Non-separating 2-factors of an even-regular graph

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Dedicated to Professor Hikoe Enomoto on his sixtieth birthday.

Abstract

For a 2-factor $F$ of a connected graph $G$, we consider $G - F$, which is the graph obtained from $G$ by removing all the edges of $F$. If $G - F$ is connected, $F$ is said to be a non-separating 2-factor. In this paper we study a sufficient condition for a $2r$-regular connected graph $G$ to have such a 2-factor. As a result, we show that a $2r$-regular connected graph $G$ has a non-separating 2-factor whenever the number of vertices of $G$ does not exceed $2r^2 + r$.

Keywords: 2-factor; Non-separating; Even-regular graph

1. Introduction

Let $G = (V(G), E(G))$ be a graph with set of vertices $V(G)$ and set of edges $E(G)$. All graphs in this paper are finite and simple. We sometimes write simply $|G|$ for the number of vertices $|V(G)|$. The neighbourhood of $x \in V(G)$ is denoted by $N_G(x)$ and the degree of $x$ by $d_G(x) = |N_G(x)|$. If $d_G(x)$ is a constant positive integer $k$ independent of the choice of $x \in V(G)$, $G$ is called a regular graph or $k$-regular; moreover, if $d_G(x)$ is a constant positive even integer, $G$ is called even-regular. A 2-factor of $G$ is a 2-regular spanning subgraph of $G$. For a subgraph $H$ of $G$, the graph such that the set of vertices is $V(G)$ and that of edges is $E(G) \setminus E(H)$ is also a subgraph of $G$. We write simply $G - H$ for this resultant graph. The subgraph of $G$ induced by a set of vertices $S \subset V(G)$ is denoted by $\langle S \rangle_G$. We let $N_G(S) = \{y \in V(G) \setminus S \mid x \in S \text{ and } xy \in E(G)\}$ denote the neighbourhood of a set of vertices $S$ in $G$. Terminology and notation not defined here can be found in [1].

There exist many kinds of sufficient conditions for a graph to have a 2-factor. The following is a famous and classical result:

\textbf{Theorem 1} (Petersen [3]). A graph $G$ is 2-factorable if and only if $G$ is $2r$-regular for some positive integer $r$. 

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Here $G$ is called 2-factorable if a family of edge-disjoint 2-factors of $G$ covers the set of edges of $G$. In general, a connected and even-regular graph $G$ has many types of 2-factors, but we do not know whether there exists a 2-factor of $G$ such that $G - F$ is connected; that is, in general, it holds that $\omega(G - F) \geq 1$, where $\omega(H)$ is the number of components of $H$.

In this paper, we study a sufficient condition for an even-regular connected graph $G$ to have a 2-factor $F$ satisfying $G - F$ is connected; we call such a 2-factor of $G$ a ‘Non-Seperating 2-Factor’ (NS2F). Let $r$ be an integer with $r \geq 2$ and $G$ a $2r$-regular graph. Our object is to determine a function $f$ on $r$, which is optimal in some sense, satisfying the following: a $2r$-regular connected graph $G$ has a NS2F if $|G| \leq f(r)$. We immediately have $f_1(r) = 4r$ as such a function $f$ by combining Petersen’s theorem and also a classical and famous result by Dirac [2]: a graph $G$ has a Hamilton cycle if $d_G(x) \geq |G|/2$ for every $x \in V(G)$. Of course, since $|G| \leq 4r = 2d_G(x)$ for any $x \in V(G)$, $G$ has a Hamilton cycle or, equivalently, a connected 2-factor $C$. We consider $G - C$ as a $(2r - 1)$-regular subgraph of $G$, then $G - C$ has a 2-factor $F$. This $F$ becomes a NS2F since $G - F$ has the connected 2-factor $C$.

Another function $f_2(r) = r^2 + 4$ was suggested in [6]. Let us sketch its proof. If $r = 2$, that is, $f_2(2) = 8$, then every 4-regular connected graph $G$ with at most 8 vertices has a Hamilton cycle; thus $G$ has a NS2F. Let us assume that every $2k$-regular connected graph $G$ with $|G| \leq f_2(k)$ has a NS2F for every $k = 2, 3, \ldots, r - 1$. Suppose $\omega(G - F) \geq 2$ for a 2-factor $F$ of a $2r$-regular connected graph $G$ with $|G| \leq f_2(r)$. Then every component $M_i$ of $G - F$ is $(2r - 1)$-regular, $|M_i| \geq 2r - 1$ and it holds that $|M_i| \leq f_2(r) - (2r - 1) = f_2(r - 1)$. By induction, there exists a NS2F, say $F_i$, of $M_i$ for every $i$. Putting $F' = \cup_i F_i$, we can see $F'$ is the desired NS2F of $G$.

We do not believe the above $f_2$ is optimal but it was pointed out in [7] that the optimal upper bound $f(r) = O(r^2)$. Our main theorem in this paper is as follows:

**Theorem 2.** Let $G$ be a $2r$-regular connected graph, where $r$ is an integer with $r \geq 2$. If $|G| \leq f(r)$, then $G$ has a non-separating 2-factor (NS2F), where $f(r) = 2r^2 + r$.

The function given above is optimal in the following sense: Let $K_{2r + 1}^-$ be the graph obtained from the complete graph $K_{2r + 1}$ with $(2r + 1)$ vertices by removing an edge $e$. Let $M_i$ be graph-isomorphic to $K_{2r + 1}^-$ for $i = 1, 2, \ldots, r$; we let $x_i$ and $y_i$ denote the non-adjacent vertices in $M_i$. We set a graph $G$ (Fig. 1.1) as $V(G) = \{v\} \cup (\cup_{i=1}^r V(M_i))$ and $E(G) = \{vx_i, vy_i | i = 1, 2, \ldots, r\} \cup (\cup_{i=1}^r E(M_i))$. Then, for every $r \geq 2$, $G$ is 2r-regular, $|G| = f(r) + 1 = 2r^2 + r + 1$ and $G$ has no NS2F.

This work is a by-product obtained in our research on the relationship between the covering structure and the properties of spectra of the discrete Laplacians [4].

2. Preliminary

We give the proof of Theorem 2 by induction on $r$. Let us put $r = 2$ at the first stage of induction. Recall $f(r) = 2r^2 + r$, hence $f(2) = 10$.

**Lemma 2.1.** Every 4-regular connected graph $G$ with $|G| \leq 10$ has a NS2F.

To prove the above, we use the following result by Bill Jackson.

**Theorem 2.2** (Jackson [5]). Let $G$ be a 2-connected $k$-regular graph with $|G| \leq 3k$. Then $G$ has a Hamilton cycle.
Proof of Lemma 2.1. It is easy to check that every 4-regular connected graph with at most 10 vertices is 2-connected. In fact, if there exists a vertex \( x_0 \in V(G) \) such that \( \omega(G-x_0) \geq 2 \), then we can find a component \( M \) such that \( |M| \leq 4 \), where \( G-x_0 = (V(G) \setminus \{x_0\})_G \). Here, for any \( x \in V(M) \), it holds that \( 3 \leq d_M(x) \leq |M| - 1 \). Thus it must hold that \( |M| = 4 \) and that \( d_M(x) = 3 \). This implies that \( N_G(x_0) = V(M) \) since \( G \) is 4-regular, which contradicts the fact that \( x_0 \) is a cut vertex of \( G \). Then Theorem 2.2 yields that \( G \) has a Hamilton cycle \( H \). The subgraph \( G-H \) is 2-regular, that is, it is a 2-factor of \( G \) and moreover it is a NS2F of \( G \). ■

Now let us set the following hypothesis for a fixed \( r \geq 3 \):

Hypothesis 2.3. For every \( k = 2, 3, \ldots, r-1 \), every \( 2k \)-regular connected graph \( G \) with \( |G| \leq f(k) \) has a NS2F.

Under Hypothesis 2.3, we only have to show that every \( 2r \)-regular connected graph \( G \) with \( |G| \leq f(r) \) has no NS2F. We will derive a contradiction when assuming that there exists a \( 2r \)-regular connected graph \( G \) with \( |G| \leq f(r) \) having no NS2F. So we suppose the following assumption under Hypothesis 2.3.

Assumption 2.4. Let \( G \) be a \( 2r \)-regular connected graph such that \( |G| \leq f(r) \) and \( G \) has no NS2F.

In Section 3, by analyzing the properties of \( G \) in the above, we show the non-existence of such a graph \( G \).

3. Graph structure under Hypothesis 2.3 and Assumption 2.4

First, let us give our notation.

Notation 3.1. For a subgraph \( F \) of \( G \), let \( G - F = \bigcup_{i=1}^{m} G_i \), where \( m = \omega(G-F) \), every \( G_i \) is a component of \( G - F \) and \( |G_i| \geq |G_j| \) for every \( 1 \leq i < j \leq m \).

Claim 3.2. For any 2-factor \( F \) of \( G \), it holds that \( G - F = \bigcup_{i=1}^{\ell} G_i \) such that \( 2 \leq \ell \leq 3 \) and \( |G_i| > f(r-1) \geq |G_j| \geq 2r-1 \) for \( j \geq 2 \). In particular, if \( \ell = 3 \), then we have \( |G_1| = f(r-1) + 1 \), \( G_2 \cong G_3 \cong K_{2r-1} \) and \( |G| = f(r) = 2r^2 + r \).

Proof. For a 2-factor \( F \), we set \( G - F = \bigcup_{i=1}^{\ell} G_i \) following Notation 3.1. By Assumption 2.4, it must hold that \( \ell \geq 2 \) for any 2-factor \( F \). Every component \( G_i \) of \( G - F \) is \((r-1)\)-regular, then \( |G_i| \geq 2r-1 \) for every \( i \).

Firstly we show \( |G_1| > f(r-1) \). Assume that \( |G_1| \leq f(r-1) \). Then \( |G_1| \leq f(r-1) \) for every \( i \) and it follows from Hypothesis 2.3 that \( G_i \) has a NS2F \( F_i \) for every \( i \). Here we put \( F' = \bigcup_{i=1}^{\ell} F_i \), then \( F' \) is a 2-factor of \( G \); moreover \( G - F' \) is connected. Thus \( G \) has a NS2F \( F' \), which contradicts Assumption 2.4. So it must hold that \( |G_1| > f(r-1) \).

Secondly we show \( |G_2| \leq f(r-1) \). Assume that \( |G_2| > f(r-1) \). We have \( |G_2| > f(r-1) = f(r) - 4r - 1 \) and then \( |G_1| \leq |G| - |G_2| < 4r - 1 \) since \( |G| \leq f(r) \). So \( |G_1| \) must satisfy \( f(r-1) < |G_1| \leq 4r - 2 \), but it contradicts \( r \geq 3 \). Thus it is shown that \( |G_2| \leq f(r-1) \).

Thirdly assume that \( \ell \geq 4 \). From the above, we have \( |G_1| > f(r-1) = f(r) - 4r - 1 \) and \( |G_2| \geq |G_3| \geq |G_4| \geq 2r - 1 \). Then

\[
f(r) \geq |G_1| + |G_2| + |G_3| + |G_4| \geq (f(r) - (4r - 1) + 1) + 3(2r - 1) = f(r) + 2r - 1.
\]

This is a contradiction. Finally if \( \ell = 3 \),

\[
f(r) \geq |G| = |G_1| + |G_2| + |G_3| \geq (f(r) - (4r - 1) + 1) + |G_2| + |G_3|.
\]

Thus we have \( |G_2| + |G_3| \leq 2(2r - 1) \), which implies that \( |G_2| = |G_3| = 2r - 1 \), that is, \( G_2 \cong G_3 \cong K_{2r-1} \). On the other hand, it holds that \( |G_1| \leq f(r) - 2(2r - 1) = f(r-1) + 1 \), thereby \( |G_1| = f(r-1) + 1 \). ■

Claim 3.3. Assume that \( G \) has a 2-factor \( F \) such that \( G - F = G_1 \cup G_2 \cup G_3 \). Let \( F_2 \) and \( F_3 \) be 2-factors of \( G_2 \) and \( G_3 \), respectively. Then for any 2-factor \( F_1 \) of \( G_1 \), \( \omega(G - (F_1 \cup F_2 \cup F_3)) = 2 \).

Proof. By Claim 3.2, we know that \( G_2 \cong G_3 \cong K_{2r-1} \); any 2-factor of \( K_{2r-1} \) is a NS2F of \( K_{2r-1} \). Then \( F_2 \) and \( F_3 \) are NS2Fs of \( G_2 \) and \( G_3 \), respectively. Moreover, for any 2-factor \( F_1 \) of \( G_1 \), \( \tilde{F} = F_1 \cup F_2 \cup F_3 \) is another 2-factor of the \( 2r \)-regular connected graph \( G \). By Claim 3.2 again, it holds that \( 2 \leq \omega(G - \tilde{F}) \leq 3 \). Assume that \( \omega(G - \tilde{F}) = 3 \). Also by Claim 3.2, \( G - \tilde{F} = N_1 \cup N_2 \cup N_3 \), where \( |N_1| = |G| - 2(2r - 1) \) and \( N_2 \cong N_3 \cong K_{2r-1} \). Remark that there
exist vertices $x \in V(G_j)$, $y \in V(G) \setminus V(G_j)$ and an edge $xy \in F$ for $j = 2, 3$. So, if $V(G_j) \cap V(N_k) \neq \emptyset$ for $j = 2, 3$ and $k = 2, 3$, then $V(N_k) \supseteq V(G_j)$, which is a contradiction. Therefore we have $V(N_1) \supseteq V(G_2) \cup V(G_3)$. Let $S = V(N_1) \setminus (V(G_2) \cup V(G_3))$. By Claim 3.2, we have $|N_1| = f(r - 1) + 1$ and $|G_2| + |G_3| = 2(2r - 1) \leq f(r - 1)$. Then $S \neq \emptyset$. Remark that $N_{G-F-F}(S) = \emptyset$. First we put $r = 3$. Then $\langle S \rangle_{G-F-F}$ is 2-regular and $|S| \geq 2$. On the other hand, $|S| \leq |G| - 4(2 \cdot 3 - 1) \leq f(3) - 20 = 1$. This is impossible. Next we put $r \geq 4$. Then $\langle S \rangle_{G-F-F}$ is $(2r - 2)$-regular and $|S| \geq 2r - 3$. Moreover $|S| \leq |G| - 4(2r - 1) \leq f(r) - 4(2r - 1) < f(r - 2)$. By Hypothesis 2.3, every component of $\langle S \rangle_{G-F-F}$ has a NS2F, which implies that $G$ has a NS2F. This is a contradiction. Therefore we obtain that $\omega(G - \hat{F}) \neq 3$, so the proof is completed. ■

Now we pursue the following procedure.

**Procedure (Step 1).** Consider an arbitrary 2-factor $F^{(1)}$ of $G$ and put $G - F^{(1)} = G_1^{(1)} \cup M^{(1)}$, where $M^{(1)} = \bigcup_{j=2}^{3} G_j^{(1)}$ and $\ell_1 = \omega(G - F^{(1)})$. According to Notation 3.1, we have $|G_1^{(1)}| > f(r - 1) \geq |G_2^{(1)}| \geq |G_3^{(1)}| \geq 2r - 1$. Now focus on those 2-factors $F^{(1)}$ whose biggest component $G_1^{(1)}$ of $G - F^{(1)}$ has minimum order. Within this range, find and fix a 2-factor $F^{(1)}_{k}$ such that $\ell_1$ has minimum value.

**Procedure (Step 2).** Consider an arbitrary 2-factor $F^{(2)}$ of $G_1^{(1)}$, which is $(2r - 1)$-regular, and put another 2-factor $F^{(2)}$ of $G$ to be $F^{(2)} = F^{(2)}_{1} \cup (\bigcup_{j=2}^{3} F_j^{(2)})$, where $F^{(2)}_{1}$ is a fixed NS2F of $G_1^{(1)}$. Moreover put $G - F^{(2)} = G_1^{(2)} \cup M^{(2)}$, where $M^{(2)} = \bigcup_{j=2}^{3} G_j^{(2)}$ and $\ell_2 = \omega(G - F^{(2)})$. Now focus on those 2-factors $F^{(2)}_{1}$ such that the biggest component $G_1^{(2)}$ of $G - F^{(2)}$ has minimum order. Within this range, find and fix a 2-factor $F^{(2)}_{1}$ such that $\ell_2$ has minimum value.

**Procedure (Step k).** Pursuing this procedure at Step $k - 1$, we have 2-factors $F^{(1)}_{k-1}$, $F^{(2)}_{k-1}$, $\ldots$, $F^{(k-1)}_{k-1}$, and also have $M^{(1)}$, $M^{(2)}$, $\ldots$, $M^{(k-1)}$, where $G - F^{(i)} = G_1^{(1)} \cup M^{(i)}$, $M^{(i)} = \bigcup_{j=2}^{3} G_j^{(i)}$ and $\ell_i = \omega(G - F^{(i)})$ for every $i = 1, 2, \ldots, k - 1$. Consider an arbitrary 2-factor $F^{(k)}_{1}$ of

$$\hat{G}^{(k-1)}_1 = \langle V(G) \setminus \bigcup_{i=1}^{k-1} V(M^{(i)}) \rangle_{G} \setminus \bigcup_{i=1}^{k-1} F^{(i)},$$

which is $(2r - k + 1)$-regular. Here we put a 2-factor $F^{(k)}$ of $G$ to be $F^{(k)} = F^{(k)}_{1} \cup (\bigcup_{j=2}^{3} F_j^{(k)})$, where $F^{(k)}_{1}$ is the fixed NS2F of $G_1^{(k-1)}$ for $i = 2, \ldots, k - 1$ and $F^{(k)}_{1}$ is a fixed NS2F of $G^{(k-1)}$. Moreover put $G - F^{(k)} = G^{(k)}_1 \cup M^{(k)}$, where $M^{(k)} = \bigcup_{j=2}^{3} G_j^{(k)}$ and $\ell_k = \omega(G - F^{(k)})$. Now focus on those 2-factors $F^{(k)}_{1}$ such that the biggest component $G_1^{(k)}$ of $G - F^{(k)}$ has minimum order. Within this range, find and fix a 2-factor $F^{(k)}_{1}$ such that $\ell_k$ has minimum value.

This procedure defined inductively is terminated at Step $r$. In the following claim, the former statement asserts the existence of the 2-factors required at Step $k + 1$ of the Procedure whenever Step $k$ is completed, whereas the latter statement follows from the former one together with Claim 3.3 and the construction of the 2-factors $F^{(k)}$.

**Claim 3.4.** $V(M^{(i)}) \cap V(M^{(k)}) = \emptyset$ for $1 \leq i < k \leq r$. In addition, if there exists an integer $i$ such that $\ell_i = 3$, then $\ell_k = 2$ for every $k \neq i$.

**Proof.** First assume that $V(M^{(1)}) \cap V(M^{(2)}) \neq \emptyset$. If $\ell_1 = 3$, then $M^{(1)} = \bigcup_{j=2}^{3} G_j^{(1)}$ and $G^{(1)} \supseteq K_{2r-1}$. Moreover, by the construction of $F^{(2)}$ and Claim 3.3, we have $M^{(2)} = G_2^{(2)}$. Since $G^{(1)}$ is the complete graph with $2r - 1$ vertices, it must hold that any edge of the 2-factor $F^{(1)}$ incident to a vertex of $G^{(1)}$ is always incident to a vertex of $V(G^{(1)} \setminus V(G_2^{(1)}))$. Then $|N_{F^{(1)}}(V(G^{(1)} \setminus V(G_2^{(1)})))| \geq 2r - 1$. Thus, in $G - F^{(2)}$, $\langle V(G^{(1)}) \cup N_G(V(G^{(1)})) \rangle_{G-F^{(2)}}$ is connected and moreover $|\langle V(G^{(1)}) \cup N_G(V(G^{(1)})) \rangle_{G-F^{(2)}}| \geq 2(2r - 1)$. Now we may set $V(G^{(1)}) \cap V(G^{(2)}) \neq \emptyset$ from the assumption that $V(M^{(1)}) \cap V(M^{(2)}) \neq \emptyset$; then we have $|G^{(2)}| \geq 2(2r - 1)$. Here the fact that $|V(M^{(1)})| = 2(2r - 1)$ and the minimality of $|G^{(1)}|$ imply $|G^{(1)}| = 2(2r - 1)$. However the minimality of $\ell_1$ contradicts $\ell_1 = 3 > \ell_2 = 2$. Then $\ell_1 = 2$. Since $G_2^{(1)} = F^{(2)}$ is connected and $N_{F^{(1)}}(V(G^{(2)})) \neq \emptyset$, we have $|V(M^{(2)})| \geq |V(G^{(1)})| + 1$. This contradicts the minimality of $|G^{(1)}|$. Then we obtain that $V(M^{(1)}) \cap V(M^{(2)}) = \emptyset$. Next assume that $2 \leq k \leq r - 1$ is fixed and that $V(M^{(i)}) \cap V(M^{(j)}) = \emptyset$ for any $i, j$ such that $1 \leq i < j \leq k$. Here remark that the hypothesis of induction guarantees the existence of the 2-factor $F^{(k+1)}$ required at Step $k + 1$. Let us show that $V(M^{(i)}) \cap V(M^{(k+1)}) = \emptyset$ for any $1 \leq i \leq k$. If there exists a $1 \leq i \leq k$ such that $V(M^{(i)}) \cap V(M^{(k+1)}) \neq \emptyset$ with
\( \ell_i = 3 \), then, by Claim 3.3 and the construction of 2-factor \( F^{(k+1)} \) at Step \( k+1 \), \( G - F^{(k+1)} = G_1^{(k+1)} \cup G_2^{(k+1)} \), that is, \( \ell_{k+1} = 2 \) and \( M^{(k+1)} = G^{(k+1)} \). We may now assume that \( V(G_2^{(i)}) \cap V(G_2^{(k)}) \neq \emptyset \), where \( G - F^{(i)} = G_1^{(i)} \cup M^{(i)} \), \( M^{(i)} = \bigcup_{j=2}^{\ell} G_j^{(i)} \) and \( G_j^{(i)} \cong K_{2r-1} \) for \( j = 2, 3 \). By a similar argument as above, we have \( |G_2^{(k+1)}| = 2(2r-1) \) and \( \ell_{k+1} = 2 \), which contradicts the choice of the 2-factor required at Step \( i \). If there exists a \( 1 \leq i \leq k \) such that \( V(M^{(i)}) \cap V(M^{(k+1)}) \neq \emptyset \) with \( \ell_i = 2 \), then it must hold that \( |V(M^{(k+1)})| \geq |V(M^{(i)})| + 1 \), since \( M^{(i)} - F^{(k+1)} \) is connected and \( N_{F^{(i)}}(V(M^{(i)})) \neq \emptyset \). This contradicts the minimality of \( |G_1^{(i)}| \) at Step \( i \). Consequently the proof is completed. ■

In addition, we also have

**Claim 3.5.** For every \( i, k \) with \( i < k \), \( N_G(V(M^{(i)})) \cap V(M^{(k)}) = \emptyset \). In other words, \( xy \notin E(G) \) for any \( x \in V(M^{(i)}) \) and any \( y \in V(M^{(k)}) \).

**Proof.** Assume that there exist \( i, k, x, y \) such that \( xy \in E(G) \), where \( i < k \), \( x \in V(M^{(i)}) \) and \( y \in V(M^{(k)}) \). We use a similar argument to that in the proof of Claim 3.4 again. First let \( \ell_i = 3 \). Then \( \ell_k = 2 \), \( G - F^{(i)} = G_1^{(i)} \cup \bigcup_{j=2}^{\ell} G_j^{(i)} \), where \( G_j^{(i)} \cong K_{2r-1} \) for \( j = 2, 3 \). As it is seen, it holds that \( |N_{F^{(i)}}(G_j^{(i)})| \geq 2r - 1 \) and \( G_j^{(i)} - F^{(k)} \) is connected. The minimality of \( |G_j^{(i)}| \) at Step \( i \) implies that \( |V(M^{(k)})| = 2(2r-1) \), but \( \ell_k < \ell_i \), which contradicts the minimality of \( \ell_i \). Next let \( \ell_i = 2 \). Then we have \( |V(M^{(k)})| \geq |V(M^{(i)})| + 1 \) according to the connectedness of \( M^{(i)} - F^{(k)} \) and the fact that \( N_{F^{(i)}}(V(M^{(i)})) \neq \emptyset \). This contradicts the minimality of \( |G_1^{(i)}| \) at Step \( i \). ■

**Definition 3.6.** When Step \( r \) of the Procedure stated above is done, we define the set of vertices \( V^r \) as

\[
V^r = V(G) \setminus \bigcup_{j=1}^{r} V(M^{(j)}).
\]

For sets of vertices \( S, T \) such that \( S, T \subset V(G) \) and \( S \cap T = \emptyset \), we define the set of edges \( e_G(S, T) \) as

\[
e_G(S, T) = \{xy \in E(G) \mid x \in S \text{ and } y \in T \}.
\]

**Claim 3.7.** \( V^r \neq \emptyset \). \( e_G(V^r, V(M^{(i)})) \neq \emptyset \) and \( e_G(V^r, V(M^{(i)})) \subset F^{(i)} \) for every \( 1 \leq i \leq r \). Moreover, for any \( 1 \leq i < j \leq r \), it holds that \( e_G(V^r, V(M^{(i)})) \cap F^{(j)} = \emptyset \).

**Proof.** Considering Claims 3.4 and 3.5 together with the construction of \( F^{(i)} \), we can easily obtain the above. ■

**Claim 3.8.** \( \ell_i = 2 \) for each \( i = 1, 2, \ldots, r \).

**Proof.** Suppose \( \ell_i = 3 \) for some \( i \). Recall Claim 3.2: \( G - F^{(i)} = G_1^{(i)} \cup M^{(i)} \), where \( M^{(i)} = \bigcup_{j=2}^{\ell} G_j^{(i)} \) and \( G_j^{(i)} \cong K_{2r-1} \) for \( j = 2, 3 \). Suppose \( |V^r| = 1 \) and we set \( \{v\} = V^r \). Claim 3.7 says \( e_G(V(M^{(i)}), V^r) \subset F^{(i)} \), thus there exist \( z_1 \in V(M^{(i)}) \) and \( z_2 \in V(M^{(i)}) \) such that \( z_1 \neq z_2 \) and \( v z_1, v z_2 \in E(G) \); it holds that \( \{v z_1, v z_2\} \neq e_G(V(M^{(i)}), V^r) \). Since all the graphs \( G_2^{(i)} \) and \( G_3^{(i)} \) are isomorphic to \( K_{2r-1} \), every edge \( xy \in E^{(i)} \) joins \( x \in G_2^{(i)} \) and \( y \in G_3^{(i)} \), where \( E^{(i)} = F^{(i)} \cap E((V(M^{(i)}))_G) \). Here we have \( |E^{(i)}| = 4r - 3 \geq 9 \). If both \( z_1 \) and \( z_2 \) are in \( G_2^{(i)} \), then we set \( E^{(i)} = E(P_2) \cup E(H_2) \), where \( P_2 \) is a Hamilton path in \( G_2^{(i)} \) from \( z_1 \) to \( z_2 \) and \( H_2 \) is a Hamilton cycle in \( G_3^{(i)} \). Considering \( E^{(i)} = (F^{(i)} \setminus E^{(i)}) \cup E^{(i)} \), we can easily see that \( E^{(i)} \) is a 2-factor and that \( \omega(G - F^{(i)}) = 2 \), which contradicts the choice of \( F^{(i)} \) at Step \( i \) of the Procedure. The same argument is valid if \( z_1 \) and \( z_2 \) are in \( G_3^{(i)} \). Thus \( z_1 \in G_2^{(i)} \) and \( z_2 \in G_3^{(i)} \). Find and fix an edge \( w_1 \) \& \( w_2 \) such that \( w_1 \in G_2^{(i)} \) and \( w_2 \in G_3^{(i)} \). Now we set \( E^{(i)} = E(P_1) \cup \{w_1 w_2\} \cup E(P_2) \), where \( P_1 \) is a Hamilton path in \( G_2^{(i)} \) from \( z_1 \) to \( w_1 \) and \( P_2 \) is also a Hamilton path in \( G_3^{(i)} \) from \( w_2 \) to \( z_2 \). Considering \( E^{(i)} = (F^{(i)} \setminus E^{(i)}) \cup E^{(i)} \), we can easily see that \( E^{(i)} \) is a 2-factor and that \( \omega(G - F^{(i)}) = 2 \), which contradicts the choice of \( F^{(i)} \) at Step \( i \) of the Procedure. Therefore we obtain \( |V^r| \geq 2 \). Now recall that \( |V(M^{(i)})| = 2(2r-1) \) and \( |V(M^{(k)})| \geq 2r - 1 \) for \( k \neq i \). Then, by Claim 3.4,

\[
|G| = |V^r| + \sum_{k=1}^{r} |V(M^{(k)})| \geq 2 + 2(2r-1) + (r-1)(2r-1) = f(r) + 1.
\]

This contradicts Assumption 2.4, thus the proof is completed. ■
From here on, we assume that $M(k) = G_2(k)$ for every $k = 1, 2, \ldots, r$. Recall that

$$G - F(k) = G_1(k) \cup M(k),$$

where $|V(M(1))| \geq |V(M(2))| \geq \cdots \geq |V(M(r))| \geq 2r - 1$. Here $V(M(1)), V(M(2)), \ldots, V(M(r))$ and $V'$ are mutually disjoint non-empty sets and $V(G) = V' \cup \left( \bigcup_{k=1}^{r} V(M(k)) \right)$.

**Claim 3.9.** If $|V(M(k))| = 2r$ for some $k$, then $|N_G(V(M(k))) \cap V'| \geq r$; in particular, $|V'| \geq r$. Furthermore, if $|V(M(k))| = 2r - 1$ for some $k$, then $|N_G(V(M(k))) \cap V'| \geq 2r - 1$; in particular, $|V'| \geq 2r - 1$.

**Proof.** Let us recall the following: $M(k)$ is $2(r - 1)$-regular in $G - F(k)$, $e_G(V(M(i)), V(M(k))) = \emptyset$ for $i \neq k$ from Claim 3.5 and $e_G(V', V(M(k))) \subseteq F(k)$ from Claim 3.7. Assume that $|V(M(k))| = 2r$. Then there exists $z_k \in V'$ such that $xz_k \in E(k)$ for every $x \in V(M(k))$. This implies that $|e_G(V', V(M(k)))| \geq 2r$. Since $F(k)$ is a 2-factor of $G$, we have $|N_G(V(M(k))) \cap V'| \geq r$. Next assume that $|V(M(k))| = 2r - 1$; then $M(k)$ becomes the complete graph with $2r - 1$ vertices in $G - F(k)$. There exist two distinct vertices $z_k^1$ and $z_k^2$ such that $z_k^j \in V'$ and $xz_k^j \in E(k)$ for $j = 1, 2$ and for every $x \in V(M(k))$; $|e_G(V', V(M(k)))| \geq 2(2r - 1)$. Thus we have $|N_G(V(M(k))) \cap V'| \geq 2r - 1$. ■

Now we distinguish three cases.

**Case I:** $|V(M(r))| \geq 2r + 1$. Then we have

$$|G| = |V'| + \sum_{k=1}^{r} |V(M(k))| \geq 1 + r(2r + 1) = f(r) + 1,$$

which contradicts $|G| \leq f(r)$. ■

**Case II:** $|V(M(r))| = 2r$. Then we have $|V'| \geq r$ by Claim 3.9. If $|V(M(1))| \geq 2r + 1$, then we obtain

$$|G| = |V'| + \sum_{k=1}^{r} |V(M(k))| \geq r + 2r + 1 + (r - 1) \cdot 2r = f(r) + 1,$$

which contradicts $|G| \leq f(r)$. Thus we may assume that $|V(M(k))| = 2r$ for every $k = 1, 2, \ldots, r$. If $|V'| \geq r + 1$, then

$$|G| = |V'| + \sum_{k=1}^{r} |V(M(k))| \geq r + 1 + r \cdot 2r = f(r) + 1,$$

so we can restrict ourselves to the following:

$$|V'| = r \text{ and } |V(M(k))| = 2r \text{ for every } k = 1, 2, \ldots, r.$$

For some $k$, assume that there exists a vertex $x_k \in V(M(k))$ such that $N_G(x_k) \cap V(M(k)) \neq V(M(k)) \setminus \{x_k\}$. Since $d_{M(k)}(x_k) = 2r - 2$ and $|V(M(k))| = 2r$, it holds that $|e_G(V', \{x_k\})| = 2$ and then it must hold that $|e_G(V', V(M(k)))| \geq 2r + 1$. On the other hand, $|e_G(V', V(M(k)))| \leq 2r$ since $|V'| = r$, a contradiction. Therefore $|V(M(k))| \leq K_2r$ for every $k = 1, 2, \ldots, r$. Let $V' = \{z_1, z_2, \ldots, z_r\}$. By the above arguments, for every $k = 1, 2, \ldots, r$ and $l = 1, 2, \ldots, r$, there exist two distinct vertices $x_{k,l}$ and $y_{k,l}$ in $V(M(k))$ such that $x_{k,l}, z_l, z_l, z_l \in e_G(V', V(M(k))) \subseteq F(k)$. Now consider the vertices $x_{k,k}$ and $y_{k,k}$ for each $k$. Since $|V(M(k))| \geq K_2r$, it is obvious that there exists a Hamilton path from $x_{k,k}$ to $y_{k,k}$, say $P^k$, and $\langle V(M(k)) \rangle_G \cong P^k$ is connected. Let $C^k$ be a cycle such that $V(C^k) = V(P) \cup \{z_k\}$ and $E(C^k) = E(P^k) \cup \{x_{k,k}, z_k, z_k, z_k\}$. Moreover setting $\tilde{F} = \bigcup_{k=1}^{r} C^k$, that is, $V(\tilde{F}) = \bigcup_{k=1}^{r} V(C^k) = V(G)$ and $E(\tilde{F}) = \bigcup_{k=1}^{r} E(C^k)$, we can see that $\tilde{F}$ is a 2-factor of $G$. As it is seen, $\langle V(M(k)) \rangle_G \cong P^k$ is connected for every $k = 1, 2, \ldots, r$ and

$$N_G(V(M(k))) \cap N_G(V(M(l))) = V', \quad V' \setminus \{z_k, z_l\} \neq \emptyset$$

for $k \neq l$ and $k, l = 1, 2, \ldots, r$. Moreover it holds that $N_G(V(M(j)) \supset \{z_j, z_j\}$ for mutually distinct $j, k, l$. Thus $G - \tilde{F}$ is connected, which implies that $\tilde{F}$ is a NS2F of $G$. This is a contradiction. ■
Case III: $|V(M^{(r)})| = 2r - 1$. Then we have $|V'| \geq 2r - 1$ by Claim 3.9. If $\sum_{k=1}^{2} |V(M^{(k)})| \geq 4r$, then we have

$$|G| = |V'| + \sum_{k=1}^{r} |V(M^{(k)})| \geq 2r - 1 + 4r + (r - 2)(2r - 1) = f(r) + 1,$$

which contradicts $|G| \leq f(r)$. So we may assume that $\sum_{k=1}^{2} |V(M^{(k)})| \leq 4r - 1$. Here we distinguish two subcases.

Case III-(i) Assume that $|V(M^{(1)})| = 2r$ and $|V(M^{(k)})| = 2r - 1$ for each $k = 2, 3, \ldots, r$. If $|V'| \geq 2r$, then we have

$$|G| = |V'| + \sum_{k=1}^{r} |V(M^{(k)})| \geq 2r + 2r + (r - 1)(2r - 1) = f(r) + 1,$$

which is a contradiction. So $|V'| = 2r - 1$. On the other hand, we have $M^{(k)} \cong K_{2r-1}$ since $|V(M^{(k)})| = 2r - 1$ and $M^{(k)}$ is $2(r-1)$-regular for $k = 2, 3, \ldots, r$; it holds that

$$|e_G(V', V(M^{(k)}))| = 2(2r - 1).$$

By using this together with the fact $|V'| = 2r - 1$, we have

$$N_G(V(M^{(k)})) = V' \text{ and } |e_G(v, V(M^{(k)}))| = 2$$

for $k = 2, 3, \ldots, r$ and for any vertex $v \in V'$. Therefore $(V' \cup V(M^{(k)}))_{F^{(k)}}$, say $C^{(k)}$, consists of some disjoint cycles: for each $k = 2, 3, \ldots, r$,

$$C^{(k)} = \bigcup_{j=1}^{m_k} C^{(k)}_j,$$

where $C^{(k)}_j$ is a cycle with positive length in $G$. On the other hand, Claim 3.9 tells us that

$$|N_G(V(M^{(1)})) \cap V'| \geq r$$

since $|V(M^{(1)})| = 2r$. So choose and fix a vertex $z_1 \in N_G(V(M^{(1)})) \cap V'$; here we may assume that $z_1 \in V(C^{(r)}_1)$. Then there exist two distinct vertices $x^{(r)}_1$ and $x^{(r)}_2$ in $V(M^{(r)})$ such that $x^{(r)}_jz_1 \in E(C^{(r)}_1)$ for $j = 1, 2$; two such vertices can be chosen by (1). Here remark $x^{(r)}_1, x^{(r)}_2 \notin E(C^{(r)}_1)$ and $x^{(r)}_1, x^{(r)}_2 \in E(G)$ since $M^{(r)} \cong K_{2r-1}$. Define a new cycle $\tilde{C}^{(r)}_1$ in $G$ by

$$V(\tilde{C}^{(r)}_1) = V(C^{(r)}_1) \setminus \{z_1\} \text{ and } E(\tilde{C}^{(r)}_1) = \left( E(C^{(r)}_1) \cup \{x^{(r)}_1, x^{(r)}_2\} \right) \setminus \{x^{(r)}_1z_1, x^{(r)}_2z_1\}.$$  

Take two distinct vertices $x^{(r-1)}_1$ and $x^{(r-1)}_2$ in $V(M^{(r-1)})$ such that $x^{(r-1)}_jz_1 \in E(G)$ for $j = 1, 2$. By the same argument as in Case II, there exists a Hamilton path, say $P^{(r-1)}$, from $x^{(r-1)}_1$ to $x^{(r-1)}_2$ since $(V(M^{(r-1)}))_G \cong K_{2r-1}$. It is obvious that $(V(M^{(r-1)}))_G \setminus P^{(r-1)}$ is connected. Define a new cycle $\tilde{C}^{(r-1)}$ by

$$V(\tilde{C}^{(r-1)}) = V(M^{(r-1)}) \cup \{z_1\} \text{ and } E(\tilde{C}^{(r-1)}) = E(P^{(r-1)}) \cup \{x^{(r-1)}_1z_1, x^{(r-1)}_2z_1\}.$$  

Moreover let $\tilde{F}^{(k)}$ be a NS2F of $M^{(k)}$ for $k = 1, 2, \ldots, r - 2$. Now we define a set of cycles $F$ by

$$F = \bigcup_{k=1}^{r-2} \tilde{F}^{(k)} \cup \tilde{C}^{(r-1)} \cup C^{(r)}_1 \cup \bigcup_{j=2}^{m_r} C^{(r)}_j,$$

which is a 2-factor of $G$. Here $(V(M^{(k)}))_{G-F}$ is connected in $G - F$ for $k = 1, 2, \ldots, r$. In addition, the following hold: $N_{G-F}(V(M^{(k)})) \cap V' = V'$ for $k = 2, 3, \ldots, r - 2$; $N_{G-F}(V(M^{(r-1)})) \cap V' = V' \setminus \{z_1\}$; $N_{G-F}(V(M^{(1)})) \cap V' \ni z_1$; $N_{G-F}(V(M^{(r)})) \cap V' = \{z_1\}$. Thus $G - F$ is connected, that is, $G$ has a NS2F, a contradiction. ■

Case III-(ii) Assume that $|V(M^{(k)})| = 2r - 1$ for each $k = 1, 2, \ldots, r$. If $|V'| \geq 2r + 1$, then we have

$$|G| = |V'| + \sum_{k=1}^{r} |V(M^{(k)})| \geq 2r + 1 + r(2r - 1) = f(r) + 1,$$
which is a contradiction. Moreover, if $|V'| = 2r - 1$, then it follows from the same argument as in Case III-(i) that $G$ must have a NS2F. This is also a contradiction. Then we may assume that $|V'| = 2r$. Again we distinguish two sub-cases.

Case III-(ii)-(a)
Assume that $N_G(V(M^{(i)})) \neq V'$ for some $i$. Recall that $M^{(k)} \cong K_{2r-1}$ and $|e_G(V', V(M^{(k)}))| = 2(2r - 1)$ for each $k$. Since $|V'| = 2r$, there exists just one vertex $z_1 \in V'$ such that $N_G(z_1) \cap V(M^{(j)}) = \emptyset$. Furthermore, we have

$$|e_G(V', \bigcup_{k=1}^{r} V(M^{(k)}))| = 2r(2r - 1),$$

then there exists a vertex $x_0 \in V'$ with $x_0 \neq z_1$ such that

$$|e_G(x_0, V(M^{(k)}))| = 2$$

for every $k = 1, 2, \ldots, r$. On the other hand, it holds that

$$e_G(\{z_1\}, \bigcup_{k=1}^{r} V(M^{(k)})) \neq \emptyset$$

since $d_G(z_1) = 2r$ and $|V'\setminus\{z_1\}| = 2r - 1$. Also it holds that

$$|N_G(z_1) \cap V'| \geq 2$$

since

$$|e_G(\{z_1\}, \bigcup_{k=1}^{r} V(M^{(k)}))| = |e_G(\{z_1\}, \bigcup_{k \neq i}^{r} V(M^{(k)}))| \leq 2r - 1.$$

Now we should remark there exists at most one vertex $x \in V'$ such that $|e_G(\{x\}, \bigcup_{k=1}^{r} V(M^{(k)}))| \leq 2$; if two distinct vertices satisfy the above, then we have

$$|e_G(V', \bigcup_{k=1}^{r} V(M^{(k)}))| \leq 2 \cdot 2 + 2r(2r - 2) = 2r(2r - 1) - 2(r - 2),$$

which contradicts (3) since $r \geq 3$. In addition, if such a vertex $x$ exists, then there exists just one $M^{(k)}$ such that $e_G(\{x\}, V(M^{(k)})) \neq \emptyset$; we replace $z_1$ in this proof with such a vertex $x$. In other words, we may say

$$e_G(\{z\} \cup_{k \neq i} V(M^{(k)})) \neq \emptyset$$

for each $z \in V' \setminus\{z_1\}$.

As is seen in (2), the subgraph of $F^{(i)}$ induced by $V(M^{(i)}) \cup (V' \setminus \{z_1\})$ consists of some cycles, that is,

$$\langle V(M^{(i)}) \cup (V' \setminus \{z_1\}) \rangle_{F^{(i)}} = \bigcup_{j=2}^{m_i} C_{j}^{(i)},$$

where $C_{j}^{(i)}$ is a cycle for each $j$. Letting $C_{1}^{(i)}$ denote the cycle in $F^{(i)}$ going through the vertex $z_1$, we have $V(C_{1}^{(i)}) \cap V' = \{z_1\}$ and $N_{C_{1}^{(i)}}(z_1) \subset V(M^{(s)})$ for some $s \neq i$. Suppose $y_1 \in V(M^{(s)})$ and $y_2 \in V(M^{(t)})$ such that $s \neq t$ and $\{y_1, y_2\} = N_{C_{1}^{(i)}}(z_1)$. Then it must hold that $(V(C_{1}^{(i)}) \setminus \{z_1\}) \cap V' \neq \emptyset$ by Claim 3.5, a contradiction. Here we set $N_{C_{1}^{(i)}}(z_1) = \{y_1^{(s)}, y_2^{(s)}\}$. It follows from Claim 3.9 and the fact $|V'| = 2r$ that

$$|N_G(V(M^{(k)})) \cap N_G(V(M^{(l)})) \cap V'| \geq 2r - 2 \geq 4$$
for every $k, l = 1, 2, \ldots, r$. By (4), we have $|N_G(x_0) \cap V(M^{(i)})| = 2$ for $t(\neq i, s)$; we set $N_G(x_0) \cap V(M^{(t)}) = \{y_1^{(t)}, y_2^{(t)}\}$. Since $M^{(i)} \cong K_{2r-1}$, there exists a Hamilton path from $y_1^{(i)}$ to $y_2^{(i)}$, say $P^{(i)}$; it is obvious that $M^{(i)} - P^{(i)}$ is connected. Let us define a cycle $C^{(i)}$ by

$$V(C^{(i)}) = V(M^{(i)}) \cup \{x_0\} \quad \text{and} \quad E(C^{(i)}) = E(P^{(i)}) \cup \{y_1^{(i)}x_0, x_0y_2^{(i)}\}.$$  

Similarly there exists a Hamilton path from $y_1^{(t)}$ to $y_2^{(t)}$, say $P^{(t)}$; it is also obvious that $M^{(t)} - P^{(t)}$ is connected. We define a cycle $C^{(t)}$ by

$$V(C^{(t)}) = V(M^{(t)}) \cup \{x_0\} \quad \text{and} \quad E(C^{(t)}) = E(P^{(t)}) \cup \{y_1^{(t)}x_0, x_0y_2^{(t)}\}.$$  

We may furthermore assume that $x_0 \in V(C^{(i)})$. Let $N_{C^{(i)}}(x_0) = \{y_1^{(i)}, y_2^{(i)}\}$. Then we define a cycle $C^{(i)}$ by

$$V(C^{(i)}) = V(C^{(i)}) \setminus \{x_0\} \quad \text{and} \quad E(C^{(i)}) = \left( E(C^{(i)}) \cup \{y_1^{(i)}y_2^{(i)}\} \right) \setminus \{y_1^{(i)}x_0, x_0y_2^{(i)}\}.$$  

Here remark that $y_1^{(i)}y_2^{(i)} \in E(G)$ and $y_1^{(i)}y_2^{(i)} \notin E(C^{(i)})$. For every $k \neq i, s, t$, let $C^{(k)}$ be a Hamilton cycle of $M^{(k)} \cong K_{2r-1}$; $M^{(k)} - C^{(k)}$ is connected. Now we define a set of cycles $F$ by

$$F = \bigcup_{k \neq i, s, t} C^{(i)} \cup C^{(s)} \cup C^{(t)} \cup C^{(i)} \cup \tilde{C}^{(i)} \bigcup_{j=3}^{m_i} C_j,$$

which is a 2-factor of $G$. Remark the following: $|N_{G-F}(V(M^{(i)}) \cap V')| = |N_{G-F}(V(M^{(i)}) \cap V')| \geq 2r - 2$ and $|N_{G-F}(V(M^{(k)}) \cap V')| \geq 2r - 1$ for $k \neq i, s, t$. Then, for every $k, l(\neq i)$,

$$|N_{G-F}(V(M^{(k)}) \cap N_{G-F}(V(M^{(l)}))) \cap V' \geq 2r - 4 \geq 2.$$  

On the other hand, for every $j(\neq t)$, $N_{G-F}(x_0) \cap V(M^{(j)}) \neq \emptyset$. Moreover, $N_{G-F}(z_1) \cap V' \neq \emptyset$ by (5); by (6), $e_G([z_1], \cup_{k \neq i} V(M^{(k)}) \neq \emptyset$ for each $z \in V(\setminus \{z_1\})$. Thus it is shown that $G - F$ is connected, a contradiction.  

Case III-(ii)-(b)

Let us assume that $N_G(V(M^{(k)})) = V'$ for each $k = 1, 2, \ldots, r$. Let $p(k)$ be

$$p(k) = \left| \left\{ x \in V' \mid |N_{F^{(i)}}(x) \cap V(M^{(i)})| = 1 \right\} \right|.$$  

Then we have

$$|e_G(V', V(M^{(k)}))| = 2(2r - p(k)) + p(k) = 4r - p(k).$$

On the other hand, it holds that $|e_G(V', V(M^{(k)}))| = 2(2r - 1)$ since $M^{(k)} \cong K_{2r-1}$. Thus we have $p(k) = 2$ for each $k$; equivalently we have

$$\left| \left\{ x \in V' \mid |N_{F^{(i)}}(x) \cap V(M^{(i)})| = 2 \right\} \right| = 2r - 2 \geq 4.$$  

Let us recall that $E((V(M^{(k)})) \cap F^{(k)} = \emptyset$ for each $k = 1, 2, \ldots, r$. It is easy to see that

$$\langle V(M^{(r)}) \cup V' \rangle_{F^{(r)}} = \bigcup_{j=1}^{m_r} C_j,$$

where $C_1, C_2, \ldots, C_{m_r}$ are mutually disjoint cycles. The 2-factor $F^{(r)}$ obtained in the Procedure can be expressed as

$$F^{(r)} = \bigcup_{k=1}^{r-1} H^{(k)} \cup \bigcup_{j=1}^{m_r} C_j,$$

where $H^{(k)}$ is a NS2F of $M^{(k)} \cong K_{2r-1}$ for $k = 1, 2, \ldots, r$. It follows from (7) that the number of vertices $v \in V'$ satisfying $|N_{F^{(r-1)}}(v) \cap V(M^{(r-1)})| = 2$ and $|N_{F^{(r)}}(v) \cap V(M^{(r)})| = 2$ is greater than or equal to $2r - 4 \geq 2$. 

Therefore there exists a vertex $z_1 \in V'$ such that
$$|N_{F(r-1)}(z_1) \cap V(M^{(r-1)})| = |N_{F(r)}(z_1) \cap V(M^{(r)})| = 2.$$ 

Now we argue similarly as in Case III-(i). Let $N_{F(r-1)}(z_1) \cap V(M^{(r-1)}) = \{x_1^{(r-1)}, x_2^{(r-1)}\}$ and $N_{F(r)}(z_1) \cap V(M^{(r)}) = \{x_1^{(r)}, x_2^{(r)}\}$. We choose a Hamilton path of $M^{(r-1)}$ from $x_1^{(r-1)}$ to $x_2^{(r-1)}$, say $P^{(r-1)}$; then $M^{(r-1)} - P^{(r-1)}$ is connected. Define a cycle $C^{(r-1)}$ by
$$V(C^{(r-1)}) = V(M^{(r-1)}) \cup \{z_1\} \quad \text{and} \quad E(C^{(r-1)}) = E(P^{(r-1)}) \cup \{x_1^{(r-1)}z_1, z_1x_2^{(r-1)}\}.$$ 

On the other hand, we may assume that $z_1 \in V(C_1^{(r)})$ since $z_1 \in V^r \subset \bigcup_{j=1}^{m_r} V(C_j^{(r)})$. Here remark that $x_1^{(r)}x_2^{(r)} \in E(G)$ and $x_1^{(r)}x_2^{(r)} \notin F^{(r)}$. So define a cycle $\tilde{C}_1^{(r)}$ by
$$V(\tilde{C}_1^{(r)}) = V(C_1^{(r)}) \setminus \{z_1\} \quad \text{and} \quad E(\tilde{C}_1^{(r)}) = \left( E(C_1^{(r)}) \cup \{x_1^{(r)}x_2^{(r)}\} \right) \setminus \{x_1^{(r)}z_1, z_1x_2^{(r)}\}.$$ 

It is obvious that $M^{(r)} - \{x_1^{(r)}x_2^{(r)}\}$ is connected. Now we define a set of cycles $F$ by
$$F = \bigcup_{k=1}^{r-2} H^{(k)} \cup C^{(r-1)} \cup \tilde{C}_1^{(r)} \cup \bigcup_{j=2}^{m_r} C_j^{(r)},$$
which is a 2-factor of $G$. We can check the following: $(V(M^{(k)}))_{G-F}$ is connected for each $k = 1, 2, \ldots, r$; $N_{G-F}(V(M^{(k)})) = V^r$ for each $k = 1, 2, \ldots, r - 2$; $N_{G-F}(V(M^{(r-1)})) = V^r \setminus \{z_1\}$; $N_{G-F}(V(M^{(r)})) = \{z_1\}$. This implies that $G - F$ is connected, which is a contradiction. \hfill \blacksquare

Consequently it is shown that Assumption 2.4 does not hold under Hypothesis 2.3. Using Lemma 2.1 together with this fact, we complete the proof of Theorem 2. \hfill \blacksquare

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References