Completion of Operator Partial Matrices to Projections

Jinchuan Hou

Department of Mathematics
Shanxi Teachers University
Linfen, 041004, People's Republic of China

Submitted by Chandler Davis

ABSTRACT

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces. For given operators $A$, $B$, and $C$ on $\mathcal{H}$ and from $\mathcal{K}$ into $\mathcal{K}$, respectively, necessary and sufficient conditions are obtained for partial matrices

\[
\begin{pmatrix}
? & C \\
C^* & B
\end{pmatrix}, \quad \begin{pmatrix}
? & C \\
C^* & ?
\end{pmatrix}, \quad \begin{pmatrix}
A & ? \\
? & B
\end{pmatrix}
\]

to have projection completions. All such completions are also characterized.

1. INTRODUCTION

Let $\mathcal{H}_i$ and $\mathcal{K}_i$ be Banach spaces, $i = 1, \ldots, n$. Let $A_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{K}_i)$, the Banach spaces of all bounded linear operators from $\mathcal{H}_j$ into $\mathcal{K}_i$. Given a subset $\Gamma \subseteq \{(i, j) : i, j = 1, \ldots, n\}$ and $A_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{K}_i)$ for $(i, j) \in \Gamma$, we get a partially specified $n \times n$ matrix $(X_{ij})$ with $X_{ij} = A_{ij}$ if $(i, j) \in \Gamma$ and $X_{ij}$ unknown if $(i, j) \notin \Gamma$, which will be denoted by $(A_{ij})_{\Gamma}$. Let $\mathcal{X} = \bigoplus_{j=1}^{n} \mathcal{H}_j$ and $\mathcal{Y} = \bigoplus_{i=1}^{n} \mathcal{K}_i$. If $Q = (Q_{ij}) \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that $Q_{ij} = A_{ij}$ whenever $(i, j) \in \Gamma$, then $Q$ is called a completion of the partial matrix $(A_{ij})_{\Gamma}$. In general, a completion problem is to find conditions on the $A_{ij}$'s so that the partial matrix $(A_{ij})_{\Gamma}$ has completions satisfying some nice properties.
recently this problem has been studied in a variety of directions by a number of authors. For a few references, see [1-11], [13], and [14].

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces. Let $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$, and $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. The well-known Parrott's theorem says that the partial 2 by 2 block operator-valued matrix

$$
\begin{pmatrix}
A & C \\
? & B
\end{pmatrix}
$$

has a contraction completion if and only if $AA^* + CC^* \leq I_{\mathcal{H}}$ and $B^*B + C^*C \leq I_{\mathcal{K}}$. This result has found several uses in dilation, interpolation theory, and systems theory, and, motivated by control theory, has been strengthened by Fois and Tannenbaum (see [5] and the references therein). Also, a number of papers have been devoted to new proofs for Parrott's theorem and the strong Parrott's theorem (see [2] and [14], for example). Note that a projection is a idempotent contraction. So it is interesting to ask when a partial operator matrix can be completed to a projection. In this note we consider this problem in the case of 2 by 2 matrix and get necessary and sufficient conditions for the partially specified 2 by 2 operator matrices

$$
\begin{pmatrix}
? & C \\
C^* & B
\end{pmatrix}, \quad \begin{pmatrix}
? & C \\
C^* & ?
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
A & ? \\
? & B
\end{pmatrix}
$$

to have projection completions. Furthermore, all such projection completions, if any exist, are characterized completely by parametric representation. In [6] and [13] Halmos and Sebestyen discussed the projection extension problem for suboperators, which is equivalent to the projection completion problem for the partial matrix of the form

$$
\begin{pmatrix}
A & ? \\
C^* & ?
\end{pmatrix}.
$$

They gave necessary and the sufficient conditions which assure the existence of projection completion without describing all such completions.

We first recall some terminology and notation. Let $T \in \mathcal{B}(\mathcal{H})$. By ker $T$, $R(T)$, $\sigma_p(T)$, and Lat $T$ we shall denoted the null space, the range, the point spectrum, and the lattice of all invariant subspaces of $T$, respectively. $T$ is called a projection if $T$ is self-adjoint and idempotent, i.e., $T^* = T$ and $T^2 = T$. We shall also write $|T| = (T^*T)^{1/2}$. For a linear manifold $\mathcal{M} \subseteq \mathcal{H}$, $P_{\mathcal{M}}$ will denote the projection onto $\mathcal{M}$ along $\mathcal{M}^\perp$, where $\mathcal{M}$ is the closure of $\mathcal{M}$, and $I_{\mathcal{H}}$ stands for the identity on $\mathcal{H}$. The operator $V \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ is said to
be a partial isometry if \( \|Vx\| = \|x\| \) for all \( x \in (\ker V)^\perp \). If \( V \) is a partial isometry, \( (\ker V)^\perp \) and \( R(V) \) are called the initial space and final space of \( V \), and denoted by \( \text{ini} \, V \) and \( \text{fin} \, V \), respectively.

2. RESULTS AND PROOFS

Let \( A \in \mathcal{B}(\mathcal{H}), B \in \mathcal{B}(\mathcal{K}), C \in \mathcal{B}(\mathcal{K},\mathcal{H}), \) and \( D \in \mathcal{B}(\mathcal{K},\mathcal{H}) \). It is clear that the operator

\[
\begin{pmatrix}
A & C \\
D & B
\end{pmatrix}
\]
on \( \mathcal{K} \oplus \mathcal{H} \)

is a projection if and only if \( A \) and \( B \) are positive, \( D = C^* \), and the following equations are satisfied:

\[
\begin{align*}
A^2 + CC^* &= A, \\
B^2 + C^*C &= B, \\
AC + CB &= C.
\end{align*}
\]

Notice that in this case we must have \( \|A\| \leq 1, \|B\| \leq 1, \) and \( \|C\| \leq \frac{1}{2} \).

In the sequel we always write \( C = V |C| \), the polar decomposition of \( C \) with \( \text{ini} \, V = R(C^*) \) and \( \text{fin} \, V = R(C) \).

Our first result concerns the projection completions of the operator partial matrices of the form \( \begin{pmatrix} ? & C \\ C^* & B \end{pmatrix} \) with \( C \) and \( B \) given.

**Theorem 1.** Let \( B \in \mathcal{B}(\mathcal{K}) \) and \( C \in \mathcal{B}(\mathcal{K},\mathcal{H}) \) be given. Then

\[
\begin{pmatrix} ? & C \\ C^* & B \end{pmatrix}
\]

has a projection completion if and only if \( B^* = B \) and \( B^2 + C^*C = B \).

Furthermore, the set of all projection completions for \( \begin{pmatrix} ? & C \\ C^* & B \end{pmatrix} \) is exactly the set

\[
\left\{ \begin{pmatrix} P_{\mathcal{K}} + V(I_{\mathcal{K}} - B)V^* & C \\
C^* & B \end{pmatrix} : \mathcal{M} \text{ is the linear subspace in } \ker C^* \right\}.
\]
proof. If there exists an operator $X \in \mathcal{B}(\mathcal{H})$ such that the completion

$$\begin{pmatrix} X & C \\ C^* & B \end{pmatrix}$$

is a projection, it is clear that $B$ and $C$ satisfy the equation $B^2 + C^*C = B$ and $B \succ 0$ by the discussion at the beginning of this section.

Conversely, assume that $B^* = B$ and $B^2 + C^*C = B$. This implies that $B \succ 0$ and $\ker C$ reduce $B$ to a projection, that is, $\ker C \in \text{Lat} B$ and $B|_{\ker C}$ is a projection. Now, for any linear subspace $\mathcal{M} \subseteq \ker C^*$, let $X = P_\mathcal{M} + V(I_{\mathcal{H}} - B)V^*$; we shall prove that $\begin{pmatrix} X & C \\ C^* & B \end{pmatrix}$ is a projection. Actually, since $X$ is self-adjoint and $P_\mathcal{M}V = 0$, we have

$$X^2 + CC^* = P_\mathcal{M} + [V(I_{\mathcal{H}} - B)V^*]^2 + VC^*CV^*$$

$$= P_\mathcal{M} + V(I_{\mathcal{H}} - 2B + B^2 + C^*C)V^* = X$$

and

$$XC + CB = V(I_{\mathcal{H}} - B)V^*C + CB = C - VBV^*C + CB$$

$$= C - VB|C| + CB = C - V|C|B + CB = C.$$

These together with the hypothesis mean that the equations (1)–(3) hold, and hence $\begin{pmatrix} X & C \\ C^* & B \end{pmatrix}$ is a projection, as desired.

Finally, to prove the last assertion of the theorem, suppose that $X \in \mathcal{B}(\mathcal{H})$ is a projection completion. We have to show that $X$ has the form of $P_\mathcal{M} + V(I_{\mathcal{H}} - B)V^*$. Since $XCC^* = CC^*X$ by (1), $\ker C^*$ reduce $X$ to a projection. So there is a subspace $\mathcal{M} \subseteq \ker C^*$ such that $P_{\ker C^*}X = XP_{\ker C^*} = P_\mathcal{M}$. By (3), we have

$$XC = C(I_{\mathcal{H}} - B) = V|C|(I_{\mathcal{H}} - B)$$

$$= V(I_{\mathcal{H}} - B)|C| = V(I_{\mathcal{H}} - B)V^*C.$$

So

$$X - V(I_{\mathcal{H}} - B)V^* = 0,$$
which implies that

\[ p_{R(C)}X - V(I_x - B)V^* = 0. \]

Therefore,

\[ X = p_{kerC^*}X + p_{R(C)}X = p_{kerC}X + V(I_x - B)V^*. \]

The proof is finished.

**COROLLARY 2.** There exists a unique projection completion for

\[
\begin{pmatrix}
? & C \\
C^* & B
\end{pmatrix}
\]

if and only if \( B \) is self-adjoint, \( B^2 + C^*C = B \), and \( C \) has dense range. Moreover, this unique projection completion is

\[
\begin{pmatrix}
I_x - VBV^* & C \\
C^* & B
\end{pmatrix}
\]

**Proof.** Obvious.

Before stating the next theorem, we need some notation.

Let \( \mathcal{A} \) be a maximal abelian self-adjoint algebra (m.a.s.a.) containing \( |C| = (C^*C)^{1/2} \). It is well known (see [12], for example) that there exists a finite measure space \( (\Omega, \mu) \) and a unitary operator \( U : \mathcal{H} \to L^2(\Omega, \mu) \) such that \( \mathcal{A} = \{ U^*M_\phi U : \phi \in L^\omega(\Omega, \mu) \} \), where \( M_\phi \) denotes the multiplication operator by \( \phi \) on \( L^2(\Omega, \mu) \). Now there is a \( \phi_0 \in L^\omega(\Omega, \mu) \) such that \( |C| = U^*M_{\phi_0}U \), since \( |C| \in \mathcal{A} \). If \( \|C\| \leq \frac{1}{2} \), then \( 0 \leq \phi_0 \leq \frac{1}{2} \mu \)-a.e. For any \( \mu \)-measurable subset \( \Delta \subseteq \Omega \), let \( \phi_\Delta \in L^\omega(\Omega, \mu) \) be defined as

\[
\phi_\Delta(\lambda) = \begin{cases} 
\frac{1}{2} \left[ 1 + \sqrt{1 - 4\phi_0^2(\lambda)} \right], & \lambda \in \Delta, \\
\frac{1}{2} \left[ 1 - \sqrt{1 - 4\phi_0^2(\lambda)} \right], & \lambda \notin \Delta,
\end{cases}
\]

and let

\[ \mathcal{S}_C(\mathcal{A}) = \{ U^*M_{\phi_\Delta}U : \Delta \subseteq \Omega \text{ is } \mu \text{-measurable} \}. \]
THEOREM 3. Let $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. The partial specified operator matrix
\[
\begin{pmatrix}
? & C \\
C^* & ?
\end{pmatrix}
\]
has a projection completion if and only if $\|C\| \leq \frac{1}{2}$.

Moreover, a completion
\[
\begin{pmatrix}
X & C \\
C^* & Y
\end{pmatrix}
\]
is a projection completion if and only if there exists a m.a.s.a. containing $|C|$ such that $Y \in \mathcal{S}_c(\mathcal{H})$ and $X = P_x + V(I_x - Y)V^*$ for some subspace $\mathcal{M} \subseteq \ker C^*$.

Proof. The necessity is clear. To prove the sufficiency, assume $\|C\| \leq \frac{1}{2}$; then by Theorem 1, it suffices to show that $Y$ is a self-adjoint solution of the equation
\[
Y^2 + C^*C = Y
\]
if and only if there is a m.a.s.a. $\mathcal{H}$ containing $|C|$ such that $Y \in \mathcal{S}_c(\mathcal{H})$.

It is easily seen that every element in $\mathcal{S}_c(\mathcal{H})$ is a self-adjoint solution to (4). Conversely, if $Y$ is a self-adjoint solution to (4), then $Y$ commutes with $|C|$. Let $\mathcal{H}$ be any m.a.s.a. containing both $|C|$ and $Y$. By the argument previous to Theorem 3, there exists a finite measure space $(\Omega, \mu)$, a unitary operator $U$ from $\mathcal{H}$ onto $L^2(\Omega, \mu)$, and functions $\phi, \phi_0 \in L^2(\Omega, \mu)$ such that $|C| = U^*M_{\phi_0}U$ and $Y = U^*M_{\phi}U$. Thus we have
\[
U^*(M_{\phi_0^2 + \phi_{\phi}^2})U = Y^2 - Y + C^*C = 0,
\]
and therefore,
\[
\phi^2(\lambda) - \phi(\lambda) + \phi_0^2(\lambda) = 0 \quad \mu\text{-a.e.},
\]
\[
\phi(\lambda) = \frac{1}{2} \left[ 1 \pm \sqrt{1 - 4\phi_0^2(\lambda)} \right] \quad \mu\text{-a.e.}
\]
Let $\Delta = \{ \lambda : \phi(\lambda) \geq \frac{1}{2} \} \subseteq \Omega$. Then it is obvious that $\Delta$ is $\mu$-measurable and $\phi = \phi_\Delta$. So $Y = U^*M_{\phi_\Delta}U \in \mathcal{S}_c(\mathcal{H})$. This completes the proof. \qed
In the case that \( \mathcal{H} \) is finite dimensional, the representation may be made more clear. Let \( \|C\| = \sum_{i=1}^{k} \lambda_i E_i \) be the spectral decomposition of \( \|C\| \), and write

\[
\mathcal{S}_C = \left\{ Y \in \mathcal{B}(\mathcal{H}) : \exists \text{ projection } F_i \leq E_i, \; i = 1, \ldots, k \text{ such that } \right. \\
\left. Y = \sum_{i=1}^{k} \left[ \frac{1}{2} \left( 1 + \sqrt{1 - 4 \lambda_i^2} \right) F_i + \frac{1}{2} \left( 1 - \sqrt{1 - 4 \lambda_i^2} \right) \left( E_i - F_i \right) \right] \right\}.
\]

**Corollary 4.** Let \( C \in \mathcal{B}(\mathcal{H}, \mathcal{H}) \) with \( \dim \mathcal{H} < \infty \). Then

\[
\begin{pmatrix} ? & C \\ C^* & ? \end{pmatrix}
\]

has a projection completion if and only if the norm of \( C \) is not greater than \( \frac{1}{2} \).

Moreover

\[
\begin{pmatrix} X & C \\ C^* & Y \end{pmatrix}
\]

is a projection completion if and only if \( Y \in \mathcal{S}_C \) and \( X = P_\mathcal{M} + V(I_\mathcal{H} - Y)V^* \) for some subspace \( \mathcal{M} \subseteq \ker C^* \).

**Proof.** Obvious from Theorem 3 and the spectral decomposition of \( \|C\| \).

**Theorem 5.** Let \( A \in \mathcal{B}(\mathcal{H}) \) and \( B \in \mathcal{B}(\mathcal{H}) \). Then the following statements are equivalent:

(i) \( \begin{pmatrix} A & ? \\ ? & B \end{pmatrix} \) has a projection completion.

(ii) \( A \) and \( B \) are positive contractions, and there exists a partial isometry \( V \in \mathcal{B}(\mathcal{H}, \mathcal{H}) \) with \( \operatorname{ini} V = R(B - B^2) \) and \( \operatorname{fin} V = R(A - A^2) \) such that \( AV + VB = V \).

(iii) \( B \) is a positive contraction, and there exists a partial isometry \( V \in \mathcal{B}(\mathcal{H}, \mathcal{H}) \) with \( \operatorname{ini} V = R(B - B^2) \) and a subspace \( \mathcal{M} \subseteq \ker V^* \) such that \( A = P_\mathcal{M} + V(I_\mathcal{H} - B)V^* \).

(iv) \( A \) is a positive contraction, and there exists a partial isometry \( V \in \mathcal{B}(\mathcal{H}, \mathcal{H}) \) with \( \operatorname{fin} V = R(A - A^2) \) and a subspace \( \mathcal{N} \subseteq \ker V \) such that \( B = P_\mathcal{N} + V^*(I_\mathcal{H} - A)V \).
Moreover, under case (ii), (iii), or (iv), $X \in \mathcal{B}(\mathcal{H})$ is an operator such that

$$
\begin{pmatrix}
    A & X \\
    X^* & B
\end{pmatrix}
$$

is a projection completion if and only if there exists a unitary operator $U$ in $\mathcal{B}(\mathcal{H})$ ($W \in \mathcal{B}(\mathcal{H})$) commuting with $A$ (with $B$) such that $X = UV(B - B^2)^{1/2} = (A - A^2)^{1/2}V^*W$.

**Proof.** (i) ⇒ (ii): Let $C \in \mathcal{B}(\mathcal{H})$ be such that

$$
\begin{pmatrix}
    A & C \\
    C^* & B
\end{pmatrix}
$$

is a projection. Then $A$, $B$, and $C$ satisfy the equations (1)–(3). $A$ and $B$ are obviously positive and contractive. By (2), there exists a partial isometry $V \in \mathcal{B}(\mathcal{H})$ with $\text{ini} V = R(B - B^2)$ and $\text{fin} V = R(C)$ such that $C = V(R - R^2)^{1/2}$ (the polar decomposition). But $CC^* = A - A^2$ by (1), so we must have $\text{fin} V = R(A - A^2)$. Now by (3),

$$AV(B - B^2)^{1/2} + V(B - B^2)^{1/2}B = V(B - B^3)^{1/2},$$

and this will imply that

$$AV + VB = V,$$

since $\ker V = R(B - B^2)^\perp \in \text{Lat } B$.

(ii) ⇒ (iii): Assume that (ii) holds true. Then $R(V)$ is a reductive subspace of $A$, since $A$ is self-adjoint and $R(V)$ is invariant under $A$. So we have

$$A|_{R(V)} = V(I_{\mathcal{H}} - B)V^*|_{R(V)}$$

and

$$A|_{R(V)^\perp} = (A|_{R(V)^\perp})^2.$$ 

Notice that $V(I_{\mathcal{H}} - B)V^*|_{R(V)^\perp} = 0$. Therefore, there must exist a subspace $\mathcal{M} \subseteq R(V)^\perp = \ker V^*$ such that

$$A = P_{\mathcal{M}} + V(I_{\mathcal{H}} - B)V^*,$$

that is, (iii) is true.
(iii) $\Rightarrow$ (i): Suppose that (iii) holds. Then $B - B^2 \geq 0$. Let $X = V(B - B^2)^{1/2}$. Because $\text{ini} V = R(B - B^2) = R(B - B^2)^{1/2}$, we have

$$B^2 - B + X^*X = B^2 - B + (B - B^2)^{1/2}V^*V(B - B^2)^{1/2}$$

$$= B^2 - B + P_{R(B - B^2)}(B - B^2) = 0,$$

$$A^2 - A + XX^* = P_{\mathcal{F}} + V(I_{\mathcal{F}} - B)V^*V(I_{\mathcal{F}} - B)V$$

$$- P_{\mathcal{F}} - V(I_{\mathcal{F}} - B)V^* + V(B - B^2)V^*$$

$$= V[(I_{\mathcal{F}} - B)^2 - (I_{\mathcal{F}} - B) + B - B^2]V^* = 0,$$

and

$$AX + XB = V(I_{\mathcal{F}} - B)V^*V(B - B^2)^{1/2} + V(B - B^2)^{1/2}B$$

$$= V(B - B^2)^{1/2} - VB(B - B^2)^{1/2} + V(B - B^2)^{1/2}B$$

$$= V(B - B^2)^{1/2} = X.$$

Therefore, by (1)–(3),

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$$

is a projection completion of

$$\begin{pmatrix} A & \gamma \\ \rho & B \end{pmatrix}.$$

(i) $\Rightarrow$ (iv) is obvious, since (iv) is just (iii) with the roles of $A$ and $B$ interchanged.

If $U \in \mathcal{B}(\mathcal{H})$ is a unitary operator commuting with $A$, let $Y = UV(B - B^2)^{1/2}$. Then a straightforward computation shows that

$$\begin{pmatrix} A & Y \\ Y^* & B \end{pmatrix}$$

is also a projection completion for

$$\begin{pmatrix} A & \gamma \\ \rho & B \end{pmatrix}.$$ The last assertion of the theorem says that each projection completion has exactly this form. In fact, if

$$\begin{pmatrix} A & Y \\ Y^* & B \end{pmatrix}$$

is also a projection completion for

$$\begin{pmatrix} A & \gamma \\ \rho & B \end{pmatrix}.$$
is an arbitrary projection completion of \( \begin{pmatrix} A & \gamma \\ \delta & B \end{pmatrix} \), then by virtue of Equations (1)-(3), \( |Y| = (B - B^2)^{1/2} \). So there exists a unique partial isometry \( V_1 \) with \( \text{ini} V_1 = R(B - B^2) \) such that \( Y = V_1(B - B^2)^{1/2} \), and there is a subspace \( \mathcal{H}_1 \) in \( \ker V_1^* \) such that \( A = P_{\mathcal{H}_1} + V_1(I_{\mathcal{X}} - B)V_1^* \).

Denote by \( \mathcal{N} \) and \( \mathcal{N}_1 \) the final spaces of \( V \) and \( V_1 \), respectively. According to the space decompositions \( \mathcal{H} = \mathcal{N} \oplus \mathcal{N}_1 = \mathcal{N}_1 \oplus \mathcal{N}_1^\perp \), \( \mathcal{H} = R(B - B^2) \oplus \ker(B - B^2) \), we have

\[
B = \begin{pmatrix} B_1 & 0 \\ 0 & P_1 \end{pmatrix}
\]

with \( P_1 \) a projection.

\[
V = \mathcal{N} \begin{pmatrix} V_0 & 0 \\ 0 & 0 \end{pmatrix} R(B - B^2) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \ker(B - B^2),
V_1 = \mathcal{N}_1 \begin{pmatrix} V_{10} & 0 \\ 0 & 0 \end{pmatrix} R(B - B^2) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \ker(B - B^2),
\]

and

\[
A = \begin{pmatrix} V_0(I - B_1)V_0^* & 0 \\ 0 & P_{\mathcal{H}_1} \end{pmatrix} \mathcal{N} = \begin{pmatrix} V_{10}(I - B)V_{10}^* & 0 \\ 0 & P_{\mathcal{H}_1} \end{pmatrix} \mathcal{N}_1.
\]

Since \( 0, 1 \notin \sigma_p(V_0(I - B_1)V_0^*) \), we have \( \mathcal{N} = \text{span}(\ker A, \ker(I_{\mathcal{X}} - A)) \). In the same way, \( \mathcal{N}_1^\perp = \text{span}(\ker A, \ker(I_{\mathcal{X}} - A)) \), too. So, \( \mathcal{N}_1 = \mathcal{N} \) and hence \( P_{\mathcal{H}_1} = P_{\mathcal{X}} \). The linear operator \( U_1 : \mathcal{N} \to \mathcal{N} \) defined by \( U_1V_0x = V_{10}x \), for every \( x \) in \( \mathcal{N} \), is clearly unitary. Let

\[
U = \begin{pmatrix} U_1 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{N}.
\]

\( U \) is unitary on \( \mathcal{H} \),

\[
UV = \begin{pmatrix} U_1V_0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} V_{10} & 0 \\ 0 & 0 \end{pmatrix} = V_1.
\]
and

\[
UAU^* = \begin{pmatrix} U_1 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} V_0(I - B_1)V_0^* & 0 \\ 0 & P_{\mathcal{I}} \end{pmatrix} \begin{pmatrix} U_1^* & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} U_1V_0(I - B_1)V_0^*U_1^* & 0 \\ 0 & P_{\mathcal{I}} \end{pmatrix} = \begin{pmatrix} V_{10}(I - B_1)V_{10}^* & 0 \\ 0 & P_{\mathcal{I}} \end{pmatrix} = A.
\]

Hence, \( UA = AU \) and \( Y = UV(B - B^2)^{1/2} \).

Similarly we can check that \( Y = (A - A^2)^{1/2}V^*W \) for some unitary operator \( W \in \mathcal{B}(\mathcal{H}) \) which commutes with \( B \), finishing the proof. \( \blacksquare \)

**Remark.** It is now trivial to check that

\[
\begin{pmatrix} ? & ? \\ ? & B \end{pmatrix},
\]

the only case remaining for the partially specified 2 by 2 operator matrices, has a projection completion if and only if \( 0 \leq B \leq I_\mathcal{F} \), and the set of all such completions consists of those operators of the form

\[
\begin{pmatrix} P_{\mathcal{I}} + V(I_\mathcal{F} - B)V^* & V(B - B^2)^{1/2} \\ (B - B^2)^{1/2}V^* & B \end{pmatrix},
\]

where \( V \) runs over all partial isometries with initial space \( R(B - B^2) \) and \( \mathcal{I} \) subspaces in \( \ker V^* \).

This work was finished during the period when the author was visiting the University of Toronto, and part of it was done when the author was visiting the Academia Sinica. He wishes to give thanks to B. Li, M.-D. Choi, P. Rosenthal, and C. Davis for hospitality, and to P. Y. Wu for kindly pointing out [6] and [13] to him. The author also thanks the referee for this suggestion concerning Theorem 5.
REFERENCES


Received 9 May 1994; final manuscript accepted 14 November 1994