MATHEMATICS

LINEAR TRANSFORMATIONS OF GENERALIZED LEGENDRE'S ASSOCIATED FUNCTIONS

BY

L. KUIPERS AND B. MEULENBELD

(Communicated by Prof. J. F. KOKSMA at the meeting of 22 February 1958)

1. *Introduction*

In the present paper we express the functions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$, two specified linearly independent solutions of the differential equation:

$$(1-z^2)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+\left\{k(k+1)-\frac{m^2}{2(1-z)}-\frac{n^2}{2(1+z)}\right\}w=0,$$

in several ways in the forms:

(I) $\quad P_k^{m,n}(z)=A_1\,F(a_1,b_1;c_1;\zeta)+A_2\,F(a_2,b_2;c_2;\zeta) \qquad |\zeta|<1,$

(II) $\quad Q_k^{m,n}(z)=A_3\,F(a_3,b_3;c_3;\zeta)+A_4\,F(a_4,b_4;c_4;\zeta) \qquad |\zeta|<1.$

ζ is a function of z, which depends on the used transformation formula. The expansions (I) and (II) are given on the following pages. The remarks in the last column in each table show the way in which the linear combination in question has been derived. The starting points for both tables are the relations (1) and (7) respectively, which have been deduced from certain integral expressions representing $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ (see [1] and [2]). The auxiliaries are listed below, from (A) to (F). Except for (D), which is Theorem 5 of [3], all these relations are well known transformation formulas of the hypergeometric function. For example, (2) results when (A) is applied to (1).

If we applied the transformation (E) to each of the expressions (1), ..., (12), then we would find 12 other hypergeometric series.

The tables of expressions (I) and (II) are not as complete as the existing corresponding tables for the associated Legendre functions (see [4], Chapter III). The reason is that the quadratic transformations cannot be applied to any of the representations (1), ..., (12). It is however possible to express our functions $P_k^{m,n}$ and $Q_k^{m,n}$ by means of infinite series of generalized hypergeometric functions. The results are not mentioned here, but will be published elsewhere.

2. *Auxiliaries*

(A)
$$\begin{cases} F(a,b;c;z) = \dfrac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\, F(a,b;a+b-c+1;1-z) + \\ \qquad + \dfrac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)}\,(1-z)^{c-a-b}\, F(c-a,c-b;\,c-a-b+1;\,1-z) \\ \qquad\qquad |\arg(1-z)| < \pi. \end{cases}$$

(B)
$$F(a,b;c;z) = (1-z)^{-a}\, F(a,c-b;c;z/(z-1)).$$

(C)
$$\begin{cases} F(a,b;c;z) = \dfrac{\Gamma(c)\,\Gamma(b-a)}{\Gamma(b)\,\Gamma(c-a)}\,(-z)^{-a}\, F(a,1-c+a;1-b+a;z^{-1}) + \\ \qquad + \dfrac{\Gamma(c)\,\Gamma(a-b)}{\Gamma(a)\,\Gamma(c-b)}\,(-z)^{-b}\, F(b,\,1-c+b;\,1-a+b;\,z^{-1}) \\ \qquad\qquad |\arg(-z)| < \pi. \end{cases}$$

(D)
$$Q_k^{m,n}(z) = -2^{n-m} e^{\pi i(\mp k+m-n)}\,\Gamma\left(k+\frac{m-n}{2}+1\right) Q_k^{n,m}(-z)/\Gamma\left(k-\frac{m-n}{2}+1\right)$$

(the upper or lower sign according as $\operatorname{Im} z \gtrless 0$).

(E)
$$F(a,b;c;z) = (1-z)^{c-a-b}\, F(c-a,c-b;c;z).$$

(F)
$$\begin{cases} F(a,b;c;z) = \dfrac{\Gamma(c)\,\Gamma(b-a)}{\Gamma(b)\,\Gamma(c-a)}\,(1-z)^{-a} F\left(a,c-b;a-b+1;\dfrac{1}{1-z}\right) + \\ \qquad + \dfrac{\Gamma(c)\,\Gamma(a-b)}{\Gamma(a)\,\Gamma(c-b)}\,(1-z)^{-b} F\left(b,c-a;b-a+1;\dfrac{1}{1-z}\right) \\ \qquad\qquad |\arg(1-z)| < \pi. \end{cases}$$

REFERENCES

1. KUIPERS, L. and B. MEULENBELD, On a generalization of Legendre's associated differential equation, I, Proc. Kon. Ned. Ak. v. Wet., Amsterdam, **60**, 4, 436–443.
2. ——— and ———, On a generalization of Legendre's associated differential equation, II, Proc. Kon. Ned. Ak. v. Wet., Amsterdam, **60**, 4, 444–450.
3. ——— and ———, Related generalized Legendre's associated functions (to be published).
4. ERDÉLYI, ARTHUR, a.o., Higher Transcendental Functions, I, McGraw–Hill (1953).

	A_1 / A_2		ζ	a_1 / a_2	b_1 / b_2	c_1 / c_2	Remarks
(1)	$(z+1)^{\frac{n}{2}}(z-1)^{-\frac{m}{2}}/\Gamma(1-m)$	(1)	$\frac{1-z}{2}$	$-k-\frac{m-n}{2}$	$k-\frac{m-n}{2}+1$	$1-m$	[1], (11)
	0			....	....	....	
(2)	$\Gamma(-n)\,(z+1)^{\frac{n}{2}}(z-1)^{-\frac{m}{2}}\Big/\Big\{\Gamma\Big(k-\frac{m+n}{2}+1\Big)\Gamma\Big(-k-\frac{m+n}{2}\Big)\Big\}$	(2)	$\frac{1+z}{2}$	$-k-\frac{m-n}{2}$	$k-\frac{m-n}{2}+1$	$1+n$	(1), (A) The upper or lower sign according as Im $z \gtrless 0$.
	$\Gamma(n)\,(z+1)^{-\frac{n}{2}}(z-1)^{\frac{m}{2}}e^{\mp m\pi i}\,2^{-m+n}\Big/\Big\{\Gamma\Big(k-\frac{m-n}{2}+1\Big)\Gamma\Big(-k-\frac{m-n}{2}\Big)\Big\}$			$-k+\frac{m-n}{2}$	$k+\frac{m-n}{2}+1$	$1-n$	
(3)	$2^{-k-\frac{m-n}{2}}(z+1)^{k+\frac{m}{2}}(z-1)^{-\frac{m}{2}}/\Gamma(1-m)$	(3)	$\frac{z-1}{z+1}$	$-k-\frac{m-n}{2}$	$-k-\frac{m+n}{2}$	$1-m$	(1), (B)
	0			....	....	....	
(4)	$-2^{k-\frac{m-n}{2}+1}\Gamma(-n)e^{\pm\pi i\left(k-\frac{m-n}{2}\right)}(z+1)^{\frac{n}{2}}(z-1)^{-k-\frac{n}{2}-1}\Big/\Big\{\Gamma\Big(k-\frac{m+n}{2}+1\Big)\Gamma\Big(-k-\frac{m+n}{2}\Big)\Big\}$	(4)	$\frac{z+1}{z-1}$	$k-\frac{m-n}{2}+1$	$k+\frac{m+n}{2}+1$	$1+n$	(2), (B) The upper or lower sign according as Im $z \gtrless 0$.
	$-2^{k-\frac{m-n}{2}+1}\Gamma(n)e^{\pm\pi i\left(k-\frac{m+n}{2}\right)}(z-1)^{-k+\frac{n}{2}-1}(z+1)^{-\frac{n}{2}}\Big/\Big\{\Gamma\Big(k-\frac{m-n}{2}+1\Big)\Gamma\Big(-k-\frac{m-n}{2}\Big)\Big\}$			$k+\frac{m-n}{2}+1$	$k-\frac{m+n}{2}+1$	$1-n$	
(5)	$2^{k-\frac{m-n}{2}+1}\Gamma(-2k-1)(z+1)^{-k+\frac{m}{2}-1}(z-1)^{-\frac{m}{2}}\Big/\Big\{\Gamma\Big(-k-\frac{m-n}{2}\Big)\Gamma\Big(-k-\frac{m+n}{2}\Big)\Big\}$	(5)	$\frac{2}{1+z}$	$k-\frac{m+n}{2}+1$	$k-\frac{m-n}{2}+1$	$2k+2$	(3), (A)
	$2^{-k-\frac{m-n}{2}}\Gamma(2k+1)(z+1)^{k+\frac{m}{2}}(z-1)^{-\frac{m}{2}}\Big/\Big\{\Gamma\Big(k-\frac{m+n}{2}+1\Big)\Gamma\Big(k-\frac{m-n}{2}+1\Big)\Big\}$			$-k-\frac{m-n}{2}$	$-k-\frac{m+n}{2}$	$-2k$	
(6)	$2^{-k-\frac{m-n}{2}}\Gamma(2k+1)(z+1)^{\frac{n}{2}}(z-1)^{k-\frac{n}{2}}\Big/\Big\{\Gamma\Big(k-\frac{m-n}{2}+1\Big)\Gamma\Big(k-\frac{m+n}{2}+1\Big)\Big\}$	(6)	$\frac{2}{1-z}$	$-k-\frac{m-n}{2}$	$-k+\frac{m+n}{2}$	$-2k$	(1), (C)
	$2^{k-\frac{m-n}{2}+1}\Gamma(-2k-1)(z+1)^{\frac{n}{2}}(z-1)^{-k-\frac{n}{2}-1}\Big/\Big\{\Gamma\Big(-k-\frac{m-n}{2}\Big)\Gamma\Big(-k-\frac{m+n}{2}\Big)\Big\}$			$k-\frac{m-n}{2}+1$	$k+\frac{m+n}{2}+1$	$2k+2$	

	A_3			a_3	b_3	c_3	Remarks
	A_4			a_4	b_4	c_4	
(7)	$e^{\pi i m}\, 2^{k-\frac{m-n}{2}}\, \Gamma\left(k+\frac{m+n}{2}+1\right)\Gamma\left(k+\frac{m-n}{2}+1\right)(z+1)^{\frac{n}{2}}(z-1)^{-k-\frac{n}{2}-1}/\Gamma(2k+2)$	(7)	$\frac{2}{1-z}$	$k+\frac{m+n}{2}+1$	$k-\frac{m-n}{2}+1$	$2k+2$	[2], (8)
	0			....	....	....	
(8)	$\dfrac{e^{\pi i m}\, 2^{-m+n-1}\, \Gamma\left(k+\frac{m+n}{2}+1\right)\Gamma\left(k+\frac{m-n}{2}+1\right)\Gamma(-m)\,(z+1)^{-\frac{n}{2}}(z-1)^{\frac{m}{2}}}{\Gamma\left(k-\frac{m-n}{2}+1\right)\Gamma\left(k-\frac{m+n}{2}+1\right)}$	(8)	$\frac{1-z}{2}$	$-k+\frac{m-n}{2}$	$k+\frac{m-n}{2}+1$	$1+m$	(7), (C)
	$\frac{1}{2} e^{\pi i m}\, \Gamma(m)\,(z+1)^{\frac{n}{2}}(z-1)^{-\frac{m}{2}}$			$-k-\frac{m-n}{2}$	$k-\frac{m-n}{2}+1$	$1-m$	
(9)	$e^{\pi i m}\, 2^{k-\frac{m-n}{2}}\, \Gamma\left(k+\frac{m-n}{2}+1\right)\Gamma\left(k+\frac{m+n}{2}+1\right)(z+1)^{-k+\frac{m}{2}-1}(z-1)^{-\frac{m}{2}}/\Gamma(2k+2)$	(9)	$\frac{2}{1+z}$	$k-\frac{m-n}{2}+1$	$k-\frac{m+n}{2}+1$	$2k+2$	(7), (D), (E)
	0			....	....	....	
(10)	$e^{\pi i m}\, 2^{-k-\frac{m-n}{2}-1}\, \Gamma(m)\,(z+1)^{k+\frac{m}{2}}(z-1)^{-\frac{m}{2}}$	(10)	$\frac{z-1}{z+1}$	$-k-\frac{m-n}{2}$	$-k-\frac{m+n}{2}$	$1-m$	(9), (A), (E)
	$\dfrac{e^{\pi i m}\, 2^{-k-\frac{m-n}{2}-1}\, \Gamma(-m)\,\Gamma\left(k+\frac{m-n}{2}+1\right)\Gamma\left(k+\frac{m+n}{2}+1\right)(z+1)^{k-\frac{m}{2}}(z-1)^{\frac{m}{2}}}{\Gamma\left(k-\frac{m-n}{2}+1\right)\Gamma\left(k-\frac{m+n}{2}+1\right)}$			$-k-\frac{m-n}{2}$	$-k-\frac{m+n}{2}$	$1+m$	
(11)	$\dfrac{-e^{\pi i\left(\mp k+m\pm\frac{m-n}{2}\right)}\,\Gamma\left(k+\frac{m+n}{2}+1\right)\Gamma(-n)\,(z+1)^{\frac{n}{2}}(z-1)^{-\frac{m}{2}}}{2\,\Gamma\left(k-\frac{m+n}{2}+1\right)}$	(11)	$\frac{1+z}{2}$	$-k-\frac{m-n}{2}$	$k-\frac{m-n}{2}+1$	$1+n$	(8), (D)
	$\dfrac{-e^{\pi i\left(\mp k+n\pm\frac{m-n}{2}\right)}\, 2^{n-m-1}\,\Gamma\left(k+\frac{m-n}{2}+1\right)\Gamma(n)\,(z+1)^{-\frac{n}{2}}(z-1)^{\frac{m}{2}}}{\Gamma\left(k-\frac{m-n}{2}+1\right)}$			$-k+\frac{m-n}{2}$	$k+\frac{m-n}{2}+1$	$1-n$	
(12)	$e^{\pi i m}\, 2^{k-\frac{m-n}{2}}\, \Gamma\left(k+\frac{m+n}{2}+1\right)\Gamma(-n)\,(z+1)^{\frac{n}{2}}(z-1)^{-k-\frac{n}{2}-1}\Big/\Gamma\left(k-\frac{m+n}{2}+1\right)$	(12)	$\frac{z+1}{z-1}$	$k-\frac{m-n}{2}+1$	$k+\frac{m+n}{2}+1$	$1+n$	(9), (F)
	$e^{\pi i m}\, 2^{k-\frac{m-n}{2}}\, \Gamma\left(k+\frac{m-n}{2}+1\right)\Gamma(n)\,(z+1)^{-\frac{n}{2}}(z-1)^{-k+\frac{n}{2}-1}\Big/\Gamma\left(k-\frac{m-n}{2}+1\right)$			$k+\frac{m-n}{2}+1$	$k-\frac{m+n}{2}+1$	$1-n$	