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ON THE COHOMOLOGY OF KÄHLER AND HYPER-KÄHLER MANIFOLDS

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0. INTRODUCTION

LET M be a compact complex manifold of complex dimension n, with real Chern classes $\mathbf{c}_1, \ldots, \mathbf{c}_n$. The Riemann–Roch theorem provides a number of relations between the Hodge numbers and the Chern numbers of M. Incorporated into these relations is the equality between the evaluation of the top Chern class $\mathbf{c}_n$ and the Euler characteristic, given that the latter equals $\chi(-1)$, where $\chi(t) = \sum_{p=0}^{n} \chi_p t^p$ and $\chi_p = \sum_{q=0}^{n} (-1)^q h^{p,q}$. In Section 3, we show more generally that for any $k > 0$, the Taylor coefficient $\chi^{(2k)}(-1)/(2k)!$ is expressible as a combination of Chern numbers in which the classes $\mathbf{c}_i$ with $2k \leqslant i \leqslant n - 2k$ have been filtered out. In particular, the class $\mathbf{c}_1\mathbf{c}_{n-1}$ may be expressed in terms of $\chi(-1)$ and $\chi''(-1)$. Such a formula was used in [25] to prove that the moduli space of stable rank 2 vector bundles (with fixed determinant of degree 1) over a Riemann surface of genus at least 3 has $\mathbf{c}_{n-1} = 0$.

The present paper arose in an attempt to understand what can be said in higher dimensions on a compact Kähler manifold M with trivial canonical bundle κ. In this case $\mathbf{c}_1 = 0$ and we obtain a non-trivial linear constraint on the Hodge numbers. By Yau's theorem, M admits a Kähler metric with zero Ricci tensor, and each non-flat factor in its universal covering is a Riemannian manifold with holonomy equal to $SU(n)$ or $Sp(m)$. A hyper-Kähler manifold is one with holonomy $Sp(m)$, or a subgroup thereof; it has complex dimension $2m$, and its Hodge numbers are "invariant by mirror symmetry" in the sense that they satisfy $h^{p,q} = h^{2m-p,q}$ (cf. [13, 29]). We show that this leads to a constraint on the Betti numbers which can be written

$$2\sum_{j=1}^{2m} (-1)^j (3j^2 - m) b_{2m-j} = m b_{2m}; \tag{0.1}$$

such a formula was first proved in [28] by modifying methods from [22]. An amusing consequence is that the middle Betti number b_{2m} of a compact hyper-Kähler manifold must be even unless its real dimension is a multiple of 32. The above results were originally found with the help of *Mathematica*, and have a computational nature to the extent that they clarify in higher dimensions facts which are well understood in complex dimension four and less.

Two families of compact irreducible hyper-Kähler manifolds were described explicitly by Beauville in [3]. A member of the first family is the Hilbert scheme $K^{[m]}$ of finite subschemes of length $m \geqslant 2$ on a K3 surface K, and is a natural resolution of the m-fold symmetric product of K. A member of the second family is a real codimension 4 factor, denoted K_{m-1}, in the de Rham decomposition of $T^{[m]}$ where $T = \mathbb{C}^2/\Lambda$ is a torus and $m \geqslant 3$.

A general formula for the Betti numbers of the Hilbert scheme $S^{[m]}$ for an algebraic surface S was discovered by Göttsche [14], and brings our results "to life". In fact, we consider the rational expression

$$\phi_2 = \frac{b''(-1)}{2b(-1)} - \frac{1}{8}d^2 \tag{0.2}$$

constructed from the Poincaré polynomial $b(t)$ of a manifold M of even real dimension d. Elementary identities in Section 2 show that ϕ_2 is additive with respect to products of manifolds, but more significantly we deduce from [14] that the "naïve" equation $\phi_2(S^{[m]}) = m\phi_2(S)$ holds for any complex surface S.

The last result implies immediately that if K is a K3 surface then the Betti numbers of $K^{[m]}$ satisfy the hyper-Kähler constraint. The case of $T^{[m]}$ is more complicated as (0.1) is disguised by cohomological reducibility; this leads us in Section 6 to analyse a variant of the quantity ϕ_2 associated to $T^{[m]}$. Using a description of the cohomology of these "higher-order Kummer varieties" from [15], we show that the constraint (0.1) nevertheless plays an important role in the theory. Replacing $b(t)$ by the Hodge polynomial in (0.2) allows ϕ_2 to be decomposed into "types" on a Kähler manifold, and the consequent theory is consistent with Hodge decompositions proved in [15]. More generally, we expect the cohomology of various moduli spaces to provide future illustrations of our results.

1. PRELIMINARIES

Throughout this section, M denotes a compact Kähler manifold of complex dimension n. The Hodge number $h^{p,q}$ denotes the dimension of the corresponding Dolbeault cohomology space $H^{p,q}$, and the well-known symmetries

$$h^{p,q} = h^{n-p,n-q} = h^{q,p}, \quad 0 \leqslant p, q \leqslant n \tag{1.1}$$

play an important role in the sequel. The integer

$$\chi_p = \sum_{q=0}^{n} (-1)^q h^{p,q}$$

may be regarded as the index of an appropriate Dolbeault complex, and

$$\sum_{p=0}^{n} (-1)^p \chi_p = \sum_{p,q=0}^{n} (-1)^{p+q} h^{p,q} = \sum_{k=0}^{2n} (-1)^k b_k$$

is the Euler characteristic of M, which we denote simply by χ. The interchange of b_k and χ_p in formulae will be a recurrent feature. (The lowered index in χ_p conflicts with [16] but should not cause confusion in combination with (2.7) below.)

Let M be a compact Kähler manifold for which $\mathbf{c}_1$ vanishes as a real cohomology class. Yau's theorem [33] implies that M has a Ricci-flat Kähler metric. Furthermore, M has a finite covering by a Riemannian product

$$T \times X_1 \times \ldots \times X_r \times Y_1 \times \cdots \times Y_s \tag{1.2}$$

where T is a complex torus with a flat metric, X_i is an irreducible simply-connected Kähler manifold with $\dim_{\mathbb{C}} X_i = n_i$ and holonomy equal to $SU(n_i)$, and Y_j is an irreducible simply-connected Kähler manifold with $\dim_{\mathbb{C}} Y_j = 2m_j$ and holonomy equal to $Sp(m_j)$. This decomposition theorem relies on the Cheeger–Gromoll theorem for metrics with non-positive Ricci tensor; see [3–5, 21].

A Riemannian manifold (Y, g) with $\dim_{\mathbb{C}} Y = 2m$ and holonomy contained in $Sp(m)$ is *hyper-Kähler*. The latter means, by definition, that Y possesses a triple of Kähler structures (J_i, ω_i, g), $i = 1, 2, 3$, compatible with the fixed metric g and satisfying $J_1 J_2 = J_3 = -J_2 J_1$. In particular, (Y, J_1) is *complex symplectic* in the sense that it admits a closed 2-form $\eta = \omega_2 + i\omega_3$ of type (2, 0) (and therefore holomorphic) relative to the complex structure J_1 with η^m nowhere zero. Conversely, a compact Kähler manifold admitting such a 2-form is hyper-Kähler. For η^m trivialises κ, and Yau's theorem implies that Y admits a Ricci-flat Kähler metric; one can then show that the latter renders η parallel and is therefore hyper-Kähler. A hyper-Kähler manifold Y possesses not only three, but a whole 2-sphere of complex structures; each of these has the form $\sum_{i=1}^{3} a_i J_i$ with $\sum_{i=1}^{3} (a_i)^2 = 1$, and gives rise to its own complex symplectic structure. Although different complex structures in the family are not in general equivalent under diffeomorphism, they all have the same Hodge numbers. We refer the reader to [9, 19] for an account of hyper-Kähler geometry.

Let M be a compact connected hyper-Kähler manifold of real dimension $4m$. By studying the action of $Sp(m)$ on spaces of harmonic forms, Wakakuwa [32] proved that $b_{2k} \geqslant \binom{k+2}{2}$ for $k \leqslant m$ and that the "odd" Betti numbers b_{2k+1} of M are all divisible by 4. These results were refined by Fujiki using Hodge decompositions relative to a choice of complex structure. Indeed, wedging with the holomorphic symplectic form η defined above induces a mapping $H^{p,q} \to H^{p+2,q}$ which is injective for $p + 1 \leqslant m$ and its $(m-p)$-fold iteration is an isomorphism. In this way, (1.1) is supplemented by the equations

$$h^{p,q} = h^{2m-p,q}, \quad 0 \leqslant p, q \leqslant 2m. \tag{1.3}$$

An efficient proof of these results has been given by Verbitskii [31] by considering the action of the Lie algebra $\mathfrak{so}(5)$ on cohomology. Fujiki also showed in [13] that $h^{p,q}$ is even and $h^{p,q} \geqslant h^{p+1,q-1}$ whenever $p \geqslant q$. Moreover,

$$b_k = \sum_{j=0}^{[k/2]} \binom{j+2}{2} \gamma_{k-2j}, \quad k \leqslant 2m$$

where

$$\gamma_k = b_k - 3b_{k-2} + 3b_{k-4} - b_{k-6} \tag{1.4}$$

(with $b_k = 0$ if $k < 0$) are integers that are non-negative in the range $k \leqslant m$.

Example. Two known irreducible hyper-Kähler 8-manifolds discussed in Sections 5 and 6 have the indicated Hodge diamonds.

$$K^{[2]}: \quad \begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 1 & & 21 & & 1 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 21 & & 232 & & 21 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 1 & & 21 & & 1 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array} \qquad K_2: \quad \begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 1 & & 5 & & 1 & & \\
 & 0 & & 4 & & 4 & & 0 & \\
1 & & 5 & & 96 & & 5 & & 1 \\
 & 0 & & 4 & & 4 & & 0 & \\
 & & 1 & & 5 & & 1 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

The full 8-fold symmetry of the Hodge diamond of a hyper-Kähler manifold is only visible when $m \geqslant 3$ or (if the odd Betti numbers are zero) when $m \geqslant 4$.

2. OPERATIONS ON POINCARÉ POLYNOMIALS

Let M be a compact oriented smooth manifold of *even* real dimension d, and let

$$b(t) = \sum_{k=0}^{d} b_k t^k$$

denote its Poincaré polynomial. In this section we shall investigate the expansion of $b(t)$ about $t = -1$.

LEMMA 2.1. *Let* $0 \leqslant k \leqslant d/2 - 1$. *Then* $b^{(2k+1)}(-1)$ *is completely determined by* $\{b^{(2i)}(-1)\colon 0 \leqslant i \leqslant k\}$.

Proof. Poincaré duality implies that $b(t^{-1}) = t^{-d} b(t)$. Replacing t by $-1 + t$ and recalling that d is even, we obtain

$$b(-1 - S) = (1 - t)^{-d} b(-1 + t)$$

where $S = \sum_{i=1}^{\infty} t^i$. For the purpose of the proof, we define

$$\bar{b}_j = \frac{1}{j!} b^{(j)}(-1)$$

so that $b(-1 + t) = \sum_{j=0}^{d} \bar{b}_j t^j$. Then

$$\bar{b}_0 - \bar{b}_1 S + \bar{b}_2 S^2 - \bar{b}_3 S^3 + \cdots = (1 + dt + \tbinom{d+1}{2} t^2 + \cdots)(\bar{b}_0 + \bar{b}_1 t + \bar{b}_2 t^2 + \cdots).$$

Comparing coefficients of t^j, we obtain

$$-\sum_{i=0}^{j-1} (-1)^i \tbinom{j-1}{i} \bar{b}_{i+1} = \sum_{i=0}^{j} \tbinom{d-1+i}{i} \bar{b}_{j-i}.$$

Rearranging the last equation when $j = 2k + 1$ is odd gives

$$-2\bar{b}_{2k+1} = \sum_{i=1}^{2k} \left((-1)^i \tbinom{2k}{i} + \tbinom{d-1+i}{i}\right) \bar{b}_{2k+1-i} + \tbinom{d+2k}{2k+1} \bar{b}_0 \tag{2.1}$$

and the result follows by induction. Q.E.D.

Suppose now that the Euler characteristic $\chi = b(-1)$ is non-zero. In order to obtain quantities which are additive with respect to products we set

$$\log b(-1 + t) - \log b(-1) = \log\left(1 + \sum_{k=1}^{d} \frac{b^{(k)}(-1)}{k!\, b(-1)} t^k\right) = \sum_{k \geqslant 1} \phi_k t^k \tag{2.2}$$

where the coefficients ϕ_k are evaluated by means of the expansion $\log(1 + x) = -\sum_{j \geqslant 1} (-x)^j / j$. From the multiplicative property of $b(t)$, this definition ensures the following.

PROPOSITION 2.2. $\phi_k(M \times N) = \phi_k(M) + \phi_k(N)$.

One of the aims of this paper is to demonstrate that geometrical significance can be attached to ϕ_2 and variants of it defined below in Kähler case (in the article [28] the symbol ϕ denotes what is here $8\phi_2$). The equations that result from (2.1) by setting $k = 0$ and $k = 1$ are

$$b'(-1) = -\tfrac{1}{2} d\, b(-1)$$

$$b'''(-1) = -\tfrac{3}{2}(d - 2) b''(-1) + \tfrac{3}{2}\tbinom{d}{3} b(-1) \tag{2.3}$$

and we may deduce Eqs (0.2) and

$$\phi_4 = \frac{b''''(-1)}{24b(-1)} - \frac{1}{8}\left(\frac{b''(-1)}{b(-1)}\right)^2 + \frac{db''(-1)}{2b(-1)} + \tfrac{1}{192}d^2(d^2 - 12d + 8).$$

On the other hand, $\phi_1 = -\frac{1}{2}d$ and $\phi_3 = \phi_2 + \frac{1}{12}d$, in accordance with Lemma 2.1. In the sequel we shall concentrate on ϕ_2, and in practice it is not necessary to exclude the case $\chi = 0$ provided we interpret the combination $\phi_2\chi$ to mean $\frac{1}{8}(4b''(-1) - d^2\chi)$.

On a Kähler manifold of real dimension $d = 2n$, the Poincaré polynomial is refined by the Hodge polynomial

$$h(s, t) = \sum_{p,q=0}^{n} h^{p,q}s^p t^q \tag{2.4}$$

which is symmetric in s, t, and $b(t) = h(t, t)$. This leads to a decomposition of ϕ_k, and we shall describe the situation when $k = 2$. Firstly,

$$b''(-1) = 2h_{ss}(-1, -1) + 2h_{st}(-1, -1),$$

since $h_{tt}(-1, -1) = h_{ss}(-1, -1)$. In analogy to the definition of ϕ_2, we set

$$\phi_{2,0} = \frac{1}{2\chi}h_{ss}(-1, -1) - \tfrac{1}{8}n^2, \qquad \phi_{1,1} = \frac{1}{\chi}h_{st}(-1, -1) - \tfrac{1}{4}n^2. \tag{2.5}$$

These are the coefficients of s^2 and st, respectively, that appear in (2.2) when $b(t)$ is replaced by $h(s, t)$, and

$$\phi_2 = 2\phi_{2,0} + \phi_{1,1}. \tag{2.6}$$

The quantity $\phi_{2,0}$ can also be derived from the well-known χ_t-characteristic, which is the one-variable polynomial that we choose to denote by $\chi(t)$ defined by

$$\chi(t) = h(t, -1) = \sum_{p=0}^{n} \chi_p t^p. \tag{2.7}$$

Indeed, as $h_{ss}(t, -1) = \chi''(t)$, it follows that $\phi_{2,0}$ is the exact analogue of ϕ_2 formed by replacing $b(t)$ by $\chi(t)$. In the process, the proof of Lemma 2.1 remains valid and, for instance, the first equation in (2.3) translates to

$$\chi'(-1) = -\tfrac{1}{2}n\chi \tag{2.8}$$

(the alternating sign in the equation $\chi_{n-p} = (-1)^n\chi_p$ ensures the validity of (2.8) when n is odd). Since $\chi(t)$ is multiplicative with respect to products of compact complex manifolds, use can be made of $\phi_{2,0}$ in non-Kähler situations.

Given importance of the polynomial $\chi(t)$, it is of value to control the term $\phi_{1,1}$ in (2.6), and we highlight two situations when this is possible.

PROPOSITION 2.3. *Let M be a compact complex n-dimensional manifold.*

(i) *If $h^{p,q} = 0$ whenever $p \neq q$ then $\phi_{1,1} = 2\phi_{2,0} + \frac{1}{2}n$.*
(ii) *If $n = 2m$ is even and* (1.3) *holds then $\phi_{1,1} = 0$.*

Proof. (i) We have $\chi_p = (-1)^p h^{p,p} = (-1)^p b_{2p}$, *and* $b(t) = \chi(-t^2)$. Thus,

$$h_{ss}(-1, -1) = \sum_{p=0}^{n} p(p-1)b_{2p} = \sum_{p=0}^{n} p^2 b_{2p} + \tfrac{1}{2}b'(-1) = h_{st}(-1, -1) - \tfrac{1}{2}n\chi$$

the last equality from (2.3). The result follows from the definitions (2.5).

(ii) By assumption,

$$\sum_{p,q=0}^{n} pq(-1)^{p+q}h^{p,q} = \sum_{p,q=0}^{n} p(n-q)(-1)^{p+q}h^{p,q}.$$

Making use of (2.8), we deduce that

$$2h_{st}(-1,-1) = -n\chi'(-1) = \tfrac{1}{2}n^2\chi$$

and once again the result follows from (2.5). Q.E.D.

The hypothesis of (i) is satisfied in particular when M is a Hermitian symmetric space or more generally a complex flag manifold $G^{\mathbb{C}}/P$. The proof of part (ii) also shows that if M, $\tilde{M}$ are two Kähler manifolds of even complex dimension n whose Hodge numbers are related by the mirror symmetry $h^{p,q} = \tilde{h}^{n-p,q}$ then $\phi_{1,1} = -\tilde{\phi}_{1,1}$. For a hyper-Kähler manifold we may take $M = \tilde{M}$, and combined with results from the next section (ii) will yield Theorem 4.1.

3. RIEMANN–ROCH THEOREMS

Let M be a compact complex manifold with holomorphic tangent bundle T and Chern classes $\mathbf{c}_i$, $1 \leqslant i \leqslant n$. The Riemann–Roch theorem expresses the indices χ_p in terms of the Chern classes of M by means of the formula

$$\chi_p = \int_M \mathbf{ch}(\textstyle\bigwedge^p T^*)\mathbf{td}(T). \tag{3.1}$$

It was first proved by Hirzebruch [16] for projective algebraic manifolds, and in the general case by Atiyah and Singer [1]. We use the symbol $\int_M$ to denote evaluation of a cohomology class on the fundamental cycle $[M]$, so that $\int_M$ annihilates $H^k(M, \mathbb{R})$ for $k < 2n$ and defines an isomorphism $H^{2n}(M, \mathbb{R}) \xrightarrow{\cong} \mathbb{R}$. The results in this section are also valid when M is a compact *almost complex manifold* provided χ_p is interpreted as the index of an appropriate 2-step elliptic complex.

Equation (3.1) can be formally combined into the expression

$$\chi(t) = (-1)^n \sum_{p=0}^{n} \chi_{n-p}t^p = (-1)^n \int_M \mathbf{ch}\left(\sum_{p=0}^{n} t^p \textstyle\bigwedge^{n-p} T^*\right)\mathbf{td}(T). \tag{3.2}$$

We now define

$$\mathbf{K}_n(t) = \sum_{k=0}^{n} \mathbf{K}_{n,k}t^k = (-1)^n \mathbf{ch}\left(\sum_{p=0}^{n} (-1-t)^p \textstyle\bigwedge^{n-p} T^*\right)\mathbf{td}(T) \tag{3.3}$$

in order to write (3.2) in the following form:

THEOREM 3.1. $\dfrac{1}{k!}\chi^{(k)}(-1) = (-1)^k \displaystyle\int_M \mathbf{K}_{n,k}.$

The definition of the series $\mathbf{K}(t)$ is similar to that of the $\tilde{T}_y$-class in [18]. Like the latter, $\mathbf{K}(t)$ may be formulated along the lines of "unstable" multiplicative sequences. Indeed, let

$$1 + \sum_{i=1}^{n} \mathbf{c}_i = \prod_{j=1}^{n} (1 + x_j)$$

be a formal factorisation of the total Chern class of T. Then $\mathbf{ch}(T) = \sum_{j=1}^{n} e^{x_j}$, and it follows that

$$(-1)^n \mathbf{ch}\left(\sum_{p=0}^{n} (-t)^p \bigwedge^{n-p} T^*\right) = \prod_{i=1}^{n} (t - e^{-x_i}).$$

Hence

$$\mathbf{K}_n(t) = \prod_{i=1}^{n} (t + 1 - e^{-x_i}) \frac{x_i}{1 - e^{-x_i}} = \prod_{i=1}^{n} (x_i + t\,\mathbf{td}_i) \tag{3.4}$$

where

$$\mathbf{td}_i = \frac{x_i}{1 - e^{-x_i}} = 1 + \tfrac{1}{2}x_i + \sum_{j \geqslant 1} \frac{B_{2j}}{(2j)!} x_i^{2j} \tag{3.5}$$

and B_{2j} are the Bernoulli numbers.

Using (3.4), we see that $\mathbf{K}_{n,0}$ must equal the top Chern class $\mathbf{c}_n$, whose integral gives the Euler characteristic. Our applications of Theorem 3.1 depend upon a generalisation of this fact, which is proved next without the use of formal factorisation.

PROPOSITION 3.2. *Let $0 \leqslant k \leqslant n$. Then $\mathbf{K}_{n,k} - \mathbf{c}_{n-k}$ belongs to the ideal in $H^*(M, \mathbb{R})$ generated by the Chern classes $\mathbf{c}_i$ with $i > n-k$.*

Proof. Using the exterior power operation of K-theory we may write

$$\bigwedge^n (T^* - t) = \sum_{i=0}^{n} (-t)^i \bigwedge^{n-i} T^*$$

where if necessary the symbol t can be thought of as a trivial line bundle. Furthermore,

$$\begin{aligned}
(-1)^k \frac{1}{k!} \frac{d^k}{dt^k}\left(\bigwedge^n (T^* - t)\right) &= \sum_{i=k}^{n} \tbinom{i}{k} (-t)^{i-k} \bigwedge^{n-i} T^* \\
&= \sum_{j=0}^{n-k} \tbinom{k+j}{j} (-t)^j \bigwedge^{n-j-k} T^* \\
&= \sum_{j=0}^{n} (-1)^j \bigwedge^{n-j-k} T^* \otimes S^j((k+1)t) \\
&= \bigwedge^{n-k} (T^* - (k+1)t).
\end{aligned}$$

It now follows from (3.3) that

$$\mathbf{K}_{n,k} = (-1)^{n-k} \mathbf{ch}\left(\bigwedge^{n-k} (W_{n-k} - 1)\right) \mathbf{td}(W_{n-k}^*) \tag{3.6}$$

where W_{n-k} denotes the virtual bundle $T^* - k$ of virtual dimension $n-k$. When $\mathbf{c}_{n-k+1}, \ldots, \mathbf{c}_n$ all vanish, the characteristic classes of W_{n-k} are identical to those of a cotangent bundle in complex rank $n-k$ to the extent that replacing n by $n-k$, T by W_{n-k}^*, and t by 0 in (3.3) must yield $\mathbf{K}_{n-k,0} = \mathbf{c}_{n-k}$. But the expression that results from these substitutions in (3.3) is identical to (3.6). Thus, $\mathbf{K}_{n,k} - \mathbf{c}_{n-k}$ belongs to the required ideal. Q.E.D.

Combining Lemma 2.1, Theorem 3.1 and Proposition 3.2, we have the following.

COROLLARY 3.3. *Let $2 \leqslant k \leqslant n$. The integer*

$$\frac{1}{k!}\chi^{(k)}(-1) = \sum_{p=k}^{n} (-1)^p \binom{p}{k} \chi_p$$

can be expressed as a linear combination of Chern numbers each of which involves at least one $\mathbf{c}_i$ *with* $i > n - 2[k/2]$.

It follows that the term $\mathbf{c}_1^n$ does not occur in the expression for $\chi^{(k)}(-1)$ if $2[k/2] < n$. In particular, if n is odd then $\mathbf{c}_1^n$ does not occur in the Todd genus $\chi_0 = (-1)^n \chi^{(n)}(-1)/n!$, even though the latter is known to be divisible by $\mathbf{c}_1$.

The following special case of Theorem 3.1 is crucial for the sequel; a version of it may also be found at the end of the paper [25].

COROLLARY 3.4. *Let M be a compact complex manifold M of complex dimension n. Then*

$$\int_M \mathbf{c}_1\mathbf{c}_{n-1} = \sum_{p=0}^{n} (-1)^p (6p^2 - \tfrac{1}{2}n(3n+1))\chi_p.$$

Proof. Since the Chern classes $\mathbf{c}_k$ are the elementary symmetric polynomials in $x_1, \ldots, x_n$, we obtain from (3.4) that $\mathbf{K}_{n,1} = \mathbf{c}_{n-1} + \frac{1}{2}n\mathbf{c}_n$ and, more to the point,

$$\mathbf{K}_{n,2} = \mathbf{c}_{n-2} + \tfrac{1}{2}(n-1)\mathbf{c}_{n-1} + \tfrac{1}{24}[2\mathbf{c}_1\mathbf{c}_{n-1} + n(3n-5)\mathbf{c}_n]. \tag{3.7}$$

Setting $k = 2$ in Theorem 3.1,

$$\int_M \mathbf{c}_1\mathbf{c}_{n-1} = 6\chi''(-1) - \tfrac{1}{2}n(3n-5)\chi(-1) \tag{3.8}$$

and the corollary follows from the equation

$$\chi''(-1) = -\tfrac{1}{2}n\chi(-1) + \sum_{p=1}^{n} (-1)^p p^2 \chi_p$$

that results from (2.8). Q.E.D.

In terms of (2.5), the right-hand side of (3.8) may be expressed as $(12\phi_{2,0} + \frac{5}{2}n)\chi$.

COROLLARY 3.5. *Let M be a compact complex n-dimensional manifold with $\mathbf{c}_1 = 0$. Then* $\phi_{2,0} = -\frac{5}{24}n$, *or else* $\chi''(-1) = 0 = \chi$.

Now suppose that $\chi = \int_M \mathbf{c}_n$ is non-zero. If we rewrite Corollary 3.4 in the form

$$\int_M \mathbf{c}_1\mathbf{c}_{n-1} \Big/ \int_M \mathbf{c}_n = 12\phi_{2,0} + \tfrac{5}{2}n \tag{3.9}$$

the left-hand quotient of Chern numbers is readily seen to share with $\phi_{2,0}$ the property of being additive with respect to products of manifolds. Of course, with hindsight, we could have adjusted our definitions so as to make $\phi_{2,0}$ *equal* to the left-hand side, although this would have obscured the role of the Index Theorem.

We may write $\mathbf{K}_{n,k} = \mathbf{K}_{n-1,k-1} + \mathbf{K}^{\cdot}_{n,k}$, where $\mathbf{K}^{\cdot}_{n,k}$ is a cohomology class in pure dimension $2n$ and can therefore replace $\mathbf{K}_{n,k}$ in Theorem 3.1. Further to (3.7), more involved calculations establish that

$$\mathbf{K}^{\boldsymbol{\cdot}}_{n,3} = \frac{1}{2^4.3}(n-2)[2\mathbf{c}_1\mathbf{c}_{n-1} + n(n-3)\mathbf{c}_n]$$

$$\mathbf{K}^{\boldsymbol{\cdot}}_{n,4} = \frac{1}{2^4.3^2.5}[-\mathbf{c}_{n-3}(\mathbf{c}_1^3 - 3\mathbf{c}_1\mathbf{c}_2 + 3\mathbf{c}_3) + \mathbf{c}_{n-2}(\mathbf{c}_1^2 + 3\mathbf{c}_2)$$

$$+ \tfrac{1}{2}(15n^2 - 85n + 108)\mathbf{c}_1\mathbf{c}_{n-1} + \tfrac{1}{8}n(15n^3 - 150n^2 + 485n - 502)\mathbf{c}_n].$$

The relatively simple form of $\mathbf{K}^{\boldsymbol{\cdot}}_{n,3}$ is predicted by Lemma 2.1; by contrast $\mathbf{K}^{\boldsymbol{\cdot}}_{n,4}$ involves all possible Chern numbers permitted by Corollary 3.3. Observe that the cofactor of $\mathbf{c}_{n-3}$ in $\mathbf{K}^{\boldsymbol{\cdot}}_{n,4}$ is a multiple of $\mathbf{s}_3$, where as usual $\mathbf{s}_k = \sum_{i=1}^n x_i^k$ equals $k!$ times the term of dimension $2k$ in $\mathbf{ch}(T)$. This leads to the following result, which was proved by Hirzebruch [17] using Steenrod powers prior to the Riemann–Roch theorem.

COROLLARY 3.6. *Let q be an odd prime number. On a compact almost complex manifold of real dimension $2n$ with $q \leqslant n + 1$,*

$$\mathbf{c}_{n-q+2}\mathbf{s}_{q-2} - \mathbf{c}_{n-q+3}\mathbf{s}_{q-3} + \cdots + \mathbf{c}_{n-1}\mathbf{c}_1 - n\mathbf{c}_n \equiv 0 \bmod q.$$

Proof. Let $k \geqslant 1$. In the notation of Section 1,

$$\mathbf{c}_{n-k}\mathbf{s}_k = \sum x_1^{k+1}x_{k+2}x_{k+3}\ldots x_n + \sum x_1^k x_{k+1}x_{k+2}\ldots x_n \tag{3.10}$$

where the sums are over orbits of the symmetric group $\mathfrak{S}_n$ (except that the second sum becomes $nx_1x_2\ldots x_n = n\mathbf{c}_n$ when $k = 1$). Theorem 3.1 tells us that $q!\mathbf{K}^{\boldsymbol{\cdot}}_{n,q-1}$ is an integral class which is zero modulo q. Next, von Staudt's theorem on the divisibility of Bernoulli numbers [6] implies that $q!\,\mathbf{td}_i$ is well defined and congruent to $-x^{q-1}$ modulo q provided we ignore powers of x_i greater than q in (3.5). We may now deduce from (3.4) that $\sum x_1^{q-1}x_qx_{q+1}\ldots x_n = 0 \bmod q$, and the result follows from repeated use of (3.10).
Q.E.D.

Example. Suppose that the anti-canonical bundle κ^{-1} of a compact Kähler manifold M is ample and that there exist divisors $D_1, \ldots, D_n$ associated to κ^{-1} such that $V_{n-k} = D_1 \cap \ldots \cap D_k$ is a smooth complete intersection for $1 \leqslant k \leqslant n$, providing a "ladder" $V_0 \subset V_1 \subset \ldots \subset V_{n-1} \subset M$. Repeated use of adjunction formulae yields

$$\int_M \mathbf{c}_1\mathbf{c}_{n-1} = \sum_{k=1}^n \chi(V_{n-k})$$

where $\chi(V_{n-k})$ is the Euler characteristic of V_{n-k}. When $n = 2$, $\chi(V_1) = 0$ and the right-hand side is just the self-intersection number of an anti-canonical divisor.

For $M = \mathbb{C}P^n$, the individual summands $\chi(V_{n-k})$ may be evaluated explicitly by means of the formula

$$\sum_{n \geqslant k} \chi(V_{n-k})x^n = \frac{(n+1)^k x^k}{(1+nx)^k(1-x)^2}$$

that is deduced from [16, Appendix 1]. In particular, the Euler characteristic of a smooth hypersurface V_{n-1} of degree $n+1$ in $\mathbb{C}P^n$ (which has $\mathbf{c}_1 = 0$) equals

$$\chi(V_{n-1}) = \frac{n(2+n-(-n)^n)}{n+1}$$

and is always even, but divisible by 3 if and only if $n - 1$ is not.

4. APPLICATIONS TO HYPER-KÄHLER MANIFOLDS

Recall that the Hodge numbers of a hyper-Kähler manifold are invariant by mirror symmetry, by which we mean that they satisfy (1.3). The results of this section are based on the following counterpart of Corollary 3.4.

THEOREM 4.1. *Let M be a compact Kähler manifold of real dimension $d = 2n = 4m$ divisible by 4 whose Hodge numbers are invariant by mirror symmetry. Then*

$$\int_M \mathbf{c}_1\mathbf{c}_{n-1} = \tfrac{1}{2}\sum_{j=0}^{d}(-1)^j(6j^2 - \tfrac{1}{2}d(3d+1))b_j.$$

Proof. Inspecting Corollary 3.4 and its proof, we see that the right-hand side of the equation to be proved would result from the right-hand side of (3.8) by replacing n by d, and $\chi(t)$ by $b(t)$. From (0.2), it therefore equals $\frac{1}{2}(12\phi_2 + \frac{5}{2}d)\chi$. (Strictly speaking we are assuming that $\chi \neq 0$, but in general $\phi_2\chi$ is well defined and the proof extends.) On the other hand, by (2.6) and Propositon 2.3(ii), $\phi_2 = 2\phi_{2,0}$, and the result is now a restatement of Corollary 3.4. Q.E.D.

A Kähler manifold of complex dimension n odd whose Hodge numbers satisfy (1.3) obviously has zero Euler characteristic. The above result therefore allows one to obtain a more sophisticated relation on the Betti numbers when n is even and (as in the hyper-Kähler case) $\mathbf{c}_1 = 0$. The resulting equation can be rearranged into the equivalent form (0.1), which is analogous to the constraint [22, Theorem 0.3(iii)] for a compact quaternion-Kähler $4m$-manifold with positive scalar curvature. There is an important difference between the two theories though that accounts for the relative elegance of (0.1): the hyper-Kähler condition is preserved under products whereas the quaternion-Kähler one is not.

We next define

$$\Phi = 3b''(-1) + m(5 - 12m)\chi = (6\phi_2 + 5m)\chi \tag{4.1}$$

this equals the right-hand side of the equation in Theorem 4.1 and, in analogy with (3.9), may be viewed as a "pseudo characteristic number".

COROLLARY 4.2. *A compact hyper-Kähler $4m$-manifold has $\Phi = 0$, and either $\phi_2 = -\frac{5}{6}m$ or else $b''(-1) = 0 = \chi$.*

The possibility that $b''(-1) = 0 = \chi$ is realised when the flat factor in the de Rham decomposition (1.2) associated to M has non-zero dimension. For in this case

$$M = (T \times M')/\Gamma \tag{4.2}$$

where $T \cong (S^1)^4$ is a 4-torus and Γ is a finite group whose elements act trivially on the cohomology of T. Therefore, the Poincaré polynomial $b(t)$ is divisible by $(1+t)^4$ and $b''(-1) = 0 = b(-1)$. In Section 6 we shall encounter a situation in which the group Γ acts non-trivially on M'.

The first few relations corresponding to the equation $\Phi = 0$ are computed most readily from (0.1); they are listed below with the assumption $b_0 = 1$. The case $m = 2$ was used in [27] to show that any compact irreducible hyper-Kähler 8-manifold has $b_3 + b_4 \geqslant 76$. Observe that b_5 does not feature in 12 dimensions.

$m=1$	$4b_1 + b_2 = 22$
2	$25b_1 + b_3 + b_4 = 46 + 10b_2$
3	$48b_1 + 16b_3 + b_6 = 70 + 30b_2 + 6b_4$
4	$143b_1 + 71b_3 + 23b_5 + 2b_8 = 188 + 104b_2 + 44b_4 + 8b_6 + b_7$
5	$476b_1 + 284b_3 + 140b_5 + 44b_7 + 5b_{10} = 590 + 374b_2 + 206b_4 + 86b_6 + 14b_8 + 4b_9$

The next result extends the fact that a K3 surface has Euler characteristic equal to 24. It is an immediate consequence of Theorem 4.1, and the previously-known result that the odd Betti numbers of a hyper-Kähler manifold are divisible by 4 (see the end of Section 1).

COROLLARY 4.3. *Let M be a compact hyper-Kähler manifold with real dimension $4m$ and Euler characteristic χ. Then $24|(m\chi)$.*

In particular, the Euler characteristic χ, the middle Betti number b_{2m} and the signature τ of M must all be even unless $8|m$.

Example. Two families of compact hyper-Kähler $4m$-manifolds $K^{[m]}$ and K_m were defined by Beauville [3] and are discussed in Sections 5 and 6, respectively. Expressions given below for their Euler characteristics then provide the following factorisations to illustrate degrees of sharpness of Corollary 4.3.

m	2	3	4	5	6	7	8
$\chi(K^{[m]})$	$2^2 3^4$	$2^7 5^2$	$2^1 3^3 5^2 19$	$2^7 3^4 17$	$2^3 5^1 17^1 1579$	$2^9 3^4 11^1 13$	$3^4 5^2 7^1 2129$
$\chi(K_m)$	$2^2 3^3$	$2^6 7$	$2^1 3^1 5^3$	$2^5 3^4$	$2^3 7^3$	$2^9 3^1 5$	$3^6 13$

Note that $K^{[8]}$ and K_8 (and therefore products of these manifolds) have χ odd. Formulae in [14, 15] also imply that $b_{16}(K^{[8]}) = 18\,669\,447$, $\tau(K^{[8]}) = 3\,355\,287$, and $b_{16}(K_8) = 67\,049$, $\tau(K_8) = 6813$. In the sequel we shall comment on the parity of $K^{[8l]}$ and K_{8l} for $l \geqslant 2$.

For curiosity value, we state without proof a version of Theorem 4.1 in terms of the numbers (1.4).

COROLLARY 4.4. *On a compact hyper-Kähler manifold,*

$$\sum_{j=1}^{m} j(j+1)(2j+4)(6j^2+12j-3-5m)\gamma_{2m-2j+1}$$

$$= \sum_{j=0}^{m} (j+1)(j+2)(2j+3)(6j^2+18j-5m)\gamma_{2m-2j}.$$

Given that $\gamma_0 = b_0$ appears on the right-hand side with coefficient

$$m(m+1)(m+2)(2m+3)(6m+13)$$

it is impossible for the γ_i, $i \geqslant 1$, to be all zero. This is itself a result undetected by the inequalities in Section 1.

5. SYMMETRIC PRODUCTS AND HILBERT SCHEMES

Let S be a compact complex algebraic surface, and let $S^{(m)}$ denote its m-fold symmetric product. An element of $S^{(m)}$ may be regarded as a 0-cycle

$$x = \sum_{i=1}^{m} i \cdot (p_{i,1} + \cdots + p_{i,\alpha_i}) \tag{5.1}$$

formed from $|\alpha| = \sum_{i=1}^{m} \alpha_i$ distinct points $p_{i,j}$ of S, with $\sum_{i=1}^{m} i\alpha_i = m$. The 0-cycles (5.1) corresponding to a fixed partition $\alpha = (\alpha_1, \ldots, \alpha_m)$ of m form a stratum $S_\alpha^{(m)}$ of $S^{(m)}$ which can be identified with a smooth subspace of the product

$$S^{(\alpha)} \cong S^{(\alpha_1)} \times \cdots \times S^{(\alpha_m)}. \tag{5.2}$$

There exists a canonical resolution $\varepsilon : S^{[m]} \to S^{(m)}$, where $S^{[m]}$ is the Hilbert scheme of closed 0-dimensional subschemes of length m on S, which is a smooth complex $2m$-dimensional manifold. The fibre over $x \in S_\alpha^{(m)}$ has the form

$$\varepsilon^{-1}(x) \cong (V_1)^{\alpha_1} \times (V_2)^{\alpha_2} \times \cdots \times (V_m)^{\alpha_m} \tag{5.3}$$

where $V_i = \mathrm{Hilb}^i(\mathbb{C}[x, y])$ is the scheme that parametrises ideals in $\mathbb{C}[x, y]$ of colength i, and is an irreducible variety of complex dimension $i - 1$. A survey of results on these schemes can be found in [11, 8, 10] and references therein.

The relevance of the above construction is explained by the following theorem of Beauville [3] which is also a consequence of more general results of Mukai [24]: if S has a complex symplectic structure then so does $S^{[m]}$ for all $m \geqslant 2$. In particular, using [30], if K is any K3 surface then $K^{[m]}$ admits a hyper-Kähler metric, which must be irreducible since $K^{[m]}$ is simply-connected. The space $K^{[2]}$ was first singled out by Fujiki [12] as a counterexample to a statement by Bogomolov, and is a $\mathbb{Z}_2$-quotient of the manifold obtained by blowing up the diagonal in $K \times K$. If T is a torus then $T^{[m]}$ is not locally irreducible, but the non-trivial factor in the de Rham decomposition of the universal covering of $T^{[m]}$ is an irreducible hyper-Kähler manifold of dimension $4m - 4$, denoted in [3] by K_{m-1} (see Section 6).

Using intersection cohomology, Göttsche and Soergel [15] have expressed the Betti numbers of $S^{[m]}$ in terms of those of (5.2) by means of the following theorem:

$$b(S^{[m]}; t) = \sum_\alpha b(S^{(\alpha)}; t) t^{2m - 2|\alpha|}. \tag{5.4}$$

The sum is over all partitions of m with notation as above. The exponent $2m - 2|\alpha|$ is the dimension of the fibre (5.3), and if we replace S by a point we obtain the Poincaré polynomial $\sum_\alpha t^{2m-2|\alpha|}$ of V_m. The latter was previously determined by Ellingsrud and Strømme [10], who also tabulated the Betti numbers of $(\mathbb{C}P^2)^{[m]}$.

The Betti numbers of $S^{(m)}$ can be computed in a more elementary way. A general formula was found by Macdonald [23]; if $b_0(S) = 1$, $b_1(S) = a$ and $b_2(S) = b$, it takes the form

$$\sum_{m \geqslant 0} b(S^{(m)}; t) x^m = \frac{(1 + tx)^a (1 + t^3 x)^a}{(1 - x)(1 - t^2 x)^b (1 - t^4 x)}. \tag{5.5}$$

With this notation, (5.4) leads to the more explicit formula

$$\sum_{m=0}^{\infty} b(S^{[m]}; -t) x^m = \exp\left(\sum_{j=1}^{\infty} \frac{x^j (1 - a t^j + b t^{2j} - a t^{3j} + t^{4j})}{j(1 - t^{2j} x^j)} \right) \tag{5.6}$$

that was first proved by Göttsche for projective surfaces using the Weil conjectures. The following consequences of (5.4) and (5.6) are worth noting. If $a = 0$ then the odd Betti numbers of $S^{[m]}$ are all zero. In general, $b_1(S^{[m]}) = a$ and $b_2(S^{[m]}) = \frac{1}{2}a(a - 1) + b + 1$ for all $m \geqslant 2$ [3]. Moreover, $b_3(S^{[2]}) = a(b + 2)$, and

$$b_3(S^{[m]}) = \binom{a}{3} + a(b + 3), \quad m \geqslant 3. \tag{5.7}$$

For all S, the Betti numbers of $S^{[m]}$ for any S stabilise according to the rule that $b_m(S^{[m+k]})$ is independent of $k \geqslant 0$.

Example. Applying (5.6) to a K3 surface K with $b(K; t) = 1 + 22t^2 + t^4$ gives

$$b(K^{[2]}; t) = 1 + 23t^2 + 276t^4 + \cdots$$
$$b(K^{[3]}; t) = 1 + 23t^2 + 299t^4 + 2554t^6 + \cdots$$
$$b(K^{[4]}; t) = 1 + 23t^2 + 300t^4 + 2852t^6 + 19298t^8 + \cdots$$
$$b(K^{[5]}; t) = 1 + 23t^2 + 300t^4 + 2875t^6 + 22127t^8 + 125604t^{10} + \cdots.$$

It follows from (5.4) that the Euler characteristic $\chi(K^{[m]})$ equals the coefficient of t^m in $\prod_{i=1}^m (1 - t^i)^{-24}$, which can be expressed as t/Δ, where Δ is the standard cusp form [18]. It is easy to check that $\binom{23+m}{m} = \chi(K^{(m)})$ is odd if and only if $m = 8l$, where l is congruent to 0 or 1 modulo 4. The parity of $\chi(K^{[8l]})$ is then determined by the number of partitions of l of the form $l = \sum i\alpha_i$ with $\alpha_i \equiv 0, 1 \bmod 4$. In particular, $\chi(K^{[8l]})$ is odd if $l = 0, 1, 2, 4, 6, 7, 9, 11, \ldots$.

If S is a complex surface with $\chi = \chi(S)$ zero then $\chi(S^{[m]}) = 0$ for all $m \geqslant 2$; this applies in particular to a complex torus or a Kodaira surface, both of which admit complex symplectic structures. When $\chi(S) \neq 0$, it makes sense to consider the invariant (0.2), and we first record a formula easily deduced from (5.5):

PROPOSITION 5.1. $\phi_2(S^{(m)}) = -m\left(1 + \dfrac{(a-4)(\chi+m)}{\chi(\chi+1)}\right).$

When $m = 1$ the right-hand side reduces to the definition of $\phi_2(S)$, and $\chi + 1$ can only vanish if S is the blow-up of a ruled surface [2]. Our next results shows, by contrast, that $S^{[m]}$ behaves like the m-fold Cartesian product relative to ϕ_2. It implies that the Betti numbers of the Hilbert scheme of points on a K3 surface satisfy the constraint (0.1), but is by no means restricted to the case in which S is hyper-Kähler.

THEOREM 5.2. *Let S be a compact complex surface. Then $\phi_2(S^{[m]}) = m\phi_2(S)$, so that if $b_0(S) = 1$ and $b_1(S) = a$,*

$$\phi_2(S^{[m]}) = -m\left(1 + \frac{a-4}{\chi}\right).$$

Proof. We derive this as a consequence of (5.6). A prime will denote differentiation with respect to t, and unless otherwise indicated sums are over the range $j \geqslant 1$. As is customary, we set $\tilde{b}(t) = b(S; -t)$, so that $\chi = \tilde{b}(1)$. Let

$$U(x,t) = \sum \frac{x^j \tilde{b}(t^j)}{j(1 - t^{2j}x^j)}.$$

Then

$$-\sum_{m \geqslant 1} b'(S^{[m]}; -t)x^m = \exp(U(x,t))U'(x,t)$$
$$= \exp(U(x,t))\left(\sum \frac{x^j t^{j-1}\tilde{b}'(t^j)}{1 - t^{2j}x^j} + 2\sum \frac{x^{2j}t^{2j-1}\tilde{b}(t^j)}{(1 - t^{2j}x^j)^2}\right).$$

Taking the second derivative, evaluating at $t = 1$ and using the equations $\tilde{b}'(1) = 2\chi$ and $\tilde{b}''(1) = 2(\phi_2(S) + 2)\chi$ gives

$$\sum_{m \geq 1} b''(S^{[m]}; -1)x^m = 2\chi E(x)F(x) \tag{5.8}$$

where $E(x) = \exp(U(x, 1))$, and

$$F(x) = \frac{1}{2\chi}(U'(x, 1)^2 + U''(x, 1))$$
$$= 2\chi\left(\sum\frac{x^j}{(1-x^j)^2}\right)^2 + \phi_2(S)\sum\frac{jx^j}{1-x^j} + \sum\frac{(3j-1)x^j + x^{2j} + jx^{3j}}{(1-x^j)^3}.$$

Now, firstly,

$$x\frac{\mathrm{d}}{\mathrm{d}x}E(x) = \chi E(x)\sum\frac{x^j}{(1-x^j)^2} = \chi E(x)\sum\frac{jx^j}{1-x^j}. \tag{5.9}$$

The second equality of (5.9) follows from an expansion of the respective denominators, namely,

$$\sum_{j \geq 1}\frac{x^j}{(1-x^j)^2} = \sum_{j,k \geq 1} kx^{jk} = \sum_{k \geq 1}\frac{kx^k}{1-x^k}. \tag{5.10}$$

Secondly,

$$x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}E(x)\right) = E(x)\left[\left(\sum\frac{x^j}{(1-x^j)^2}\right)^2 + \sum\frac{jx^j + jx^{2j}}{(1-x^j)^3}\right]$$
$$= \chi E(x)\left[\chi\left(\sum\frac{x^j}{(1-x^j)^2}\right)^2 + \sum\frac{(3j-1)x^j + x^{2j} + jx^{3j}}{2(1-x^j)^3}\right]$$

the second equality following from similar tricks to (5.10). From the expression above for $F(x)$, we see that

$$2\chi E(x)F(x) = 2\phi_2(S)x\frac{\mathrm{d}}{\mathrm{d}x}E(x) + 4x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}E(x)\right).$$

Now the coefficient of x^m in the right-hand side equals $2m\phi_2(S) + 4m^2$ times the Euler characteristic $b(S^{[m]}; -1)$. Thus, from (5.8),

$$\phi_2(S^{[m]}) = \frac{b''(S^{[m]}; -1)}{2b(S^{[m]}; -1)} - 2m^2 = m\phi_2(S). \qquad \text{Q.E.D}$$

The identities inherent in the above proof are also consistent with the formula

$$\sum_{m=0}^{\infty} h(S^{[m]}; -s, -t)x^m = \exp\left(\sum_{j=1}^{\infty}\frac{x^j h(S; -s^j, -t^j)}{j(1 - s^j t^j x^j)}\right) \tag{5.11}$$

for the Hodge polynomial (2.4) of $S^{[m]}$, which was conjectured in [14] and proved in [15]. Indeed, using (5.11) and interchanging the roles of $b(t)$ and $\chi(t)$ in the proof of Theorem 5.2 (paying attention to the halving of polynomial degrees) shows that

$$\phi_{2,0}(S^{[m]}) = m\phi_{2,0}(S).$$

By (3.9), this is equivalent to asserting that

$$\frac{\int_{S^{[m]}} \mathbf{c}_1(S^{[m]})\mathbf{c}_{2m-1}(S^{[m]})}{\chi(S^{[m]})} = m\frac{\int_S \mathbf{c}_1(S)^2}{\chi(S)}.$$

For a surface S of general type, the Bogomolov–Miyaoka–Yau inequality [2] says that the right-hand side is no greater than $3m$.

Example. When $S = K$ is a K3 surface, the last equation is consistent with the vanishing of $\mathbf{c}_1$ for $S^{[m]}$. The Hodge numbers of $K^{[2]}$ are completely determined by its Betti numbers and the identities (1.1), (1.3), and provide the first diamond's entries in Section 1. The equation $b_4 = 276$ may be deduced from Corollary 4.2 once one knows that $b_1 = 0 = b_3$ and $b_2 = 23$. As for $K^{[3]}$, (5.11) implies that

$$h^{1,1} = 21 = h^{5,1}, \qquad h^{2,2} = 253 = h^{4,2}, \qquad h^{3,1} = 22, \qquad h^{3,3} = 2004.$$

Suppose that S is a hyper-Kähler 4-manifold. The moduli space of anti-self-dual connections over S admits a complex symplectic structure, and in various situations this is known to extend to an appropriate compactification [24, 20]. For example, suppose that K is a K3 surface with an ample divisor of degree $2m$, and let $\mathcal{M}_m$ denote the corresponding moduli space of stable rank 2 locally-free sheaves with $\mathbf{c}_1 = 0$ and $\mathbf{c}_2 = 2m + 3$. Then $\mathcal{M}_m$ admits a smooth compactification which is both birational to $K^{[4m+3]}$, and complex symplectic [26]. Analogous results hold for moduli spaces over tori, although (as in the next section) one needs to factor out various symmetries to arrive at an irreducible space. The resulting theory is explained in [7] with reference to a moduli space of stable bundles over the product T of a pair of elliptic curves, represented by an open set of $T \times \hat{T}^{[2]}$, where $\hat{T}$ is the dual of T. These constructions promise to provide a source of new diffeomorphism classes of compact hyper-Kähler manifolds in higher dimensions. In view of Theorem 5.2, one might speculate that the constraint (0.1) is validated by some general principle for these moduli spaces.

6. CALCULATIONS FOR THE TORUS

This section specialises the above discussions to the case in which $S = T$ is a complex 2-dimensional torus. It follows from (5.5) and (5.4) or (5.6) that the Poincaré polynomials of both $T^{(m)}$ and $T^{[m]}$ are divisible by $(1 + t)^4$ for any integer $m \geqslant 1$. Accordingly, we write

$$b(T^{(m)}; t) = (1 + t)^4 \hat{b}(T^{(m)}; t), \qquad b(T^{[m]}; t) = (1 + t)^4 \hat{b}(T^{[m]}; t). \tag{6.1}$$

The second factorisation reflects the fact that we are in the situation of (4.2) where $M = T^{[m]}$, $\Gamma \cong (\mathbb{Z}_m)^4$ and $M' = K_{m-1}$ is a simply-connected irreducible hyper-Kähler manifold of real dimension $4m - 4$. Indeed, Beauville defines K_{m-1} as $\sigma^{-1}(0)$, where $\sigma: T^{[m]} \to T$ factors through the "centre of mass" map that interprets (5.1) as a point of the abelian group T. The natural action of T on $T^{[m]}$ gives rise to a commutative diagram

$$\begin{array}{ccc} T \times K_{m-1} & \to & T^{[m]} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \sigma} \\ T & \to & T, \end{array}$$

in which π_1 is the projection and the horizontal maps are coverings associated to Γ, which may now be defined to be the subgroup of T of m-division points.

It follows that $\hat{b}(T^{[m]}; t)$ is equal to the Poincaré polynomial of the singular space K_{m-1}/Γ. As K_{m-1} is simply-connected we have $b_1(K_{m-1}) = 0 = \hat{b}_1(T^{[m]})$, and from the line preceding (5.7) we deduce that

$$\hat{b}_2(T^{[m]}) = \tfrac{1}{2} b_1(T)(b_1(T) - 1) + 1 = 7.$$

When $m = 2$, K_1 is identified with the Kummer surface associated to T, and the above diagram resolves the mapping $T \times (T/\mathbb{Z}_2) \to T^{(2)}$ which sends $(s, \pm t)$ to the 0-cycle consisting of the points $s + t$ and $s - t$ of T. Since $b_2(K_1) = 22$, one has $\hat{b}_2(T^{[2]}) = b_2(K_1) - 15$, though $b_2(K_{m-1}) = \hat{b}_2(T^{[m]}) = 7$ for all $m \geqslant 3$ [3].

More generally, we may apply the formula

$$\hat{b}_k(T^{[m]}) = \dim H^k(K_{m-1}, \mathbb{R})^\Gamma = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \operatorname{tr}(g^* | H^k(K_{m-1}, \mathbb{R})). \tag{6.2}$$

As a simple application of this, we have the following.

PROPOSITION 6.1. *The hyper-Kähler 8-manifold K_2 has Poincaré polynomial*

$$b(K_2; t) = 1 + 7t^2 + 8t^3 + 108t^4 + 8t^5 + 7t^6 + t^8.$$

Proof. The action of Γ on $K_2 = \sigma^{-1}(0) \subset T^{[3]}$ covers its action on $\varepsilon(K_2) \subset T^{(3)}$. The induced action of Γ on $H^k(K_2, \mathbb{R})$ can be inferred by counting distinct homology classes in the fibres $\varepsilon^{-1}(\gamma(x))$, where $\gamma \in \Gamma$ and $x \in \varepsilon(K_2)$. Each non-trivial representation of Γ on $H^k(K_2, \mathbb{R})$ must have dimension 3^4, and the trace of any non-identity element of Γ equals $b_k(K_2) - 3^4 N_k$ for some non-negative integer N_k. Therefore $\hat{b}_k(T^{[3]}) = b_k(K_2) - 80N_k$ for $k = 2, 3, 4$, though from a remark above we already know that $N_2 = 0$.

Applying (0.1) to K_2 gives $116 = \hat{b}_3(T^{[3]}) + \hat{b}_4(T^{[3]}) + 80(N_3 + N_4)$, and we claim

$$\hat{b}(T^{(3)}; t) = 1 + 6t^2 + 4t^3 + 21t^4 + 4t^5 + 6t^6 + t^8$$

$$\hat{b}(T^{[3]}; t) = 1 + 7t^2 + 8t^3 + 28t^4 + 8t^5 + 7t^6 + t^8.$$

The first line follows from (5.5), and the second may be deduced either from (5.4) or from first principles (it would in fact suffice to prove (5.7)). We may conclude that $N_3 = 0$ and $N_4 = 1$ by observing that Γ must preserve the $Sp(2)$ decomposition of $H^k(K_2, \mathbb{R})$ which forces $4 | N_3$, just as $4 | b_3(K_2)$. Q.E.D.

We shall now proceed to consider the higher-dimensional situation more systematically. Although $\chi(T^{(m)})$ vanishes and $\phi_2(T^{(m)})$ is indeterminate, we are at liberty to consider the quantities

$$\hat{\chi}(T^{(m)}) = \hat{b}(T^{(m)}; -1)$$

$$\hat{\Phi}(T^{(m)}) = 3\hat{b}''(T^{(m)}; -1) + (m-1)(17 - 12m)\hat{b}(T^{(m)}; -1)$$

and analogous ones defined by replacing $T^{(m)}$ by $T^{[m]}$. (Compared to (4.1), m has been replaced by $m - 1$ which is one quarter the degree of $\hat{b}(T^{(m)}; t)$ or $\hat{b}(T^{[m]}; t)$.) Direct calculations from (5.5) yield the following.

PROPOSITION 6.2. $\hat{\chi}(T^{(m)}) = m^3$, $\hat{\Phi}(T^{(m)}) = (m-1)m^4$.

The next result is presented as a corollary of (5.4) and Proposition 6.2, although it may also be deduced from (5.6) by exploiting properties of Lambert series of the type that feature in (5.10). We shall use the following terminology. Each divisor d of m gives rise to a partition $\alpha(d)$ of m with $\alpha(d)_i = d$ for $i = m/d$. Conversely, we shall call a partition α of m *exact* if α_i is non-zero for only one value of i.

PROPOSITION 6.3. $\hat{\chi}(T^{[m]}) = \sum_{d|m} d^3$, $\hat{\Phi}(T^{[m]}) = \sum_{d|m} \left(d - \frac{m}{d}\right) d^4$.

Proof. Further to the notation (5.2) and (6.1), for each partition α of m we write $b(T^{(\alpha)}; t) = (1+t)^4 \hat{b}(T^{(\alpha)}; t)$, so that (5.4) becomes

$$\hat{b}(T^{[m]}; t) = \sum_\alpha \hat{b}(T^{(\alpha)}; t) t^{2m-2|\alpha|}. \tag{6.3}$$

Hence $\hat{\Phi}(T^{[m]}) = \sum_\alpha \hat{\Phi}_\alpha$, where

$$\hat{\Phi}_\alpha = 3\frac{\mathrm{d}}{\mathrm{d}t}(\hat{b}(T^{(\alpha)}; t)t^{2m-2|\alpha|})|_{t=-1} + (m-1)(17-12m)\hat{b}(T^{(\alpha)}; -1).$$

Since $b(T^{(\alpha_i)}; t)$ is divisible by $(1+t)^4$ whenever $\alpha_i \geqslant 1$, it follows that $\hat{b}(T^{(\alpha)}; t)$ has $(1+t)^4$ as a factor and $\hat{\Phi}_\alpha = 0$ unless α is exact. Consequently, we need only sum over the divisors d of m, replacing the superscript α by $\alpha(d)$ or d. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\hat{b}(T^{(d)}; t)t^{2m-2d})|_{t=-1} &= \hat{b}''(T^{(d)}; -1) - 2(2m-2d)\hat{b}'(T^{(d)}; -1)\\ &\quad + (2m-2d)(2m-2d-1)\hat{b}(T^{(d)}; -1)\\ &= \hat{b}''(T^{(d)}; -1) + (4m^2 - 4d^2 - 10m + 10d)\hat{b}(T^{(d)}; -1)\end{aligned}$$

using (2.3) (in which the symbol d has a different meaning). Hence,

$$\begin{aligned}\hat{\Phi}_{\alpha(d)} &= 3\hat{b}''(T^{(d)}; -1) + (30d - m - 12d^2 - 17)\hat{b}(T^{(d)}; -1)\\ &= \hat{\Phi}(T^{(d)}) + (d-m)\hat{b}(T^{(d)}; -1)\\ &= d^5 - md^3\end{aligned}$$

and the result follows. Q.E.D.

The key point is that when building up the cohomology of $T^{[m]}$ from $T^{(m)}$, both $\hat{\chi}$ and $\hat{\Phi}$ are sensitive only to exact partitions of m. In particular, if m is a prime or the square of a prime then $\hat{\Phi}(T^{[m]}) = \hat{\Phi}(T^{(m)}) - m + 1$.

For each partition α of m, the number of distinct elements in the orbit of Γ on the top homology class of the fibre (5.3) must equal $g(\alpha)^4$ for some divisor $g(\alpha)$ of m. Referring to the proof of Proposition 6.1 and applying (6.2) to (6.3) one would therefore expect that

$$b(K_{m-1}; t) = \sum_\alpha g(\alpha)^4 \hat{b}(T^{(\alpha)}; t) t^{2m-2|\alpha|}. \tag{6.4}$$

In fact, Göttsche and Soergel [15] prove that (6.4) is valid with

$$g(\alpha) = \gcd\{i\colon \alpha_i \neq 0\}.$$

On the other hand, given (6.4), Corollary 4.2 implies that

$$0 = \Phi(K_{m-1}) = \sum_\alpha g(\alpha)^4 \hat{\Phi}_\alpha = \sum_{d|m}\left(d - \frac{m}{d}\right)(dg(\alpha(d)))^4 \tag{6.5}$$

using notation from the previous proof. The correct solution $g(\alpha(d)) = m/d$ to (6.5) is justified by the fact that the connected components of $\varepsilon(K_{m-1}) \cap \overline{T^{(m)}_{\alpha(d)}}$ are in bijective correspondence with the elements of $\Gamma/\Gamma(d)$, where $\Gamma(d)$ denotes the subgroup of size d^4 consisting of d-division points. Moreover, each component admits an action by a connected group extending $\Gamma(d)$, which therefore acts trivially on $H^k(K_{m-1}, \mathbb{R})$. The easiest situation is that in which m is prime, for then the action of Γ identifies the top homology classes in the m^4 distinct fibres $\varepsilon^{-1}(m \cdot p)$ lying in K_{m-1}, and

$$b(K_{m-1}; t) = \hat{b}(T^{[m]}; t) + (m^4 - 1)t^{2m-2}.$$

Proposition 6.1 illustrated a particular case of this.

Example. Further to Proposition 6.1, (6.4) implies that

$$b(K_3; t) = 1 + 7t^2 + 8t^3 + 51t^4 + 56t^5 + 458t^6 + \cdots$$

$$b(K_4; t) = 1 + 7t^2 + 8t^3 + 36t^4 + 64t^5 + 168t^6 + 288t^7 + 1046t^8 + \cdots$$

$$b(K_5; t) = 1 + 7t^2 + 8t^3 + 36t^4 + 64t^5 + 191t^6 + 344t^7 + 915t^8 + 1312t^9 + 3748t^{10} + \cdots$$

Analogues of Propositions 6.2, 6.3 for K_{m-1} are completed by the calculation

$$\chi(K_{m-1}) = \sum_{d \mid m} g(\alpha(d))^4 \hat{\chi}(T^{(d)}) = m^3 \sum_{d \mid m} d$$

that is a close companion of (6.5). Further to Corollary 4.3, the Euler characteristic of K_{m-1} is odd if and only if m is the square of an odd integer which means that $m - 1 = 8l$, where $l = 0, 1, 3, 6, 10, 15, \ldots$

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REFERENCES

1. M. F. Atiyah and I. M. Singer: The index theory of elliptic operators: III, *Ann. Math.* **87** (1968), 546–604.
2. W. Barth, C. Peters and A. Van de Ven: *Compact complex surfaces*, Springer, Berlin (1984).
3. A. Beauville: Variétés Kähleriennes dont la première classe de Chern est nulle, *J. Diff. Geometry* **18** (1983), 755–782.
4. A. Besse: *Einstein manifolds*, Springer, Berlin (1987).
5. F. A. Bogomolov: On the decomposition of Kähler manifolds with trivial canonical class, *Math. USSR-Sb.* **22** (1974), 580–583.
6. Z. I. Borevich and I. R. Shafarevich: *Number theory*, Academic Press, New York (1966).
7. P. J. Braam, A. Maciocia and A. Todorov: Instanton moduli as a novel map from tori to K3 surfaces, *Invent. Math.* **108** (1992), 419–451.
8. J. Briancon: Description de Hilb$^n\mathbb{C}\{x, y\}$, *Invent. Math.* **41** (1977), 45–89.
9. E. Calabi: Isometric families of Kähler structures, in *The Chern symposium*, W.-Y. Hsiang et al., Eds., Springer, Berlin (1980).
10. G. Ellingsrud and S. A. Strømme: On the homology of the Hilbert scheme of points in the plane, *Invent. Math.* **87** (1987), 343–352.
11. J. Fogarty: Algebraic families on an algebraic surface, *Amer. J. Math.* **10** (1968), 511–521.
12. A. Fujiki: On primitively symplectic Kähler V-manifolds of dimension four, in *Classification theory of algebraic and analytic manifolds, 1982* (Progress in Mathematics **39**), Birkhäuser (1983).
13. A. Fujiki: On the de Rham cohomology group of a compact Kähler symplectic manifold, in *Algebraic geometry*, Sendai, 1985 (Advanced Studies in Pure Mathematics **10**), T. Oda, Ed., North-Holland, Amsterdam (1987).
14. L. Göttsche: The Betti numbers of the Hilbert scheme of points on a smooth projective surface, *Math. Ann.* **286** (1990), 193–207.
15. L. Göttsche and W. Soergel: Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces, *Math. Ann.* **296** (1993), 235–245.
16. F. Hirzebruch: *Topological methods in algebraic geometry*, 3rd Edn, Springer, Berlin (1966).
17. F. Hirzebruch: The index of an oriented manifold and the Todd genus of an almost complex manifold, in *Friedrich Hirzebruch, Collected Papers*, Band I, Springer, Berlin (1987).
18. F. Hirzebruch and D. Zagier: *The Atiyah–Singer theorem and elementary number theory*, Math. Lect. Series **3**, Publish or Perish, Berkeley (1974).
19. N. J. Hitchin: Hyperkähler manifolds, Séminaire N. Bourbaki, no. 748 (1991).

20. S. KOBAYASHI: On moduli of vector bundles, in *Complex geometry and analysis*, V. Villani, Ed., Lect. Notes. Math. **1422**, Springer, Berlin (1990).
21. H. B. LAWSON and M.-L. MICHELSOHN: *Spin geometry*, Princeton University Press, Princeton (1988).
22. C. R. LEBRUN and S. M. SALAMON: Strong rigidity of positive quaternion-Kähler manifolds, *Invent. Math.* **118** (1994), 109–132.
23. I. G. MACDONALD: The Poincaré polynomial of a symmetric product, *Proc. Camb. Phil. Soc.* **58** (1962), 563–568.
24. S. MUKAI: Symplectic structure of the moduli space of sheaves on an abelian or K3 surface, *Invent. Math.* **77** (1984), 101–116.
25. M. S. NARASIMHAN and S. RAMANAN: Generalized Prym varieties as fixed points, *J. Indian Math. Soc.* **39** (1975), 1–19.
26. K. G. O'GRADY: Donaldson's polynomials for K3 surfaces, *J. Diff. Geometry* **35** (1992), 415–427.
27. S. SALAMON: *Riemannian geometry and holonomy groups*, Pitman Research Notes in Math. **201**, Longman Scientific, New York (1989).
28. S. M. SALAMON: Spinors and cohomology, *Rend. Sem. Mat. Univ. Pol. Torino* **50** (1992), 393–410.
29. C. VAFA: Topological mirrors and quantum rings, in *Essays on mirror manifolds*, S-T. Yau, Ed., International Press, London (1992).
30. J. VAROUCHAS: Stabilité de la classe des variétés kählériennes par certaines morphismes propres, *Invent. Math.* **77** (1984), 117–127.
31. M. S. VERBITSKII: Action of the Lie algebra of SO(5) on the cohomology of hyper-Kähler manifolds, *Functional Anal. Appl.* **24** (1991), 229–230.
32. H. WAKAKUWA: On Riemannian manifolds with homogeneous holonomy group $Sp(n)$, *Tôhoku Math. J.* **10** (1958), 274–303.
33. S. T. YAU: On the Ricci-curvature of a complex Kähler manifold and the complex Monge–Ampère equations, Comment. *Pure Appl. Math.* **31** (1978), 339–411.

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Added in proof

An earlier treatment of Corollaries 3.3 and 3.4 was given in the paper.

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