad \text{for } x \in \Omega_{s - s_0}, \]  

(6.4)

To complete the proof, suppose for contradiction that \( u^m(x, T^*) \notin K \). Then for each integer \( p \geq 1 \) there exists a point \( x_p \in \Omega \) such that

\[ 0 < u^m(x_p, T^*) \leq (1/p) e(x_p). \]  

(6.5)

Without loss of generality we can assume that \( x_p \to x^* \in \overline{\Omega} \) as \( p \to +\infty \). If \( x^* \in \Omega \) then (6.5) implies that \( u^m(x^*, T^*) = 0 \), which contradicts the positivity of \( u^m(\cdot, T^*) \) in \( \Omega \). On the other hand, if \( x^* \in \partial \Omega \) then (6.5) implies that

\[ \lim_{p \to +\infty} \frac{u^m(x_p, T^*)}{d(x_p)} = 0. \]

However, it follows from (6.4) that

\[ \lim_{p \to +\infty} \frac{u^m(x_p, T^*)}{d(x_p)} \geq \omega > 0 \]
so that we again have a contradiction. We therefore conclude that there is a constant \( k \in \mathbb{R}^+ \) such that \( u^m(\cdot, T^*) \geq ke(\cdot) \) in \( \Omega \). Note that \( k \) depends only on the data.

We are now in position to state and prove the main results of this section. The first of these is a lower bound for \( u \).

**Theorem 2.** Assume that \( \Omega \) and \( u_0 \) satisfy (H). If \( u \) denotes the solution of Problem (I) then there exist constants \( T^* \in [0, +\infty) \) and \( \tau_0 \in \mathbb{R}^+ \) which depend only on the data such that

\[
 u(x, t) \geq (\tau_0 + t)^{-\gamma}f(x) \quad \text{in} \quad \Omega \times \{T^*, +\infty) . \tag{6.6}
\]

**Proof.** By Proposition 4, there exist constants \( T^* \in [0, +\infty) \) and \( k_1 \in \mathbb{R}^+ \) which depend only on the data such that \( u^m(\cdot, T^*) \geq k_1 e(\cdot) \) in \( \Omega \). On the other hand, \( f \) is bounded. Thus there exists a constant \( k_2 \in \mathbb{R}^+ \) such that

\[
 -\Delta(k_2 f^m) = k_2 g f \leq 1 = -\Delta e \quad \text{in} \quad \Omega .
\]

Then since \( k_2 f^m = e = 0 \) on \( \partial\Omega \), it follows from the Maximum Principle for elliptic equations that \( k_2 f^m \leq e \) in \( \Omega \). Therefore

\[
 u(\cdot, T^*) \geq (k_1 k_2)^{1/m} f(\cdot) \quad \text{in} \quad \Omega .
\]

If we now define \( \tau_0 \) by the relation \( (\tau_0 + T^*)^{-\gamma} = (k_1 k_2)^{1/m} \) then (6.6) follows from the Comparison Principle.

Finally, we combine Theorems 1 and 2 to obtain the asymptotic form of \( u \).

**Theorem 3.** Assume that \( \Omega \) and \( u_0 \) satisfy (H), and let \( u \) denote the solution of Problem (I). There exists a constant \( \mathcal{B} \in \mathbb{R}^+ \) which depends only on the data such that

\[
 |(1 + t)^\gamma u(x, t) - f(x)| \leq \mathcal{B} f(x)(1 + t)^{-1} \quad \text{in} \quad \Omega \times [0, +\infty) . \tag{6.7}
\]

**Proof.** Define the function \( \phi : [0, +\infty) \times \mathbb{R}^+ \to \mathbb{R} \) by

\[
 \phi(t, \tau) = 1 - \left( \frac{1 + \tau}{\tau + t} \right)^\gamma .
\]

Then, by Theorems 1 and 2

\[
 -|\phi(t, \tau_0)| f(x) \leq (1 + t)^\gamma u(x, t) - f(x) \leq |\phi(t, \tau_1)| f(x) \quad \text{in} \quad \Omega \times [T^*, +\infty) . \tag{6.8}
\]
By the Theorem of the Mean, 
\[ \phi(t, \tau) = \gamma(\tau - 1) \left( \frac{1 + t}{\tau' + t} \right)^{1 + \gamma} (1 + t)^{-1} \]
for some \( \tau' \) between 1 and \( \tau \). Thus
\[ |\phi(t, \tau)| \leq c(\tau)(1 + t)^{-1}, \]
where 
\[ c(\tau) = \gamma |\tau - 1| \max\{1, \tau^{-1 - \gamma}\}. \]
Set \( \mathcal{E}' = \max\{c(\tau_0), c(\tau_1)\} \). Then (6.8) implies that
\[ |(1 + t)^\gamma u(x, t) - f(x)| \leq \mathcal{E}' f(x)(1 + t)^{-1} \quad \text{in } \Omega \times [T^*, +\infty). \]
According to Theorem 1,
\[ |(1 + t)^\gamma u(x, t) - f(x)| \leq \left\{ 1 + \left( \frac{1 + t}{\tau_1 + t} \right)^\gamma \right\} f(x) \quad \text{in } \Omega \times \mathbb{R}^+. \]
Therefore, in particular,
\[ |(1 + t)^\gamma u(x, t) - f(x)| \leq \mathcal{E}'' f(x)(1 + t)^{-1} \quad \text{in } \Omega \times [0, T^*], \]
where
\[ \mathcal{E}'' = 1 + T^* + \tau_1^{-\gamma}(1 + T^*)^{\gamma + 1}. \]
Thus (6.7) holds with \( \mathcal{E} = \max(\mathcal{E}', \mathcal{E}''). \)

By considering the particular solution
\[ u(x, t) = f(x)(\tau + t)^{-\gamma} \]
of Problem (I) for arbitrary \( \tau \in \mathbb{R}^+ \), it is easy to see that the estimate (6.7) is sharp. Indeed this remark follows from the estimate for \( |\phi(t, \tau)| \) in the proof of Theorem 3.

7. An Application

In Ref. [13], Gurtin and MacCamy propose and analyse a model for the spatial spread of biological populations which leads to an equation of the form
\[ \rho_t = \Delta \phi(\rho) + \sigma(\rho), \]
where \( \rho \) denotes the population density and \( \sigma(\rho) \) denotes the net growth rate of the population, that is, \( \sigma(\rho) \) is the difference between the birth and death rates. For most of their analysis Gurtin and MacCamy take \( \phi(\rho) = \rho^m \) for \( m \in [2, +\infty) \) and the Malthusian growth law \( \sigma(\rho) = \mu \rho \), where \( \mu \in \mathbb{R} \).

If we consider a population living in a bounded habitat \( \Omega \subset \mathbb{R}^+ \) with the exterior of \( \Omega \) completely hostile, we are led to the problem

\[
\begin{align*}
\rho_t &= \Delta (\rho^m) + \mu \rho & \text{in } \Omega \times \mathbb{R}^+, \\
\rho &= 0 & \text{in } \partial \Omega \times \mathbb{R}^+, \\
\rho &= \rho_0 & \text{in } \bar{\Omega} \times \{0\}. 
\end{align*}
\]

(V)

In the birth dominant case, that is, when \( \mu \in \mathbb{R}^+ \), Problem (V) can be transformed to Problem (I) by the change of dependent and independent variables \([13]\)

\[
\rho(x, t) = e^{\mu t} u(x, t) \quad \text{and} \quad \tau = \frac{\gamma}{\mu} (e^{\omega \tau} - 1). 
\]

We shall regard \( \rho(x, t) \) as the weak solution of Problem (V) if and only if \( U(x, \tau) = u(x, t(\tau)) \) is the weak solution of Problem (I) with initial function \( U(x, 0) = \rho_0(x) \). By a straightforward but tedious translation of estimate (6.7) we obtain the following result on the asymptotic form of the solution of Problem (V). In stating the result we shall use \( f(\cdot; \mu) \) to denote the nontrivial solution of Problem (II) with \( \gamma \) replaced by \( \mu \). As we observed in Section 2, \( f(\cdot; \mu) = (\mu/\gamma)^{\gamma} f(x) \).

**Theorem 4.** Assume that \( \Omega \) and \( \rho_0 \) satisfy (H) and let \( \rho \) denote the solution of Problem (V) for some \( \mu \in \mathbb{R}^+ \). Then there exists a constant \( \mathcal{C}^* \) which depends only on \( m, n, \mu, \rho_0, \) and \( \Omega \) such that

\[
|\rho(x, t) - f(x; \mu)| \leq \mathcal{C}^* f'(x; \mu) e^{-\omega \tau} \quad \text{in } \bar{\Omega} \times [0, +\infty).
\]

In the one-dimensional case, MacCamy \([16]\) has proved the convergence of \( \rho \) to \( f(\cdot; \mu) \), by means of an argument based on a Liapunov function. However, he does not get an estimate for the rate of convergence.

In the death dominant case, \( \mu < 0 \), we can show by means of a slight extension of the Comparison Principle that \( 0 \leq \rho \leq u \) in \( \bar{\Omega} \times [0, +\infty) \), where \( \rho \) denotes the solution of Problem (V) and \( u \) denotes the solution of Problem (I) with \( u_0 = \rho_0 \). Thus, according to Theorem 1,

\[
0 \leq \rho(x, t) \leq f(x)(\tau + t)^{-\gamma} \quad \text{in } \bar{\Omega} \times [0, +\infty)
\]

so that

\[
\lim_{t \uparrow +\infty} \rho(\cdot, t) = 0
\]
uniformly in $\Omega$. However, in this case we cannot use Theorem 3 to derive the asymptotic form of $\rho$ as we did in proving Theorem 4. The difficulty stems from the observation that if $\mu < 0$ then

$$
\tau(t) = \frac{\gamma}{|\mu|} (1 - e^{((\mu \gamma^2)t)}) \to \frac{\gamma}{|\mu|} < +\infty \quad \text{as} \quad t \uparrow +\infty.
$$

Thus if we use (7.1) to transform Problem (V) the resulting problem is not Problem (I) but rather the problem

$$
U_t = A(U^m) \quad \text{in} \quad \Omega \times \left(0, \frac{\gamma}{|\mu|}\right],
$$

$$
U = 0 \quad \text{in} \quad \partial \Omega \times \left(0, \frac{\gamma}{|\mu|}\right],
$$

$$
U = \rho_0 \quad \text{in} \quad \bar{\Omega} \times \{0\}.
$$

It is still possible to obtain a sharp asymptotic result provided we assume that the solution $U$ of Problem (VI) has a uniform modulus of continuity $\omega$, with respect to $\tau$ in $\bar{\Omega} \times [0, \gamma/|\mu|]$ (in this connection, see the remarks following Proposition 3 in Section 3). Under this assumption, if $\tau < \gamma/|\mu|$ then

$$
\left| U(\cdot, \tau) - U\left(\cdot, \frac{\gamma}{|\mu|}\right) \right| \leq \omega \left(\tau - \frac{\gamma}{|\mu|}\right) \quad \text{in} \quad \bar{\Omega}.
$$

Since, $U(\cdot, \tau) = e^{\mu t} \rho(\cdot, t)$, it follows that

$$
\left| e^{\mu t} \rho(\cdot, t) - U\left(\cdot, \frac{\gamma}{|\mu|}\right) \right| \leq \omega \left(\frac{\gamma}{|\mu|} e^{-\gamma/|\mu|} t\right), \quad \text{in} \quad \bar{\Omega}. \quad (7.2)
$$

Observe that (7.2) implies that in the death dominant case, the asymptotic form of $\rho$ depends on the initial function $\rho_0$. To see that this is reasonable one need only contrast the case in which $\text{supp } \rho_0$ is so small that $\text{supp } U(\cdot, \gamma/|\mu|) \subset \Omega$ with the case in which $\text{supp } \rho_0 = \bar{\Omega}$. Theorem 4 shows that the asymptotic form of $\rho$ is independent of $\rho_0$ in the birth dominant case.

**APPENDIX**

In this Appendix we shall prove Propositions 2 and 3 of Section 3; that is, we shall establish the existence and uniqueness of the weak solution of Problem (I), and derive the Comparison Principle. For this purpose it is
convenient to follow the procedure used in Ref. [18] and transform Problem (I) by setting \( u^m = v \). Thus, formally, Problem (I) can be written as

\[
\begin{align*}
\frac{\partial v}{\partial t} &= a(v) \Delta v & \text{in } Q_T, \\
v &= 0 & \text{on } S_T, \\
v &= v_0 & \text{in } \Omega \times \{0\},
\end{align*}
\]  

(I')

where \( v_0 = u_0^m \) and

\[ a(v) = m(v)^{(m-1)/m}. \]

The following Comparison Lemma will be used repeatedly in the proofs of Propositions 2 and 3. Let \( \Sigma \subset \mathbb{R}^n \) be a bounded connected domain. We shall use the notation \( C_T = \Sigma \times (0, T] \) and

\[ \Gamma_T = [\partial \Sigma \times [0, T]] \cup [\Sigma \times \{0\}] \]

for arbitrary \( T \in \mathbb{R}^+ \). The set \( \Gamma_T \) is often called the parabolic boundary of the cylinder \( C_T \).

**Comparison Lemma.** Suppose that \( w \) and \( v \) are \( C^{2,1}(C_T) \) functions such that \( w \) and \( v \) are continuous and positive in \( C_T \), and either \( \Delta w \leq k \) or \( \Delta w \leq k \) in \( C_T \) for some constant \( k \in \mathbb{R} \). If

\[ w \geq v \quad \text{on } \Gamma_T \]

then

\[ w \geq v \quad \text{in } \bar{C}_T. \]

**Proof.** Set \( z = w - v \) and suppose that \( \Delta w \leq k \). Then \( z \geq 0 \) in \( \Gamma_T \) and, by the Theorem of the Mean,

\[ z_t - A(x, t) \Delta z - B(x, t) z \geq 0, \]

where

\[ A(x, t) = a(v(x, t)) \]

and

\[ B(x, t) = \Delta w(x, t) a'(\theta w(x, t) + (1 - \theta) v(x, t)) \]
for some \( \theta = \theta(x, t) \in (0, 1) \). Let

\[
\sigma \equiv \min\left( \min_{\xi_T} w, \min_{\xi_T} v \right).
\]

By hypothesis, \( \sigma > 0 \). Thus \( A(x, t) \geq a(v) > 0 \) and

\[
0 < a'(\theta w + (1 - \theta) v) \leq (m - 1) \sigma^{-1/m}
\]

so that \( B(x, t) \) is bounded above in \( C_T \). The assertion now follows from the standard maximum principle for parabolic inequalities [20].

To prove Proposition 2 and 3 we shall construct the weak solution of Problem (I) as the limit of a sequence of functions which are, in turn, obtained by solving Problem (I') for a sequence of strictly positive data functions. The properties of this sequence of functions are described in the following lemma.

**Lemma A.** Let \( \Omega \) and \( u_0 \) satisfy (H) and set \( v_0 = u_0^* \). There exists a sequence \( \{v_{0p}(x): p = 1, 2, \ldots\} \) of \( C^\infty(\mathbb{R}^n) \) functions with the following properties:

(i) \( 1/p \leq v_{0p}(x) \leq M \equiv \max\{v_0(x): x \in \partial \Omega\} + 1 \)

(ii) \( v_{0p}(x) \downarrow v_0(x) \) as \( p \to +\infty \)

(iii) \( v_{0p} = 1/p \) and \( \Delta v_{0p} = 0 \) in a neighbourhood of \( \partial \Omega \).

(iv) \( |\nabla v_{0p}|(x) = \max \left\{ \left| \frac{\partial v_{0p}}{\partial x_1}(x) \right|, \ldots, \left| \frac{\partial v_{0p}}{\partial x_n}(x) \right| \right\} \leq L \).

**Corollary.** Let \( u_0 \) and \( u_0^* \) satisfy (H2). If \( u_0 \geq u_0^* \) in \( \partial \Omega \) then the corresponding sequence \( \{v_{0p}\} \) and \( \{v_{0p}^*\} \) can be chosen so that \( v_{0p} \geq v_{0p}^* \) in \( \mathbb{R}^n \) for all \( p \geq 1 \).

We defer the rather technical proof of this lemma and its corollary until the end of this Appendix and proceed directly to the proofs of the propositions.

**Proof of Proposition 2.** For each integer \( p \geq 1 \) consider the problem

\[
v_t = a(v) \Delta v \quad \text{in} \quad Q_T, \quad v_t = 1/p \quad \text{on} \quad S_T, \quad v = v_{0p} \quad \text{on} \quad \Omega \times \{0\},
\]

where \( \{v_{0p}\} \) is the sequence whose existence is assured by Lemma A. Since \( v_{0p} \) is strictly positive we can apply the standard theory of quasilinear
parabolic equations [9] to conclude that for each integer $p \geq 1$ Problem (I$_p$) possesses a unique solution $v_p \in C^{2,1}(\overline{Q}_T)$. Moreover, the sequence $\{v_p\}$ has the following properties:

\begin{align*}
1/p &\leq v_p \leq M & \text{in } \overline{Q}_T \text{ for all } p \geq 1, \\
v_p &\geq v_{p+1} & \text{in } \overline{Q}_T \text{ for all } p \geq 1, \\
\|\nabla v_p(\cdot, t)\|_{L^2(\Omega)} &\leq J/|\Omega|^{1/2} & \text{for all } t \in [0, T] \text{ and } p \geq 1,
\end{align*}

for all $p \geq 1$, where $J = n^{1/2}L$.

Properties (A.1) and (A.2) follow from the Comparison Lemma and Lemma A. To prove (A.3) we multiply both sides of the equation

$$v_{pt} = a(v_p) \Delta v_p$$

by $\{a(v_p)^{-1} v_p\}$ and integrate over the cylinder $\Omega \times (0, t)$ for $t \in (0, T]$. Since $v_{pt} = 0$ on $S_T$, we obtain

$$0 \leq \int_0^t \int_\Omega \frac{(v_{pt})^2}{a(v_p)} \, dx \, dt = - \int_\Omega \int_0^t \nabla v_p \cdot \frac{\partial}{\partial \tau} \nabla v_p \, dx \, dt$$

$$= - \frac{1}{2} \int_\Omega (\nabla v_p \cdot \nabla v_p)(x, t) \, dx + \frac{1}{2} \int_\Omega (\nabla v_p \cdot \nabla v_p)(x, 0) \, dx$$

and (A.3) follows immediately from (iv) of Lemma A.

The sequence of positive functions $\{v_p\}$ is decreasing in $\overline{Q}_T$. We may therefore define a function $v: \overline{Q}_T \to [0, M]$ by

$$v(x, t) = \lim_{p \to \infty} v_p(x, t)$$

and a function $u: \overline{Q}_T \to [0, M^{1/m}]$ by

$$u(x, t) = \{v(x, t)\}^{1/m}.$$

We shall show that this function $u$ is a weak solution of Problem (I).

By (A.3), the sequence $\{\nabla v_p\}$ is bounded in $\{L^2(Q_T)\}$. Hence there exists a subsequence, which we again denote by $\{\nabla v_p\}$, converging weakly to an element $\psi \in \{L^2(Q_T)\}$. For $\phi \in C_0^\infty(Q_T)$

$$\int_{Q_T} \phi \nabla v_p \, dx \, dt = - \int_{Q_T} v_p \nabla \phi \, dx \, dt.$$

Thus if we let $p \to \infty$ and use the Dominated Convergence Theorem we obtain

$$\int_{Q_T} \phi u \, dx \, dt - \int_{Q_T} u \nabla \phi \, dx \, dt.$$
that is, \( \psi = \nabla v \) in the sense of distributions. It is now an easy matter to show that the function \( u = v^{1/m} \) satisfies conditions (ii) and (iii) of the definition of a weak solution of Problem (I).

It remains to be shown that \( u \) tends to zero on the lateral boundary of \( Q_T \). We shall do this by constructing a barrier function for Problem (I'). Let \( k_i \), for \( i = 1, \ldots, n - 1 \), denote the principal curvatures of \( \partial \Omega \) and set

\[
K = \max \max_{x \in \partial \Omega} \{|k_i(x)| : i = 1, \ldots, n - 1 \}.
\]

Fix \( s \in \mathbb{R}^+ \) and set

\[
\mathcal{C} = \max \left\{ \frac{M}{1 - e^{-nsK}}, \frac{L}{nK} e^{nsK} \right\}.
\]

Then it is easily verified that the function \( h: [0, s] \to [0, \mathcal{C}] \) given by

\[
h(x) = \mathcal{C} (1 - e^{-nkx})
\]

is such that

\[
h(0) = 0, \quad h(s) \geq M, \quad (A.4)
\]

and

\[
h'(x) \geq L \text{ for } x \in [0, s]. \quad (A.5)
\]

According to Lemma 1, there is a constant \( \sigma \in \mathbb{R}^+ \) such that \( d(x) \in C^2(\Omega_\sigma) \), where \( d(x) \) denotes the distance from \( x \) to \( \partial \Omega \) and

\[
\Omega_s = \{ x \in \tilde{\Omega} : 0 < d(x) < s \}
\]

for any \( s \in \mathbb{R}^+ \). Moreover, for each \( x \in \Omega_\sigma \) there exists a unique \( z(x) \in \partial \Omega \) such that \( d(x) = |x - z(x)| \). Fix \( s \in (0, \sigma) \). Then \( h \circ d \in C^3(\tilde{\Omega}_s) \) so that \( \Delta(h \circ d) \) is well defined in \( \tilde{\Omega}_s \). Serrin [22] has shown that

\[
\Delta(h \circ d)(x) = (h'' \circ d)(x) + \left( \sum_{i=1}^{n-1} \frac{k_i}{1 - k_i d(x)} \right) (h' \circ d)(x) \leq (h'' \circ d)(x) + Kn(h' \circ d)(x)
\]

for \( x \in \tilde{\Omega}_s \), where the boundary curvatures \( k_i \) are evaluated at \( y(x) \). Therefore

\[
\Delta(h \circ d)(x) \leq 0 \quad \text{for } x \in \tilde{\Omega}_s. \quad (A.6)
\]
For each integer \( p \geq 1 \) define
\[
w_p(x) = 1/p + (h \circ d)(x) \quad \text{for } x \in \bar{Q}_T.
\]
Let \( Q_T^s = \{ x \in \Omega : 0 < d(x) < s \} \times (0, T) \). In view of (A.6),
\[
w_{pt} - a(w_p) \Delta w_p \geq 0 = v_{pt} - a(v_p) \Delta v_p
\]
with \( w_p \) and \( v_p \) positive on \( \bar{Q}_T^s \) and \( \Delta w_p \leq 0 \) in \( Q_T^s \). Moreover, it follows from (A.1) and (A.4) that
\[
w_p(x) \geq 1/p - v_p(x, t) \quad \text{for } (x, t) \in \bar{S}_T
\]
and
\[
w_p(x) \geq 1/p + M > v_p(x, t) \quad \text{for } (x, t) \in \{ x : d(x) = s \} \times [0, T].
\]
Finally, by (iv) of Lemma A, (A.5), and the Theorem of the Mean
\[
w_p(x) \geq 1/p + L > v_p(x, t) \quad \text{for } (x, t) \in \{ x : d(x) = s \} \times [0, T].
\]
Therefore, by the Comparison Lemma, \( w_p \geq v_p \) in \( \bar{Q}_T^s \) for each \( p \geq 1 \). In the limit as \( p \to +\infty \) we obtain
\[
(h \circ d)(x) \geq v(x, t) \geq 0 \quad \text{in } \bar{Q}_T.
\]
Since \( (h \circ d)(x) \to 0 \) as \( x \) approaches \( \partial \Omega \) it follows that \( v(x, t) \to 0 \) as \( (x, t) \to (x_0, t_0) \) for any \( (x_0, t_0) \in \bar{S}_T \).

The proof that the weak solution of Problem (I) is unique is essentially the same as the corresponding proof in the case \( n = 1 \) given by Oleinik et al. in Ref. [18]. We shall therefore omit it.

**Proof of Proposition 3.** As in the proof of Proposition 2, we approximate the initial and boundary values by smooth positive functions and consider the corresponding smooth positive solutions \( u_p \) and \( u^*_p \) of the initial boundary value problem. That is, \( u_p = v_p^{1/m} \), where \( v_p \) is the solution of Problem (I_0^p) and similarly for \( u^*_p \). Since \( u_0 \geq u^*_0 \) in \( \bar{Q} \), it follows from the Corollary to Lemma A that we can arrange things so that for each \( p \geq 1 \), \( u_p \geq u^*_p \) on the parabolic boundary of \( Q_T \). Hence, by the standard maximum principle, \( u_p \geq u^*_p \) in \( \bar{Q}_T \). As was shown in the proof of Proposition 2, \( u_p \) and \( u^*_p \) tend, respectively, to the unique weak solutions \( u \) and \( u^* \) of Problem (I) with intitial functions \( u_0 \) and \( u^*_0 \). Therefore \( u \geq u^* \) in \( Q_T \).

**Proof of Lemma A.** We extend the domain of \( v_0 \) to all of \( \mathbb{R}^n \) by setting \( v_0(x) \equiv 0 \) in \( \mathbb{R}^n \setminus \bar{Q} \). Then \( v_0 \in C(\mathbb{R}^n) \) but \( v_0 \not\in C^1(\mathbb{R}^n) \). However, in view of (H2), \( v_0 \in C^1(\bar{Q}) \) so that
\[
L = \max_{\bar{Q}} |\nabla v_0| < +\infty.
\]
Choose the nonnegative integer \( p_0 \) so large that

\[
\frac{1}{L(p_0 + 1)} < \sigma
\]

and set \( q = q(p) = p_0 + p \). Here \( \sigma \) is the constant whose existence is asserted in Lemma 1. Let \( k_\delta(x) \) denote a \( C^\infty(\mathbb{R}^n) \) function such that \( k_\delta \geq 0 \), \( \text{supp} k_\delta = R_\delta(0) \), and

\[
\int_{\mathbb{R}^n} k_\delta(x) \, dx = 1.
\]

Define the sequence \( \{v_{0p}(x) : p = 1, 2, \ldots\} \) of functions on \( \mathbb{R}^n \) by

\[
v_{0p}(x) = \int_{\mathbb{R}^n} k_\delta(y)(x - y) \, \bar{v}_{0p}(y) \, dy,
\]

where

\[
\bar{v}_{0p}(x) = \max \left\{ v_0(x), \frac{1}{2p}, \frac{1}{2p} \right\} + \frac{1}{2p}
\]

and

\[
\delta(p) = \frac{1}{4Lq(q + 1)}, \quad q = p_0 + p.
\]

Observe that \( \bar{v}_{0p}(x) \equiv 1/p \) for \( x \in \tilde{\Omega} \) and, indeed,

\[
1/p \leq \bar{v}_{0p}(x) \leq M \quad \text{in} \quad \mathbb{R}^n.
\]

Thus (i) follows immediately from the properties of the kernel \( k_\delta \).

It is easily verified that the sequences \( \{v_{0p}\} \) and \( \{\bar{v}_{0p}\} \) have the following properties:

\[
\frac{1}{2p} \leq \bar{v}_{0p}(x) - v_0(x) \leq \frac{1}{p}, \quad (A.7)
\]

\[
\bar{v}_{0p}(x) - \bar{v}_{0p+1}(x) \geq \frac{1}{2p(p + 1)}, \quad (A.8)
\]

and

\[
|\bar{v}_{0p}(y) - \bar{v}_{0p}(x)| \leq |v_0(y) - v_0(x)|. \quad (A.9)
\]

Note that each of these inequalities holds throughout \( \mathbb{R}^n \) for all \( p \geq 1 \).
To prove (ii) we shall need the estimate
\[ \Gamma(x) \equiv |v_{0p}(x) - \bar{v}_{0p}(x)| \leq \frac{1}{4q(q + 1)} \text{ in } \bar{\Omega} \text{ for all } p \geq 1. \] (A.10)

In view of (A.9)
\[
\Gamma(x) = \left| \int_{\mathbf{R}^n} k_{\delta(p)}(x - y) |\bar{v}_{0p}(y) - \bar{v}_{0p}(x)| \, dy \right| \\
\leq \int_{|y - x| < \delta(p)} k_{\delta(p)}(y - x) |v_0(x) - v_0(y)| \, dy.
\]

Let \( \zeta(x, y) = \{ z \in \mathbf{R}^n : z = \theta x + (1 - \theta) y, \theta \in [0, 1] \} \) and write \( B_{\delta(p)}(x) = S_1 \cup S_2 \cup S_3 \), where
\[
S_1 = \{ y \in B_{\delta(p)}(x) : y \in \Omega \}, \\
S_2 = \{ y \in B_{\delta(p)}(x) : y \in \Omega, \zeta(x, y) \not\subseteq \Omega \},
\]
and
\[
S_3 = \{ y \in B_{\delta(p)}(x) : y \in \Omega, \zeta(x, y) \subseteq \Omega \}.
\]

Since \( v_0 \in C^1(\bar{\Omega}) \) it follows from the Theorem of the Mean that
\[
\int_{S_3} k_{\delta(p)}(x - y) |v_0(y) - v_0(x)| \, dy \leq L \int_{S_3} |y - x| k_{\delta(p)}(x - y) \, dy \\
\leq L \delta(p) \int_{S_3} k_{\delta(p)}(x - y) \, dy.
\]

For \( y \in S_1 \), \( v_0(y) = 0 \) so that
\[
\int_{S_1} k_{\delta(p)}(x - y) |v_0(y) - v_0(x)| \, dy = |v_0(x)| \int_{S_1} k_{\delta(p)}(x - y) \, dy.
\]

Note that in this case \( \partial \Omega \) must intersect the closed segment \( \zeta(x, y) \) at least once. Thus \( d(x) \leq |x - y| < \tilde{\delta}(p) < \sigma/4 \) and, by Lemma 1, there exists a unique \( z(x) \in \partial \Omega \) such that \( d(x) = |x - z(x)| \). By the Theorem of the Mean applied on the line segment joining \( x \) and \( z(x) \)
\[
|v_0(x)| \leq L d(x) < L \delta(p),
\]
and hence
\[
\int_{S_1} k_{\delta(p)}(x - y) |v_0(y) - v_0(x)| \, dy < L \delta(p) \int_{S_1} k_{\delta(p)}(x - y) \, dy.
\]
Finally, if \( y \in S_2 \) then again \( \partial \Omega \) must intersect the segment \( \zeta(x, y) \) at least once for \( \theta \in (0, 1) \). In this case, \( d(x) + d(y) \leq |x - y| < \delta(p) \) and by an argument similar to the one used for \( y \in S_1 \)

\[
\int_{S_2} k_{\delta(p)}(x - y) |v_0(y) - v_0(x)| \, dy \\
\leq \int_{S_2} k_{\delta(p)}(x - y) (|v_0(y)| + |v_0(x)|) \, dy \\
\leq L \int_{S_2} k_{\delta(p)}(x - y) (d(y) + d(x)) \, dy \leq L \delta(p) \int_{S_2} k_{\delta(p)}(x - y) \, dy.
\]

Adding the three estimates we obtain

\[
\Gamma(x) \leq L \delta(p) = \frac{L}{4Lq(q + 1)} = \frac{1}{4q(q + 1)}.
\]

Write

\[
u_{op} - v_{op+1} = v_{op} - \bar{v}_{op} + \bar{v}_{op+1} + \bar{v}_{op+1} - v_{op+1}.
\]

Then, by (A.8) and (A.10)

\[
v_{op} - v_{op+1} \geq -\frac{1}{4q(q + 1)} + \frac{1}{2p(p + 1)} - \frac{1}{4(q + 1)(q + 2)} \geq 0
\]

so that the sequence \( \{v_{op}\} \) is decreasing. Moreover, in view of (A.7) and (A.10),

\[
|v_{op}(x) - v_0(x)| \leq |v_{op}(x) - \bar{v}_{op}(x)| + |\bar{v}_{op}(x) - v_0(x)| \\
\leq \frac{1}{4q(q + 1)} + \frac{1}{p}.
\]

Therefore (ii) holds.

Let

\[
\beta(p) = \frac{2q + 1}{4Lq(q + 1)}, \quad q = p_0 + p.
\]

Note that, in view of the definition of \( p_0 \),

\[
\beta(p) < \frac{1}{2Lq} \leq \frac{1}{2L(p_0 + 1)} < \frac{\sigma}{2}.
\]
If \( x \in \Omega_{\beta(p)} \) and \( y \in B_{\delta(p)}(x) \cap \Omega \) then

\[
d(y) = \min \{|y-z|: z \in \partial \Omega\} \leq \delta(p) + d(x) < \frac{1}{2\text{Lq}}.
\]

By the Theorem of the Mean, for \( y \in B_{\delta(p)}(x) \cap \Omega \)

\[
v_0(y) < Ld(y) < \frac{1}{2q} \leq \frac{1}{2p}
\]

so that

\[
\bar{v}_{op}(y) = 1/p.
\]

But since \( \bar{v}_{op}(y) = 1/p \) for all \( y \notin \Omega \), it follows that \( \bar{v}_{op}(y) = 1/p \) throughout \( B_{\delta(p)}(x) \) whenever \( x \in \Omega_{\beta(p)} \). Thus \( x \in \Omega_{\beta(p)} \) implies

\[
v_{op}(x) = 1/p
\]

and, clearly,

\[
\Delta v_{op}(x) = 0.
\]

Let \( \text{e} \) denote an arbitrary unit vector in \( \mathbb{R}^n \) and let \( h \in \mathbb{R}^+ \) be such that \( h < \sigma \). Consider

\[
v_{op}(x + he) - v_{op}(x) = \int_{\mathbb{R}^n} k_{\delta(p)}(x-y)\{\bar{v}_{op}(y + he) - \bar{v}_{op}(y)\} \, dy.
\]

By (A.9)

\[
|v_{op}(x + he) - v_{op}(x)| \leq \int_{|x-y| < \delta(p)} k_{\delta(p)}(x-y) |v_0(y + he) - v_0(y)| \, dy.
\]

Let \( \zeta = \{z \in \mathbb{R}^n: z = y + \theta he, \theta \in [0, 1]\} \), and write \( B_{\delta(p)}(x) = \bigcup_{j=1}^4 S_j \), where

\[
S_1 = \{y \in B_{\delta(p)}(x): \zeta \in \Omega\},
S_2 = \{y \in B_{\delta(p)}(x): y \in \Omega, y + he \notin \Omega, \zeta \notin \Omega\},
S_3 = \{y \in B_{\delta(p)}(x): y \in \Omega, y + he \notin \Omega\},
S_4 = \{y \in B_{\delta(p)}(x): y \notin \Omega, y + he \in \Omega\}.
\]
and

\[ S_1 = \{ y \in B_{\delta(y)}(x): y \in \Omega, y + he \notin \Omega \}. \]

For \( y \in S_1 \), \( D \equiv |v_0(y + he) - v_0(y)| = 0 \). For \( y \in S_1 \) we can apply the Theorem of the Mean on the segment \( \zeta \) to obtain \( D \leq Lh \). If \( y \in S \), then \( \partial \Omega \) intersects the segment \( \zeta \) for some \( \theta \in (0, 1] \) so that \( d(y) \leq h < \sigma \). Therefore, by the Theorem of the Mean, \( D = |v_0(y)| < Ld(y) \leq Lh \). A similar argument yields the same estimate for \( y \in S_1 \). Finally, if \( y \in S_2 \) then again \( \partial \Omega \) intersects the segment \( \zeta \) between \( y \) and \( y + he \) so that \( d(y) + d(y + he) \leq h \). By the Theorem of the Mean, we obtain the estimate

\[ D \leq |v_0(y + he)| + |v_0(y)| \leq Ld(y + he) + Ld(y) \leq Lh. \]

Therefore

\[ |v_{op}(x + he) - v_{op}(x)| \leq Lh \]

and it follows that

\[ |e \cdot \nabla v_{op}(x)| \leq L \]

in \( \mathbb{R}^n \). By choosing \( e \) to be the various unit coordinate vectors in \( \mathbb{R}^n \) we obtain (iv).

**Proof of the Corollary.** In this case take

\[ L \equiv \max \{ \| \max_\Omega |\nabla v_0|, \max_\partial |\nabla v_0^*| \} \]

and define the sequences \( \{v_{op}\} \) and \( \{v_{op}^*\} \) as was done in the proof of Lemma A, using the same kernel \( k_\delta \) for both sequences. Then each sequence has all of the properties listed in Lemma A and, as is easily verified, \( v_{op} \geq v_{op}^* \).

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**REFERENCES**