JOURNAL OF FUNCTIONAL ANALYSIS **140**, 300–311 (1996)
ARTICLE NO. 0109

Dilations and Commutant Lifting for Jointly Isometric Operators—A Geometric Approach

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Received February 2, 1993; revised October 18, 1994

A tuple of commuting contractions $\mathbf{T}=(T_1, T_2, ..., T_n)$ is called a joint-isometry if $\sum T_j^* T_j = I$. We give a geometric proof that joint isometries have a regular unitary dilation and that its commutant lifts. We also show that $\mathbf{T}$ is subnormal and that its minimal normal extension is also jointly isometric. © 1996 Academic Press, Inc.

1. INTRODUCTION

Let H be a Hilbert space and $\mathbf{T}=\{T_j\}$ be a system of commuting contractions on H. It said to be a joint-isometry if $\sum_i \|T_i(x)\|^2 = \|x\|^2$ for every $x \in H$, or, equivalently, if $\sum_i T_i^* T_i = I$ in the WOT. Athavale [2], using his previous work on operator-valued kernels, proved that finite jointly isometric systems are subnormal, and that the elements in their commutant lift to the commutatnt of the commuting normal extension. Proposition 3, which may be viewed as a commutant lifting theorem [13, p. 96], follows easily from Athavale's results in the case of finite systems, however, our approach is purely geometric, and conceptually more elementary than Athavale's work; moreover, it holds for infinite systems. Our techniques also give a direct proof of the subnormality of jointly-isometric systems.

2. DILATIONS OF JOINT-ISOMETRIES

A commutative system, finite or infinite, $\mathbf{S}=\{S_j\}_{j\in J}$ of bounded linear operators on a Hilbert space $K \supset H$ is said to be a dilation of a commutative system $\mathbf{T}=\{T_j\}_{j\in J}$ of bounded linear operators on a Hilbert space H if

$$T_{j_1}^{n_1}\cdots T_{j_r}^{n_r} = P(S_{j_1}^{n_1}\cdots S_{j_r}^{n_r})$$

for all choices of integers $n_i \geqslant 0$, $i=1, ..., r$ and for every finite set of subscripts $j_i \in J$, where P denotes the orthogonal projection of K onto H. The

0022-1236/96 $18.00

index set J may be, a usually unstated, $\{1, 2, ..., m\}$ for some positive integer m or the entire set of positive integers. The above dilation $\mathbf{S}$ is called unitary if each $S_j \in \mathbf{S}$ is unitary, and it is well-known that a tuple of two contractions has a unitary dilation, but there exist tuples of three contractions for which no unitary dilation exists. Commutant lifting theorems concern operators on H commuting with $\mathbf{T}$ that have dilations on K commuting with $\mathbf{S}$. Sz-Nagy and Foias's book, [13], provides an excellent general reference for these topics.

For the rest of this section given multi-index $I = (n_1, ..., n_r)$, $|I|$ will denote the sum $|n_1| + \cdots + |n_r|$, $I! = n_1! \cdots n_r!$, $I^+ = (n_1^+, ..., n_r^+)$, $I^- = (n_1^-, ..., n_r^-)$, where $n_i^+ = \sup(n_i, 0)$ and $n_i^- = \sup(-n_i, 0)$; ε_k denotes the multi-index whose kth entry is 1 and all other entries are 0. Given r fixed, we denote $\mathbf{I} = \{I = (n_i, ..., n_r) : n_i \text{ is } 0 \text{ or } 1\}$. For a finite set u of positive integers, $I < u$ will mean $I \in \mathbf{I}$ with $r = \max u$ and $n_i = 0$ for all $i \notin u$. Given a commuting system $\{T_j\}$ operators, denote $T^I = T_1^{n_1} \cdots T_r^{n_r}$.

DEFINITION. A unitary dilation $\mathbf{S}$ of a commutative tuple of contractions $\mathbf{T}$ is called *regular* if for all multi-indices I of integers

$$(T^{I^-})^* \, (T^{I^+}) = PS^I.$$

Our proofs heavily rely upon the following theorem.

THEOREM 1 [13, p. 37]. *$\{T_j\}_{j \in J}$ has a regular unitary dilation if and only if*

$$S(u) = \sum_{I < u} (-1)^{|I|} (T^I)^* \, T^I \geqslant 0$$

for every finite subset u of J.

We may require that the regular dilation be minimal, i.e., that the subspaces $(S^I H)$, $I \in Z^J$, span K; in this case, the regular dilation is unique up to an isomorphism.

A corollary of this result states [13, p. 39] that $\mathbf{T}$ has a regular unitary dilation if $\sum \|T_i\|^2 \leqslant 1$. We begin with a straightforward generalization of this result.

PROPOSITION 2. *Let $\mathbf{T} = \{T_j\}_{j \in J}$ be a commutative system of joint-contractions, i.e., $\sum \|T_j(x)\|^2 \leqslant \|x\|^2$ for every $x \in H$. Then $\mathbf{T}$ has a regular unitary dilation.*

Proof. Let u be a finite subset of J, and without loss of generality, let $u = \{1, 2, ..., r\}$. For $I \in \mathbf{I}$, $h \in H$, and $p = 1, 2, ..., r$, let

$$
\begin{aligned}
a_p(h) &= \sum_{|I|=p} \|T^I h\|^2 \\
&\leqslant \sum_{j=1}^{r} \sum_{|I|=p-1} \|T_j T^I h\|^2 \\
&= \sum_{|I|=p-1} \left(\sum_{j=1}^{r} \|T_j T^I h\|^2 \right) \\
&\leqslant \sum_{|I|=p-1} \|T^I h\|^2 \\
&= a_{p-1}(h),
\end{aligned}
$$

where the second inequality follows because $\mathbf{T}$ is an joint contraction. Hence,

$$
\begin{aligned}
(S(u)h, h) &= \left(\sum_{I \in \mathbf{I}} (-1)^{|I|} (T^I)^* T^I h, h \right) \\
&= \sum_{p=0}^{r} (-1)^p a_p(h) \\
&= [a_0(h) - a_1(h)] + [a_2(h) - a_3(h)] + \cdots \\
&\geqslant a_0(h) - a_1(h) \\
&= \|h\|^2 - \sum_{j=1}^{r} \|T_j h\|^2 \\
&\geqslant 0.
\end{aligned}
$$

Thus, $S(u) \geqslant 0$ which implies $\mathbf{T}$ has a regular unitary dilation. ∎

Since von Neumann's inequality, $\|p(T_1, ..., T_r)\| \leqslant \sup\{|p(z)| : |z_i| \leqslant 1\}$ for all r-variable polynomials p, follows from the existence of a unitary dilation for $\mathbf{T}$, we have as an immediate corollary to Proposition 1 the result proven in [7] that joint contractions satisfy von Neumann's inequality. (Compare this with [12].)

Our next proposition requires a combinatoric lemma essentially concerned with partitioning the set of combinations of r objects $p+1$ at a time into classes each element of which contains a common combination of the r objects taken p at a time. This is probably well-known but our lemma will establish the notation used in the proof. Note that we use multi-index notation in lieu of combinations.

LEMMA 3. *Let* $I_p = \{I \in I : |I| = p\}$ *for* $p < r$. *Then there are* $C_p = \{L_1, ..., L_{n_p}\} \subset I_p$ *and* $D_{L_1}, ..., D_{L_{n_p}} \subset \{1, 2, ..., r\}$ *such that*

$$I_{p+1} = \bigcup_{i=1}^{n_p} B_{L_i}$$

where $B_{L_i} = \{I \in I_{p+1} : I = L_i + \varepsilon_k, k \in D_{L_i}\}$ *and* $\{B_{L_i}\}$ *are pairwise disjoint.*

Proof. Let $L_1 \in I_p$ be arbitrary and let

$$D_{L_1} = \{k\colon L_1 \text{ has 0 in the } k\text{th coordinate}\}.$$

Note that this determines B_{L_1}. Suppose $L_1, ..., L_l, D_{L_1}, ..., D_{L_l}$ have been chosen such that $\{B_{L_i}\colon i = 1, ..., l\}$ are pairwise disjoint and each

$$D_{L_i} = \left\{k\colon L_i \text{ has 0 in the } k\text{th coordinate and } (L_i + \varepsilon_k) \notin \bigcup_{j=1}^{i-1} B_{L_j}\right\}.$$

If $\bigcup_{i=1}^{l} B_{L_i} = I_{p+1}$, let $n_p = l$ and the construction is complete. Otherwise, choose $I \in I_{p+1}$, $I \notin \bigcup_{i=1}^{l} B_{L_i}$ and let $L_{(l+1)} = I - \varepsilon_{k_0}$ for any ε_{k_0} such that I has 1 in that coordinate. $D_{L_{(l+1)}}$ defined as above contains at least the one element k_0, and hence our process must terminate after finitely many steps. ∎

Note that the construction is not unique even up to permutations and the cardinalities of the D_{L_i} in general differ.

The following theorem, proved by different means, appears in [3, Prop. 8] for finite tuples.

PROPOSITION 4. *Let* $\mathbf{T} = \{T_j\}_{j \in J}$ *be a joint-isometry on* H. *Let* A *be a contraction on* H *such that* $AT_j = T_j A$ *for all* j. *Then* $\mathbf{T}_0 = (\mathbf{T} \cup \{A\})$ *has a regular unitary dilation.*

Proof. Denote $A = T_0$ and $J_0 = \{0\} \cup J$ and let u be a finite subset of J_0. If $0 \notin u$, then, using the previous notation, $S(u) \geqslant 0$ by Proposition 8. Hence without loss of generality, let $u = \{0, 1, ..., r\}$ and $h \in H$. Then

$$\begin{aligned}(S(u)\,h, h) = \|h\|^2 &- \left(\sum_{j=1}^{r} \|T_j h\|^2 + \|Ah\|^2\right) \\ &+ \left(\sum_{1 \leqslant j_1 < j_2 \leqslant r} \|T_{j_1} T_{j_2} h\|^2 + \sum_{j=1}^{r} \|T_j A h\|^2\right) + \cdots \\ &+ (-1)^{r+1} \|T_1 \cdots T_r A h\|^2\end{aligned}$$

$$= \|h\|^2 + \sum_{p=1}^{r} (-1)^p \left[\sum_{I \in I_p} \|T^I h\|^2 + \sum_{I \in I_{p-1}} \|T^I A h\|^2 \right]$$

$$+ (-1)^{r+1} \|T_1 \cdots T_r A h\|^2$$

$$= (\|h\|^2 - \|Ah\|^2) + \sum_{p=1}^{r} (-1)^p \left(\sum_{I \in I_p} \|T^I h\|^2 - \sum_{I \in I_p} \|T^I A h\|^2 \right)$$

$$= \sum_{p=0}^{r} (-1)^p Q_p$$

$$= \sum_{p=0}^{s} (Q_{2p} - Q_{2p+1}) + Q_{2s+2}$$

$$\geqslant \sum_{p=0}^{s} (Q_{2p} - Q_{2p+1}),$$

where s is the greatest integer strictly less than $r/2$ and

$$Q_p = \sum_{I \in I_p} (\|T^I h\|^2 - \|T^I A h\|^2), \qquad p = 0, 1, ..., r$$

and Q_{2s+2} is either Q_r or 0 depending upon r is even or odd.

We note that $Q_p \geqslant 0$ for all p since

$$\|T^I A h\|^2 = \|A T^I h\|^2 \leqslant \|A\|^2 \, \|T^I h\|^2$$

and $\|A\| \leqslant 1$. We will conclude the proof by showing that

$$Q_{2p} - Q_{2p+1} \geqslant 0 \qquad \text{for} \quad p = 0, 1, ..., s.$$

We use the notation from the previous lemma and let C'_p be the relative complement $C'_p = (I_p \backslash C_p)$, $p = 0, ..., 2s$; also let $D'_L = J \backslash D_L$ (note that $D_L \subset u = \{1, ..., r\} \subset J$). Note that for any $x \in H$ and any p,

$$\sum_{I \in I_{2p+1}} \|T^I x\|^2 = \sum_{I \in C_{2p}} \sum_{K \in B_I} \|T^K x\|^2 = \sum_{I \in C_{2p}} \sum_{l \in D_I} \|T_l T^I x\|^2$$

by the partitioning of I_{2p+1} in the lemma and that

$$\|x\|^2 = \sum_{j \in J} \|T_j x\|^2 = \sum_{j \in D_L} \|T_j x\|^2 + \sum_{j \in D'_L} \|T_j x\|^2.$$

Now,

$$
\begin{aligned}
&Q_{2p} - Q_{(2p+1)} \\
&\quad = \left(\sum_{I \in I_{2p}} \|T^I h\|^2 - \sum_{I \in I_{(2p+1)}} \|T^I h\|^2 \right) \\
&\qquad - \left(\sum_{I \in I_{2p}} \|T^I A h\|^2 - \sum_{I \in I_{2p}} \|T^I A h\|^2 \right) \\
&\quad = \left(\sum_{I \in C_{2p}} \|T^I h\|^2 + \sum_{I \in C'_{2p}} \|T^I h\|^2 - \sum_{I \in C_{2p}} \sum_{l \in D_I} \|T_l T^I h\|^2 \right) \\
&\qquad - \left(\sum_{I \in C_{2p}} \|T^I A h\|^2 + \sum_{I \in C'_{2p}} \|T^I A h\|^2 - \sum_{I \in C_{2p}} \sum_{l \in D_I} \|T_l T^I A h\|^2 \right) \\
&\quad = \left[\sum_{I \in C_{2p}} \left(\|T^I h\|^2 - \sum_{l \in D_I} \|T_l T^I h\|^2 \right) + \sum_{I \in C'_{2p}} \|T^I h\|^2 \right] \\
&\qquad - \left[\sum_{I \in C_{2p}} \left(\|T^I A h\|^2 - \sum_{l \in D_I} \|T_l T^I A h\|^2 \right) + \sum_{I \in C'_{2p}} \|T^I A h\|^2 \right] \\
&\quad = \left(\sum_{I \in C_{2p}} \left(\sum_{l \in D'_I} \|T_l T^I h\|^2 \right) - \sum_{I \in C_{2p}} \left(\sum_{l \in D'_I} \|T_l T^I A h\|^2 \right) \right) \\
&\qquad + \left(\sum_{I \in C'_{2p}} \|T^I h\|^2 - \sum_{I \in C'_{2p}} \|T^I A h\|^2 \right) \\
&\quad = \sum_{I \in C_{2p}} \sum_{l \in D'_I} (\|T_l T^I h\|^2 - \|A T_l T^I h\|^2) \\
&\qquad + \sum_{I \in C'_{2p}} (\|T^I h\|^2 - \|A T^I h\|^2) \\
&\quad \geqslant (1 - \|A\|^2) \left[\sum_{I \in C_{2p}} \sum_{l \in D'_I} \|T_l T^I h\|^2 + \sum_{I \in C'_{2p}} \|T^I h\|^2 \right] \\
&\quad \geqslant 0,
\end{aligned}
$$

and the theorem is established. ▮

3. SUBNORMALITY

A commutative system $\mathbf{T}$ of operators on a Hilbert space is said to be subnormal if there exists a system of commuting normal operators $\mathbf{N}$ on a Hilbert space $K \supseteq H$ such that $\mathbf{N} = \mathbf{T}$ on H. A single isometry is, clearly, subnormal. We show that joint-isometries are also subnormal; see Athavale papers [2, 3] for a proof of this and a discussion of joint-isometries albeit from a different view point from ours.

Now a word about notation: Let n and m be positive integers, m is fixed for Proposition 1, and $J=(j_1, j_2, ..., j_n)$ be a multi-index where $1 \leqslant j_i \leqslant m$. The collection of all such multi-indices J will be denoted by $F(n)$ (we have suppressed writing m). Then $\mathbf{T}_J$ stands for the product of operators $T_{j_1} T_{j_2} \cdots T_{j_n}$.

To prove the subnormality of joint-isometries, we will use Agler's [1] criterion: *A contraction S is subnormal if and only if*

$$(I - S^*S)^{[n]} = \sum_{k=0}^{n} (-1)^k \, {}^nC_k S^{*^k} S^k \geqslant 0, \qquad (n \geqslant 1).$$

PROPOSITION 5. *Let $(T_1, T_2, ..., T_m)$ be a joint-isometry. Then,*

$$(I - T_i^* T_i)^{[n]} = \sum_{J \in F'(n)} T_J^* T_J,$$

where the prime over $F(n)$ indicates that $i \notin J$ for $J \in F(n)$.

Proof. The proof is by induction on n, but first notice that:

$${}^{n+1}C_k = {}^nC_k + {}^nC_{k-1} \quad n \geqslant k \geqslant 1.$$

Assume the claim of the proposition holds for some $n \geqslant 1$, then

$$\begin{aligned}
(1 - T_i^* T_i)^{[n+1]} &= (1 - T_i^* T_i)^{[n]} - T_i^* (1 - T_i^* T_i)^{[n]} \, T_i \\
&= \sum_{J \in F'(n)} T_J^* T_J - T_i^* \left(\sum_{J \in F'(n)} T_J^* T_J \right) T_i \\
&= \sum_{J \in F'(n)} T_J^* (1 - T_i^* T_i) \, T_J \\
&= \sum_{j \neq i} \sum_{J \in F'(n)} T_J^* T_j^* T_j T_J \\
&= \sum_{J \in F'(n+1)} T_J^* T_J,
\end{aligned}$$

establishing the assertion for $n+1$. ∎

We may note that for the validity of the proof, the commutativity of the tuple $\mathbf{T}$ is not necessary; the proof still holds if

$$T_{\tilde{J}}^* T_{\tilde{J}} = T_J^* T_J,$$

where J is any multi-index in $F(n)$ and $\tilde{J}$ is any permutation of J.

COROLLARY 6. *If* $(T_1, T_2, ..., T_m)$ *is a joint-isometry, then* T_i *are subnormal.*

We may note that Proposition 4 holds in a C^* algebra, and since Agler's criterion is valid in the C^* algebra setting, Corollary 2 holds in the C^* algebra setting as well.

The commutant of a subnormal operator cannot be, in general, lifted to the commutant of its minimal normal extension, but in the case of joint-isometries it can be done:

PROPOSITION 7. *Let* $(T_1, T_2, ..., T_m)$ *be a joint-isometry. Then there exists normal extensions* M_i *of* T_i *such that* $(M_1, M_2, ..., M_m)$ *is a joint-isometry.*

Proof. Suppose $(T_1, T_2, ..., T_m, S)$ is a joint-isometry and let $N\colon K \to K$ be the minimal normal extension of S. Consider the standard extension of T_i, $1 \leqslant i \leqslant m$, to K:

$$N_i\left(\sum_k N^{*k} f_k\right) = \left(\sum_k N^{*k} T_i f_k\right) \quad f_1, f_2, ..., f_n \in H.$$

We will verify the criterion [5], or [6, 10.4 Theorem, Chap. 2, p. 80], for N_i to be well-defined

$$\exists c > 0 \ni \sum_{j,k=0}^{n} \langle S^j T_i f_k, S^k T_i f_j \rangle \leqslant c \sum_{j,k=0}^{n} \langle S^j f_k, S_k f_j \rangle, \tag{1}$$

and that

$$\sum_{i}^{m} N_i^* N_i + N^* N = I. \tag{2}$$

Since S and the T_1 $1 \leqslant i \leqslant m$ commute, the left-hand-side of Eq. (1) is

$$\begin{aligned} &\sum_{j,k=0}^{m} \langle T_i^* T_i S_j f_k, S^k f_j \rangle \\ &\quad = \sum_{j,k=0}^{m} \left\langle \left(1 - \sum_{q \neq i} T_q^* T_q - S^* S\right) S_j f_k, S_k f_j \right\rangle \\ &\quad = \sum_{j,k=0}^{n} \langle S^j f_k, S^k f_j \rangle \\ &\qquad - \sum_{j,k=0}^{n} \left\langle \sum_{q \neq i} S^j T_q f_k, S^k T_q f_j \right\rangle - \langle S^{j+1} f_k, S^{k+1} f_j \rangle \end{aligned} \tag{3}$$

By Halmos's criterion for subnormality [8], or [6, Theorem 1.9, Chap. II, p. 30], the last two expressions of (3) are non-negative; thus inequality (1) holds with $c=1$.

We will now verify Eq. (2):

$$\sum_i \left\|N_i \sum_k N^{*k} f_k\right\|^2 + \left\|N \sum_k N^{*k} f_k\right\|^2$$

$$= \sum_i \left\|\sum_k N^{*k} T_i f_k\right\|^2 + \left\|\sum_k N^{*k} N f_k\right\|^2$$

$$= \sum_i \sum_{k,l} \langle N^{*k}_{\;i} T_i f_k, N^{*l} T_i f_l\rangle + \sum_{k,l} \langle N^{*k} N f_k, N^{*l} N f_l\rangle.$$

Since N is normal, this is

$$= \sum_i \sum_{k,l} \langle N^l T_i f_k, N^k T_i f_l\rangle + \langle N^l N f_k, N^k N f_l\rangle$$

$$= \sum_i \sum_{k,l} \langle S^l T_i f_k, S^k T_i f_l\rangle + \langle S^l S f_k, S^k S f_l\rangle$$

$$= \sum_i \sum_{k,l} \langle T_i S^l f_k, T_i S^k f_l\rangle + \langle S S^l f_k, S S^k f_l\rangle$$

$$= \sum_i \sum_{k,l} \langle T_i^* T_i S^l f_k, S^k f_l\rangle + \langle S^* S S^l f_k, S^k f_l\rangle$$

$$= \sum_{k,l} \langle S^l f_k, S^k f_l\rangle = \sum_{k,l} \langle N^l f_k, N^k f_l\rangle.$$

Since N is normal the last expression is $\|\sum N^k f_k\|^2$, which verifies Eq. (2).

Moreover, Bram's theorem [5, Theorem 8] or [6, Theorem 10.5, Chap. II, p. 81], implies that the lifting of a normal operator (to the commutant of some minimal normal extension) remains normal. It is a triviality to verify that $(N_1, N_2, ..., N_m, N)$ is a commuting tuple, so it is, in fact, a joint-isometry.

Thus given a joint-isometry $(T_1, T_2, ..., T_m)$ we can now initiate an inductive process where at each stage the operators get lifted to a joint isometry; at the $m-i$th stage the $m-i$, $m-i+1, ..., m$ operators are normal. This proves the proposition. ∎

The M_i are the minimal jointly-isometric normal extensions of $(T_1, T_2, ..., T_m)$.

COROLLARY 8. *If* $(T_1, T_2, ..., T_m)$ *is an joint-isometry, then*

$$\sum_i T_i T_i^* \leqslant I.$$

Proof. Let $(M_1, M_2, ..., M_m T_m)$. Let P be the projection from K to H. Then use $T_i^* = PM_i^*$, and $\sum_i M_i M_i^* = I$. ▌

4. DILATIONS OF JOINT-ISOMETRIES

Recall that a commutative system $\mathbf{T} = \{T_j\}$, finite or infinite, is called a joint-contraction if $\Sigma T_i^* T_i \leqslant I$. The reader may want to see the example in the following section, in the context of the following proposition.

PROPOSITION 9. *Suppose* $(T_1, T_2, ..., T_m)$ *is a joint-contraction on H. Then the commuting tuple* $(T_1, T_2, ..., T_m)$ *is the dilation of a joint-isometry if and only if the unit ball in* C^m *is a complete spectral set for* $(T_1, T_2, ..., T_m)$.

Proof. Suppose $\mathbf{T} = (T_1, T_2, ..., T_n)$ has a jointly-isometric dilation $\mathbf{N} = (N_1, N_2, ..., N_m)$ on a space K, then, by Proposition 4, we may assume that the dilation is also normal. The Waelbroeck spectrum of $\mathbf{T}$, $\sigma_R(\mathbf{T})$, is contained in the polynomially convex hull of $\sigma(\mathbf{N})$, the spectrum of $\mathbf{N}$ in the C^* algebra generated by $\mathbf{N}$—here is the standard argument: Write P for the projection of K to H, and let $(\lambda_1, \lambda_2, ..., \lambda_m) = \lambda \in \sigma_R(\mathbf{T})$, and let p be a m variate polynomial. Then, by the spectral mapping theorem [4],

$$p(\lambda) \in \sigma(\mathbf{p}(\mathbf{T})).$$

Hence

$$\begin{aligned} |p(\lambda)| &\leqslant \|p(\mathbf{T})\| = \|P(p(\mathbf{N}))_{|H}\| \\ &\leqslant \|p(\mathbf{N})\| = \|p\|, \end{aligned}$$

where the last norm is the sup norm over $\sigma(\mathbf{N})$, the algebraic spectrum of $\mathbf{N}$.

Thus, in particular, the $\sigma_R(\mathbf{T})$ is contained in the unit ball of C^m, and by Averson's dilation theorem [4] the unit ball is a complete spectral set.

Conversely, if the unit ball in C^m is a complete spectral set for $\mathbf{T}$, then, there exists a commuting tuple of normal dilation $\mathbf{N}$ of $\mathbf{T}$ such that

$$\sigma_R(\mathbf{N}) \subseteq \partial B,$$

where B is the unit ball [4]. Thus, the C^* algebra spectrum of $\mathbf{N}$ is also contained in the unit sphere, so $\mathbf{N}$ is a joint-isometry. ▌

5. EXAMPLES

If $(T_1, ..., T_m)$ is any tuple of commuting bounded linear operators on H, then $(cT_1, ..., cT_m)$ is a joint-contraction for c sufficiently small. (In fact, if $\{T_j\}_{j \in J}$ is a commutive system, then $\{c_j T_j\}$ will be an joint contraction if $\{c_j\}$ is appropriately chosen e.g., if $|c_j| < (2^{-j} \|T_j\|^{-1})$. Thus, joint contractions are exceedingly general. We give below some examples of joint-isometries and discuss a case in which a joint-contraction is induced by a joint-isometry.

Let $\mathbf{S}$ be a joint-isometry on H and let $M \subset H$ be a joinly invariant subspace for S_j. Let $T_j = PS_j|_{M^\perp}$ be the compression of S_j to $M^\perp$, where P is the orthogonal projection of H onto $M^\perp$. Then $= P\mathbf{S}|_{M^\perp}$ is a joint contraction having $\mathbf{S}$ as a jointly isometric dilation. The following is an example of a joint-contraction which is not compression of a joint-isometry.

Example. Let $Z^n_+ = \{I = (i_1, ..., i_n)\}$ be the set of multi-indices of non-negative integers and let $B = \{e_I : I \in Z^n_+\}$ be an orthornormal basis for the (abstract) Hilbert space H. Let $\{w_{I,k} : I \in Z^n_+, k = 1, ..., n\}$ be a bounded net of complex numbers such that

$$w_{I,k} w_{I+\varepsilon_k, l} = w_{I,l} w_{I+\varepsilon_l, k}$$

for all $I \in Z^n_+$, $k, l = 1, ..., n$. Define T_k on H by

$$T_k e_I = w_{I,k} e_{I+\varepsilon_k},$$

$k = 1, ..., n$. Then $\mathbf{T} = (T_1, ..., T_n)$ is a system of n-variable commuting weighted shifts as discussed in [11]. Note that $\mathbf{T}$ can also be considered as shifts on a weighted sequence space. Clearly, $\mathbf{T}$ is a joint-isometry if $\sum_{k=1}^n |w_{I,k}|^2 = 1$ for all I and many systems $\{w_{I,k}\}$ satisfying this condition can be easily be constructed.

Consider weights such that $T_k^* e_I = (i_k/|I|)^{1/2} e_{I-\varepsilon_k}$ or 0 depending upon if $i_k > 0$ or $i_k = 0$. Note that $\{T_k^*\}$ is the system of n-variable weighted shifts that appeared in [10–11].

If $I \neq 0$, $\sum_{j=1}^n \|T_j^* e_I\|^2 = \sum_{j=1}^n i_j/|I| = 1 = \|e_I\|^2$ and $T_j^* e_0 = 0$, $j = 1, ..., n$. Hence, $\{T_j^*\}$ is jointly-contractive.

Suppose $\{S_j\}$ is a jointly-isometric dilation of $\mathbf{T}^*$ on $K \supset H$. Since $PS_j = T_j$, $\|S_j x\| \geqslant \|T_j x\|$ for all $x \in H$ and hence if $I \neq 0$, $\|e_I\|^2 = \sum_{j=1}^n \|S_j e_I\|^2 \geqslant \sum_{j=1}^n \|T_j^* e_I\|^2 = \|e_I\|^2$. Thus, for $j = 1, ..., n$, $\|S_j e_I\| = \|T_j^* e_I\|$ and consequently $S_j e_I = T_j^* e_I$ for $I \neq 0$, $j = 1, ..., n$. However, for $1 \leqslant k \leqslant n$, $k \neq j$,

$$S_k e_0 = S_k T_j^* e_{\varepsilon_j} = S_k S_j e_{\varepsilon_j} = S_j S_k e_{\varepsilon_j} = S_j 0 = 0.$$

Hence, $\{S_j\}$ cannot be jointly isometric. Note that this example is valid for $n \geqslant 2$.

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