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CHROMATIC POLYNOMIALS AND NETWORK RELIABILITY

A. SATYANARAYANA and R. TINDELL
Department of Computer Science, Stevens Institute of Technology, Hoboken, NJ, U.S.A.

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In this paper, we introduce and study an extension of the chromatic polynomial of a graph. The new polynomial, determined by a graph G and a subset K of points of G, coincides with the classical chromatic polynomial when K is the set of all points of G. The main theorems in the present paper include analogues of the standard axiomatic characterization and Whitney's topological characterizations of the chromatic polynomial, and the theorem of Stanley relating the chromatic polynomial to the number of acyclic orientations of G. The work in this paper was stimulated by important connections between the chromatic polynomial and the all-terminal network reliability problem, and by recent work of Boesch, Satyanarayana, and Suffel on a graph invariant related to the K-terminal reliability problem. Several of the results of Boesch, Satyanarayana, and Suffel are derived as corollaries to the main theorems of the present paper.

1. Introduction

As a result of work on network reliability analysis, a graph invariant called *domination* was introduced in [14] for directed graphs and subsequently in [10] for undirected graphs. This invariant has since been explored and applied by several authors [1, 2, 6, 7, 11–13] and shown to possess many interesting properties. The original definition of the domination of a graph employs the concept of *formations* of a graph. Let $G = (V, E)$ be a graph and $K \subseteq V$ be a specified subset of V. A *K-tree* is a minimal connected subgraph H of G which contains all points of K; minimality, as usual, means that no connected proper subgraph of H contains all points of K. It is obvious that a K-tree is a tree which contains K and has all its degree-one points in K. A *formation* of G is a set of K-trees of G whose union yields the edge set of G. A formation is said to be *odd* if the number of K-trees in the formation is odd, and is said to be *even* otherwise. The *signed K-domination* $d_K(G)$ is the number of odd minus the number of even formations of G. If $K = V$, the invariant $d_V(G)$ is known as the *all-terminal domination* of G.

The results to be presented here were motivated by the recent paper of Boesch, Satyanarayana, and Suffel [4], in which several alternate characterizations for $d_K(G)$ were established. Of particular relevance is the connection they discovered between the chromatic polynomial of G and $d_V(G)$, namely that $d_V(G)$ has the same absolute value as the coefficient of λ in the chromatic polynomial $P(G, \lambda)$ of G. This observation follows from one of the characterizations of $d_K(G)$ in [4] and

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Whitney's expression for the coefficients of the chromatic polynomial in terms of the spanning subgraphs of G. It is well known that the coefficients of $P(G;\lambda)$ have several fascinating properties. For example, in the influential paper of Whitney [17], another characterization is given for the coefficients of $P(G;\lambda)$ in terms of spanning forests which do not contain what Whitney called "broken cycles". Another example is the remarkable result of Stanley [15] that the sum of the absolute values of the coefficients of $P(G,\lambda)$ is just the number of acyclic orientations of G.

In this paper we introduce a polynomial $P(G, K;\lambda)$ in λ determined by a graph G and a specific subset K of vertices of G. Like the classical chromatic polynomial $P(G;\lambda)$, this new polynomial has integer coefficients that alternate in sign, and is defined in terms of vertex colorings. However, in the case of the extended polynomial, only the points in K are colored. The concept of a proper coloring of K within G, as defined in Section 4, is intended to account for connections between points of K that involve points of $V-K$. The extended chromatic polynomial $P(G, K;\lambda)$ is then defined to be plus or minus the number of proper colorings of K within G. The sign of $P(G, K;\lambda)$ is determined by the parity of $|V|-|K|-i$, where i is the number of isolated points of G which are not in K. The definitions are such that $P(G, K;\lambda)=P(G;\lambda)$ if K is the entire vertex set of G.

The remainder of the paper is organized as follows. We first review in Section 2 the main results of Birkhoff, Whitney, Tutte and Stanley on the chromatic polynomial. In Section 3, we introduce a generalization of the concept of domination and review its connection with the chromatic polynomial. Section 4 contains the definition of the extended chromatic polynomial and the statement and proof of a fundamental axiomatic characterization for $P(G, K;\lambda)$. Section 4 concludes with an extension of Stanley's Theorem which states that the sum of the absolute values of the coefficients of $P(G, K;\lambda)$ is the number of acyclic orientations of G having all sources and sinks in K. Section 5 is devoted to various topological interpretations of the coefficients of the extended chromatic polynomial, including analogues of Whitney's results on $P(G;\lambda)$. We show that $d_K(G)$ is, up to sign, the coefficient of λ in $P(G, K;\lambda)$ and from this obtain as corollaries several of the alternate characterizations of $d_K(G)$ given in [4]. Finally, Section 6 provides computational formulas for $P(G, K;\lambda)$ for special cases of G, including cycles, trees, and complete graphs.

2. Preliminaries

Unless defined otherwise, graph-theoretic terminology used here follows Harary [5]. One exception is that we allow multiple edges and self-loops in a graph, so that by *graph* we mean what Harary calls a *pseudograph*. If m is a positive integer, we will denote by $[m]$ the set $\{1,\ldots,m\}$. If $G=(V,E)$ is a graph, then we denote by $n(G)$ the number $|V|$ of points of G and by $e(G)$ the number $|E|$ of edges of G. A *λ-coloring* of a graph $G=(V, E)$ is a mapping $\alpha: V \rightarrow [\lambda]$. The integers $1, 2, \ldots, \lambda$ are called *colors*. A λ-coloring is *proper* if no two adjacent points of G are assigned the same color. If λ is sufficiently large, G can in general be properly colored in many different ways. Birkhoff [3] noticed that the number of distinct proper λ-colorings of a given graph G may be expressed elegantly as a polynomial in λ, now well-known as the *chromatic polynomial of* G and denoted by $P(G; \lambda)$. Clearly, if c_j is the number of proper j-colorings of G in which all j colors are used, then

$$P(G; \lambda)=\sum_{j=1}^{n}\binom{\lambda}{j} c_j.$$

The first two propositions of this section are well-known.

Proposition 2.1. *The function $P(G; \lambda)$ is a polynomial of degree $n(G)$ that has integer coefficients that alternate in sign, has leading coefficient* 1, *and has constant term zero.*

Proposition 2.2. *The chromatic polynomial is uniquely determined by the following two conditions. Let $G=(V, E)$ be a graph,*

(C_1) $P(G; \lambda)=\lambda^{|V|}$ *if G has no edges,*

(C_2) *If $G-x$ and $G \mid x$ are the graphs obtained from graph G by deleting and contracting, respectively, an edge x of G, then*

$$P(G; \lambda)=P(G-x; \lambda)-P(G \mid x; \lambda).$$

We will refer to condition (C1) as the *discrete graph condition* and to condition (C_2) as the *pivot condition.* Whitney [17] provided a topological interpretation for the coefficients of the chromatic polynomial which is embodied in the following result.

Theorem 2.1 (Whitney). $P(G; \lambda)=\sum_s (-1)^{e(S)} \lambda^{c(S)}$, *where the summation is over the set of spanning subgraphs S of G, $e(S)$ denotes the number of edges of S, and $c(S)$ denotes the number of connected components of S.*

Whitney then observed that in most cases there are many paris of terms of the above summation that cancel each other. For example, if G is the triangle, then the contributions of G and $G-x$, x an edge of G, are $-\lambda$ and λ, respectively. This observation led Whitney to the following notion and the remarkable result given in Theorem 2.2.

Definition 2.1 (External activity). Suppose $G=(V, E)$ is a graph and $<$ is a strict linear order on E. An edge x with endpoints u, v is said to be *externally active* relative to a set X of edges of G if there is a path P in G between u and v

which uses only edges from $X - \{x\}$ and has the property that $x < y$ for all edges y on P. (Whitney called such a path a *broken cycle*.) The number of edges which are externally active relative to X is called the *external activity* of X in G.

Theorem 2.2 (Whitney). *Let $G = (V, E)$ be a graph with n points, let $<$ be a strict linear order on E, and let $m_j(G)$ denote the number of spanning forests of G having j connected components and external activity zero. Then*

$$P(G; \lambda) = \sum_{j=1}^{n} (-1)^{n-j} m_j(G) \lambda^j.$$

Subsequently, Tutte [16] introduced the notion of internal activity relative to a set of edges and showed that the chromatic polynomial can be expressed in terms of certain spanning trees of G.

Definition 2.2 (Internal activity). Let G, X, and $<$ be as in Definition 2.1. An edge $x \in X$ is *internally active* in X if

(i) x lies on no cycle of X, and

(ii) if $y \in E - X$ and $y < x$, then $X - \{x\}$ contains a path connecting endpoints of y.

The number of edges of X which are internally active is called the *internal activity* of X.

Notice that in the case where X constitutes a spanning subgraph of G, condition (ii) of Definition 2.2 is equivalent to saying that $x < y$ for any $y \in E - X$ such that X and $(X - \{x\}) \cup \{y\}$ have the same number of connected components. Further, if X constitutes a spanning tree of G, Definition 2.2 is equivalent to saying that an edge $x \in X$ is internally active in X if and only if $x < y$ for all $y \in C - x$, where C is the fundamental cut defined by x in G relative to the spanning tree X. Similarly, if X is a spanning tree of G, Definition 2.1 reduces to the fact that an edge $x \notin X$ is externally active relative to X if and only if $x < y$ for all $y \in C - x$, where C is the fundamental cycle defined by x in G relative to X.

Theorem 2.3 (Tutte). *Let $G = (V, E)$ be a connected graph with n points, let $<$ be a strict linear order on E, and let $t_{j0}(G)$ denote the number of spanning trees of G with internal activity j and external activity* 0. *Then*

$$P(G; \lambda) = (-1)^{n-1} \lambda \sum_{j=1}^{n-1} t_{j0}(G)(1 - \lambda)^j.$$

Finally, we note an important connection between $P(G; \lambda)$ and orientations of G. An *orientation* of a graph is an assignment of a direction to each edge of the graph. An orientation is *acyclic* if the resulting digraph has no directed cycles, and is *cyclic* otherwise. The number of acyclic orientations of G is denoted $N(G)$. Stanley [15] discovered a remarkable connection between $N(G)$ and $P(G; \lambda)$.

Theorem 2.4 (Stanley). *If G has n points, then $P(G;\lambda)\,|_{\lambda=-1}=(-1)^n N(G)$.*

3. The chromatic polynomial and all-terminal domination

Suppose $G=(V,E)$ is a graph and K is a specified subset of V. A forest H of G is called a *K-forest* if H contains all points of K and every point of degree one or zero in H belongs to K. Let $F(G,K,j)$ denote the collection of all the K-forests of G having exactly j components. A subset F of $F(G,K,j)$ is called a *j-formation* of G if every edge of G is in some K-forest from F. A j-formation is *odd* or *even* depending upon whether the number of forests in the formation is odd or even, respectively.

Definition 3.1 (Domination). The *signed domination* $d(G,K,j)$ of G, with respect to a subset K of vertices and an integer j, is the number of odd minus the number of even j-formations of G. The *domination* $D(G,K,j)$ is the absolute value of the signed domination.

The notion of the domination of a graph, originally defined only in the case $j=1$, arose in connection with a problem in network reliability [14]. In this special case the notations $d_K(G)$ and $D_K(G)$ were used for $d(G,K,1)$ and $D(G,K,1)$, respectively. Subsequently, these invariants have been studied by several authors [1,2,6,7,10–13]. In a recent work, Boesch, Satyanarayana, and Suffel [4] established several alternate characterizations of $d_K(G)$. They observed two interesting connections between $d_K(G)$ and $P(G;\lambda)$ which are of particular interest here.

Proposition 3.1 (Boesch, Satyanarayana, and Suffel). *If $S_O(G)$ and $S_E(G)$ are the number of spanning connected subgraphs of G having, respectively, an odd and even number of edges, then*

$$D_V(G)=|S_O(G)-S_E(G)|$$

Since the above expression for $D_V(G)$ is precisely the same as that given in Theorem 2.1 for the absolute value of the coefficient of λ in the chromatic polynomial, we have the following corollary:

Corollary 3.1. $|(P(G;\lambda)/\lambda)|_{\lambda=0}\,|=D_V(G)$.

Recall that a K-tree of G is a tree whose vertex set contains K and whose leaves are all contained in K. If T is a spanning tree of G, then clearly T contains exactly one K-tree; we denote this K-tree by T^K. Now let $<$ be a strict linear order on the edge set of G. Denote by $\tau_{*0}(G,K)$ the collection of all spanning trees T of G which have no externally active edges and have all internally active edges contained in T^K. Finally, let $t_{*0}(G, K)$ be the number of elements in $\tau_{*0}(G, K)$.

Proposition 3.2 (Boesch, Satyanarayana and Suffel). $D_K(G) = t_{*0}(G, K)$.

When K is the set of endpoints of an edge of G, Proposition 3.2 yields an important special case which we state as a corollary.

Corollary 3.2. *If u and v are adjacent points of G, then $D_{\{u,v\}} = t_{*0}(G, \{u, v\}) = t_{10}(G)$, where $t_{10}(G)$ is the number of spanning trees of G with internal activity* 1 *and external activity zero.*

It is easy to see from Theorem 2.3 that if G is connected, then

$$|P(G; \lambda)/(1-\lambda)|_{\lambda=1}| = t_{10}(G),$$

which with Corollary 3.2 leads to the following

Corollary 3.3. *If u and v are adjacent points in a connected graph G, then*

$$|P(G; \lambda)/(1-\lambda)|_{\lambda=1}| = D_{\{u,v\}}(G).$$

The following proposition, due to Rodriguez [9], clearly exhibits the interplay between $P(G; \lambda)$ and $d(G, V, i)$. Indeed, this result is a special case of a more general result of Rodriguez dealing with $d(G, K, i)$ and the extended chromatic polynomial introduced in Section 4 (see Theorem 5.5).

Proposition 3.3 (Rodriguez). *If $G = (V, E)$ is a graph with n points and $e > 0$ edges, then*

$$P(G; \lambda) = (-1)^e(1-\lambda) \sum_{j=1}^{n-1} d(G, V, j)\lambda^j.$$

We conclude this section with a "reverse" pivot equation for $d(G, K, j)$ useful in Section 5. A proof may be found in [4].

Proposition 3.4. *If $G = (V, E)$ is a graph, K is a subset of V, and x is an edge of G, then*

$$d(G, K, j) = d(G \mid x, K \mid x, j) - d(G - x, K, j).$$

4. An extended chromatic polynomial

In this section, we first define an extension of the chromatic polynomial and then provide a characterization analogous to Proposition 2.2. We conclude the section with a generalization of Stanley's result on acyclic orientations.

If D is a directed graph and u is point of D, then we say that u is a *source* of D if its indegree is zero and its outdegree is positive. Similarly, we say that u is a *sink* of D if its outdegree is zero and its indegree is positive. Note that an isolated point of D is neither a source nor a sink.

Definition 4.1 (K-acyclic orientation). Given a graph $G = (V, E)$ and a subset K of V, a *K-acyclic orientation* of G is an acyclic orientation of G with all sources and sinks in K. The number of K-acyclic orientations of G will be denoted by $N_K(G)$.

Notice that if $K = V$, then all acyclic orientations are K-acyclic, and $N_K(G) = N(G)$.

Definition 4.2 (Proper λ-coloring of K within G). Given a graph $G = (V, E)$ and a subset K of V, a *proper λ-colouring of* K within G is a pair (D, α) where $\alpha : K \to [\lambda]$ is a coloring and D is a K-acyclic orientation of G such that if u, v are in K and there is a direct path in D from u to v, then $\alpha(u) > \alpha(v)$.

Definition 4.3 (Extended chromatic polynomial $P(G, K; \lambda)$). Let $G = (V, E)$ be a graph and K a subset of V. The *extended chromatic polynomial* $P(G, K; \lambda)$ is defined to be $(-1)^{|V|-|K|-i}$ times the number of distinct proper λ-colourings of K within G, where i is the number of isolated points of G which are not in K.

Since a graph containing a self-loop has no acyclic orientations, $P(G, K; \lambda) = 0$ for such graphs. It is also immediate from the definition that if x and y are distinct edges of G with the same endpoints, then $P(G, K; \lambda) = P(G - x, K; \lambda)$.

Proposition 4.1. *For any graph $G = (V, E)$, $P(G, V; \lambda) = P(G; \lambda)$.*

Proof. As noted by Stanley [15], every proper λ-coloring α of G determines a unique acyclic (and hence V-acyclic) orientation D_α in which each edge $x = \{u, v\}$ is oriented from u to v if and only if $\alpha(u) > \alpha(v)$. Clearly (D_α, α) is thus a proper λ-coloring of V within G. It is equally clear that if (D, α) is a proper coloring of V within G, then no two adjacent vertices of G receive the same color under α and thus α is a proper λ-coloring of G. □

Our next objective is a characterization of the extended chromatic polynomial analogous to that given for the classical chromatic polynomial in Proposition 2.2. The discrete graph condition and the pivot condition of Proposition 2.2 have natural analogues for the extended chromatic polynomial which will be shown to characterize $P(G, K; \lambda)$. Specifically, let $f(G, K; \lambda)$ be a function defined for graphs G, subsets K of the vertex set of G and integers λ. We say that f satisfies the *pivot condition* if for any graph G and edge x of G,

$$f(G, K; \lambda) = f(G - x, K; \lambda) - f(G \mid x, K \mid x; \lambda).$$

The equation of the pivot condition will be referred to as the *pivot equation* for f on G with *pivot* x. The next lemma may be proved by an easy induction on the number of edges of G.

Lemma 4.1. *Let f and g be functions defined on graphs G, subsets K of the vertex set of G, and integers λ. Let m be a nonnegative integer. If f and g satisfy the pivot equation and $f(G, K; \lambda) = g(G, K; \lambda)$ for all graphs G having exactly m edges, then $f(G, K; \lambda) = g(G, K; \lambda)$ for all graph with m or more edges.*

We then have as an immediate corollary the following

Corollary 4.1. *There is at most one integer-valued function $f(G, K; \lambda)$, defined for graphs G and integers λ, which satisfies the following two conditions*:

(E_1) *If G has no edges, then $f(G, K; \lambda) = \lambda^{|K|}$,*

(E_2) *If x is an edge of G, then*

$$f(G, K; \lambda) = f(G - x, K; \lambda) - f(G \mid x, K \mid x; \lambda).$$

As indicated above condition (E_2) will be refered to as the pivot condition. The condition (E_1) will be henceforth referred to as the *discrete graph condition.*

Recall from Harary [5] the definition of an elementary contraction: if x is an edge of G with endpoints u and v, then $G \mid x$ is the graph obtained from G by deleting x and identifying the points u and v to a single point. Note that all other edges with endpoints u, v become self-loops. If K is a subset of the vertex set of G, we shall denote by $K \mid x$ the corresponding subset of the vertex set of $G \mid x$. Each edge of G other than x can, in a natural way, be regarded as an edge of $G \mid x$. This identification, which we henceforth assume, constitutes a bijection between the edge sets of $G - x$ and $G \mid x$.

At this point it is useful to introduce some notation regarding orientations. If u and v are points of a directed graph, an arc from u to v will be said to be of the form $u \to v$. We shall view an *orientation* of an undirected graph G as a directed graph D obtained from G by replacing each edge x with a distinct arc $\vec{x}$ connecting the endpoints of x. Note that D has exactly as many arcs as G has edges. If D is an orientation of the graph G and $\vec{x}$ is an arc of D, the contraction $D \mid \vec{x}$ is defined as for the undirected case. If D is an orientation of G and x is an edge of G, then $D \mid \vec{x}$ is clearly an orientation of $G \mid x$. We shall in this case simplify our notation and write $D \mid x$ instead of $D \mid \vec{x}$.

Our next goal is to show that $P(G, K; \lambda)$ satisfies the pivot condition. In order to provide a generalization of Stanley's theorem (Theorem 2.4) to K-acyclic orientations, we also wish to prove that the function $\nu(G, K) = (-1)^{n-i} N_K(G)$ satisfies the pivot equation. Our approach will be to develop these two results in a unified setting. To this end, let $G = (V, E)$ be a graph, $A \subseteq K$ be subsets of V, and s be a function from A into the positive integers. An orientation D' of G will be said to be *s-compatible* if $s(a) > s(b)$ for any points a, b of A for which there is a path in D' from a to b. We denote by $O(G, K, s)$ the set of all K-acyclic, s-compatible orientations of G. Notice that if s is the empty function, then $O(G, K, s)$ is just the set of K-acyclic orientations of G. Our immediate goal is to prove that $|O(G, K, s)|$ satisfies the additive pivot condition for certain special edges x of G. One requirement is that at least one endpoint of x is in $V - A$. The other requirement is that x be a *K-proper edge* of G, by which we mean that if x has an endpoint u in $V - K$, then u is not a degree-one point of G. The following proposition is immediate from definitions.

Proposition 4.2. *Let $G = (V, E)$ be a graph and $A \subseteq K$ subsets of V and let i denote the number of isolated points of G not in K. Then*

$$P(G, K; \lambda) = (-1)^{|V|-|K|-i} \sum_{s} |O(G, K, s)|,$$

where the summation is over all functions s from K to $[\lambda]$.

The proof of the desired additive pivot equation for $|O(G, K, s)|$ will require a sequence of preliminary results. Lemma 4.2 consists of nine observations, all of which follow easily from definitions or preceding observations in the lemma. Notice that if x is an edge of G with at least one endpoint in $V - A$, then any function s defined on A may also be viewed as a function on $A \mid x$, since we identify A and $A \mid x$ in this situation. If u, v are points of a directed graph D, we shall use the notation $D \cup \{u \to v\}$ to denote the results of adding to D a new arc from u to v.

Lemma 4.2. *Let $G = (V, E)$ be a graph, let $A \subseteq K$ be subsets of V, let s be a function from A into the positive integers, let x be a K-proper edge of G having at least one of its endpoint u, v in $V - A$, and let D be an orientation of $G - x$. Then*

$D \mid x$ is acyclic iff $D \cup \{u \to v\}$ and $D \cup \{v \to u\}$ are acyclic, (4.2.1)

$D \mid x$ is s-compatible iff $D \cup \{u \to v\}$ and $D \cup \{v \to u\}$ are s-compatible, (4.2.2)

$D \mid x$ is $(K \mid x)$-acyclic iff at least one of $D \cup \{u \to v\}$, $D \cup \{v \to u\}$ is K-acyclic, (4.2.3)

If $D \mid x \in O(G \mid x, K \mid x, s)$, then at least one of $D \cup \{u \to v\}$, $D \cup \{v \to u\}$ is in $O(G, K, s)$, (4.2.4)

D is acyclic iff at least one of $D \cup \{u \to v\}$, $D \cup \{v \to u\}$ is acyclic, (4.25)

D has all sinks and all sources in K iff both $D \cup \{u \to v\}$ and $D \cup \{v \to u\}$, have all sinks and all sources in K, (4.2.6)

If either of $D \cup \{u \to v\}$, $D \cup \{v \to u\}$ is s-compatible, then D is s-compatible, (4.2.7)

($D \in O(G-x, K, s)$ and $D \mid x \in O(G \mid x, K \mid x, s)$) iff both $D \cup \{u \to v\}$ and $D \cup \{v \to u\}$ are in $O(G, K, s)$, (4.2.8)

If D is s-compatible and contains a path from u to v, then $D \cup \{u \to v\}$ is s-compatible. (4.2.9)

It may be of interest to note that (4.2.3), and hence each of (4.2.4) and (4.2.8), fails when x is not K-proper.

Lemma 4.3. *Let G, K, A, s, x, and D be as in Lemma* 4.2. *If D is s-compatible, then at least one of $D \cup \{u \to v\}$, $D \cup \{v \to u\}$ is s-compatible.*

Proof. Suppose that D is s-compatible and that $D \cup \{u \to v\}$ is not s-compatible. Then there exists vertices a_0, $b_0 \in A$ with $s(a_0) \leqslant s(b_0)$ and a path T in $D \cup \{u \to v\}$ from a_0 to b_0. Since D is s-compatible, T must contain the arc $u \to v$. Let T_1 be the segment of T from a_0 to u and T_2 the segment of T from v to b_0. Note that T_1 and T_2 are paths in D. We wish to show that $D \cup \{v \to u\}$ is s-compatible. To that end, let T' be a path in $D \cup \{v \to u\}$ from a vertex a of A to a vertex b of A. If T' does not contain the arc $v \to u$, then T' is a path in D and therefore $s(a) > s(b)$. Thus we assume that T' contains the arc $v \to u$, and set T_1' equal to the segment of T' from a to v and T_2' equal to the segment of T' from u to b. It then follows that T_1 followed by T_2' is a path in D from a_0 to b and T_1' followed by T_2 is a path in D from a to b_0. Since D is s-compatible, we conclude that $s(a) > s(b_0)$ and $s(a_0) > s(b)$. Since $s(a_0) \leqslant s(b_0)$, we have $s(a) > s(b)$ and the proof is complete. □

Lemma 4.4. *Let G, K, A, s, x, and D be as in Lemma* 4.2. *If $D \in O(G-x, K, s)$, then at least one of $D \cup \{u \to v\}$, $D \cup \{v \to u\}$ is in $O(G, K, s)$.*

Proof. Assume that $D \in O(G-x, K, s)$ and $D \cup \{v \to u\} \notin O(G, K, s)$. We must show that $D \cup \{v \to u\} \in O(G, K, s)$. If follows from (4.2.6) that both $D \cup \{u \to v\}$ and $D \cup \{v \to u\}$ have all sources and all sinks in K. Thus we know that $D \cup \{u \to v\}$ either is not s-compatible or is cyclic, and must prove that $D \cup \{v \to u\}$ is s-compatible and acyclic. If $D \cup \{u \to v\}$ is cyclic, then D is acyclic by (4.2.5). Moreover, D must contain a path from v to u and hence, by (4.2.9), $D \cup \{v \to u\}$ is also s-compatible. Thus we may assume that $D \cup \{u \to v\}$ is acyclic and not s-compatible. By (4.2.9), D contains no path from u to v and thus $D \cup \{v \to u\}$ is acyclic. By Lemma 4.3, $D \cup \{v \to u\}$ must be s-compatible, and the proof of Lemma 4.4 is complete. □

The desired additive pivot equation for the function $|O(G, K, s)|$ is an immediate consequence of (4.2.8) and Lemmas 4.3 and 4.4

Corollary 4.2. *If G, K, A, s, and x are as in Lemma* 4.2, *then*

$$|O(G, K, s)| = |O(G-x, K, s)| + |O(G \mid x, K \mid x, s)|.$$

If we consider the special case where $A = \emptyset$, then every edge has an endpoint which is not in A. Thus, as a corollary to the previous result, we obtain an additive pivot equation for the number $N_K(G)$ of K-acyclic orientations.

Corollary 4.3. *If G is a graph, K is a subset of the vertex set of G, and x is a K-proper edge of G, then $N_K(G) = N_K(G - x) + N_K(G \mid x)$.*

Let $v(G, K) = (-1)^{n-i} N_K(G)$, where n is the number of points of G and i is the number of siolated points of G lying in $V - K$.

Proposition 4.3. *Let $G = (V, E)$ be a graph, K a subset of V, and x an edge of G. Then*

$$v(G, K) = v(G - x, K) - v(G \mid x, K \mid x).$$

Proof. The pivot equation holds, by Corollary 4.3, if x is a K-proper edge. Suppose that x is not K-proper. Then clearly $v(G, K) = 0$ as G has no K-acyclic orientations. It is also simple to see that when isolated points not in K are removed from $G - x$ and isolated points not in $K \mid x$ are removed from $G \mid x$, the resulting graphs are the same. Thus the pivot equation holds with pivot x. □

The next theorem follows easily from Proposition 4.2 and Corollary 4.2 by noticing that, if x is an edge with at least one endpoint in $V - K$, then $K = K \mid x$ and $G \mid x$ has one less vertex than G.

Theorem 4.1. *Let $G = (V, E)$ be a graph, K a subset of V, and x a K-proper edge of G with at least one endpoint in $V - K$. Then $P(G, K; \lambda) = P(G - x, K; \lambda) - P(G \mid x, K \mid x; \lambda)$.*

The next lemma, though cumbersome, is used repeatedly in inductive proofs below that a given function f satisfies the pivot condition.

Lemma 4.5. *Suppose f is a function defined for graphs G, subsets K of the vertex set of G, and an integer parameter λ. Let G be a graph and x and y be distinct edges of G. If the pivot equation is valid for f with pivot y on each of G, $G - x$, and $G \mid x$ and on each of $G - y$ and $G \mid y$ with pivot x, then the pivot condition holds for f on G with pivot x.*

Proof. We first note the following simple identities, valid for distinct edges x, y of a graph G: $(G - x) - y = (G - y) - x$, $(G \mid x) \mid y = (G \mid y) \mid x$ and $(G - x) \mid y = (G \mid y) - x$. Using these observations and the assumption that the pivot equation holds for f on each of $G - y$ and $G \mid y$ with pivot x, we obtain

$$\begin{aligned} f(G - y, K; \lambda) &= f((G - y) - x, K; \lambda) - f((G - y) \mid x, K \mid x; \lambda) \\ &= f((G - x) - y, K; \lambda) - f((G \mid x) - y, K \mid x; \lambda), \end{aligned}$$

and

$$f(G \mid y, K \mid y; \lambda) = f((G \mid y) - x, K \mid y; \lambda) - f((G \mid y) \mid x, (K \mid y) \mid x; \lambda)$$
$$= f((G - x) \mid y, K \mid y; \lambda) - f((G \mid x) \mid y, (K \mid x) \mid y; \lambda).$$

If we subtract the second equation from the first then, since the pivot equation holds for f on G with pivot y, the resulting equation has left-hand side equal to $f(G, K; \lambda)$. If we rearrange the terms of the right-hand side of the difference, we obtain

$$f(G, K; \lambda) = (f((G - x) - y, K; \lambda) - f((G - x) \mid y, K \mid y; \lambda))$$
$$- (f((G \mid x) \mid y, (K \mid x) \mid y; \lambda) - f((G \mid x) - y, K \mid x; \lambda)).$$

By assumption, the pivot equation holds for f with pviot x on both $G - y$ and $G \mid y$. Thus the right hand side of the previous equation is equal to $f(G - x, K; \lambda) - f(G \mid x, K \mid x; \lambda)$. □

We now come to the main theorem of this section.

Theorem 4.2. *$P(G, K; \lambda)$ is the unique function, defined on graphs G and integers λ, which satisfies the pivot and discrete graph conditions.*

Proof. Since we know, by Corollary 4.1, that there is at most one function that satisfies the pivot and discrete graph condition, all that remains is to show that $P(G, K; \lambda)$ satisfies these two conditions. It is simple to verify that $P(G, K; \lambda)$ satisfies the discrete graph condition, so we turn to the pivot condition.

It is immediate from definitions that isolated points of G which lie in $V - K$ may be deleted without changing the value of $P(G, K; \lambda)$. Thus we assume that all isolated points of G are in K. Further, it follows from the proof of Proposition 4.3, that the pivot equation holds when the edge x is not K-proper. We therefore prove that $P(G, K; \lambda)$ satisfies the pivot condition for all K-proper edges. The proof is by induction on the number of K-proper edges of G having at least one endpoint in $V - K$. For the basis step, suppose that all edges have both endpoints in K. Then, since all isolated points of G are in K, we have $K = V$. Since $P(G, V; \lambda) = P(G; \lambda)$, the pivot condition holds in this case by Proposition 2.2, and the basis step is established.

For the inductive step, let G be a graph such that at least one K-proper edge of G has an endpoint in V-K, and assume that the Pivot Theorem holds for all graphs with fewer such edges than G. Now let x be a K-proper edge of G. By Theorem 4.1, the pivot equation is valid if x has at least one endpoint in $V - K$. Thus assume that x has both endpoints in K. Since not every K-proper edge of G has both endpoints in K, we may choose a K-proper edge y with at least one endpoint in $V - K$. Clearly $y \neq x$. Note that since x has both endpoints in K, the edge y, viewed as an edge in $G \mid x$, has at least one endpoint which is not in $K \mid x$. Thus the pivot equation holds for $G \mid x$ with pivot y. Similarly, the pivot equation holds for $G-x$ with pivot y. We may now conclude from Lemma 4.5 that the pivot equation holds for G with pivot x, and the proof is complete. □

It can be easily shown, by induction on the number of edges in G, that the discrete graph condition and the pivot condition imply the next two results.

Corollary 4.4. *Let $G=(V, E)$ be a graph and K a subset of V. If G is the disjoint union of graphs G_1 and G_2 then*

$$P(G, K; \lambda)=P(G_1, K \cap V(G_1); \lambda) \cdot P(G_2, K \cap V(G_2); \lambda).$$

Corollary 4.5. *Let $G=(V, E)$ be a graph and K a subset of V. If G has a degree-one point which is not in K, then $P(G, K; \lambda)=0$.*

We also note the following elementary consequence of the definition of $P(G, K; \lambda)$.

Corollary 4.6. *Let $G=(V, E)$ be a graph and K a subset of V. If v is an isolated point of G which is not in K, then $P(G, K; \lambda)=P(G-v, K; \lambda)$.*

We conclude this section with an extension of Stanley's result on acyclic orientations.

Theorem 4.3. *Let $G=(V, E)$ be a graph with n points and K a subset of V. If i is the number of isolated points of G lying in $V-K$, then $P(G, K; \lambda)\,|_{\lambda=-1}=(-1)^{n-i} N_K(G)$.*

Proof. By Proposition 4.3, $\nu(G, K)=(-1)^{n-i} N_K(G)$ satisfies the pivot equation. Since $P(G, K; \lambda)\,|_{\lambda=-1}$ satisfies the pivot condition, by Lemma 4.1, the two functions coincide if they have the same value on edgeless graphs. When G is edgeless, $n-i=|K|$, so that $\nu(G, K)=(-1)^{|K|}$. Since $P(G, K; \lambda)=\lambda^{|K|}$ when G is edgeless, the proof is complete. □

5. Topological interpretations of $P(G, K; \lambda)$

In this section we derive topological interpretations for the coefficients of the extended chromatic polynomial. Two of these characterizations generalize Whitney's theorems on the classical chromatic polynomial (Theorems 2.1 and 2.2). We also show that the results presented here have, as corollaries, several of the alternate characterizations for the invariant $d_K(G)$ originally established in [4].

Several of the formulations for $P(G, K; \lambda)$ given in this section are expressed directly as polynomials. Thus we note the following simple proposition.

Proposition 5.1. *The polynomial $f(G, K; \lambda) = \sum_{j=1}^{m} b_j(G, K)\lambda^j$ satisfies the pivot condition if and only if $b_j(G, K)$ satisfies the pivot condition for every $j = 1, \ldots, m$.*

For any graph H and set K, we denote by $c(H, K)$ the number of connected components of H which contain at least one point of K. Recall that $e(H)$ denotes the number of edges in H.

Theorem 5.1. *If $G = (V, E)$ is a graph and $K \subseteq V$, then*

$$P(G, K; \lambda) = \sum_{S} (-1)^{e(S)} \lambda^{c(S, K)},$$

where the summation is over all spanning subgraphs S of G.

Proof. Let $f(G, K; \lambda)$ be the function defined by the right side of the equation in the statement of the theorem. If G has no edges, then the only spanning subgraph of G is G itself; since $e(G) = 0$ and $c(G, K) = |K|$, $f(G, K; \lambda) = \lambda^{|K|}$. Since the preceding shows that f satisfies the discrete graph condition, we may conclude the proof by showing that f satisfies the pivot condition.

Let G be a graph and x an edge of G. Clearly the spanning subgraphs of $G - x$ are precisely the spanning subgraphs of G which do not contain the edge x. Thus the value of $f(G, K; \lambda) - f(G - x, K; \lambda)$ is given by the summation in the statement of the theorem with S running over the spanning subgraphs of G which contain the edge x. If we map each such subgraph S to $S \mid x$, the result is a bijection from the set of spanning supgraphs of G containing x to the set of all spanning subgraphs of $G \mid x$. If S is a spanning subgraph of G containing x, then $c(S, K) = c(S \mid x, K \mid x)$ and $e(S) = e(S \mid x) - 1$. Thus

$$f(G, K; \lambda) - f(G - x, K; \lambda) = -f(G \mid x, K \mid x; \lambda)$$

and the proof is complete. □

The terms of the expression for $P(G, K; \lambda)$ given in Theorem 5.1 may be rearranged to yield expressions for the coefficients of $P(G, K; \lambda)$ Let $S_0(G, K, j)$ denote the number of spanning subgraphs S of G with $e(S)$ odd and $c(S, K) = j$. Similarly, $S_E(G, K, j)$ is the number of spanning subgraphs with $e(S)$ even and $c(S, K) = j$.

Corollary 5.1 *If $G = (V, E)$ is a graph and $K \subseteq V$ with $k = |K|$, then*

$$P(G, K; \lambda) = \sum_{j=1}^{k} (S_E(G, K, j) - S_0(G, K, j))\lambda^j.$$

By Proposition 5.1, the coefficients of the extended chromatic polynomial are functions which satisfy the pivot equation. Since each such function is determined by its value on edgeless graphs, we may easily deduce as a corollary the following result of the paper [4].

Corollary 5.2. *If $G = (V, E)$ is a graph and $K \subseteq V$, then $D_K(G) = |S_O(G, K, 1) - S_E(G, K, 1)|$.*

The next therorem is immediate from Corollaries 5.1 and 5.2.

Theorem 5.2. *If $G = (V, E)$ is a graph and K is a subset of V, then*

$$P(G, K; \lambda)/\lambda|_{\lambda=0} = (-1)^{|E|} d_K(G).$$

Definition 5.1 (K-covering subgraph). Let $G = (V, E)$ be a graph and K a subset of V. A *K-covering subgraph* of G is a subgraph H of G such that H contains all points of K, every connected component of H contains a point of K, and every edge of G has at least one endpoint in H.

Notice that if G has no edges, K is the only K-covering subgraph of G. With this observation as the starting point, the arguments in the proof of Theorem 5.1 can be used to prove the following

Theorem 5.3. *If $G = (V, E)$ is a graph and $K \subseteq V$, then $P(G, K; \lambda) = \sum_H (-1)^{e(H)} \lambda^{c(H)}$, where the summation is over all K-covering subgraphs H of G, $e(H)$ is the number of edges in H, and $c(H)$ is the number of connected components of H.*

The next result may be derived from Theorem 5.3 in exactly the same manner as Corollary 5.2 was derived from Theorem 5.1.

Corollary 5.3. *If $G = (V, E)$ is a graph and $K \subseteq V$, then $D_K = |H_O(G, K) - H_E(G, K)|$, where $H_O(G, K)$ and $H_E(G, K)$ are the number of connected K-covering subgraphs of G with an odd and even number of edges, respectively.*

Our next objective is a topological characterization for the extended chromatic polynomial analogous to Theorem 2.2. As in the case of that theorem, the characterization involves spanning forests having external activity zero. However, the spanning forests must be properly related to the set K. This is made precise in the next two definitions.

Definition 5.2. (K-proper subgraph). Let $G = (V, E)$ be a graph and K a subset of V. A subgraph H of G is a *K-proper subgraph* of G if every connected component of H which is not an isolated point of G contains at least one point of K.

We next introduce an extended notion of Tutte's internal activity (Definition 2.2). If x is an edge of a K-proper spanning forest F of G, then we say that x is *replaceable in* F by an edge y if y is not in $F-x$ and $(F-x)+y$ is a K-proper forest of G. Note that if x is replaceable in F by y, then F and $(F-x)+y$ have the same number of components. The set of all edges y such that x is replaceable in F by y is denoted $R(F, x)$. Note that $x \in R(F, x)$.

Definition 5.2 (K-internal activity). Let $<$ be a strict linear order on the edge set E of a graph $G=(V, E)$ and let K be a subset of V. If x is an edge of a spanning K-proper forest F of G, then we say that x is a *K-internally active edge of F in G* if $F-x$ is not K-proper and $x \leqslant y$ for all y in $R(F, x)$. The number of K-internally active edges of F in G is called the *K-internal activity* of F in G.

Note that if $K=V$, then every spanning forest F of G is K-proper and clearly has K-internal activity zero.

Lemma 5.1. *Let G be a graph and K a subset of the vertex set of G. If G has a degree-one point u which is not in K, then every K-proper spanning forest of G has a K-internally active edge.*

Proof. Let F be a K-proper spanning forest of G. If F does not contain the unique edge z of G incident with u, then u is a component of F which contains no point of K. Since u is not an isolated point in G and F is K-proper, this is impossible. Thus z is an edge of F. As in the previous case, $F-z$ is not K-proper. Since u has degree-one in G, $R(F, z)=\{z\}$ and thus z is an internally active edge of F in G. □

Theorem 5.4. *Let $<$ be a strict linear order on the edge set E of graph $G=(V, E)$ and let K be a subset of V. If $n=|V|$, $k=|K|$, and i is the number of isolated points of G which are not in K, then*

$$P(G, K; \lambda)=\sum_{j=i+1}^{i+k}(-1)^{n-j} m_j(G, K, <) \lambda^{j-i},$$

where $m_j(G, K, <)$ is the number of K-proper spanning forests of G which have exactly j connected components and have external activity and K-internal activity zero.

Proof. Let $f(G, K; \lambda)$ be the polynomial given in the right side of the equation above. Suppose u is an isolated point of G. Then $m_j(G, K)=m_{j-1}(G-u, K)$ for $i+1 \leqslant j \leqslant i+k$. Since $G-u$ has one fewer point and one fewer isolated point than G, it is clear that $f(G, K; \lambda)=f(G-u, K; \lambda)$. By Corollary 4.6, the same relation holds for the extended chromatic polynomial, and thus we may assume that all isolated points of G are in K. By Lemma 5.1, $f(G, K; \lambda)$ is zero if G

contains a degree-one point which is not in K. Since $P(G, K; \lambda)$ is also zero in this case, we henceforth assume that all degree-one points of G are in K.

It is a simple matter to see that $f(G, K; \lambda)$ agrees with $P(G, K; \lambda)$ when G is edgeless. Thus, by Theorem 4.2, we need only show that f satisfies the pivot condition. The proof that f satisfies the pivot condition is by induction on the number of edges of G. The basis step, when G is edgeless, is vacuously true, so we proceed to the inductive step. Let $G = (V, E)$ be a graph, K a subset of V, and $<$ a strict linear order on E. Assume that f satisfies the pivot condition for all graphs with fewer edges than G. We must show that the pivot condition holds for f on G. We first show that the pivot equation for f on G holds when the pivot is the maximum edge x of G in the order $<$. Recall that the pivot equation with pivot x holds for the polynomial f on G if and only if it holds for each of the coefficients of f. Moreover, since $G \mid x$ has one fewer point than G, the pivot equation for the coefficient of λ^j in f is equivalent to the following additive pivot equation for $m_j(G, K)$:

$$m_j(G, K) = m_j(G - x, K) + m_j(G \mid x, K \mid x).$$

We now prove the validity of the additive pivot equation for m_j with pivot x. Let F be a spanning forest of G which does not contain x. Since x is greater than all other edges of G in the order $<$, it is easy to see that F is a K-proper spanning forest of G with external and K-internal activity zero if and only if the same holds for F in $G - x$. Thus $m_j(G, K) - m_j(G - x, K)$ is equal to the number of K-proper spanning forests of G with external and K-internal activity zero and with exactly j connected components. Therefore, we must show that this is also the number of $(K \mid x)$ - proper spanning forests of $G \mid x$ with external and K-internal activity zero. We do this by showing that if x is an edge of F, then F is a K-proper spanning forest with external and K-internal activity zero in G if and only if $F \mid x$ is a $(K \mid x)$-proper spanning forest with external and K-internal activity zero in $G \mid X$. The only-if portion of the preceding statement is immediate from the fact that x is the maximum edge of G. Thus suppose that $F \mid x$ is a $(K \mid x)$-proper spanning forest with external and K-internal activity zero in $G \mid x$. It is obvious that F must then be a spanning forest of G with external activity zero. It is also clear that x is the only edge of F which could be a K-internally active edge of F in G. Assume that this is the case. Then, since x is the maximum edge of G, $R(F, x) = \{x\}$. Let W be the vertex set of the tree T of F containing x, let T_1 and T_2 be the trees resulting when x is deleted from T, and let W_1 and W_2 be the vertex sets of T_1 and T_2, respectively. Since $F - x$ is not K-proper, one of the sets W_1, W_2 must be disjoint from K. Since $R(F, x) = \{x\}$, there are no edges of G with an endpoint in each of W and $V - W$ and x is the only edge of G with an endpoint in each of W_1 and W_2. Thus suppose that, say, W_2 contains no point of K. Since G is assumed to have no degree-one points not in K, T_2 must contain at least one edge. Let y be the least edge among those with both endpoints in W_2. Then y is not an edge of F, since if it were it would be a K-internally active edge of F in G. On the other hand, y cannot be outside F, since if it were it would be externally active relative to F in G. From this contradiction we conclude that $F \mid x$ being a $(K \mid x)$-proper spanning forest with external activity zero in $G \mid x$ implies that F is K-proper spanning forest with external activity zero in G. This completes the proof that f satisfies the pivot equation on G with pivot the maximum edge of G in the order $<$. We may now appeal to Lemma 4.5 to conclude that the pivot equation holds for f on G with any edge as the pivot. □

We note that Whitney's Theorem (Theorem 2.2) is an immediate corollary of Theorem 5.4, since $P(G, V; \lambda) = P(G; \lambda)$ and every spanning forest of $G = (V, E)$ is V-proper and has K-internal activity zero.

Another important consequence of Theorem 5.4 is the expression for the K-domination discovered in [4] and stated in Proposition 3.4. This is immediate, since by Theorem 5.2, the coefficient of λ in $P(G, K; \lambda)$ has absolute value equal to the K-domination $D_K(G)$.

The following Theorem of Rodriguez [9] may now be derived from Theorem 4.2 and a result from [4].

Theorem 5.5 (Rodriguez). *Let $G = (V, E)$ be a graph with n points and e edges, let K be a subset of V, and let $k = |K|$. If $e > 0$, then*

$$P(G, K; \lambda) = (-1)^e (1 - \lambda) \sum_{j=1}^{n-1} d(G, K, j) \lambda^j.$$

Proof. Let $f(G, K; \lambda)$ be the polynomial given by the expression on the right side of the equation in the statement of Theorem 5.5. First we note that $f(G, K; \lambda) = 0$ if G has a self-loop or degree-one point lying in $V - K$. Since this agrees with the extended chromatic polynomial, we assume that G has no self-loops or degree-one points lying in $V - K$. We also note that deletion of isolated points of G which are in $V - K$ leaves the values of both $f(G, K; \lambda)$ and $P(G, K; \lambda)$ unchanged. Thus we assume that all isolated points of G are in K.

By Lemma 4.1, we can show that f agrees with the extended chromatic polynomial for graphs with nonempty edge set by proving the following two claims:

(1) $f(G, K; \lambda) = (P(G, K; \lambda)$ if G has exactly one edge, and

(2) f satisfies the pivot condition.

Thus let G be a graph with a single edge. Since we assume that G has no self-loops, $n \geqslant 2$. Also, since G has no isolated or degree-one points outside K, K is the vertex set V of G. It is easy to verify that $P(G, K; \lambda) = P(G; \lambda) = \lambda^{n-1}(\lambda - 1)$. It is also easy to see that $d(G, K, n-1) = 1$ and $d(G, K, i) = 0$ for all $i < n - 1$, so that $f(G, K; \lambda)$ also has value $\lambda^{n-1}(\lambda - 1)$. Since this establishes claim (1), we turn to claim (2).

We now prove, by induction on e, that f satisfies the pivot equation. By

Proposition 5.1, we need only show that for each j, $1 \leq j \leq k$, the coefficient $b_j(G, K)$ of λ^j in $f(G, K; \lambda)$ satisfies the pivot equation. Note that $b_j(G, K)$ is given by:

$$b_1(G, K) = (-1)^e d(G, K, 1), \qquad b_k(G, K) = (-1)^{e+1} d(G, K, k-1),$$

and

$$b_j(G, K) = (-1)^e [d(G, K, j) - d(G, K, j-1)], \quad \text{for } 2 \leq j < k.$$

From the above it is clear that we need only show that the function $(-1)^{e(G)} d(G, K, j)$ satisfies the pivot equation. Multiplying both sides of the "reverse" pivot equation for $d(G, K, j)$ given in Proposition 3.4 by $(-1)^{e(G)}$, we have

$$\begin{aligned}(-1)^{e(G)} d(G, K, j) &= (-1)^{e(G)} d(G \mid x, K \mid x, j) - (-1)^{e(G)} d(G - x, K, j)\\ &= (-1)^{e(G)-1} d(G - x, K, j) - (-1)^{e(G)-1} d(G \mid x, K \mid x, j)\\ &= (-1)^{e(G-x)} d(G - x, K, j) - (-1)^{e(G \mid x)} d(G \mid x, K \mid x, j). \quad \square\end{aligned}$$

6. Computational formulas for $P(G, K; \lambda)$

In this section we present explicit formulas for the extended chromatic polynomial of graphs having special structures. These results are analogous to some of the well-known results concerning $P(G; \lambda)$ [8]. We first prove the following useful lemma.

Lemma 6.1. *Let $G = (V, E)$ be a graph and K a subset of V. Suppose $u \in V - K$ is a degree-two point adjacent to two distinct points v and w in G. Let G' be the graph obtained from $G - u$ by adding an edge between v and w. Then*

$$P(G, K; \lambda) = -P(G', K; \lambda).$$

Proof. Let x be the edge with endpoints u and v in G. First note that G' is isomorphic to $G \mid x$. By the pivot condition,

$$P(G, K; \lambda) = P(G - x, K; \lambda) - P(G \mid x, K \mid x; \lambda).$$

Since $G - x$ has a degree-one point not in K, we have by Corollary 4.5 that $P(G - x, K; \lambda) = 0$. $\square$

The following result can be easily proved using Lemma 6.1 and the pivot equation.

Theorem 6.1. *If $G = (V, E)$ is a cycle on n points and $K \subseteq V$ is such that $k = |K| > 0$, then $P(G, K; \lambda) = (-1)^{n-k}[(\lambda - 1)^k + (-1)^k(\lambda - 1)]$.*

Theorem 6.2. *Let $H = (V, E)$ be a graph and K a subset of V. Let G be the graph obtained from H by adding a new point u and edges connecting u to every point of K. Then $P(G, K \cup \{u\}; \lambda) = \lambda P(H, K; \lambda - 1)$.*

Proof. One may prove this theorem by direct appeal to the definition. Suppose that j is an integer in $[\lambda]$ and (D, α) is a proper λ-coloring of K within H such that, for all v in V, $\alpha(v) \neq j$. Notice that there are exactly as many such pairs as there are proper $(\lambda - 1)$-colorings of K within H. We claim that for each such pair (D, α) there is a unique proper λ-colouring (D', α') of $K \cup \{u\}$ within G such that D is a subdigraph of D' and the restriction of α' to K is α. Notice that, since u is adjacent to every point of K, such an α' would have to send u to j. If D' is α'-compatible and x is an edge of G connecting u to some point v of K, the arc of D' corresponding to x is of the form $u \to v$ if $\alpha(v) < j$ and $v \to u$ if $\alpha(v) > j$. Thus we see that α' and D' are uniquely determined by D and α. If we define α' and D' as stated above, it is clear that D' is α'-compatible. Moreover, D' is acyclic, since otherwise D is not α-compatible. Thus (D', α') is a proper λ-coloring of $K \cup \{u\}$ within G. It is simple to see that every proper λ-coloring of $K \cup \{u\}$ within G is obtained in the above fashion. Noting that the sign of $P(G, K \cup \{u\}; \lambda)$ is the same as that of $P(H, K; \lambda)$, the theorem then follows. □

If G is a graph and u is a point of G, then the *closed neighborhood* of u in G is the subgraph of G induced by u and the points adjacent to u in G. The next theorem provides a reduction formula for points of G which are not in K and whose closed neighborhoods are complete graphs. A *complete graph* is a graph that has no self-loops or multiple edges, and is such that every pair of distinct points are adjacent.

Theorem 6.3. *Let $G = (V, E)$ be a graph, K a subset of V, and $u \in V - K$ be such that the closed neighborhood of u in G is a complete graph. If d is the degree of u in G, then $P(G, K; \lambda) = -(d-1)P(G - u, K; \lambda)$.*

Proof. The proof is by induction on the degree d of u in G. If $d = 0$, then u is an isolated point of G and $P(G - u, K; \lambda) = P(G, K; \lambda)$ by Corollary 4.6. Since this establishes the basis step, we assume $d > 0$ and that the theorem holds when u has degree less than d. Let x be an edge of G incident with u. Since G has no multiple edges incident with u, $G \mid x$ does not have a self-loop at u. It then follows that $G \mid x$ is just $G - u$ with possibly some replicated edges. Since u is not in K, $K \mid x$ may be identified with K. Since replication of edges does not affect the value of the extended chromatic polynomial, $P(G - u, K; \lambda) = P(G \mid x, K \mid x; \lambda)$. Note that the closed neighborhood of u in $G - x$ is a complete graph and that u has degree $d - 1$ in $G - x$. By the pivot condition and the inductive hypothesis we have

$$\begin{aligned} P(G, K; \lambda) &= P(G - x, K; \lambda) - P(G \mid x, K \mid x; \lambda) \\ &= P(G - x, K; \lambda) - P(G - u, K; \lambda) \\ &= -(d-2)P((G - x) - u, K; \lambda) - P(G - u, K; \lambda) \\ &= -(d-2)P(G - u, K; \lambda) - P(G - u, K; \lambda) \\ &= -(d-1)P(G - u, K; \lambda) \quad \square \end{aligned}$$

We may use Theorem 6.3 to reduce the computation of $P(G, K; \lambda)$ for G a complete graph to the corresponding computation for the classical polynomial. The result is the following

Theorem 6.4. *Let $G = (V, E)$ be a complete graph on n points and let K be a subset of V with $|K| = k \geq 2$. Then $P(G, K; \lambda) = (-1)^{n-k}[(n-2)!/(k-2)!]\lambda(\lambda - 1) \cdots (\lambda - k + 1)$.*

Theorem 6.5. *Let $G = (V, E)$ be a graph and K a subset of V. Suppose that G is the union of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ whose intersection $G_3 = (V_3, E_3)$ is either a point of K or a complete graph containing at least two points of K. Then $P(G, K; \lambda) = [P(G_1, K \cap V_1); \lambda) \cdot P(G_2, K \cap V_2); \lambda)]/P(G_3, K \cap V_3); \lambda)$.*

We omit the proof of Theorem 6.5, as it follows the same outline as several proofs in this paper. Suffice it to say that the proof uses induction on the number of edges of G having at least one endpoint in $V - V_3$.

The value of the classical polynomial on trees is particularly simple, being independent of the structure of the tree: if G is a tree on n points, then $P(G, \lambda) = \lambda(\lambda - 1)^{n-1}$. The case for the extended chromatic polynomial is a good deal more involved. The value of $P(G, K; \lambda)$ for G a tree does however show some degree of independence from the structure of G in that it depends solely on the degrees in G of the points in $V - K$.

Theorem 6.6 *Let $T = (V, E)$ be a tree with $n \geq 2$ points, and let K be a nonempty subset of V. If $d_1, d_2, \ldots, d_s$ are the degrees in T of the points of $V - K$ and σ is the sum of those degrees, then*

$$P(T, K; \lambda) = (\lambda - 1)^{n-1+s-\sigma} \prod_{m=1}^{s} [(\lambda - 1)^{d_m - 1} - \lambda^{d_m - 1}].$$

Proof. Let $\varphi(G, K; \lambda)$ be the polynomial given by the right side of the equation in the statment of the theorem. For any tree $T = (V, E)$ and $K \subseteq V$, denote by $\sigma(T, K)$ the sum of the degrees in T of the points in $V - K$. Assuming that any product from 1 to 0 has value 1, we see that $\varphi(T, V; \lambda) = \lambda(\lambda - 1)^{n-1}$, and hence that the theorem holds in this case. We will prove the result for the cases where

$\sigma(T, K) \geqslant 1$ by induction on $\sigma(T, K)$. First note that if T has a degree-one point which is not in K, then one of the degrees d_j used in forming the above product is 1. In this case, the corresponding term of the product, and hence the entire product, is zero. Since, by Corollary 4.5, $P(T, K; \lambda)$ also has value zero in this case, the theorem holds when G has an endpoint which is not in K. This immediately yields the basis step of our inductive proof, since $\sigma = 1$ implies that $d_1 = 1$. We henceforth assume that T has no endpoint in $V - K$.

Let T be a tree and K be a subset of the vertex set of T such that $\sigma(T, K) > 1$. Assume that the theorem holds for all trees T' and associated vertex sets K' with $\sigma(T', K') < \sigma(T, K)$. Since K and $V - K$ are nonempty, the tree T must contain an edge x with an endpoint u in $V - K$ and another endpoint v in K. It is simple to see that we may choose x so that the connected component of v in $T - x$ contains only points from K. Since all degree-one points of T are in K, the connected component of u in $T - x$ must contain a point w of K. Let x' be the edge with endpoints v and w. Clearly, x' is not an edge of T, and $T' = (T - x) + x'$ is a tree with the same vertex set as T. Applying the pivot equation to $P(T, K; \lambda)$ with pivot x and to $P(T', K; \lambda)$ with pivot x', we obtain

$$P(T, K; \lambda) = P(T - x, K; \lambda) - P(T \mid x, K \mid x; \lambda),$$
$$P(T', K; \lambda) = P(T' - x', K; \lambda) - P(T' \mid x', K \mid x'; \lambda).$$

Let the degrees in T of the points of $V - K$ be given as $d_1, \ldots, d_s$, with d_s the degree of u. Then the degrees on the non-K points of T' are the same as the degrees of the non-$(K \mid x')$ points of $T' \mid x'$, namely $d_1, \ldots, d_s - 1$. Moreover, the degrees of the non-$(K \mid x)$ points of $T \mid x$ are $d_1, \ldots, d_{s-1}$. Thus $\sigma(T', K) = \sigma(T' \mid x', K \mid x') = \sigma(T, K) - 1$, and $\sigma(T \mid x, K) = \sigma(T, K) - d_s$. Thus we may apply the inductive hypothesis to each of these cases. Since $T - x = T' - x'$, we arrive at the following:

$$\begin{aligned} P(T, K; \lambda) &= P(T', K; \lambda) + P(T' \mid x', K \mid x'; \lambda) - P(T \mid x, K \mid x; \lambda) \\ &= \varphi(T', K; \lambda) + \varphi(T' \mid x', K \mid x'; \lambda) - \varphi(T \mid x, K \mid x; \lambda). \end{aligned}$$

It is now a matter of algebraic manipulation to show that the immediately preceding equation simplifies to $P(T, K; \lambda) = \varphi(T, K; \lambda)$. $\square$

Conclusions

In this paper we have introduced an extended chromatic polynomial $P(G, K; \lambda)$ and shown that this polynomial possesses many of the interesting properties of the classical chromatic polynomial. As already noted, this work was stimulated by a connection between $P(G; \lambda)$ and the all-terminal network reliability problem, namely that the coefficient of λ in $P(G; \lambda)$ and $d_V(G)$ have the same absolute value. In this paper we have established a similar connection between $P(G, K; \lambda)$ and $d_K(G)$, which is of interest in the K-terminal reliability problem. Let $G=(V, E)$ be a graph whose edges fail with known probabilities and let K be a subset of V. The K-terminal reliability problem for G is the problem of computing the probability that the points of K lie in a connected component of G. The invariant $d_K(G)$ was shown in [12] to play a key role in the computational complexity of a class of solutions for this problem. In Section 3 we introduced the extension $d(G, K, j)$, $1 \leqslant j \leqslant |K|$, of $d_K(G)$ which naturally leads to an extension of the K-terminal reliability problem. We can now define a reliability problem for graph G, set K, and positive integer j as follows: the *(K, j)-reliability* of G is the probability that the points of K lie in at most j connected components of G. While there are many known results and applications for the K-terminal reliability problem, this new reliability measure remains to be explored. Another open problem arising from this paper is to find a way to generalize Tutte's characterization of $P(G; \lambda)$ to $P(G, K; \lambda)$.

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