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Valuation Methods in Division Rings*

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1. INTRODUCTION

The first main goal of this paper is to give a new proof, simpler than the original one, of the following theorem of Cohn [2]: *If R is a filtered ring such hat the associated graded ring is an Ore domain, then R can be imbedded in a (skew) field.* The important example of a ring which satisfies the conditions of Cohn's Theorem is the universal enveloping algebra $U(L)$ of an arbitrary Lie algebra L; here the graded ring, associated to the canonical filtration is a polynomial ring (see Jacobson [5]) and hence Cohn's theorem implies that $U(L)$ is imbedded into a field D. (The term "field" will be used throughout the paper in a sense of "skew field").

We give a short proof of Cohn's Theorem and of some related results in Sections 2 and 3. Like Cohn, we make an essential use of the valuation function in R, related to the given filtration; the main step in our proof is the following theorem.

THEOREM 1. *Let R be a domain and let t be a central element. Assume that $\bigcap_{n=1}^{\infty}(t)^n = 0$ and that $R/(t)$ is an Ore domain. Then R can be imbedded in a division ring D, the valuation function $v(x)$ is extended to D and the subset RR^{-1} is dense in D. If $S = \{s \in D \mid v(s) \geq 0\}$, then $S/tS \simeq \Delta$, the field of fractions of $R/(t)$.*

We then apply Theorem 1 to obtain simple proofs of Cohn's Theorem and of the following theorem whose first part is a corollary of Cohn's Theorem [2, Sect. 5] and the second part is proven in Lichtman [8, Prop. 3].

THEOREM 2. (i) *The algebra $U(L)$ is embedded into a field D with a discrete valuation function $\rho(x)$, which extends the canonical valuation $v(x)$ of $U(L)$; the field D is complete in the topology defined by $\rho(x)$.*

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(ii) *If S is the valuation ring of $\rho(x)$ i.e., $S = \{x \in D \mid v(x) \geq 0\}$ and $J(S)$ is the maximal ideal of S then $S/J(S)$ is a commutative field, purely transcendental over K.*

We now describe the idea of the proof of Theorem 1. To obtain an imbedding of R into a field D we consider the inverse system of rings $R_n = R/(t)^n$ $(n = 1, 2, \dots)$; every ring R_n contains a nilpotent ideal tR_n, and we prove in Proposition 1 that this ideal can be localized in R_n (more precisely, it is its complement $R_n \backslash (tR_n)$ which is localized in R_n.) We denote the ring of fractions of R_n with respect to the set $R_n \backslash (tR_n)$ by S_n and obtain an inverse system of rings S_n. The inverse limit $S = \lim_{\leftarrow} S_n$ contains R and we prove the following fact.

PROPOSITION 2. *S is a local domain with radical tS, the quotient ring $S/(t)$ is isomorphic to Δ, the field of fractions of $R/(t)$. Furthermore, $tS \cap R = tR$, and the ideal tS defines a t-adic valuation in S; this valuation extends the valuation $v(x)$ of the ring $R \subseteq S$ and the ring S is complete in the topology defined by it.*

Finally, to obtain the field D from S we take the ring of fractions of S with respect to the central subsemigroup $\langle t \rangle$, generated by the powers of t.

Propositions 1 and 2 (see Section 2.1) are the core of our method; they allow us to simplify the original method of Cohn, which can be outlined in the following way. It begins with a construction of an inverse system of semigroups, then each of these semigroups is imbedded into a group of right quotients, and then a group G which is an inverse limit of these groups is considered (see [2, Sect. 2); Cohn obtains the field D from this group G (see [2, Sect. 3, 4).

Although we do not prove this fact in the paper, we would like to point out that if R is a ring which satisfies the conditions of Cohn's theorem then both the methods, the original method of Cohn and the method of this paper, yield the same field D containing R.

The second goal of this article in the study of some classes of fields which includes the ones constructed in Section 2 for imbedding of universal enveloping algebras $U(L)$. Let D be a (skew) field with a discrete valuation function $v(x)$. The valuation that $v(x)$ is quasiabelian if $v(xy - yx) > v(x) + v(y)$ or, equivalently, the graded ring $\mathrm{gr}(D)$, associated to the valuation $v(x)$, is commutative. (See Cohn [3, 4]). In Sections 3 and 4 of the paper we develop a method for lifting different valuation functions from the ring $\mathrm{gr}(D)$ to the field D and then obtain the main technical result of these sections.

THEOREM 4. *Let D be a countable (skew) field with a discrete quasiabelian valuation $v(d)$, $T = \{d \in D \mid v(d) \geq 0\}$, $J(T) = \{d \in D \mid v(d) > 0\}$, and $\overline{T} = T/J(T)$ be the residue field of $v(d)$. Assume that there exists a* *central element t such that $v(t) = 1$. Let K be a commutative subfield of D, $f[X] = X^n + \lambda_1 X^{n-1} + \cdots + \lambda_n$ be a polynomial with coefficients $\lambda_\alpha \in K$ $(\alpha = 1, 2, \ldots, n)$, where $v(\lambda_\alpha) \geq 0$ $(\alpha = 1, 2, \ldots, n-1)$; $v(\lambda_n) = 0$. Assume that the valuation $v(d)$ is quasiabelian (i.e., the associated graded ring is commutative) and that at least one of the following two conditions hold*

(I) char $D = 0$

(II) *At least one of the images $\bar{\lambda}_\alpha$ of the elements λ_α $(\alpha = 1, 2, \ldots, n)$ in the quotient field $\bar{T} = T/J(T)$ is transcendental over the prime field Π.*

Then there exists in D an infinite system of quasiabelian valuations $\Phi_i(x)$ $(i \in I)$ such that

(1) *For every $i \in I \Phi_i(\lambda_\alpha) \geq 0$ $(\alpha = 1, 2, \ldots, n)$ and $\Phi_i(\lambda_\alpha) = 0$ if $v(\lambda_\alpha) = 0$; in particular, $\Phi_i(\lambda_n) = 0$ $(i \in I)$.*

(2) *For every $i \in I$ the restriction $\Psi_i(x)$ of $\Phi_i(x)$ on the subfield $R = \Pi(t, \lambda_1, \lambda_2, \ldots, \lambda_n)K$ is Henselian on the completion of this subfield (we will call such valuations quasihenselian)*

(3) *$f[X]$ has a root in the residue field of every $\Psi_i(x)$.*

We apply Theorem 4 to derive Theorems 5 and 6 (see Section 5), which are our main results about fields with a discrete quasiabelian valuation.

THEOREM 5. *Let D be a (skew) field of characteristic zero. Assume that there exists a discrete quasiabelian valuation $v(d)$ of D, which is trivial on the prime subfield of D, i.e., $v(q) = 0$ for every rational number $q \neq 0$. Then the center Z of D is algebraically closed in D, i.e., if an element $a \in D$ is algebraic over Z then $a \in Z$.*

Before formulating the other results of the paper we would like to describe briefly the main idea of the proof of Theorem 5. (We will present it now in a form a bit different from our proof.) We reduce first the proof to the case when $v(a) = 0$ and the extension $Z(a)$ is unramified over Z. Let $T = \{d \in D \mid v(d) \geq 0\}$, $J(T)$ be the radical of T, $\Delta = T/J(T)$ be the residue field which is commutative because $v(d)$ is quasiabelian. Let $f[X]$ be the minimal polynomial of a over Z, $\overline{f[X]}$ be the image of $f[X]$ modulo $J(T)$. We can find a valuation $\rho(r)$ of Δ such that $\overline{f[X]}$ has a root in the residue field of $\rho(r)$ and hence by Hensel's Lemma it has a root in the field $\tilde{\Delta}$, the completion of Δ. One can use then the valuation $v(d)$ of D and $\rho(r)$ of Δ to construct a new valuation function $\Phi(x)$ on D, as in Theorem 4. Let $\hat{D}$ and $\hat{Z}$ be the completions of D and Z with respect to $\Phi(x)$; then one of the generalizations of Hensel's Lemma implies that $f[X]$ has a root in $\hat{Z}$. This is impossible because $\hat{D}$ is a (skew) field.

As an immediate corollary of Theorem 1, we obtain the following fact.

COROLLARY OF THEOREM 5. *Let D be as in Theorem 5 and let Δ be a subfield of D. Then the center of Δ is algebraically closed in Δ. In particular, if Δ is finite dimensional over its center then it must be commutative.*

We obtain from this (see Lichtman [9]) that if $\operatorname{char} K = 0$ then a PI-subring of D or of $U(L)$ must be commutative; the center of $U(L)$ is algebraically closed in $U(L)$.

We consider also the case when $\operatorname{char} D = p$ or, more generally, when the residue field of the valuation $v(d)$ has a finite characteristic:

THEOREM 6. *Let $v(d)$ be a discrete quasiabelian valuation of D, $T = \{d \in D \mid v(d) \geq 0\}$, $J(T) = \{d \in D \mid v(d) > 0\}$. Assume that the residue field $\overline{T} = T/J(T)$ has a finite characteristic p. Let Z be the center of D and $E \supseteq Z$ be a commutative finite dimensional subfield of D. Then $\dim(E : Z)$ is a power of p.*

Once again the proof of this theorem is based on Theorem 4 and the following fact which might be of independent interest.

PROPOSITION 5. *Let θ be a non-central element algebraic over Z; assume that $v(\theta) = 0$. Then the minimal polynomial of θ over Z has a form*

$$f[X] = X^{p^m} + z + f_1[X],$$

where $m \geq 1$, $z \in Z$ is an element such that $v(z) = 0$ and the image $\bar{z}$ of z in $T/J(T)$ is transcendental over the prime subfield Z_p (and hence z is transcendental over the prime subfield of D), $f_1[X]$ is a polynomial with coefficients from $J(T)$, i.e.,

$$f[X] \equiv X^{p^m} + z (\operatorname{mod} J(T)).$$

Theorem 6 has the following corollary.

COROLLARY OF THEOREM 6. *Let D be as in Theorem 6, Δ be a (skew) subfield of D with center Z_1, a be an element of D algebraic over Z_1. Then $\dim(Z_1(a) : Z_1)$ is a power of p. In particular, if Δ is finite dimensional over Z_1 then $\dim(\Delta : Z_1)$ is a power of p.*

We conclude from this easily that if R is a PI-subring of D then its PI-degree is a power of p; this implies that if $\operatorname{char} K = p$ then a PI-subring D or of $U(L)$ must have its PI-degree a power of p. (See Lichtman [9].)

Now let D be the field constructed by Cohn in [2] for imbedding of a universal enveloping algebra $U(L)$ (this field is obtained also in Theorem 2 of the paper). Lemma 2 in [8] (or Corollary 2 of Theorem 2) imply that the valuation $\rho(x)$ on D is quasiabelian and hence Theorems 4–6 and their corollaries hold for this field; the results of this paper give therefore generalizations and simpler proofs of results in Lichtman [9]; we point out also that Lemma 4 in [9] is incorrect.

If L is a soluble-by-finite dimensional Lie algebra its universal enveloping algebra $U(L)$ is an Ore ring (see Lichtman [10, Prop. 4.1]); hence, the field of fractions of $U(L)$ is contained in any field generated by $U(L)$. We see that Theorems 4–6 and their corollaries hold for the fields of these algebras; similarly, they are true for the fields of fractions of Weyl algebras.

The author is preparing a paper for publication where some of the valuation methods developed in the current paper will be applied to the study of division rings generated by group rings of torsion free nilpotent and residually torsion free nilpotent groups; in particular, we will consider the relation between the properties of the fields generated by the groups rings and the fields generated by the universal enveloping algebras of the Lie algebras of these groups.

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2.1. Let R be a domain, t be a central element such that $\bigcap_{n=1}^{\infty}(t)^n = 0$ and the quotient ring $R/(t)$ is a domain. We conclude immediately that the ideal $T = (t)$ defines in R a t-adic valuation by the rule

$$v(x) = n \text{ if } x \in T^n \setminus T^{n+1}, \qquad v(0) = \infty.$$

Our first main result is the following theorem.

THEOREM 1. *Let R be a domain, t be a central element. Assume that $\bigcap_{n=1}^{\infty}(t)^n = 0$ and R/T is an Ore domain. Then R can be imbedded into a division ring D and the valuation function $v(x)$ is extended to D and the subset RR^{-1} is dense in D. Let $S = \{s \in D \mid v(s) \geq 0\}$. Then $S/tS \simeq \Delta$, the field of fractions of R/T.*

Our proof is based on Propositions 1 and 2 below; we keep in these propositions the notations of Theorem 1 and assume that the assumptions of this theorem are satisfied.

PROPOSITION 1. *Let an arbitrary natural n be given and let $\overline{X}$ be the image of a subset $X \subseteq R$ under the natural homomorphism $R \to R/T^n$. Then the subset $\overline{R} \setminus \overline{T}$ is a right Ore set in $\overline{R}$.*

Proof. Since the powers of the ideal (t) define a valuation in R a routine argument shows that the elements of $\overline{R} \setminus \overline{T}$ are not zero divisors in $\overline{R}$. Now let $x \in R \setminus T$, $y \in R$ be given. Since the conditions of Theorem 1 imply that the assertion is true for $n = 1$ we can assume that $n > 1$ and there exists $u \in R$, $v \in R \setminus T$ such that

$$xu - yv = z \in T^{n-1}. \tag{2.1}$$

We can assume that $z \notin T^n$, otherwise the result is proved; hence, $z = z_0 t^{n-1}$, $z_0 \in R \setminus T$ and we rewrite now (2.1) in the form

$$xu - yv - z_0 t^{n-1} = 0. \tag{2.1'}$$

On the other hand, for $x \in R \setminus T$, $z_0 \in R \setminus T$ we can find $v_1 \in R \setminus T$, $u_1 \in R$ such that

$$xu_1 - z_0 v_1 \in T$$

and hence

$$xu_1 t^{n-1} - z_0 v_1 t^{n-1} \in T^n. \tag{2.2}$$

We multiply (2.1′) on the right by $-v_1$ and add to (2.2) to obtain

$$x\left(u_1 t^{n-1} - uv_1\right) + yvv_1 \in T^n. \tag{2.3}$$

However, $v \in R \setminus T$, $v_1 \in R \setminus T$ and hence $vv_1 \in R \setminus T$. We see now that (2.3) implies that the set $\overline{R} \setminus \overline{T}$ is localizable in $\overline{R} = R/T^n$ and the assertion follows.

For every given $n = 1, 2, \ldots$ we now denote $R_n = R/T^n$ and S_n the ring of fractions of R_n with respect to the set $M_n = R_n \setminus (t)$. Clearly, the ideal $(t) \subseteq R_n$ is nilpotent, $(t)^n = 0$, and there exists a natural homomorphism $R_{n+1} \to R_n$ with kernel $(t^n)R_{n+1}$. A routine argument now implies the following corollary of Proposition 1.

COROLLARY. *The ideal $(t) \subseteq S_n$ is nilpotent, $(t)^n = 0$, and the quotient ring $S_n/(t)$ is isomorphic to the field Δ, the field of fractions of $R/(t)$. Furthermore, $(t^n S_{n+1}) \cap R_{n+1} = (t^n)R_{n+1}$ and the homomorphism $R_{n+1} \to R_{n+1}/(t^n) \simeq R_n$ is extended to the homomorphism $S_{n+1} \to S_{n+1}/(t) \simeq S_n$.*

We have now an inverse system of rings $R_n \subseteq S_n$ with a system of epimorphisms $S_{n+1} \xrightarrow{\varphi_n} S_n$ such that $\varphi_n(R_{n+1}) = R_n$. Let $S = \lim_{\leftarrow} S_n$, $\tilde{R} = \lim_{\leftarrow} R_n$ be the inverse limits of these systems. Clearly $S \supseteq \tilde{R} \supseteq R$.

PROPOSITION 2. *S is a local domain with radical tS, the quotient ring $S/(t)$ is isomorphic to Δ, the field of fractions of $R/(t)$, $tS \cap R = tR$, and the ideal tS defines a t-adic valuation in S; this valuation extends the valuation $v(x)$ of the ring $R \subseteq S$ and the ring S is complete in the topology defined by it.*

Proof. Every homomorphism $S_{n+1} \xrightarrow{\varphi_n} S_n$ maps the radical tS_{n+1} on the radical tS_n of S_n and induces an epimorphism of the quotient rings: $S_{n+1}/(tS_{n+1}) \xrightarrow{\varphi_n} S_n/(tS_n)$. These two quotient rings are in fact isomorphic, via Corollary of Proposition 1, to the field Δ. We conclude therefore that the ring S contains an ideal $tS = \lim_{\leftarrow} (tS_n)$ such that $S/(t) \simeq \Delta$. Every element $s \in S\setminus(t)$ must be invertible in S since its images under all the epimorphisms $S \xrightarrow{\gamma_n} S_n$ are in $S_n\setminus(tS_n)$ and hence are invertible in every S_n ($n = 1,2,\dots$). This shows that S is a local ring with a radical tS and $S/tS \simeq \Delta$.

We observe now that $\bigcap_{n=1}^{\infty}(tS)^n = 0$. This follows from the fact that the ideal tS is mapped on nilpotent ideals $tS_n \subseteq S_n$ under the epimorphisms $S \xrightarrow{\gamma_n} S_n$ ($n = 1,2,\dots$). It is important also for every n that $(\gamma_n(t))^n = 0$ but $(\gamma_n(t))^{n-1} \neq 0$ and hence the element $t \in S$ is not nilpotent.

Since $S/tS \simeq \Delta$ is a field a routine argument now yields that S is a domain and that the powers of the ideal (t) define a t-adic valuation $\rho(x)$ in S; the topology defined by this valuation is in fact the topology of the inverse limit $S = \lim_{\leftarrow} S_n$. It is known (see [7]) that S is complete in this topology.

Finally, we prove that the restriction of the valuation $\rho(x)$ on R coincides with $v(x)$. Let $0 \neq x \in R$ and $\rho(x) = k$. Then $x \in (tS)^k \setminus (tS)^{k+1} = (t^kS)\setminus(t^{k+1}S)$. Hence, there exists a number N such that for every $n > N$ we have $\gamma_n(x) \in (t^kS_n)\setminus(t^{k+1}S_n)$. Since S_n is the ring fractions of R_n and $\gamma_n(x) \in R_n$ we obtain easily that $\gamma_n(x) \in (t^kR_n)\setminus(t^{k+1}R_n)$ for $n > N$ and hence $x \in (t^kR)\setminus t^{k+1}R$, i.e., $v(x) = k$. This completes the proof.

Now let $\overline{X}$ be an arbitrary system of coset representatives of $S/(tS) = \Delta$; Proposition 1 implies that we can pick $\overline{X}$ from the elements of the set RR_1^{-1}, where $R_1 = R\setminus(t)$. A routine argument now yields the following corollary of Proposition 2.

COROLLARY. *An arbitrary element $s \in S$ has a unique representation*

$$s = \sum_{i=0}^{\infty} \alpha_i t^i \left(\alpha_i \in \overline{X}, i = 0,1,\dots\right) \tag{2.4}$$

and hence the set RR_1^{-1} is dense in S.

Proof of Theorem 1. The proof is now easily obtained. Take the ring S and observe that every element $s \in S$ has a unique representation $s = t^k s_1$ where $k = \rho(s) \geq 0$ and s_1 is invertible. Let D be the ring of fractions of S with respect to the central subsemigroup $\langle t \rangle$, generated by the element t. It is easy to verify that D is a field and that every element $d \in D$ has a unique representation

$$d = \sum_{i=n}^{\infty} d_i t^i \left(d_i \in \overline{X}, i \geq n\right), \tag{2.5}$$

where n can be an arbitrary integer.

It is well known and can be verified easily that a valuation of a ring is extended in a unique way a valuation of its ring of fractions, and hence, the valuation $\rho(x)$ is extended to D; this follows also from the representation (2.5) if we define $v(d) = n$. Finally, the representation (2.5) shows that the set $X = RR_1^{-1}$ is dense in D and the assertion follows.

2.2. Let L be an arbitrary Lie algebra over a commutative field K, $U(L)$ be its universal envelope; let e_i $(i \in I)$ be an ordered basis of L. It is well known (see [5, V.3]) that $U(L)$ has a filtration

$$U^{-1} = 0; U^i = K + L + L^2 + \cdots + L^i (i = 0, 1, \dots)$$

whose associated graded algebra is a polynomial algebra. It is easy to verify (see [1, p. 524]) that this filtration defines a valuation function $v(x)$ by the following rule:

$$v(0) = \infty, \qquad v(x) = -i \text{ if } x \in U^i \setminus U^{i-1}. \tag{2.6}$$

THEOREM 2. *The algebra $U(L)$ is embedded into a field D with a discrete valuation function $\rho(x)$, which extends the canonical valuation $v(x)$ of $U(L)$. The field D is complete in the topology defined by $\rho(x)$. Furthermore, if S is the valuation ring of $\rho(x)$, i.e., $S = \{x \in D \mid v(x) \geq 0\}$ and $J(S)$ is the maximal ideal of S then $S/J(S)$ is a commutative field, purely transcendental over K.*

Proof. Consider the direct sum of Lie algebras $L_1 = L + Ke$, where Ke is a one dimensional Lie algebra; then $U(L_1)$ is isomorphic to the polynomial ring $U(L)[e]$. We have a valuation function on $U(L_1)$ whose restriction on $U(L)$ coincide with $v(x)$; we will denote this valuation function on $U(L_1)$ also by $v(x)$; it is worth remarking that $v(e) = -1$.

Now consider the ring of fractions R_1 of $U(L_1)$ with respect to the central subsemigroup E generated by the element e, and extend the valuation $v(x)$ on R_1 by the natural way, i.e., if $x = re^{-k}$, $r \in R$, $k \geq 0$ then we define $v(x) = v(r) + k$. Let $R = \{x \in R_1 \mid v(x) \geq 0\}$.

We see that R contains a central element $t = e^{-1}$ such that the valuation function $v(x)$ on R is defined by the powers of the ideal (t). It is easy to see also that R_1 is isomorphic to the ring of fractions of R with respect to the subsemigroup generated by e^{-1}.

In order to complete the proof we need the following fact.

PROPOSITION 3. *The quotient ring $R/(t)$ is isomorphic to a polynomial ring $K[\bar{t}_i]$ where $\bar{t}_i$ is the image of the element $t_i = e_i e^{-1}$ $(i \in I)$.*

The proof of Theorem 2 is now completed easily. Theorem 1 and Proposition 2 imply that R is imbedded into a field D with a discrete valuation function $\rho(x)$ which coincides with $v(x)$ on R. We recall that

R_1 is isomorphic to a ring of fractions of R. Hence, D contains a subring isomorphic to R_1 and $U(L_1) \subseteq R \subseteq D$. This completes the proof of Theorem 2.

Proof of Proposition 3. First observe that every element $x \in R_1$ has a unique representation as a sum of elements of a type πe^{-k}, where $k \geq 0$ and π is a standard monomial in $U(L)$, say $\pi = e_1^{\alpha_1} e_2^{\alpha_2} \dots e_s^{\alpha_s}$, (we remind that the set e_i $(i \in I)$ is ordered) and $x \in T$ if we have for all of these monomials $\alpha_1 + \alpha_2 + \cdots + \alpha_s + k \geq 0$; furthermore, if $v(x) = 0$ then we have for all of these elements

$$\pi e^{-k} = \left(e_1 e^{-1}\right)^{\alpha_1} \left(e_2 e^{-1}\right)^{\alpha_2} \cdots \left(e_s e^{-1}\right)^{\alpha_s}; \tag{2.7}$$

this implies that the elements $t_i = e_i e^{-1}$ $(i \in I)$ generate the quotient ring $R/(t)$. This quotient ring is commutative: indeed, we have for arbitrary two elements t_{i_1} and t_{i_2}

$$v\left(\left[t_{i_1}, t_{i_2}\right]\right) = v\left(\left[e_{i_1}, e_{i_2}\right] e^{-2}\right) = v\left(\left[e_{i_1}, e_{i_2}\right]\right) + 2 \geq 1$$

because $v([e_{i_1}, e_{i_2}]) \geq -1$.

It remains to prove that the elements $\bar{t}_i$ $(i \in I)$ are algebraically independent modulo (t); this is equivalent to the fact that lexicographically different monomials of the type (2.7), together with 1, are linearly independent modulo (t); it is worth remarking that the monomial (2.7) is obtained from the standard monomial $\pi = e_1^{\alpha_1} e_2^{\alpha_2} \dots e_k^{\alpha_k}$ if we replace e_i by $e_i e^{-1}$ $(i = 1, 2, \dots, s)$.

Assume that there exist standard monomials $\pi_1, \pi_2, \dots, \pi_m$ on the set of elements $t_i = e_i e^{-1}$ $(i \in I)$ and elements $\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_m \in K$ such that

$$\lambda_0 + \lambda_1 \pi_1 + \lambda_2 \pi_2 + \cdots + \lambda_m \pi_m \in (t). \tag{2.8}$$

We assume that the degrees of the monomials (2.10) are

$$0 < r_1 \leq r_2 \leq \cdots \leq r_m.$$

Now multiply (2.8) by e^{r_m} and obtain

$$\left(\lambda_0 e^{r_m} + \lambda_1 \Pi_1 e^{(r_m - r_1)} + \lambda_2 \Pi_2 e^{r_m - r_2} + \cdots + \lambda_{m-1} \Pi_{m-1} e^{r_m - r_{m-1}} + \lambda_m \Pi_m\right) \in \left(t^{1 - r_m}\right), \tag{2.9}$$

where for a given $j \in \{1, 2, \dots, m\}$ Π_j is a standard monomial obtained from π_j by substitution of the element e_i instead of $e_i e^{-1}$ $(i \in I)$. The element in the left side of (2.9) belongs to $U(L_1)$; moreover, if the coefficients λ_j $(j = 0, 1, \dots, m)$ are not all equal to zero then this element is in fact homogeneous of degree r_m in the canonical filtration of $U(L_1)$ and hence its norm in R_1 is $-r_m$; however, this contradicts the fact that the norm of the right side is greater than or equal to $1 - r_m$. This contradiction shows that all the coefficients $\lambda_0, \lambda_1, \dots, \lambda_m$ in (2.8) are equal to zero and the assertion follows.

COROLLARY OF THEOREM 2. *The set $U(L_1)(U(L_1))^{-1}$ is dense in D.*

2.3. We consider now once again a field D with a discrete valuation function $v(x)$; let $S = \{d \in D \mid v(d) \geq 0\}$, $J(S) = \{d \in D \mid v(s) > 0\}$. We will assume that there exists a central element $t \in D$ such that $v(t) = 1$ and hence $J(S) = tS$; this condition holds in the fields obtained in Theorems 1 and 2.

PROPOSITION 4. *Let $\mathrm{gr}(D)$ be the graded ring of D, associated to the valuation $\rho(x)$ of D. Then $\mathrm{gr}(D)$ is isomorphic to the group ring $\bar{S}\langle t\rangle$ of an infinite cyclic group over the commutative field $\bar{S} = S/J(S)$.*

Proof. We give only a sketch of the proof since the argument is routine. Let for a given integer k $D_k = \{d \in D \mid v(d) \geq k\}$. Then $D_0/D_1 \simeq S/J(S) \simeq \bar{S}$ is a subring of $\mathrm{gr}(D)$ and it is easy to see that $D_k/D_{k+1} = \bar{S}\bar{t}^k$, where $\bar{t} = t + D_2$ and the map $D_k/D_{k+1} \to \bar{S}\bar{t}^k$ defines an isomorphism between $\mathrm{gr}(D)$ and $\bar{S}\langle t\rangle$.

We shall now state and prove Theorem 3, which will be later applied in the proof of Theorem 6. We denote, as usual, by D^* the multiplicative group of D and by $\gamma_n(D)$ the nth term of the lower central series of D^*.

THEOREM 3. *Assume that there exists a central element t such that $v(t) = 1$ and that $S/(tS)$ is commutative. Then $\gamma_n(D^*) \subseteq 1 + t^nS$ ($n = 2, 3 \dots$) and the group D^* is residually nilpotent.*

Proof. Since $v(x)$ is a discrete valuation on D we obtain for every element $d \in D$ a representation similar to (2.5)

$$d = t^n d_1, \tag{2.5'}$$

where $n = \rho(d)$, $\rho(d_1) = 0$ and hence $d_1 \in S^*$, the group of units of S. The representation (2.5′) implies immediately that $\gamma_n(D^*) = \gamma_n(S^*)$ ($n = 2, 3, \dots$). On the other hand the quotient ring S/tS is commutative; hence, for arbitrary two elements $x, y \in S^*$ we have

$$1 - x^{-1}y^{-1}xy = x^{-1}y^{-1}(yx - xy) \in tS, \tag{2.10}$$

which implies that $(S^*)' \subseteq 1 + tS$. A standard argument now implies that $\gamma_n(S^*) \subseteq 1 + t^nS$ ($n = 2, 3, \dots$) and hence

$$\gamma_n(D^*) \subseteq 1 + t^nS (n = 2, 3, \dots). \tag{2.11}$$

Since $\bigcap_{n=1}^{\infty} t^n S = 0$ we conclude that the group D^* is residually nilpotent and the proof is complete.

COROLLARY 1. (i) *If* char $D = 0$ *then the group* $(D^*)'$ *is residually torsion free nilpotent.*

(ii) *If* char $D = p$ *then the group* $(D^*)'$ *is residually "a nilpotent p-group of bounded exponent."*

We recall that a group G is residually torsion free nilpotent (residually a nilpotent p-group of bounded exponent") if for every element $1 \neq g \in G$ a homomorphism $G \xrightarrow{\varphi} H$ can be found such that $\varphi(g) \neq 1$ and H is torsion free nilpotent (a nilpotent p-group of bounded exponent).

The assertion follows from Theorem 3 by a routine argument.

COROLLARY 2. *Assume that D is as in Theorem* 3. *Then*

(i) *If* char $D = 0$ *then the group* $(D^*)'$ *is torsion free.*

(ii) *If* char $D = p$ *then the elements of finite order in* $(D^*)'$ *have orders powers of p.*

The assertion follows immediately from Corollary 1.

Remark. Let L be a Lie algebra over a commutative field K, D be the field obtained in Theorem 2 for the imbedding of the universal enveloping algebra $U(L)$ (this field is isomorphic to the field constructed by Cohn in [2]). The following result, more precise than Theorem 3, was obtained in [8] for the multiplicative group D^* of D:

(i) *If* char $K = 0$ *then* $D^* \simeq K^* \times D_1$, *where the group* D_1 *is residually torsion free nilpotent.*

(ii) *If* char $K = p$ *then* $D^* \simeq K^* \times D_1$, *where* D_1 *is a residually "nilpotent p-group of bounded exponent."*

3

Let R be a ring, Γ be a totally ordered abelian group. Assume that a valuation function $v(r)$: $R \to \Gamma \cup \infty$ is defined on R, i.e.,

$$v(r) = \infty \qquad \text{iff } r = 0$$

$$v(1) = 0$$

$$v(x+y) \geq \min\{v(x), v(y)\}$$

$$v(xy) = v(x) + v(y).$$

We would like to point out that in this paper we will consider only the valuations with an abelian group of values. A valuation function defines a filtration in R if we set $R_\gamma = \{x \in R \mid v(x) \geq \gamma\}$; let $R'_\gamma = \{x \in R \mid v(x) > \gamma\}$. Then the graded ring $S = \mathrm{gr}(R)$ associated to this filtration (and to the valuation $v(x)$) is defined in the following way. The additive group of S is $\Sigma_{\gamma \in \Gamma} R_\gamma / R'_\lambda$; if $\bar{x} = (x + R'_\alpha) \in R_\alpha / R'_\alpha$, $\bar{y} = (y + R'_\beta) \subseteq R_\beta / R'_\beta$ then the product $\bar{x}\bar{y}$ is the element $(xy + R'_{\alpha+\beta}) \in R_{\alpha+\beta}/R'_{\alpha+\beta}$. It is not difficult to verify that this operation is well defined for the homogenous elements and then it can be extended by the distributivity law to arbitrary elements of $\Sigma_{\gamma \in \Gamma} R_\gamma / R'_\gamma$ and that $\mathrm{gr}\, R$ is a ring without zero divisors. We need the following concept (see Cohn [2, 3]).

Definition. The valuation $v(r)$ on R is called quasiabelian if the ring $\mathrm{gr}(R)$ is commutative.

The proof of the following lemma is straightforward and we omit it.

LEMMA 1. *The valuation $v(r)$ is quasiabelian iff for arbitrary non-zero $a, b \in R$ we have*

$$v(ab - ba) > v(a) + v(b). \tag{3.1}$$

LEMMA 2. *Let R be a ring with a valuation function $v(r)$: $R \to \Gamma \cup_\infty$, M be a right denominator set in R. Then the valuation $v(r)$ is extended to the valuation of the ring of fractions RM^{-1} by the rule*

$$v(ab^{-1}) = v(a) - v(b)(a \in R, b \in M).$$

Furthermore, if the valuation $v(r)$: $R \to \Gamma \cup \infty$ is quasiabelian then the extended valuation is also quasiabelian.

Proof. The first statement is a well known fact. The second statement is Lemma 2 in [8].

Let R be a ring with a discrete valuation function $v(x)$; to extend the valuation $v(x)$ from R to the polynomial ring $R[t]$ we define for an element $f[t] = \alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n$

$$v(f[t]) = \min_i \{v(\alpha_i) + i\}. \tag{3.2}$$

It is well known (see Bourbaki [4], V1.10.1) that (3.2) defines a valuation function on $R[t]$ which extends $v(x)$; *we denote this extended valuation functon also by $v(x)$.*

Now assume that the valuation $v(x)$ on R is quasiabelian. We show that the extended valuation is quasiabelian on $R[t]$. Indeed, let $f[t], g[t] \in R[t]$,

$v(f[t]) = k_1$, $v(g[t]) = k_2$. We have to show that

$$v(f[t]g[t] - g[t]f[t]) > k_1 + k_2. \tag{3.3}$$

It is easy to see from the definition (3.2) that we can assume that in fact that $f[t] = \sum_i \alpha_i t^i$ where $v(\alpha_i) + i = k_1$ for all i; similarly, $g[t] = \sum_j \beta_j t^j$ with $v(\beta_j) + j = k_2$ for all j. Hence,

$$f[t]g[t] - g[t]f[t] = \sum_{i,j} (\alpha_i \beta_j - \beta_j \alpha_i) t^{i+j}. \tag{3.4}$$

Since $v(\alpha_i \beta_j - \beta_j \alpha_i) > v(\alpha_i) + v(\beta_j)$ we conclude that in (3.4)

$$\begin{aligned} v\big((\alpha_i \beta_j - \beta_j \alpha_i) t^{i+j}\big) &> v(\alpha_i) + v(\beta_j) + i + j \\ &> k_1 + k_2 + i + j \geq k_1 + k_2 \end{aligned} \tag{3.5}$$

and (3.3) follows now from (3.5). We proved the following fact.

LEMMA 3. *Let R be a ring with a discrete valuation function $v(x)$. Then $v(x)$ is extended to the polynomial ring $R[t]$ by formulae (3.2). If $v(x)$ is quasiabelian on R then the extended valuation is quasiabelian on $R[t]$.*

We consider now the ring of fractions of $R[t]$ with respect to the multiplicative semigroup generated by the element t. Since this ring of fractions is isomorphic to the group ring of an infinite cyclic group we obtain the following corollary of Lemma 3.

COROLLARY. *Let R be a ring with a discrete valuation $v(x)$, $R\langle t\rangle$ be the group ring of an infinite cyclic group $\langle t\rangle$. Then the valuation $v(x)$ is extended to $R\langle t\rangle$, the extended valuation is quasiabelian if $v(x)$ is quasiabelian.*

LEMMA 4. *Let R be a domain, $R\langle t\rangle$ be the group ring of an infinite cyclic group. Assume that there exists in $R\langle t\rangle$ a discrete valuation $v(x)$ such that $v\langle t\rangle = 1$; let $T = \{x \in R\langle t\rangle \mid v(x) \geq 0\}$, $T_0 = \{x \in R\langle t\rangle \mid v(x) > 0\}$. Then the quotient ring T/T_0 is isomorphic to the graded ring $\mathrm{gr}(R)$ of R with respect to $v(x)$.*

Proof. The quotient ring T/T_0 is generated by the images of the elements of the form αt^k $(\alpha \in R)$ with $v(\alpha) = -k$. Let $R_{-k} = \{x \in R \mid v(x) \geq -k\}$. We consider now the image of the set $(R_{-k})t^k$ under the homomorphism $\theta\colon T \to T/T_0$. We see that two elements $\lambda_1 t^k$ and $\lambda_2 t^k$ $(\lambda_1, \lambda_2 \in R_{-k})$ are congruent modulo T_0 iff $v(\lambda_1 - \lambda_2) \geq -k + 1$. This implies that there exists a one-to-one additive correspondence between $\theta(R_{-k}t^k)$ and R_{-k}/R_{-k+1}. Furthermore, if $x = \alpha_{-i_1}t^{i_1} + \alpha_{-i_2}t^{i_2} + \cdots + \alpha_{-i_n}t^{i_n}$, where $\alpha_{-i_j} \in R_{-i_j}$ $(j = 1, 2, \ldots, n)$ and $i_1 < i_2 < \cdots < i_n$, then it is easy to see that $x \notin T_0$. This implies that θ is extended to a one-to-one additive correspondence between $\mathrm{gr}(R)$ and T/T_0 and it is not difficult to verify that θ preserves also the multiplication. This completes the proof.

Remark. We will prove in our next paper [11] that more general facts are true for arbitrary valuations. At this moment our aim to obtain in Lemmas 3 and 4 some simple facts which are needed for a proof of Cohn's theorem.

We can give now a simple proof of Cohn's theorem.

Proof of Cohn's Theorem. Consider the group ring $R\langle t\rangle$ and extend the valuation $v(x)$ to $R\langle t\rangle$ by Lemma 3 and its Corollary. Let $T = \{x \in R\langle t\rangle \mid v(x) \geq 0\}$ and $T_0 = \{x \in R\langle t\rangle \mid v(x) > 0\}$ (see Lemma 4). Clearly $T_0 = tT$ and the restriction of $v(x)$ on T is a p-adic valuation defined by the powers of the ideal (t). Since $T/T_0 \simeq \mathrm{gr}(R)$ is an Ore domain we obtain from Theorem 1 that T is imbedded into a field D. Finally, $R\langle t\rangle$ is isomorphic to ring of fractions of T with respect to the multiplicative semigroup generated by the element t and hence is imbedded in D. This completes the proof.

4

Now assume that a valuation function $\rho(s)$: $S \to \Gamma_1 \cup \infty$ is defined on the ring $S = \mathrm{gr}(R)$. We will define a new valuation $w(x)$ on R. Take the direct sum $\Gamma + \Gamma_1$ and order this group lexicographically. Let $0 \neq x \in R$, $v(x) = \alpha \in \Gamma$ and denote $\bar{x} = (x + R_\alpha) \in \mathrm{gr}(R)$. Assume that $\rho(\bar{x}) = \beta$ and define

$$w(x) = (\alpha, \beta) \in \Gamma + \Gamma_1; \qquad w(0) = \infty \tag{4.1}$$

LEMMA 5. (i) *The function $w(x)$ defined by* (4.1) *is a valuation function on R*

(ii) *If at least one of the valuations $v(r)$ or $\rho(s)$ is quasiabelian then so is $w(x)$.*

Proof. Let x, y be non-zero elements of R, $w(x) = (v(x), \rho(\bar{x}))$, $w(y) = (v(y), \rho(\bar{y}))$. We have $w(xy) = (v(xy), \rho(\overline{xy}))$; however, $\overline{xy} = \bar{x}\bar{y}$, hence,

$$\begin{aligned} w(xy) &= (v(xy), \rho(\overline{xy})) = (v(x) + v(y), \rho(\bar{x}) + \rho(\bar{y})) \\ &= w(x) + w(y). \end{aligned} \tag{4.2}$$

A similar argument shows that

$$w(x+y) \geq \min\{w(x), w(y)\}$$

and the proof of the statement (i) is completed.

To prove (ii) we have to prove that the relation

$$w(ab - ba) > w(a) + w(b) \qquad (4.3)$$

holds for two arbitrary non-zero elements $a, b \in R$. If $(ab - ba) = 0$ then (4.3) is true; we can assume therefore that $(ab - ba) \neq 0$ and hence

$$w(ab - ba) = \left(v(ab - ba), \rho(\overline{ab - ba})\right) \qquad (4.4)$$

On the other hand,

$$w(a) + w(b) = (v(a), \rho(\bar{a})) + (v(b), \rho(\bar{b}))$$

$$= \left(v(a) + v(b), \rho(\bar{a}) + \rho(\bar{b})\right). \qquad (4.5)$$

Consider first the case when $v(ab - ba) > v(a) + v(b)$ (this holds, in particular, when $v(r)$ is quasiabelian); then (4.3) follows from (4.4) and (4.5) because of the lexicographic order in $\Gamma + \Gamma_1$.

Now assume that $v(ab - ba) = v(a) + v(b)$; in this case $v(r)$ is not quasiabelian and the conditions of the lemma imply that $\rho(s)$ must be quasiabelian. We have now $(ab - ba) \in R_{(v(a)+v(b))} \backslash R'_{(v(a)+v(b))}$. Since ab and ba also belong to $R_{(v(a)+v(b))} \backslash R'_{(v(a)+v(b))}$ we obtain

$$\overline{ab - ba} = \overline{ab} - \overline{ba} = \bar{a}\bar{b} - \bar{b}\bar{a}$$

and hence

$$\rho(\overline{ab - ba}) = \rho(\bar{a}\bar{b} - \bar{b}\bar{a}) \qquad (4.6)$$

and

$$w(ab - ba) = \left(v(a) + v(b), \rho(\bar{a}\bar{b} - \bar{b}\bar{a})\right) \qquad (4.7)$$

Since $\rho(s)$ is quasiabelian we conclude now that (4.3) follows from (4.7) and (4.5) and the proof is completed.

Remark. Consider the important special case when the valuation $v(r)$ of R is discrete. Let t be an element such that $v(t) = 1$ and $x \neq 0$ be an arbitrary element of R. If $v(x) = \alpha$ then $x = t^{\alpha} y$, where $y \in R_0 \backslash R_1$ and we obtain from (4.1) that $w(x) = (\alpha, \beta)$ where $\beta = \rho(\bar{y})$. *We see that in this case the valuation $w(x)$ on R is constructed in fact from the valuation $v(x)$ of R and the valuation $\rho(s)$ of the quotient ring R_0/R_1.* We will use this observation later in Lemma 8.

Now let D be a (skew) field with a valuation $v(r)$: $D \to \Gamma \cup \infty$, $T = \{r \in D \mid v(r) \geq 0\}$, $J(T) = \{r \in D \mid v(r) > 0\}$ be the radical of T. Let $S = \mathrm{gr}(D)$; clearly the field $\Delta = T/J(T)$ is isomorphically imbedded in the ring S. Consider a valuation $\rho(s)$: $S \to \Gamma_1 \cup \infty$ and then define a valuation $D \to (\Gamma + \Gamma_1) \cup \infty$ by (4.1). Let $U = \{x \in D \mid w(x) \geq 0\}$ be the valuation ring of $w(x)$. We see that if $x \in U$ then it must be $v(x) \geq 0$ and if $v(x) = 0$ then $\rho(\bar{x}) \geq 0$. Hence U is a subring of T and $T \supseteq U \supseteq J(T)$, and if V is the valuation ring of $\rho(s)$, i.e., $V = \{s \in \Delta \mid \rho(s) \geq 0\}$, then $U/J(T) \simeq V$. We obtained the following lemma.

LEMMA 6. *Let φ denote the natural homomorphism $T \to T/J(T) = \Delta$. Then $\varphi(U) = V$ and $\varphi(J(U)) = J(V)$ and hence $U/J(U) \simeq V/J(V)$.*

Now let $R = \Pi(t_0, t_1, t_2, \dots, t_k)$ be a commutative field, finitely generated over the prime subfield Π. Assume that the element $t_0 = t$ is transcendental over Π and defines a t-adic valuation $v(x)$ in R. Replacing, if necessary, the elements t_i $(i = 1, 2, \dots, k)$ by new elements $t_i t^{-v(t_i)}$ we can assume that $v(t_i) = 0$.

Let $\tilde{R}$ be the v-completion of R, T be the ring of integers of $\tilde{R}$, $\bar{T} = T/(tT)$ be the residue field.

LEMMA 7. *Assume that the field $\bar{T}$ is infinite. Then one of the following two alternatives holds.*

(i) *The field $\bar{T}$ is transcendental over Π. Then there exists a purely transcendental subfield $\Pi(u_1, u_2, \dots, u_r) \subseteq T$ such that T is a finite separable extension $T = T_1(\theta)$, where T_1 is isomorphic to the Laurent power series ring $\Pi(u_1, u_2, \dots, u_r)[[t]]$. The residue field $\bar{T} = T/(tT)$ is isomorphic to the field $\Pi(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_r, \bar{\theta})$, where $\Pi(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_r) \simeq \Pi(u_1, u_2, \dots, u_r)$ is a purely transcendental field, $\bar{\theta}$ is algebraic, separable and integral over the polynomial ring $\Pi[\bar{u}_1, \bar{u}_2, \dots, \bar{u}_r]$. Furthermore, if $f[X]$ is the minimal polynomial of θ over $T_1 = \Pi(u_1, u_2, \dots, u_r)[[t]]$ then the minimal polynomial of $\bar{\theta}$ over $\Pi(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_r)$ is $\overline{f[X]}$, the reduction of $f[X]$ modulo (t).*

(ii) *$\bar{T}$ is algebraic over the field of rational numbers Π. In this case $T = T_1[\theta]$, where T_1 is the power series ring $\Pi[[t]]$, θ is algebraic (and separable) over T_1. The residue field $\bar{T}$ is isomorphic to the algebraic number field $\Pi(\bar{\theta})$, where $\bar{\theta}$ is an algebraic integer and once again its minimal polynomial is obtained by the reduction modulo (t) of the minimal polynomial of θ over T_1.*

Proof. We give a proof of statement (i); the proof of statement (ii) is obtained by the same argument with obvious changes and simplifications.

Let $t, t_1, t_2, \dots, t_r$ be a transcendency basis of R. Then R is a finite algebraic extension of $\Pi(t, t_1, t_2, \dots, t_r)$ and hence $\tilde{R}$ is a finite algebraic extension of $\tilde{R}_1$, the completion of $R_1 = \Pi(t, t_1, t_2, \dots, t_r)$; this completion is isomorphic to the Laurent power series field $\Pi(t_1, t_2, \ldots, t_r)[[t]]$. Since R is unramified over R_1 there exists a basis $\theta_1, \theta_2, \ldots, \theta_s$ of T over $T_1 = J(T) \cap R_1$ such that the elements $\bar{\theta}_i$, the images of θ_i in $\bar{T} = T/J(T)$ $(i = 1, 2, \ldots, s)$, form a basis of $\bar{T}$ over the subfield $\bar{T}_1 = \Pi(t_1, t_2, \ldots, t_r)$. Since the field $\bar{T}$ is finitely generated over the prime subfield Π it is separately generated (see [3, 5.13]). We can pick in $\bar{T}$ a system of generators $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_r, \bar{\theta}$ such that the subfield $\Pi(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_r)$ is purely transcendental and $\bar{\theta}$ is separable over $\Pi(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_r)$; clearly, we can assume that $\bar{\theta}$ belongs to the integral closure of the polynomial ring $\Pi[\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_r]$. We take now the elements $u_1, u_2, \ldots, u_r, \theta$, the inverse images of $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_r, \bar{\theta}$, in T and these element satisfy all the conclusions of the lemma. This completes the proof.

We will construct now a valuation function on the field $\bar{T}$ and then extend it to the ring T. *Once again, assume that the field $\bar{T}$ is infinite.* If the alternative i) of Lemma 7 holds we denote $K = \Pi(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{r-1})$, the subfield generated by $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{r-1}$; let $\bar{u}_r = \bar{u}$ and $u_r = u$; we take in the polynomial ring $K[\bar{u}]$ a maximal ideal A which is not ramified in the integral closure of $K[\bar{u}]$ in the field $\bar{T} = K(\bar{u}, \bar{\theta})$; (hence A is also unramified in the ring $(K[\bar{u}])[\bar{\theta}]$). The ideal A defines a p-adic valuation of $K[\bar{u}]$, we extend it to a valuation of the field of rational functions $K(\bar{u})$ and then to its algebraic extension $\bar{T}$. In the second case, we take a p-adic valuation of the rationals which is unramified in $\Pi(\theta)$. We obtained in both cases a p-adic valuation $\rho(x)$ on $\bar{T}$ and we define a valuation $w(x)$ on T by (4.1). (See the remark after the proof of Lemma 5.) We need the following fact about these valuations.

LEMMA 8. *Assume that the field $\bar{T}$ is infinite and let $\rho(x)$ be a p-adic valuation of $\bar{T}$ which is defined by an unramified p-adic valuation of $K[\bar{u}]$ or of Π, $(\hat{\bar{T}})_\rho$ be the completion of $\bar{T}$, $\hat{T}_w$ be the completion of T. Then the ideal $t\hat{T}_w$ is the radical of $\hat{T}_w$, the quotient ring $(\hat{T}_w)/(t\hat{T}_w)$ is isomorphic to the field $(\hat{\bar{T}})_\rho$ and the powers of the ideal $t\hat{T}_w$ define a t-adic valuation in $\hat{T}_w$; the ring $(\hat{T})_w$ is complete with respect to this valuation.*

Let U, V be the valuation rings of the valuations $w(x)$ and $\rho(s)$, respectively. Then

$$(\hat{U}_w)/(t\hat{U}_w) \simeq \hat{V}_\rho. \tag{4.8}$$

Proof. Let T_1 be as in Lemma 7, $(\hat{T}_1)_w$ be the completion of T_1 in the topology defined by $w(x)$, $(\hat{\bar{T}}_1)_\rho$ be the completion of $\bar{T}_1$ in the topology defined by $\rho(s)$. We observe first of all that $(\hat{\bar{T}}_1)_\rho$ is isomorphic to either $(\widehat{K(\bar{u})})_\rho$, the ρ-completion of the field $K(\bar{u})$, or to the field $\hat{\Pi}_\rho$, the p-adic completion of the rationals. A straightforward verification shows that the ring $(\hat{T}_1)_w$, the w-completion of T_1, is isomorphic either to the power series ring over $(\widehat{K(\bar{u})})_\rho$, or to the ring $\hat{\Pi}[[t]]$. We obtain from this that $(\hat{T}_1)_w/(t)$ is isomorphic either to $(\widehat{K(\bar{u})})_\rho$ or to $\hat{\Pi}_\rho$; i.e., it is isomorphic to $(\hat{\bar{T}}_1)_\rho$, and the ideal $t(\hat{T}_1)_w$ defines a t-adic valuation in $(\hat{T}_1)_w$ and $(\hat{T}_1)_w$ is complete.

Now let $V_1 = V \cap \bar{T}_1, U_1 = U \cap T_1$. The same straightforward argument as in Lemma 6 implies that $U_1/tU_1 \simeq V_1$ and

$$(\hat{U}_1)_w/t(\hat{U}_1)_w \simeq (\hat{V}_1)_\rho. \tag{4.9}$$

We consider now the rings $(\hat{T}_1)_w[\theta] \simeq (\hat{T}_1)_w \otimes_{T_1} T_1[\theta]$ and $(\hat{\bar{T}}_1)_\rho[\bar{\theta}] \simeq (\hat{\bar{T}}_1)_\rho \otimes_{\bar{T}_1} \bar{T}_1[\bar{\theta}]$; they are isomorphic to $(\hat{T}_1)_w[X]/(f[X])$ and $t(\hat{\bar{T}}_1)_\rho[X]/(\overline{f[x]})$, respectively. (We recall that $f[X]$ is the minimal polynomial of θ over (T_1), $\overline{f[X]}$ is the minimal polynomial of $\bar{\theta}$ over $\bar{T}_1$, and $\overline{f[X]}$ is obtained by reduction of $f[X]$ modulo (t); see Lemma 7.)

We recall now that the valuation $\rho(x)$ of $\bar{T}_1$ is unramified in $\bar{T}_1[\theta]$ and that θ is separable over $\bar{T}_1$. Hence the factorization of $\overline{f[X]}$ in $\bar{T}_1[X]$ has a form

$$\overline{f[X]} = \prod_{\alpha=1}^{m} \overline{f_\alpha[X]},$$

where $\overline{f_\alpha[X]}$ $(\alpha = 1, 2, \ldots, m)$ are distinct irreducible polynomials. The classical Hensel's Lemma now implies that

$$f[X] = \prod_{\alpha=1}^{m} f_\alpha[X],$$

where $f[X]$ $(\alpha = 1, 2, \ldots, m)$ are distinct irreducible polynomials and every $\overline{f_\alpha[X]}$ is obtained by reduction of $f_\alpha[X] \operatorname{mod}(t)$.

We obtain therefore that the homomorphism π: $(\hat{T}_1)_w \to (\hat{T}_1)_w/(t) \simeq (\hat{\bar{T}}_1)_\rho$ is extended to a homomorphism $(\hat{T}_1)_w[X]/(f[X]) \to (\hat{\bar{T}}_1)_\rho[X]/(\overline{f[X]})$ (the kernel of this homomorphism is generated by the element t) and then to homomorphisms

$$(\hat{T}_1)_w[X]/(f_\alpha[X]) \to (\hat{\bar{T}}_1)_\rho/(\overline{f_\alpha[X]})(\alpha = 1, 2, \ldots, m).$$

However, $(\hat{T}_1)_w[X]/(f(X))$ is a direct sum of integral domains

$$\begin{aligned}(\hat{T}_1)_w[X]/(f[X]) \simeq (\hat{T}_1)_w[X]/(f_1[X]) + (\hat{T}_1)_w[X]/(f_2[X]) \\ + \cdots + (\hat{T}_1)_w[X]/(f_m[X])\end{aligned}$$

as well as $(\hat{\bar{T}}_1)_p[X]/(\overline{f[X]})$ is a direct sum of fields

$$(\hat{\bar{T}}_1)_p[X]/(\overline{f[X]}) \simeq (\hat{\bar{T}}_1)_p[X]/(\overline{f_1[X]}) + (\hat{\bar{T}}_1)_p[X]/(\overline{f_2[X]}) + \cdots (\hat{\bar{T}}_1)_p[X]/(\overline{f_m[X]}).$$

and for every given $\alpha = 1, 2, \ldots, m$ there exists a homomorphism

$$\pi_\alpha: (\hat{T}_1)_w[X]/(f_\alpha[X]) \to (\hat{\bar{T}}_1)_p/(\overline{f_\alpha[X]}),$$

which extends the homomorphism $\pi: (\hat{T}_1)_w \to (\hat{\bar{T}}_1)_p$.

Finally, the domain $(\hat{T})_w$ is isomorphic to one of the domains $(\hat{T}_1)_w[X]/(f_\alpha[X])$ and we obtain a homomorphism of $(\hat{T})_w$ on the ring $(\hat{\bar{T}}_1)_p/(\overline{f_\alpha[X]}) \simeq (\hat{\bar{T}}_1(\theta))_p$, i.e., we obtained a homomorphism $\hat{T}_w \to (\hat{T}_w)/(t) \simeq (\hat{\bar{T}})_p$ which extends the homomorphism $(\hat{T}_1)_w \to (\hat{T}_1)_w/(t) \simeq (\bar{T}_1)_p$. The ideal (t) defines a t-adic valuation in the ring $(\hat{T}_1)_w$ and does not ramify in its integral algebraic extension $(\hat{T})_w$; it is important that $(\hat{T}_1)_w$ is complete with respect to this t-adic valuation. This implies that the ideal $t\hat{T}_w$ defines a valuation of $\hat{T}_w$ and $\hat{T}_w$ is also complete. The relation (4.8) follows now easily from (4.9) or from the relation $\hat{T}_w/t\hat{T}_w \simeq (\hat{\bar{T}})_p$. This completes the proof.

The following corollary follows now immediately from Lemma 8.

COROLLARY. *Assume that the conditions of Lemma* 8 *are satisfied. Then*

$$\hat{U}_w/J(\hat{U}_w) \simeq \hat{V}_p/J(\hat{V}_p) \simeq V_p/J(V_p)$$

We recall now the following fact from the classical valuation theory. Let F be a commutative field with a discrete valuation $v(x)$: $F \to \Gamma \cup \infty$; let $U = \{x \in F \mid v(x) \geq 0\}$ be the valuation ring of $v(x)$. Let $\hat{F}$ and $\hat{U}$ be the completions of F and U respectively, $J(\hat{U}) = \{x \in \hat{U} \mid v(x) > 0\}$ be the radical of $\hat{U}$, $\bar{U} = \hat{U}/J(\hat{U})$ be the residue field of $v(x)$. Let $f[X] \in \hat{U}[X]$ be a monic polynomial, $\overline{f[X]}$ be its image modulo $J(\hat{U})$. Assume that in $\bar{U}[X]$

$$\overline{f[X]} = \overline{f_1[X]}\,\overline{f_2[X]}, \tag{4.10}$$

where $\overline{f_1[X]}$, $\overline{f_2[X]}$ are relatively prime. The classical Hensel's Lemma states that there exist monic polynomials $f_\alpha[X] \in \hat{U}[X]$ $(\alpha = 1, 2)$ such that

$$f[X] = f_1[X]f_2[X] \tag{4.11}$$

and $f_\alpha[X]$ coincides with $f_\alpha[X]$ ($\alpha = 1,2$) modulo the ideal $J(\hat{U})$.

Now let R be a commutative field, $v(r)$: $R \to \Gamma \cup \infty$ be an arbitrary, not necessarily discrete, valuation; we will call it quasihenselian if the analog of Hensel's Lemma holds in the ring $\hat{R}[X]$, where $\hat{R}$ is the completion of R.

We consider now a quasihenselian valuation $v(r)$: $R \to \Gamma \cup \infty$ of a commutative field R; let $S = \mathrm{gr}(R)$; clearly the residue field $\overline{T} = T/J(T)$ is naturally imbedded in the ring S. Once again, we assume that there exists a valuation $\rho(s)$: $S \to \Gamma_1 \cup \infty$ and we construct a valuation $w(x)$ on R by (4.1).

PROPOSITION 5. *Let R be a finitely generated commutative field. Assume that the valuation $v(r)$ is discrete on R and $\rho(x)$ be a valuation on $\overline{T}$ which was constructed in Lemma* 8. *Then the valuation $w(x)$: $R \to (\Gamma + \Gamma_1) \cup \infty$ is quasihenselian on R.*

Proof. We will use the same notations as in Lemma 8. Let $f[X] \in U[X]$ be a monic polynomial; assume that (4.10) holds modulo $J(\hat{U})$, where $\overline{f_1[X]}$ and $\overline{f_2[X]}$ are relatively prime. Lemma 8 and its Corollary imply that the homomorphism α: $\hat{U} \to \hat{U}/J(\hat{U})$ can be factored through the homomorphism β: $\hat{U} \to \hat{V}$ which is induced by the homomorphism $\hat{\pi}$: $\hat{T}_w \to (\hat{\overline{T}})_\rho$ with kernel $t\hat{T}$; hence, $\ker\beta = \hat{U} \cap t\hat{T}$. Let $\widetilde{f[X]}$ be the image of $f[X]$ under the homomorphism β. Since the valuation ρ is quasihenselian and we have for the polynomial $\widetilde{f[X]} \in \hat{V}[X]$ factorization (4.10) modulo $J(\hat{V})$ we obtain that in $\hat{V}[X]$

$$\widetilde{f[X]} = \widetilde{f_1[X]}\,\widetilde{f_2[X]}, \tag{4.12}$$

where $\widetilde{f_\alpha[X]}$ coincides with $\overline{f_\alpha[X]}$ ($\alpha = 1,2$) modulo $J(\hat{V})$. Lemma 8 implies that the ring $\hat{T}$ is complete in the t-adic topology defined by the powers of the ideal $t\hat{T}$ and hence $\hat{U}$ is complete in the topology defined by the powers of the ideal $\ker\beta = \hat{U} \cap t\hat{T}$. Since $f[X]$ has a factorization (4.12) modulo the ideal $\ker\beta$ and $\hat{U}$ is complete in the topology defined by this ideal we conclude by one of the versions of Hensel's Lemma (see Bourbaki [1, Th. III.4.1]) that there exist in $\hat{U}[X]$ monic polynomials $f_\alpha[X]$ ($\alpha = 1,2$) such that (4.11) holds and $f_\alpha[X]$ consides with $\widetilde{f_\alpha[X]}$ modulo $\ker\beta$ for $\alpha = 1,2$ and the assertion now follows.

LEMMA 9. *Let K be a commutative field, $f[X] = \lambda_0 X^n + \lambda_1 X^{n-1} + \cdots + \lambda_n$ be a polynomial with coefficients $\lambda_j[t] \in K[t]$ ($j = 0, 1, \ldots, n$; $n \geq 1$). Then the system of all the polynomials $f[\lambda[t]]$ ($\lambda[t] \in K[t]$) includes an infinite number of distinct prime divisors $p_i[t]$ ($i \in I$).*

Proof. In fact, assume that there exists only a finite number of such divisors; let $p_1[t], p_2[t], \ldots, p_r(t)$ be all of them. We apply now a method similar to [12, Problem 108]. Pick an arbitrary power of t such that $f[t^m] \neq 0$. We obtain

$$f[t^m] = s[t] = p_1[t]^{\alpha_1} p_2[t]^{\alpha_2} \dots p_r[t]^{\alpha_r}.$$

Now let

$$\varphi[X] = \frac{1}{s[t]} f[t^m + p_1[t] p_2[t] \dots p_r[t] s[t] X].$$

It is easy to verify that $\varphi[X]$ is a non-constant polynomial over $K[t]$ and $\varphi[X] \equiv 1 \operatorname{mod}(p_1[t]p_2[t]\dots p_r[t])$. Hence, there exists $\lambda[t] \in K[t]$ such that $\varphi[\lambda[t]]$ is a non-constant polynomial in $K[t]$ whose prime factors differ from $p_i[t]$ $(i = 1, 2, \dots, r)$; clearly, these prime factors are also divisors of $f[t^m + p_1[t]p_2[t]\dots p_r[t]s[t]\lambda[t]$. This contradiction shows that the set $p_i[t]$ $(i = 1, 2, \dots, r)$ cannot be finite and the proof is completed.

COROLLARY. *There exists an infinite set of irreducible polynomials $p_i[t] \in K[t]$ $(i \in I)$ such that $f[X]$ has a root modulo $p_i[t]$ $(i \in I)$.*

Let K be a commutative field, $f[X] = X^n + \lambda_1 X^{n-1} + \dots + \lambda_n$ be a polynomial where $\lambda_\alpha \in K(\alpha = 1, 2, \dots, n;\ n \geq 1)$; assume that at least one of the coefficients λ_α is transcendental over the prime subfield Π. Let $u_1, u_2, \dots, u_r$, θ be a separating transcendency basis for the field $S = \Pi(\lambda_1, \lambda_2, \dots, \lambda_n)$; i.e., θ is algebraic and separable over the purely transcendental subfield $\Pi(u_1, u_2, \dots, u_r)$. We denote $S_1 = \Pi(u_1, u_2, \dots, u_{r-1})$, the purely transcendental subfield generated by $u_1, u_2, \dots, u_{r-1}$, and let $u_r = u$; so $S = S_1(u, \theta)$; let T be the integral closure of the polynomial ring $S_1[u]$ in $S_1(u, \theta)$.

LEMMA 10. *There exists an infinite number of maximal ideals $A_i \subseteq T$ $(i \in I)$ which are unramified over $S_1(u)$ and define p-adic valuations ρ_i on S such that $\rho_i(\lambda_\alpha) = 0$ $(i \in I;\ \alpha = 1, 2, \dots, n)$ and $f[X]$ has a root in the residue field of every ρ_i.*

Proof. Let $\tilde{S}$ be the normal extension of $S_1(u)$, containing $S_1(u, \theta)$, $\tilde{T}$ be the integral closure of $S_1[u]$ in $\tilde{S}$. Assume first that $f[X] \in T[X]$, i.e. $\lambda_\alpha \in T$ $(\alpha = 1, 2, \dots, n)$. Let $f_1[X] = f[X], f_2[X], \dots, f_m[X]$ be all the polynomials obtained from $f[X]$ by action of the Galois group $\operatorname{Gal}(\tilde{S}\colon S_1(u))$ and let $\varphi[X] = \Pi_{j=1}^m f_j[X]$; the polynomial $\varphi[X]$ has all its coefficients in the polynomial ring $S_1[u]$. By Lemma 9 there exists an infinite number of polynomials among the divisors of all the polynomials $\varphi[\lambda[u]]$ $(\lambda[u] \in K[u])$. Hence there exists also an infinite number of maximal ideals $B_j \subseteq \tilde{T}$ $(j \in J)$ which occur in the factorization of the principal ideals $(\varphi[\lambda[u]])$ $(\lambda[u] \in K[u])$. This implies easily that there is an infinite subset $J_1 \subseteq J$ such that the maximal ideals B_j $(j \in J_1)$ occur in the factorization of the principal ideals $(f[\lambda[u]])\tilde{T}$ $(\lambda[u] \in K[u])$. Finally, taking the intersections $A_j = B_j \cap T$ $(j \in J_1)$ we obtain an infinite set of maximal ideals $A_i \subseteq T$ $(i \in I)$, which occur in the factorization of the principal ideals $(f[\lambda[u]])T$ $(\lambda[u] \in K[u])$. Every ideal A_i defines a p-adic valuation ρ_i in T and $f[X]$ has a root in the residue field of ρ_i. Since the system of ideals A_i $(i \in I)$ is infinite we can assume that $\rho_i(\lambda_\alpha) = 0$ $(\alpha = 1, 2, \dots, n;\ i \in I)$. The proof is completed for the case when $\lambda_\alpha \in T$ $(\alpha = 1, 2, \dots, n)$.

If now λ_α $(\alpha = 1, 2, \dots, n)$ are arbitrary elements of $S_1(u, \theta)$ we can represent them in a form $\lambda_\alpha = \mu_\alpha b^{-1}, 0 \neq b \in T$, $\mu_\alpha \in T$ $(\alpha = 1, 2, \dots, n)$ and observe that the assertion has already been proven for the polynomial $\rho[X] = \mu_1 X^n + \mu_2 X^{n-1} + \cdots + \mu_n$. Clearly there exists a cofinite subset $I_1 \subseteq I$ such that $\rho_i(b) = 0$ for every $i \in I_1$ and the proof is completed.

LEMMA 11. *Let K be an algebraic number field, $f[X] = \lambda_1 X^n + \lambda_2 X^{n-1} + \cdots + \lambda_n$ be a non-constant polynomial, $S = \Pi(\lambda_1, \lambda_2, \dots, \lambda_n)$. Then there exists in S an infinite system of p-adic valuations ρ_i $(i \in I)$ such that $\rho_i(\lambda_\alpha) = 0$ $(i \in I;\ \alpha = 1, 2, \dots, n)$, $f[X]$ has a root in the residue field of every ρ_i and ρ_i is unramified over the rationals.*

Proof. The proof differs from the proof of Lemma 10 only in one point: for a given polynomial $f[X] \in Z[X]$ the set of all the integers $f[\lambda]$ $(\lambda \in Z)$ is composed from an infinite set of prime numbers (see [11, Problem 108]) and the proof can be completed by the same argument as in Lemma 10.

Remark. It is worth remarking that the valuations ρ_i $(i \in I)$ constructed in Lemmas 10 and 11 satisfy all the conditions of Lemma 8.

LEMMA 12. *Let K be a finitely generated commutative field, $f[X] = X^n + \lambda_1 X^{n-1} + \cdots + \lambda_n$ be a polynomial $(\lambda_\alpha \in K,\ \alpha = 1, 2, \dots, n;\ n \geq 1)$. Assume that at least one of the following two conditions hold.*

(1) char $K = 0$

(2) *At least one of the coefficients λ_α $(\alpha = 1, 2, \dots, n)$ of $f[X]$ is transcendental over the prime subfield of K.*

Then every valuation ρ_i $(i \in I)$ of the ring $S = \Pi(\lambda_1, \lambda_2, \dots, \lambda_n)$ which was constructed in Lemmas 10 *and* 11 *is extended to a p-adic valuation on K.*

Proof. Let as in Lemmas 10 and 11 $S = \Pi(\lambda_1, \lambda_2, \dots, \lambda_n)$. Since K is finitely generated we can assume without loss of generality that K is in fact either a simple algebraic or a simple transcendental extension of S. In the first case ρ_i is extended to a p-adic valuation of K by Coro. 2 of Th.

VI.8.2. in Bourbaki [1]; in the second case, we apply Prop. V.10.2 in [1]. This completes the proof.

COROLLARY 1. *Let K be a countable commutative field, $f[X] = X^n + \lambda_1 X^{n-1} + \cdots + \lambda_n n$ be a polynomial ($\lambda_\alpha \in K$, $\alpha = 1,2,\ldots,n$; $n \geq 1$). Assume that at least one of the conditions* 1) *or* 2) *of Lemma* 12 *hold. Then every valuation ρ_i ($i \in I$) which is constructed in Lemmas* 10 *and* 11 *is extended to a valuation on K.*

Proof. Let $K_0 = \Pi(\lambda_1, \lambda_2, \ldots, \lambda_n) \subseteq K_1 \subseteq K_2 \subseteq \cdots$ be a system of finitely generated subfields such that $K = \cup_{j=1}^{\infty} K_j$. We begin as in Lemma 12 with a system of p-adic valuations ρ_i ($i \in I$) in K_0. We extend then each ρ_i inductively to K_j ($j = 1,2,\ldots$); since $K = \bigcup_{j=1}^{\infty} K_j$, we obtain eventually that every valuation ρ_i ($i \in I$) is extended to K.

COROLLARY 2. *Let C be a countable commutative domain with a unit, K be its subfield, $f[X] = X^n + \lambda_1 X^{n-1} + \cdots + \lambda_n$ be a polynomial with coefficients from K ($n \geq 1$). Assume that at least one of the conditions* (1) *or* (2) *of Lemma* 12 *hold. Then every valuation ρ_i ($i \in I$) on $\Pi(\lambda_1, \lambda_2, \ldots, \lambda_n)$ which was constructed in Lemma* 10 *and* 11 *is extended to a valuation on C.*

Proof. Replace C by its field of fractions F. The assertion now follows from Corollary 1.

THEOREM 4. *Let D be a countable (skew) field with a discrete quasiabelian valuation $v(d)$, $T = \{d \in D \mid v(d) \geq 0\}$, $J(T) = \{d \in D \mid v(d) > 0\}$, and $\bar{T} = T/J(T)$ be the residue field of $v(d)$. Assume that there exists a central element t such that $v(t) = 1$. Let K be a commutative subfield of D, $f[X] = X^n + \lambda_1 X^{n-1} + \cdots + \lambda_n$ be a polynomial with coefficients $\lambda_\alpha \in K$ ($\alpha = 1,2,\ldots,n$), where $v(\lambda_\alpha) \geq 0$ ($\alpha = 1,2,\ldots,n-1$); $v(\lambda_n) = 0$. Assume that the valuation $v(d)$ is quasiabelian and that at least one of the following two conditions hold*

(I) char $D = 0$

(II) *At least one of the images $\bar{\lambda}_\alpha$ of the elements λ_α ($\alpha = 1,2,\ldots,n$) in the quotient field $\bar{T} = T/J(T)$ is transcendental over the prime field Π. Then there exists in D an infinite system of quasiabelian valuations $\Phi_i(x)$ ($i \in I$) such that*

(1) *For every $i \in I$ $\Phi_i(\lambda_\alpha) \geq 0$ ($\alpha = 1,2,\ldots,n$) and $\Phi_i(\lambda_\alpha) = 0$ if $v(\lambda_\alpha) = 0$; in particular, $\Phi(\lambda_n) = 0$.*

(2) *For every $i \in I$ the restriction $\Psi_i(x)$ of $\Phi_i(x)$ on the field $R = \Pi(t, \lambda_1, \lambda_2, \ldots, \lambda_n)$ is quasihenselian.*

(3) *$f[X]$ has a root in the residue field of every $\Psi_i(x)$.*

Proof. Let $T_1 = T \cap R$ and $\bar{R} = T_1/J(T_1)$; clearly, $\bar{R}$ is a subfield of gr(D). Let $\overline{f[X]}$ be the image of $f[X]$ under the natural homomorphism $T_1 \to \bar{R}$. The ring gr(D) is countable and commutative, we take the infinite system of valuations of $\bar{R}$ which was constructed in Lemmas 10 and 11 and then apply Corollary 2 of Lemma 12 to extend them to a system of valuations $\rho_i(s)$ $(i \in I)$ on the ring gr(D) such that for each of them the coefficients of $\overline{f[X]}$ belong to the valuation ring V_i of $\rho_i(s)$ and $\overline{f[X]}$ has a root in the residue field of $\rho_i(s)$. We apply now Lemma 5 and (4.1) and construct from each $\rho_i(s)$ a quasiabelian valuation $\Phi_i(x)$ of D. The restriction of $\Phi_i(x)$ on R, which we denote by $\Psi_i(x)$, is composed from two quasihenselian valuations: the restriction on R of the discrete valuation $v(d)$ and the valuation $\rho_i(s)$ on $\bar{R}$. Proposition 5 implies that every $\Psi_i(x)$ is quasihenselian. Finally, for a given $i \in I$ let $U_i = \{x \in R \mid \Psi_i(x) \geq 0\}$. Then as in Lemma 6 $U_i/J(U_i) \simeq V_i/J(V_i)$ and hence $f[X]$ has a root in $U_i/J(U_i)$. This completes the proof.

Remark. The condition of countability in Theorem 4 can be removed by applying some additional argument based on ultraproduct machinery. Since this theorem is a technical result and its current version is sufficient for the proofs of our main results, Theorems 5 and 6, we prefer to leave it in its present form.

5

We prove in this section Theorems 5 and 6; our proofs will be based on Theorem 4.

Let D be an arbitrary field with center Z, $D[t]$ be the polynomial ring, $D[t]$ be its field of fractions, i.e., the field of rational functions over D, and let Z_1 be the center of $D(t)$. We don't know whether or not $Z_1 = Z(t)$ (it seems reasonable to believe that this is so), but the following weaker fact will be sufficient for our further arguments.

LEMMA 13. *Let $a \in D$ be an element algebraic over Z, $f[X]$ be its minimal polynomial over Z. Then $f[X]$ is also the minimal polynomial of a over Z_1.*

Proof. We consider the field of Laurent power series $D[[t]]$. Clearly, $D(t) \subseteq D[[t]]$ and a straightforward argument shows that the center of $D[[t]]$ coincides with the power series field $Z[[t]]$; since Z_1 centralizes D we obtain that $Z_1 \subseteq Z[[t]]$. On the other hand, the polynomial $f[X]$ is irreducible over Z and it follows easily that it must be irreducible over $Z[[t]]$; hence, $f[X]$ is irreducible over Z_1 and the assertion follows.

THEOREM 5. *Let D be a (skew) field of characteristic zero. Assume that there exists a discrete quasiabelian valuation $v(d)$ of D such that $v(q) = 0$ for every rational number $q \neq 0$. Then the center Z of D is algebraically closed in D; i.e., if an element $a \in D$ is algebraic over Z then $a \in Z$.*

Proof. We observe first that we can assume that D is countable. Indeed, let D_1 be an arbitrary finitely generated subfield of D, containing a. If we prove that the condition of the theorem imply that a belongs to the center of D_1 then a must commute with all elements of D; i.e., $a \in Z$. We show then that we can assume without loss of generality that $Z(a)$ is unramified over Z. In fact, consider the field of rational functions $D(t)$. Lemmas 2 and 3 imply that we can extend $v(d)$ to a discrete quasiabelian valuation of $D(t)$. Let Z_1 be the center of $D(t)$. The subfield $Z_1(a)$ is algebraic and unramified over Z_1 since Z_1 contains an element t with $v(t) = 1$. If we assume that the assertion has already been proven for the unramified extensions then $a \in Z_1$ and hence $a \in Z_1 \cap D = Z$. We proved therefore that we can assume that $Z(a)$ is unramified over Z.

Taking, if necessary the element a^{-1} we can deal with the case when $v(a) \geq 0$. Furthermore, if $v(a) > 0$ then $v(1 + a) = 0$ and we can assume in fact that $v(a) = 0$. Let $f[X]$ be the minimal polynomial of a over Z; we can assume that $f[X]$ is not linear. Let $\tilde{D}$ and $\tilde{Z}$ be the completions of D and Z respectively. Since $v(d)$ is discrete and $Z(a)$ is unramified over Z we conclude that $f[X]$ is also the minimal polynomial of a over $\tilde{Z}$. This, together with the fact that $v(a) = 0$ implies that $f[X]$ has a form

$$f[X] = X^n + \lambda_1 X^{n-1} + \cdots + \lambda_n$$
$$\times (n > 1;\ \lambda_\alpha \in Z(\alpha = 1, 2, \ldots, n);$$
$$v(\lambda_\alpha) \geq 0(\alpha = 1, 2, \ldots, n-1);\ v(\lambda_n) = 0) \quad (5.1)$$

Let K be the subfield of Z generated by the elements $t, \lambda_1, \lambda_2, \ldots, \lambda_n$. We apply Theorem 4 (in fact, a special case of it when $K = R$ and is in the center of D) and find a quasiabelian valuation $\Phi(x)$ of D such that its restriction $\Psi(x)$ on K is quasihenselian, $\Psi(\lambda_\alpha) \geq 0$ $(\alpha = 1, 2, \ldots, n)$ and $f[X]$ has a root in the residue field of $\Psi(x)$.

Let $\hat{D}$ and $\hat{Z}$ be the completions of D and Z, respectively, with respect to the valuation $\Phi(x)$ and $\hat{K}$ be the completion of K with respect to $\Psi(x)$; clearly, $\hat{D}$, $\hat{K}$ and $\hat{Z}$ are fields and $\hat{Z} \supseteq \hat{K}$. Let D_1 be the subring of $\hat{D}$ generated by $\hat{Z}$ and D; clearly, D_1 is a homomorphic image of $\hat{Z} \otimes D$. But $\hat{Z} \otimes D$ is a central simple algebra; hence, $D_1 \simeq \hat{Z} \otimes D$. This implies in particular that $f(X)$ is the minimal polynomial of a over $\hat{Z}$ and hence over $\hat{K}$.

On the other hand, $\Psi(x)$ is quasihenselian and $f[X]$ has a root in the residue field of $\Psi(x)$ which has characteristic zero. This implies that $f[X]$

has a root in $\hat{K}$. This contradiction shows that in fact $a \in Z$ and the assertion follows.

COROLLARY. *Let D be a skew field of characteristic zero with a quasiabelian valuation $v(d)$ such that $v(d) = 0$ for every non-zero element $d \in Q$. If Δ is a subfield of D finite dimensional over its center than Δ is commutative.*

Proof. We can consider if necessary the subfield Δ_1, generated by Δ and Z; this subfield is finite dimensional over its center $Z_1 \supseteq Z$. If $(\Delta_1 : Z_1) = m^2$ then Δ_1 contains a maximal commutative subfield $S \supseteq Z_1$ with $\dim(S : Z_1) = m$. Hence by Theorem 5 $S = Z_1$ and $\Delta = Z_1$.

We consider now the second case when the residue field of the valuation has a finite characteristic. In particular, this holds in the case when $\operatorname{char} D = p$. Let $v(d)$ be discrete quasiabelian valuation in, $T = \{d \in D \mid v(d) \geq 0\}$, $J(T) = \{d \in D \mid v(d) > 0\}$. Let once again Z be the center of D.

THEOREM 6. *Let $v(d)$ be a discrete quasiabelian valuation in D, such that the residue field $\overline{T} = T/J(T)$ has a finite characteristic p. Let $E \supseteq Z$ be a commutative finite dimensional subfield. Then* $\dim(E : Z)$ *is a power of p.*

Proof. We have $E \simeq E_1 \otimes E_2$ where E_1 is separable over Z and E_2 is purely inseparable. Since the dimension of E_2 is a power of p we can consider only the case when $E = E_1$ is separable. In this case there exists an element θ such that $E = Z(\theta)$. Once again, as in Theorem 5, we can assume that $v(\theta) = 0$. The theorem will follow immediately if we prove the following fact.

PROPOSITION 6. *Let θ be a non-central element algebraic over Z; assume that $v(\theta) = 0$. Then the minimal polynomial of θ over Z has a form*

$$f[X] = X^{p^m} + z + f_1[X], \tag{5.2}$$

where $m \geq 1$, $z \in Z$ is an element such that $v(z) = 0$ and the image $\bar{z}$ of z in $T/J(T)$ is transcendental over the prime subfield Z_p (and hence z is transcendental over the prime subfield of D), $f_1[X]$ is a polynomial with coefficients from $J(T)$, i.e.,

$$f[X] \equiv X^{p^m} + z (\operatorname{mod} J(T)) \tag{5.3}$$

and the degree of $f_1[X]$ is less than n.

Proof. Once again, as in the proof of Theorem 5, an easy argument implies that we can assume that D is countable. We reduce first the proof to the case when $Z(\theta)$ is unramified over Z. Once again as in the proof of Theorem 5 consider the field of rational functions $D(t)$ and extend the valuation $v(d)$ to this field by (3.2). Since the center Z_1 of $D(t)$ contains the element t we see that $Z_1(\theta)$ is unramified over Z_1; however, Lemma 13 implies that $f[X]$ is the minimal polynomial of θ over Z_1. Let $T_1 = \{d \in D(t) \mid v(d) \geq 0\}$, $J(T_1) = \{d \in D(t) \mid v(d) > 0\}$. Since $T_1/J(T_1) \simeq T/J(T)$ we see that if the assertion is proven for unramified extensions, and in particular for the subfield $Z_1(a) \supseteq Z_1$ of $D(t)$, then (5.2) will follow. We showed that we can assume that $Z(\theta)$ is unramified over Z.

We assume from this moment that $Z(\theta)$ is unramified over Z. Since $v(x)$ is discrete and $v(\theta) = 0$ the minimal polynomial of θ has the form

$$f[X] = X^n + \lambda_1 X^{n-1} + \cdots + \lambda_n, \tag{5.4}$$

where $v(\lambda_\alpha) \geq 0$ ($\alpha = 1, 2, \ldots, n$), $v(\lambda_n) = 0$. Let for every $\alpha = 1, 2, \ldots, n$ $\bar{\lambda}_\alpha$ denote the image of λ_α in the residue field of $\bar{T} = T/J(T)$. We will give the proof in two steps.

Step 1. Prove first that at least one of the elements $\bar{\lambda}_\alpha$ ($\alpha = 1, 2, \ldots, n$) must be transcendental over the prime subfield $Z_p \subseteq \bar{T}$. Indeed, assume that all these elements are algebraic over the prime subfield $Z_p \subseteq T$. We will show that this assumption leads to a contradiction.

Indeed, under this assumption the element $\bar{\theta}$, the image of θ in $\bar{T}$, must be algebraic over Z_p. This implies that $\bar{\theta}^k = 1$ for some k. We consider once again the fields $\tilde{D}$ and $\tilde{Z}$, the completions of D and Z respectively. Since $v(d)$ is discrete we obtain by Hensel's Lemma that there exists $u \in \tilde{Z}(\theta)$ such that $u^k = 1$. Let D_1 be the subring of $\tilde{D}$, generated by $\tilde{Z}$ and D; like in the proof of Theorem 5 we obtain $D_1 \simeq \tilde{Z} \otimes D$. Since $u \in \tilde{Z}(\theta)$ and $\theta \in D$ we obtain that $u \in D_1$; furthermore, $u \notin \tilde{Z}$, i.e., u is non-central in D_1, which implies that u is non-central in $\tilde{D}$. Coro. 2 of Th. VII.4.1 in [6] implies that we can find an element $v \in \tilde{D}$ such that $v^{-1}uv = u^{-1}$ and hence the element $u^{-2} = v^{-1}uvu^{-1}$ is in the commutator subgroup of the group $\tilde{D}^*$. Since u^{-2} has a finite order we conclude from Corollary 2 of Theorem 3 that the order of u^{-2} must be a power of p. But we have in fact $(u^{-2})^k = 1$, $(k, p) = 1$; hence, $u^{-2} = 1$, i.e., $u = \pm 1$, which contradicts the relation $u \notin \tilde{Z}$. This contradiction shows that at least one of the coefficients $\bar{\lambda}_\alpha$ ($\alpha = 1, 2, \ldots, n$) must be transcendental over Z_p.

Step 2. We obtained that at least one of the coefficients $\bar{\lambda}_\alpha$ ($\alpha = 1, 2, \ldots, n$) is transcendental over the prime subfield Z_p. We will prove that this implies that the minimal polynomial of $\bar{\theta}$ over $\bar{Z}$ has a form $\overline{f[X]} = X^{p^m} + \bar{z}$ ($\bar{z} \in \bar{Z}$, $m \geq 1$) (and $p^m = n$ by (5.4)). Since $Z(\theta)$ is unramified over Z this is equivalent to the relations (5.3) and (5.2); furthermore, since at least one of the elements $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n$ must be transcendental this will imply that $\bar{z} = \bar{\lambda}_n$ is transcendental over Z_p; the proof therefore will be complete.

We assume that $\overline{f[X]} \neq X^{p^m} + \bar{z}$, (where $p^m = n$), i.e., there exists an element $\bar{\lambda}_k \neq 0$ $(k < n)$. Once again as in the proof of Theorem 5, let K be the subfield of Z generated by $t, \lambda_1, \lambda_2, \ldots, \lambda_n$; let $\Phi(x)$ be a quasiabelian valuation of D such that its restriction $\Psi(x)$ on K is quasihenselian, $\Psi(\lambda_\alpha) \geq 0$ $(\alpha = 1, 2, \ldots, n)$, $f[X]$ has a root α in the residue field of $\Psi(x)$ and in addition $\Psi(\lambda_k) = 0$. We denote by $\bar{\bar{K}}$ the residue field of $\Psi(x)$, by $\overline{\overline{f[X]}}$ the image of the polynomial $f[X]$ in $\bar{\bar{K}}$, and by $\bar{\bar{\lambda}}_k$ the image of λ_k. Since $\bar{\bar{\lambda}}_k \neq 0$ $(k < n)$ and the degree of $\overline{\overline{f[X]}}$ is n we conclude that the multiplicity of the root α is less than n; i.e., there exists a monic polynomial $\overline{\overline{g[X]}} \in \bar{\bar{K}}[X]$ such that

$$\overline{\overline{f[X]}} = (X - \alpha)^r \overline{\overline{g(X)}}, (r < n), \tag{5.5}$$

where $X - \alpha$ is relatively prime with $\overline{\overline{g[X]}}$.

Now let as in the proof of Theorem 5 $\hat{D}$ and $\hat{Z}$ be the completions of D and Z, respectively, with respect to $\Phi(x)$, and $\hat{K}$ be the completion of K with respect to $\Psi(x)$. Let $\hat{U} = \{x \in K \mid \Psi(x) \geq 0\}$. Since $\Psi(x)$ is quasihenselian we conclude from (5.5) that there exist two monic polynomials $\varphi[X], g[X] \in \hat{U}[X]$ such that

$$f[X] = \varphi[X]g[X] \tag{5.6}$$

and $\varphi[X] \equiv (X - \alpha)^r (\operatorname{mod} J(\hat{U}))$, $g[X] \equiv \overline{\overline{g[X]}} (\operatorname{mod} J(\hat{U}))$. But once again, as in the proof of Theorem 5, $f[X]$ must be irreducible over $\hat{K}$. This contradiction shows that in fact $\overline{f[X]} = X^{p^m} + \bar{z}$, and the proof is completed.

COROLLARY. *Let D be as in Theorem 4 and Δ be a (skew) subfield finite dimensional over its center Z_1. Then* $\dim(\Delta : Z_1)$ *is a power of p.*

The proof does not differ from the proof of the Corollary of Theorem 5.

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